

HELPING CHILDREN LEARN

mathematics

2ND AUSTRALIAN EDITION

REYS | LINDQUIST | LAMBDIN | SMITH
ROGERS | COOKE | EWING | ROBSON | BENNETT

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Helping children learn mathematics

SECOND EDITION

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WILEY

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Kylie Robson leads numeracy instruction at Charles Weston School Coombs in the ACT as a Curriculum Focused Executive Teacher. Kylie works with educators to design, implement and evaluate high-quality

mathematics programs and provides professional support to individual teachers. Initially a primary school teacher with a passion for mathematics, she has taught students from Preschool to Year 6 and been an educator for over 15 years. Prior to this, she held the position of Clinical Teaching Specialist at the University of Canberra, where she taught a series of Mathematics Education units to undergraduate students as part of the Bachelor of Education degree. Through this role, her pedagogical content knowledge combined with her understanding of the complexities of teaching provided students with a unique perspective into Mathematics education. Kylie's research interests and publications include the use of quality literature in the mathematics classroom, embedding meaningful discourse into mathematics teaching and the impact of teacher efficacy on student performance.

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Bronwyn Ewing is Senior Lecturer in the School of Curriculum at QUT. Bronwyn graduated with a PhD in the area of early school leavers' experiences of mathematics education in 2009. She takes a transdisciplinary approach to research to integrate disciplines and address the teaching and learning of mathematics to students and adults from low SES background, Indigenous students, students with disability and students in youth detention. Before commencing her academic career, Bronwyn was a primary school teacher, fulfilling a number of extra curriculum roles.

PREFACE

Welcome to the second Australian edition of *Helping Children Learn Mathematics*. This text reflects the ever-changing world of learning and teaching in early childhood to primary level mathematics. Teachers of mathematics not only experience change but, more importantly, have an opportunity to lead the way. This text empowers pre-service teachers to successfully navigate the Australian Curriculum and NAPLAN testing environment with practical, engaging strategies to help children learn mathematics in tomorrow's classroom.

Helping Children Learn Mathematics is built around three main themes:

1. helping children make sense of mathematics
2. incorporating practical experiences and research to guide teaching
3. emphasising major ideas discussed in detail in the Australian Curriculum.

Helping Children Learn Mathematics, 2nd Australian edition, is designed to empower you to help children learn mathematical concepts and skills, as well as important problem-solving strategies. In the process, it will challenge your thinking and, we hope, will further stimulate your interest in learning and teaching mathematics.

The textbook consists of two main parts.

The first part (chapters 1–6) provides a basis for understanding the changing mathematics curriculum in Australia and how children learn mathematics. It offers some guidelines for planning and evaluating instruction. It directs attention to problem solving and assessment, both of which have profound implications for mathematics teaching at all levels, the importance of which is reflected by their integration throughout the book.

The second part (chapters 7–18) discusses teaching strategies and techniques, as well as learning activities, related to specific mathematical topics. The emphasis is on using models and materials to develop concepts and understanding, so that mathematics learning is indeed meaningful.

We believe that meaning is most effectively established by helping students discuss mathematics as they move from concrete materials and examples to generalisations and abstractions.

Anna Rogers
Sue Bennett
Audrey Cooke
Kylie Robson
Bronwyn Ewing
August, 2016

FEATURES OF THIS TEXT

We revised this resource in order to meet the changing needs of students preparing to become teachers, or teachers who are continuing to learn. This edition continues the rich tradition of this book with contemporary ideas interspersed. It has been updated to reflect current recommendations from the Australian Curriculum as well as recent research findings relevant to teaching mathematics, while maintaining the characteristics and features that made the US version of the text such a popular choice for education academics. A useable text for instructors, it is also readable and understandable by students who are being introduced to teaching elementary mathematics. Yet, its depth also makes it appropriate for teachers to use as they continue to learn about teaching mathematics.

Snapshot of a lesson features are based on local videos illustrating best practice in mathematics education. The chapter-opener snapshots are excerpts from actual mathematics lessons in Australian classrooms to show the vital role teachers play in helping students talk and learn about mathematics. Featuring student–teacher and student–student dialogue, they cover a wide range of age groups and mathematical topics, demonstrate many effective classroom practices, and provide a smooth, practical segue into the body of the chapter. In addition to strategies woven into the text, the following two sections will help you practise translating theory into practice as you guide and teach children in their study of mathematics.

A wealth of real-world examples from the Australian Curriculum provide a diverse coverage of the most up-to-date issues in the contemporary Australian mathematics classroom.

Concept maps provide clear visual overviews of the content explored in each chapter.

In the classroom activities provide a wealth of ideas and strategies for teaching children the topics of each chapter.

Spotlight on early childhood focuses on the specific teaching and learning needs of early childhood students, providing a comprehensive application of mathematical concepts in real-world classroom scenarios for this critical stage of learning.

Making connections provides a broad scope of connections for pre-service teachers, including a platform for students to explore important cross-cultural issues in the mathematics classroom, as well as coverage of essential material relating to the Australian Curriculum such as proficiency standards, diversity and general capabilities.

A glance at where we've been provides a concise summary of suggested responses to the focus questions posed at the beginning of each chapter.

Things to do embody the text's active learning and teaching approach to mathematics. Divided into two parts, they are designed to engage you in inquiring and thinking about mathematics — to offer you experiences and introduce you to investigations that will help you achieve the understanding and insight you need to be a successful teacher.

Children's literature connections include relevant texts referenced at various points within the text. An annotated list of useful books and websites is provided at the end of each chapter.

ABOUT THIS TEXT

Helping Children Learn Mathematics, 2nd Australian edition, is an ideas text. We believe that you will learn much from reading it and from talking about what you have read. The Australian Curriculum highlights communication as an important part of mathematics learning, and this text is designed to encourage and facilitate communication.

It is not possible — or desirable — to specify the exact steps to follow in teaching mathematics. Too much depends on what is being taught, to whom, and at what levels. In your classroom, it is you who will ultimately decide what to teach, to whom to teach it, how to teach it, and the amount of time to spend. This text will not answer all of these questions for you, but we think you will find it very helpful in making wise decisions as you guide your students in their learning of mathematics. We believe this text will be a valuable teaching resource that you can use again and again in your classroom long after your course has been completed.

CHAPTER 1

School mathematics in a changing world

LEARNING OUTCOMES

1.1 Connecting your disposition towards mathematics with your mathematical experiences.

1.2 Recognising factors that determine the mathematics that is currently taught, and the role, rationale and aims of the Australian Curriculum in contemporary mathematics teaching and learning.

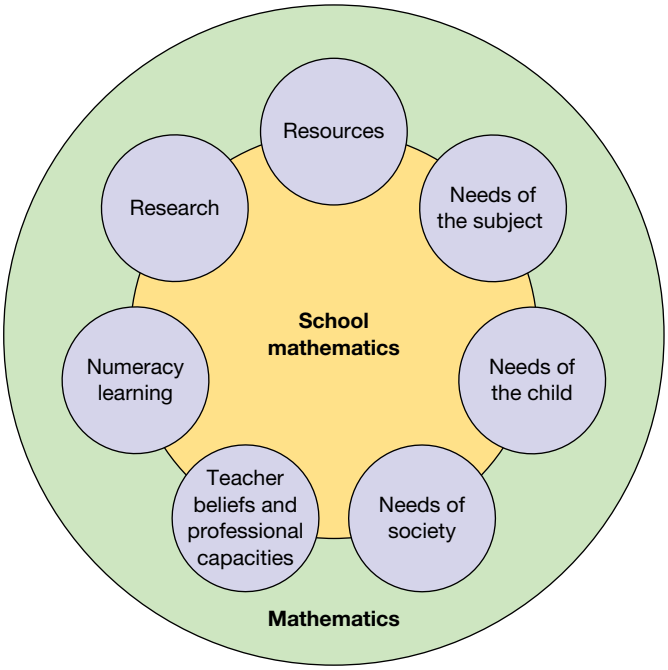
1.3 Evaluating resources to determine those of use in your continuing development of mathematics, mathematical learning and teaching mathematics.



‘Education is the most powerful weapon which you can use to change the world.’

Nelson Mandela

Chapter 1 concept map



Introduction

Mathematics remains an essential tool for understanding the world, so mathematics education needs to keep pace with the advancements that our society makes at an ever-increasing rate. Taken from a mathematics learning and teaching perspective, the words of Nelson Mandela in the opening quote are very relevant. As a teacher of mathematics, you have the opportunity to provide a positive and meaningful experience for those you help to learn mathematics.

What is your vision of the mathematics you will be teaching? What is your vision of the classroom? Many of you will remember your experiences in primary school. Some of you will remember memorising multiplication tables, operating with fractions or doing long division. Others may remember exploring patterns, doing geometry projects or solving problems. Mathematics in primary school may have been a positive experience for some of you, but for others it was filled with anxiety and frustration. Why were you learning mathematics and when would you use it? Teachers want students to learn mathematics and to recognise that it is a useful subject. How can they make sure this happens?

This text is designed to expand your vision of mathematics teaching and learning and to help you help students learn mathematics. The text interweaves 3 main themes.

- *Theme 1 — recommendations from national professional organisations and from research on learning and teaching mathematics.* These recommendations and research provide a basis for you to understand what mathematics children are expected to learn and how children learn mathematics.
- *Theme 2 — sense making*. Mathematics must make sense to children. If children make sense of the mathematics they are learning, they can build on this understanding to learn more mathematics and use the mathematics to solve problems.
- *Theme 3 — practical suggestions*. Learning to teach mathematics requires experience. This theme is explicated by including many suggestions from teachers and our own experiences as well as many ideas for you to use as you learn to teach and later when you become a teacher.

Learning to teach is a lifelong journey. During that journey, you will often ask questions such as those below.

- What mathematical knowledge and understandings does each student bring to the class?
- What mathematics do students need to learn?
- How can I teach each unique child so that he or she will learn?
- What is the influence of my own attitude toward mathematics?

Your answers to these questions will influence what you do when you are teaching. No matter what the age of the children you teach, you will probably have several general *goals* — for example:

- to help children make sense of specific mathematical content, including both procedures and concepts
- to help children learn how to apply mathematical ideas to solve problems
- to foster positive dispositions, such as persistence, flexibility, willingness to learn and an appreciation of the value of mathematics.

Developing ways to help you reach these 7 goals is considered in later chapters of this text. This first chapter focuses on what mathematics is and what determines the mathematics that is taught in schools. We also share suggestions as to where you can turn for additional help.

1.1 What is mathematics?

LEARNING OUTCOME 1.1 Connecting your disposition towards mathematics with your mathematical experiences.

Frequently, people equate mathematics with arithmetic. In school mathematics, arithmetic is about numbers. At times, arithmetic, especially computation with numbers, has been the only focus in primary school. Currently, however, this limited view of mathematics is not prevalent. Table 1.1 provides an outline of the Australian Curriculum, Assessment and Reporting Authority's (ACARA) *The Australian Curriculum: Mathematics* content strands and sub-strands.

TABLE 1.1 Australian Curriculum content strands and sub-strands for mathematics

Number and Algebra	Measurement and Geometry	Statistics and Probability
Number and place value (F–8)	Using units of measurement (F–10)	Chance (1–10)
Fractions and decimals (1–6)	Shape (F–7)	Data representation and interpretation (F–10)
Real numbers (7–10)	Geometric reasoning (3–10)	
Money and financial mathematics (1–10)	Location and transformation (F–7)	
Patterns and algebra (F–10)	Pythagoras and trigonometry (9–10)	
Linear and nonlinear relationships (7–10)		

Source: © Australian Curriculum, Assessment and Reporting Authority 2015.

The 3 strands — Number and Algebra, Measurement and Geometry, and Statistics and Probability — are each composed of sub-strands. Most of these sub-strands are emphasised in the early school years. Within the Number and Algebra strand, the focus progresses from simple to more abstract number representations and computations, and then into algebraic relationships. In the Measurement and Geometry strand, a range of basic concepts and skills are developed across the primary school years, with more

abstract and complex concepts related to reasoning and trigonometry beginning to be attended to in the middle school years. For the Statistics and Probability sub-strands, the emphasis across all of primary school through to the end of lower secondary school reflects an increased awareness of the importance of these areas of mathematics in daily life.

Although we can consider mathematics as a collection of separate areas such as geometry and algebra, this may not be the best way of looking at it. It may be helpful to broaden your view of mathematics. The 5 views below may help you think of mathematics differently than that of being a collection of subjects.

1. *Mathematics is a study of patterns and relationships*. Children need to become aware of recurring ideas and of relationships between mathematical ideas. These relationships and ideas provide a unifying thread throughout the curriculum. Children should come to see how one idea is like or unlike other ideas. For example, children in Year 1 can see how one basic fact (say, $3 + 2 = 5$) is related to another basic fact (say, $5 - 3 = 2$). Older children can relate measuring to the nearest centimetre to rounding to the nearest hundred.
2. *Mathematics is a way of thinking*. Mathematics provides people with strategies for organising, analysing and synthesising information. Often symbolising a real-life problem reduces it to a well-known mathematical procedure, making the problem easier to solve.
3. *Mathematics is an art, characterised by order and internal consistency*. Many children think of mathematics as a confusing set of discrete facts and skills that must be memorised. Teachers tend to focus on developing the skills required to 'do' mathematics, and in doing so they may forget that children need guidance to recognise and appreciate the underlying orderliness and consistency as they construct their own understanding of mathematics.
4. *Mathematics is a language that uses carefully defined terms and symbols*. Learning these terms and symbols enhances our ability to communicate about science, real-life situations and mathematics itself. Like any language, you need to understand the meaning of these words and when it is appropriate to use them.
5. *Mathematics is a tool*. Not only mathematicians use mathematics, but everyone in the course of daily life uses it. Seeing this will help children appreciate why they are learning mathematics. They, too, will be able to use mathematics to solve both abstract and practical problems, just as mathematicians and other people do. Mathematics has become an essential part of our world, both in everyday life and in the workplace.

1.2 What determines the mathematics being taught?

LEARNING OUTCOME 1.2 Recognising factors that determine the mathematics that is currently taught, and the role, rationale and aims of the Australian Curriculum in contemporary mathematics teaching and learning.

Mathematics is highly valued in society by parents, politicians and employers. However, many students do not share this view of mathematics due to a lack of interest and engagement in the subject. Often this is because students are unable to see the relevance or purpose of the subject in a real-life or meaningful way. The *National Numeracy Review Report* (Council of Australian Governments 2008, p. xii) suggests that this scenario is widely held and a risk to Australia achieving its human capital goals. Further, due to the high stakes nature of mathematics achievement as a selection and employment criterion, there is a direct relationship between the level of mathematics studied and students' aspirations and career prospects.

In December 2010, ACARA released *The Australian Curriculum: Mathematics* for implementation in all Australian schools by 2013. This marked the first time in Australia's history that there was a national approach to teaching and learning mathematics, from Foundation to Year 10, across all states and

territories. However, in their review of the Australian Curriculum, Donnelly and Wiltshire (2014) stated 'the implementation picture is extremely confusing and provides little assurance that the Australian curriculum is being implemented, as intended, across the nation' (p. 106). They proposed several reasons for this, including disagreement between ACARA and state and territory governments, with 'a strident argument from states and territories that implementation is their domain and ACARA is not an accountable body for implementation' (p. 107). Donnelly and Wiltshire did find that the Australian Curriculum documents were being incorporated or supplemented by resources developed by the government body responsible for education in that state or territory.

In England, the national curriculum must be delivered by all schools that are maintained by the local authority (Department of Education, 2014). Mathematics, together with English and Science, is considered a core subject (Department of Education, 2014). Teachers are required to 'develop pupils' numeracy and mathematical reasoning in all subjects so that they understand and appreciate the importance of mathematics' (para. 5.2).

In the United States, the National Council of Teachers of Mathematics (NCTM), the largest professional organisation of teachers of mathematics, developed standards for curriculum and for evaluation, teaching and assessment (NCTM 1989, 1991, 1995). Because states and localities in the United States have the right to determine their own school policies, these standards are not prescriptive, but they have provided vision and direction for schools in developing mathematics curricula. In Australia, the national professional body, the Australian Association of Mathematics Teachers (AAMT), has developed several statements and policies pertaining to mathematics learning and teaching. These include *Mathematical Knowledge and Understanding for Effective Participation in Australian Society* (1996a)*, Statement on the Use of Calculators and Computers for Mathematics in Australian Schools* (1996b), *Policy on Numeracy Education in Schools* (1998), *Quality Mathematics in the Middle Years Communiqué* (2006a), *Standards for Excellence in Teaching Mathematics in Australian Schools* (2006b), *Position Paper on Early Childhood Mathematics* (2006), *Position Paper on the Practice of Assessing Mathematics Learning* (2008) and the *Position Paper on Consumer and Financial Literacy in Schools* (2009).

In 2000, the NCTM published an update of the standards in a document titled *Principles and Standards for School Mathematics* (NCTM 2000). The principles represent fundamental beliefs about the characteristics of a high-quality, equitable mathematics program. The standards describe the mathematical content and mathematical processes that should be taught in school mathematics. Combined, the principles and standards present a vision for mathematics education programs in a changing world.

In 2006, the NCTM published *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics* as one possible response to the question of how to organise curriculum standards within a coherent, focused curriculum by showing how to 'build on important mathematical content and connections identified for each grade level' (NCTM 2006, p. 3). The tables of focal points and connections are reprinted in appendix B. The rationale, aims and content framework of *The Australian Curriculum: Mathematics* (ACARA 2012) align with those of the NCTM *Principles and Standards for School Mathematics* (2000). Similarly, the guidelines in the *Position Paper on the Practice of Assessing Mathematics Learning* (AAMT 2008) are consistent with those of the NCTM.

It is beneficial to consider the curriculum and standards documents developed in Australia, England, and the US as this will enable you to develop a greater understanding of the content being addressed. It will also enable you to access resources created for each of these 3 locations and to use them in your mathematics programs. As you think about mathematics programs, you should give careful attention to 3 general factors: the needs of the subject, the needs of the child and the needs of society. Consideration of these factors highlights the importance of the guiding principles or the rationale of both the Australian (see figure 1.1) and North American systems. They have many similarities in approach and design.

Content strands
- Number and Algebra
- Measurement and Geometry
- Statistics and Probability

Proficiency strands
- Understanding
- Fluency
- Problem Solving
- Reasoning

Achievement standards
- Indicate the quality of learning that students should typically demonstrate by a particular point in their schooling

Diversity of learners
- Special education needs
- English as an additional language or dialect
- Gifted and talented

General capabilities
- Literacy
- Numeracy
- Information and communication technology (ICT) capability
- Critical and creative thinking
- Ethical understanding
- Personal and social capability
- Intercultural understanding

Cross-curriculum priorities
- Aboriginal and Torres Strait Islander histories and cultures
- Asia and Australia's engagement with Asia
- Sustainability

FIGURE 1.1 Features of *The Australian Curriculum: Mathematics*

Source: © Australian Curriculum, Assessment and Reporting Authority 2015b.

Needs of the subject

The nature of mathematics helps determine what is taught and when it is taught in primary school years. For example, whole numbers are the basis for many mathematical ideas; moreover, experiences with whole numbers arise long before children come to school. Building on children's experiences with counting, the earlier years emphasise whole numbers. Work with fractions and decimals logically follows the work with whole numbers. Such seemingly natural sequences are the result of long years of curricular evolution. This process has involved much analysis of what constitutes a progression from easy to difficult, based in part on what is deemed necessary at one level for the development of ideas at later levels. Once a curriculum is in place for a long time, however, people tend to consider it the only proper sequence. Thus, omitting a topic or changing the sequence of topics often involves a struggle for acceptance. However, research shows that all students do not always learn in the sequence that has been ingrained in our curriculum. You need to be open to change so that each child's needs are met.

Mathematics is continually in flux; new mathematics is created, and new uses of mathematics are discovered. As part of this change, technology has made some mathematics obsolete and has opened the door for other mathematics. The influence of technology and the importance of a flexible approach to curriculum are emphasised in *The Australian Curriculum: Mathematics* (ACARA 2012, p. 11) as part of a student's information and communication technology (ICT) capability:

> Students develop ICT capability as they learn to use ICT effectively and appropriately to access, create and communicate information and ideas, solve problems and work collaboratively in all learning areas at school, and in their lives beyond school. ICT capability involves students in learning to make the most of the technologies available to them, adapting to new ways of doing things as technologies evolve.

You will teach at a time when technology dominates activities both in and out of school. Technology will continue to be important in teaching and learning mathematics, as long as it enhances what is being learned and how it is being taught. As you teach your classes, you should keep asking 3 questions.

1. How can I help children use technology appropriately?
2. What mathematics do children need in order to use technology wisely?
3. What mathematics is no longer necessary because of technology?

Some parents continue to be concerned about the use of calculators in learning mathematics in primary schools. A meta-analysis of 54 research studies on the use of and attitudes toward calculators (Ellington 2003) suggests that using calculators does not hinder the development of mathematical skills and that students who used calculators had better attitudes toward mathematics than those who did not. Of course, children need to learn to use calculators appropriately, as they do any other tool.

No one knows exactly what mathematics will be needed as the twenty-first century progresses, but it is clear that students will need to know how to reason mathematically and how to apply mathematical thinking to a wide range of situations. How you view mathematics will determine how you view teaching mathematics. If you view mathematics as a collection of facts to learn and procedures to practise, then you will teach that to your students. If you view mathematics as a logical body of knowledge, you will adopt teaching strategies that let you focus on guiding children to make sense of mathematics.

Children need a curriculum that does more than represent mathematics as a collection of isolated skills and fun activities. The Australian Curriculum is designed so that 'links between various components of mathematics, as well as the relationship between mathematics and other disciplines, are made clear' (ACARA 2012, p. 3).

Throughout the remaining chapters of this text, you will investigate topics that you can incorporate into the mathematics curriculum for your students.

Needs of the child

The mathematics curriculum has been influenced by beliefs about how children learn and, ultimately, about how they should be taught. Before the early years of the twentieth century, mathematics was taught to train 'mental faculties' or provide 'mental discipline'. Struggling with mathematical procedures was thought to exercise the mind (like muscles are exercised), helping children's brains work more effectively. Around the turn of the twentieth century, 'mental discipline' was replaced by *connectionism*, the belief that learning established bonds, or connections, between a stimulus and responses. This led teachers to the endless use of drills aimed at establishing important mathematical connections.

In the 1920s, the Progressive movement advocated *incidental learning*, reflecting the belief that children would learn as much arithmetic as they needed and would learn it better if it was not systematically taught. The teacher's role was to take advantage of situations as they occurred to teach arithmetic, as well as to create situations in which arithmetic would arise.

During the late 1920s, the Committee of Seven, a committee of school superintendents and principals from midwestern US cities, surveyed pupils to find out when they mastered various topics (Washburne 1931). Based on that survey, the committee recommended teaching mathematics topics according to students' mental age. For example, subtraction facts under 10 were to be taught to children with a mental age of 6 years 7 months, and facts over 10 at 7 years 8 months; subtraction with borrowing or carrying was to be taught at 8 years 9 months. The recommendations of the Committee of Seven had a strong impact on the sequencing of the curriculum in the United States and elsewhere for years afterward.

Another change in thinking occurred in the mid-1930s, under the influence of *field theory*, or *Gestalt theory*. With William A. Brownell (2006) as a prominent spokesperson, this approach placed greater emphasis on a planned program to encourage the development of insight and the understanding of relationships, structures, patterns, interpretations and principles. It contributed to an increased focus on learning as a process that led to *meaning and understanding*. The value of drill was acknowledged, but it was given less importance than understanding; drill was no longer the major means of providing instruction.

The relative importance of drill and understanding is still debated today. In this debate, people often treat understanding and learning skills as if they were opposites, but this is not the case. Clearly, drill is necessary to build speed and accuracy and to make skills automatic. However, equally clearly, you need to know *why* as well as *how*. Both skills and understanding must be developed, and they can be developed together.

Changes in the field of psychology have continued to affect education. During the second half of the twentieth century, educators came to understand that the developmental level of the child is a major factor in determining the sequence of the curriculum. Topics cannot be taught until children are developmentally ready to learn them. Or, from another point of view, topics must be taught in such a way that children at a given developmental level are ready to learn them.

Additionally, educators' attention is being drawn to the increasing evidence that children *construct* their own knowledge. Thus, helping children learn mathematics means being aware of how children construct mathematics from their experiences both in and out of school. Such considerations are now routinely taken into account in developing mathematics curricula.

The Australian Curriculum proficiency strands 'describe the actions in which students can engage when learning and using the content' of mathematics and 'indicate the breadth of mathematical actions that teachers can emphasise' (ACARA 2012, p. 4). Table 1.2 outlines the key student actions that comprise the foci of these strands. These strands should be viewed as interdependent and hence they need to be taught in an interwoven manner.

TABLE 1.2 **Student actions as they relate to proficiency strands**

Proficiency strand	Key ideas
Understanding	• Developing a robust knowledge of adaptable and transferable mathematical concepts • Making connections between related concepts and applying the familiar to develop new ideas.
Fluency	• Developing skills in choosing appropriate procedures • Carrying out procedures flexibly, accurately, efficiently and appropriately • Recalling factual knowledge and concepts readily
Problem Solving	• Making choices • Interpreting, formulating, modelling and investigating problem situations • Communicating solutions effectively
Reasoning	• Developing a capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising

Source: Adapted from © Australian Curriculum, Assessment and Reporting Authority 2015c.

What it means to learn mathematics has changed a great deal over the past century. In a changing world, learning mathematics with understanding is essential in order to meet the goal of mathematical proficiency. Students must actively build knowledge of mathematics from their personal experiences and prior knowledge. Research has shown that if children are able to make sense of the mathematics they are learning, they can build on this understanding to learn more mathematics and use that mathematics to solve problems in order to become mathematically proficient.

Ideas about developing mathematical proficiency are considered in more depth in the next chapter, but here you can consider some initial important questions. What does it mean to learn mathematics with understanding? How did you learn mathematics? How can you learn to understand what children know? Where can you turn to find out what mathematics children need to learn and how you can challenge and support them in learning that mathematics?

To teach mathematics effectively, teachers must know more than just mathematics. They need to know their students as learners and they must adjust their pedagogical strategies in response to students' varying experiences. Teachers must design lessons that reveal to them what students already know, that reveal students' misunderstandings and that guide students to construct more complex understandings of mathematics. Teachers must create challenging and supportive classroom learning environments that help children make sense of mathematics. Teachers must also encourage students to think, question, solve problems and discuss their ideas. 'Effective mathematics teaching requires understanding what students

know and need to learn and then challenging and supporting them to learn it well' (NCTM 2000, p. 16). Teachers can help children make sense of mathematics in many ways. Chapter 3 initiates the discussion of teaching and subsequent chapters focus on ways to teach and on useful types of activity.

Needs of society

The usefulness of mathematics in everyday life and in many vocations has also affected what is taught and when it is taught. In earlier times, mathematics was considered necessary primarily for clerks and bookkeepers. The curriculum was limited to counting; the simpler procedures for addition, subtraction and multiplication; and some facts about measures and fractions. By the late nineteenth century, business and commerce had advanced to the point where mathematics was considered important for everyone. The arithmetic curriculum expanded to include such topics as percentage, ratio and proportion, powers, roots and series.

This emphasis on *social utility*, on teaching what was needed for use in occupations, continued into the twentieth century. The proponents of social utility approaches focused on the essential skills and arithmetic used by carpenters, shopkeepers and other workers as the foundation for teaching school mathematics. They thought that school mathematics programs should be limited to teaching only these skills.

The outburst of public concern in the 1950s over the 'space race' resulted in a wave of research and development in mathematics curricula. Much of this effort was focused on teaching the mathematically talented student. By the mid-1960s, however, concern was also being expressed for the disadvantaged student, as society renewed its commitment to equality of opportunity. With each of these changes, more and better mathematical achievement was promised.

In the 1970s, when it became apparent that the promise of greater achievement had not fully materialised, another swing occurred in curriculum development. Emphasis was again placed on the skills needed for success in the real world. The minimal competency movement stressed the basics. As embodied in sets of objectives and in tests, the basics were considered to be primarily addition, subtraction, multiplication and division with whole numbers and fractions. Thus, the skills needed in colonial times were again being considered by many to be the sole necessities, even though children were now living in a world with calculators, computers and other features of a much more technological society.

By the 1980s, it was acknowledged that no one knew exactly what skills were needed for the future but that everyone needed to be able to solve problems. The emphasis on problem solving matured through the last 20 years of the century to the point where problem solving was not seen as a separate topic but as a way to learn and to use mathematics. In the 1990s, the standards movement became a major focus. Along with standards came an emphasis on assessment, accountability and equity.

Historically, in Australia each state and territory has taken responsibility for developing a K–12 mathematics curriculum program. More recently, guided by the *Melbourne Declaration on Educational Goals for Young Australians* (Ministerial Council on Education, Employment, Training and Youth Affairs [MCEETYA] 2008), Australia has moved to implementation of a national curriculum from Foundation to Year 12 in specified learning areas.

Mathematics is also considered essential for a competitive society. As the acronym shows, mathematics is an essential element of Science, Technology, Engineering, and Mathematics (STEM) and, as such, is inherent in the strong focus on the importance of STEM education and careers (Office of the Chief Scientist, 2014a). The report on Australia's STEM future by the Office of the Chief Scientist (2014b) recommended that opportunities be provided to enable the development of lifelong skills such as quantitative skills and critical thinking skills (p. 23). In their report on the skills required to ensure Australia is positioned to be both productive and competitive, the Australian Industry Group [AIG] (2013) reiterate the importance of increasing the participation of the population in STEM-related activities.

Assessment

People often think of assessment as testing to see what students have learned. However, it is much broader in scope. Helping all students learn mathematics requires that assessment be an integral part of the teaching program. Assessment should not be something that is done *to* students; rather, a mathematics program must

include assessments that are done *for* students, to guide and enhance their learning (NCTM 2000). In order to develop a program that provides for the strengths and needs of individual students, teachers must continue to gather information about their students' progress. Useful and effective assessment is more than a test at the end of a unit. Asking students questions during lessons helps them describe their thinking and understanding; their responses help teachers make the decisions they must make to meet students' needs. Observing students as they solve problems or build ideas provides a different look at student understanding. Ultimately, 'Assessment should become a routine part of the ongoing classroom activity rather than an interruption' (NCTM 2000, p. 23). (Assessment is considered in more detail in a later chapter.)

Today's society seems focused on assessments in the form of tests that are used to compare students' performance across schools, states and nations. In Australia in July 2006, MCEETYA declared that national testing in literacy and numeracy would proceed. Consequently in 2008, the National Assessment Program — Literacy and Numeracy (NAPLAN) commenced in Australian schools. Every year, all students in Years 3, 5, 7 and 9 are assessed on the same days using national tests. NAPLAN results are reported using 4 scales, one for each of the following domains: Reading, Writing, Language Conventions (spelling, grammar and punctuation) and Numeracy. These reporting scales apply to each of the year levels tested and are accompanied by descriptions of the skills and understandings typical of students at various proficiency levels along each scale.

Accountability

An effort to hold schools accountable for student learning began with the new century. In Australia the federal government recently adopted *My School* (www.myschool.edu.au) as a reporting and accountability resource for parents. *My School* is a web-based information service developed by ACARA. ACARA is an independent authority with functions including the publishing of nationally comparable data on all Australian schools. This responsibility is derived from the *Australian Curriculum, Assessment and Reporting Authority Act 2008* (the ACARA Act) and through the decisions of the Ministerial Council for Education, Early Childhood Development and Youth Affairs (MCEECDYA).

Equity

Equity relates to ensuring that all students learn mathematics. The Australian Curriculum (ACARA 2012, p. 8) emphasises the need to cater for the diversity of learners, recognising that:

> Australian students have multiple, diverse, and changing needs that are shaped by individual learning histories and abilities as well as personal, cultural language backgrounds and socio-economic factors.

This vision can be realised only if each person involved in education firmly believes that all children can learn mathematics and that each child should be expected to do so. Every child must be given the opportunity to learn worthwhile mathematics. This means designing instructional programs that can encompass all the different interests, strengths, needs, cultures and mathematical backgrounds of students. Plenty of evidence supports the idea that all students can learn mathematics. High-quality instructional programs are needed that let well-prepared teachers and other school personnel respond to students' varied strengths and needs.

Our schools are characterised by diversity — students from many different cultures and languages; from many different economic and home backgrounds; with many different strengths, ways of learning mathematics and past experiences with mathematics. Equity *does* mean that all children must learn worthwhile mathematics, but it *does not* mean that all should have the same instruction. In fact, it means that children can reach the high expectations set for them only if we meet the individual needs of each child. Your repertoire of ways to reach children will grow as you teach and learn. At this point, you can begin by challenging the popular belief that only some children can learn mathematics. This is an important first step in becoming a teacher who can help every child learn.

Your role

Before reading the previous section, you might have responded with a blank look to the question, 'What determines the mathematics being taught?' Hopefully, this discussion has helped you see that answering

this question leads us to consider the needs of the subject, the child and society. These needs have always been considered, but the relative importance assigned to them has changed over time.

Your actions within your role as an educator are influenced by your disposition towards mathematics. Cooke (2015) proposed that disposition towards mathematics incorporated 4 measurable components — attitudes towards mathematics, mathematics anxiety, confidence with mathematics, and conceptualisation of mathematics. In addition, your mathematical knowledge and understandings need to be sufficiently developed and usable when engaging with children. Your disposition towards and competency with mathematics will impact on what you will do as a teacher in a mathematics lesson.

If your goal is to help each child become mathematically proficient, you will have to consider many factors, some of which you can control and some of which you cannot. Certainly, you can hold high expectations for each child by believing that every child can learn mathematics. Equally certainly, you cannot control whether your students will have to take mandated tests. This text focuses on the things you can control — the things that you can do in the classroom to provide a worthwhile mathematics program for each child.

1.3 Where can you turn?

LEARNING OUTCOME 1.3 Evaluating resources to determine those of use in your continuing development of mathematics, mathematical learning and teaching mathematics.

There are many places you can turn to for help with the curriculum, with helping students learn and with continuing to develop your knowledge of mathematics and of mathematics learning and teaching. There are many resources available to support you. In this section, we discuss a few of the resources that we reference throughout this text.

National guidelines for school mathematics

In this chapter we have briefly outlined the content and proficiency strands of *The Australian Curriculum: Mathematics* (ACARA, 2012, 2015a, 2015c). This curriculum document, along with the accompanying website www.australiancurriculum.edu.au, will be a primary source of information for your broad curriculum planning. Its scope and sequence charts will provide details of specific content and sequencing within the content strands and sub-strands.

Professional organisations also have recommendations for school mathematics. Of particular value are the standards and policy statements of the AAMT. These complement both the Australian Curriculum documents and previous and current curriculum documents of the states and territories.

State and local guidelines

Although the Australian education ministers endorsed *The Australian Curriculum: Mathematics* in 2015, the curriculum documents of the Australian states and territories cannot be ignored.

The Australian Curriculum website provides implementation advice for states and territories (www.australiancurriculum.edu.au/curriculum/overview). The advice provides information regarding the education authority within each state that is responsible for implementing the curriculum.

Research

Research is referenced throughout this text, not only to acquaint you with research in mathematics education, but also to illustrate or support discussions in the text. There is a substantial body of research in mathematics education, both about children's learning and about teaching. The Mathematics Education Research Group of Australasia (MERGA) (www.merga.net.au) is a leading research organisation in Australia. MERGA publishes 2 international journals, *Mathematics Education Research Journal* (MERJ) and *Mathematics Teacher Education and Development* (MTED), both of which are excellent

sources of current research in mathematics learning, teaching and assessment, as well as mathematics teaching professional learning. In addition, MERGA publishes 4-year reviews of mathematics education research in Australasia. Another relevant source of research information is the *Second Handbook for Research on Mathematics Teaching and Learning* (Lester 2007).

Research reports and related articles appear in many teacher professional journals, making these journals very valuable resources for effective research-derived mathematics learning and teaching activities. For use in primary school mathematics in Australia, the AAMT publishes *Australian Primary Mathematics Classroom* and *Australian Mathematics Teacher*. In the United States, the NCTM publishes *Teaching Children Mathematics* and *Mathematics Teaching in the Middle School*.

Cultural and international resources

The Trends in Mathematics and Science Study (TIMSS) is an international study that focuses on students in Years 4 and 8. The results of the study in 2003 with 45 participating countries showed that Australian Year 4 students performed at approximately the international mean, while the Year 8 students performed significantly better than the international mean. There is also evidence that the performance of Australian students has remained relatively static over the period 1995 to 2003 (Stacey & Stephens 2008). Performance varies greatly within Australia and is closely linked to economic status.

The TIMSS also collects information about curriculum, teaching and teachers. Video studies of classrooms suggest that Japanese teaching in Year 8 more closely resembles the recommendations of the NCTM's Standards than does teaching in either Australia or the United States. These results suggest that both Australian and US educators can learn much from analysing how other countries teach mathematics.

Many of the other chapters in this text suggest resources that will help you understand how mathematics is taught and learned in other cultures and that show you ways to use culturally oriented activities as you strive to teach each child.

History

Before 1900, arithmetic was taught in later school years, and it then became a primary school subject. You can gain some historical perspective on mathematics education, and have an enjoyable experience at the same time, by looking at mathematics textbooks from the 1800s and early 1900s. Read the prefaces of the books, note their lengths and look at the exercises and the activities. Are the activities relevant for today's children in learning mathematics? If you are a student of history, you may want to examine the trends in mathematics education over the decades. How has the emphasis changed and how has it remained the same? Did students spend more time on learning procedures or on problem solving? Were students provided with manipulatives to help them grasp the ideas?

Textbooks and other materials

Many teachers use a textbook when teaching primary and middle school mathematics. Today's textbooks provide supplementary materials such as assessments, problems to solve and extra practice. Teachers' manuals also provide a wealth of materials and teaching suggestions.

There are many different types of textbook. Some provide a lot of drill and practise but offer little help in developing understanding and using mathematics. Others do the opposite: providing a great deal of help in developing students' understanding but falling short on practise of necessary skills. As you gain experience, you will be better able to judge the quality of a textbook and better able to depart from it as appropriate. If you have not had much experience, you may want to stick with the textbook until you become more comfortable with teaching. Be sure to always ask why you need to teach a given lesson. Does it help children develop the mathematics they need? Does it achieve this goal in a way that makes sense to children? Does it help children make sense of the mathematics?

Electronic materials

The kind and quantity of materials readily available are increasing every day. The web provides immediate access to lesson plans, help with the mathematics itself, assessment items and information that can be used in teaching mathematics. Additionally, many sites have videos of mathematics classes. Links to a number of useful websites have been included in the resources at the end of each chapter.

Professional organisations

Being a part of a profession opens many opportunities and imposes many responsibilities. Professional association with others and the support you can find from being a part of a professional organisation will enhance your teaching career. Journals, conferences and other materials of a professional organisation are often available through schools and even to individuals who are not members of the organisation.

The professional journals of the AAMT and the NCTM have already been listed for you. These professional organisations also sponsor conferences. The state-based affiliated organisations of the AAMT (see table 1.3) provide local conferences, professional development activities and teaching resources.

TABLE 1.3 **Affiliated associations of the Australian Association of Mathematics Teachers**

State/territory	Mathematics association
New South Wales	Mathematical Association of New South Wales (MANSW), www.mansw.nsw.edu.au
Victoria	Mathematical Association of Victoria (MAV), www.mav.vic.edu.au
South Australia	Mathematical Association of South Australia (MASA), www.masa.on.net
Australian Capital Territory	Canberra Mathematical Association (CMA), www.canberramaths.org.au
Western Australia	Mathematical Association of Western Australia (MAWA), www.mawainc.org.au
Northern Territory	Mathematics Teachers Association of the Northern Territory (MTANT), www.mtant.wikispaces.com
Queensland	Queensland Association of Mathematics Teachers (QAMT), www.qamt.org
Tasmania	Mathematical Association of Tasmania (MAT), www.mat.aamt.edu.au

Professional development

Start taking advantage of professional development opportunities. Some of these will be formal, such as workshops, university courses and conferences; others will include informal study groups. Your school, region or state will provide some opportunities for you. Others will be commercially sponsored or sponsored by a professional organisation. An increasing number of opportunities on the web are designed so you can participate as your schedule permits.

Often, schools or regions have funds set aside for professional development that are available on request. Some districts have teachers design their own professional development plans and support them in carrying out those plans.

Other teachers

Teachers learn from each other. You will learn from your school experiences, but do not let learning stop there. Look for schools where sharing ideas about helping students learn mathematics and sharing

teaching tips and materials are the norm. Look for teachers in other schools, either near you or far away but connected electronically, who are willing to discuss and to share. A good teacher who is willing to work with you is an invaluable resource.

MAKING CONNECTIONS

As you prepare for teaching mathematics, be sure to think about the broader context of your work and carefully consider these 3 challenges.

1. *Examine your own disposition towards mathematics and your beliefs about who can learn mathematics*. Be ready to question your beliefs, to evaluate proposed changes and to make a difference in helping children learn mathematics.
2. *Take seriously the title of this text.* Teaching means helping students learn, not merely giving out information. As you begin working with children, stop and listen to them, individually and collectively. Reflect on what you are hearing, and learn with and from the children.
3. *Realise that doing mathematics and teaching mathematics are different*. Teaching mathematics requires a depth of understanding about mathematics (Ma 1999), about students, about schools, about curriculum and about pedagogy. If you come to this realisation and actively seek knowledge and experiences that integrate these areas, you are well on your way to becoming a good teacher.

A GLANCE AT WHERE WE'VE BEEN

Teaching mathematics in a changing world means that the curriculum and instruction must change to reflect the needs of the subject, the child and society. In this chapter you have been challenged to consider your view of mathematics as a subject. You have seen a glimpse of the changes through the past as well as recommendations for teaching mathematics in the twenty-first century. The aim and rationale of *The Australian Curriculum: Mathematics* along with the principles put forth by the NCTM underpin many of the recommendations for today. Resources have been identified to support your study throughout the rest of this text and, more importantly, as you teach. The challenge is to keep an open mind and continue your own learning about teaching children mathematics. Prepare to help your students make sense of mathematics.

THINGS TO DO: FROM WHAT YOU'VE READ

1. What are the 3 general goals mentioned in the introduction? Which do you think is the most important? Explain why.
2. Give an example (different from those in this chapter) of how mathematics is a study of patterns and relationships, a way of thinking, an art and a language.
3. Which of the resources discussed in this chapter have you already used? Which ones do you think will be most helpful to you?
4. Explain in your own words the aims and rationale of *The Australian Curriculum: Mathematics*.
5. In the Australian Curriculum, 4 proficiency strands are highlighted. Describe the 4 strands using specific examples from your personal experiences. Explain how these strands are interwoven and interdependent.

THINGS TO DO: GOING BEYOND THIS TEXT

IN THE FIELD

1. *Links between mathematics at home and mathematics at school.* What does mathematics look like at school? Consider how the mathematics might link back to the experiences children bring with them to

the classroom. What might the teacher do to make connections between the mathematical knowledge and understandings children have from home and the mathematical experiences created at school?

2 *Equity*. What does equity look like in the classroom? How might the ideas raised by the *Australian Curriculum: Mathematics* be enacted? How does focusing on equity in a mathematical experience look different to focusing on equality?

IN YOUR JOURNAL

3 The Australian Curriculum recommends that technology should support effective mathematics teaching. Write a statement of your experience using calculators and describe your philosophy regarding calculators in learning primary mathematics.

4 Several educators have noted that the curriculum is in a continuous process of change in order to maintain balance as the needs of the subject, the child and society pull it first one way and then another. Discuss this comment.

5 React to the 3 challenges presented in the close of the chapter.

WITH ADDITIONAL RESOURCES

6 Describe the 4 strands of mathematical proficiency and how they are intertwined and interdependent. What practices are needed if all children are to become mathematically proficient?

7 Find a recent issue of *Australian Primary Mathematics Classroom* (*APMC*) or *Australian Mathematics Teacher* (*AMT*). Select an article that describes a classroom application. Describe the principle(s) that are considered in the recommendations for this classroom application.

WITH TECHNOLOGY

8 Access the *My School* website (www.myschool.edu.au). After looking through the website, identify how your perspective as a pre-service teacher may have impacted on what you reviewed. Now, consider what might be focused on if you are a teacher or a parent — how do these perspectives change what is viewed and how it is viewed? Does focusing on mathematics change the similarities and differences of these perspectives?

9 There are many websites that provide mathematical activities, but you need to be selective in using them. Have a look at 3 — www.connectwith.earlyyears.aamt.edu.au from Australia, www.illuminations.nctm.org from the US, and www.nrich.maths.org/frontpage from the UK. Explore the websites and then compare what each offers. Consider how you might use the resources from these websites when you teach

10 Go to the National Literacy and Numeracy Week in NSW website (www.nlnw.nsw.edu.au/numvideos.htm) and select a video that corresponds to a mathematical area you might have struggled with at some time (for example, decimals or fractions). How might the points raised in the video connect to your difficulties?

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CHAPTER 2

Helping children learn mathematics with understanding

LEARNING OUTCOMES

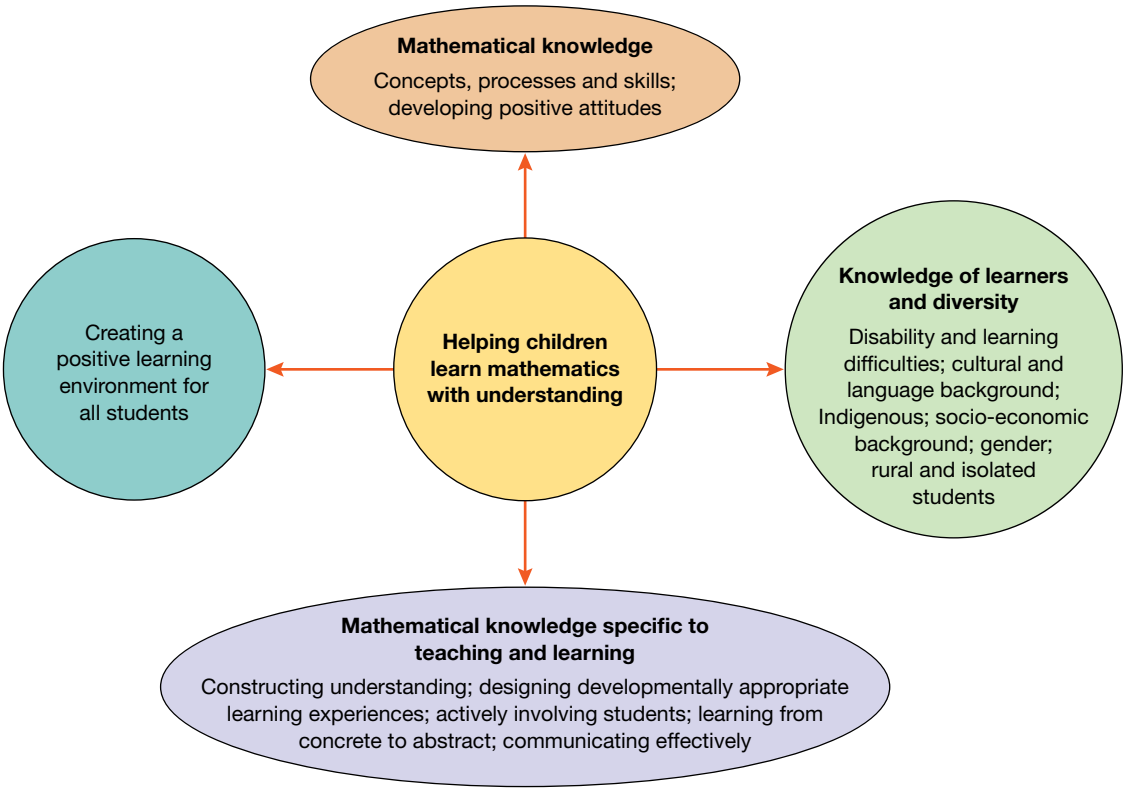
2.1 Understanding the 5 strategies that will help all students learn mathematics.
2.2 Articulating the differences between procedural knowledge and conceptual knowledge.
2.3 Describing behaviourist and constructivist approaches to learning.
2.4 Making connections between the 4 recommendations and constructivist approaches to learning.



'In mathematics the art of proposing a question must be held of higher value than solving it.'

Georg Cantor (mathematician)

Chapter 2 concept map



Introduction

How do children learn mathematics? This important question has no simple answer. Primary classrooms are filled with children who have different backgrounds, interests, strengths and needs. It is important for teachers not only to know the children in their classroom well but also to answer the question, 'How will my students best learn mathematics?' Teachers provide their answers through classroom practices. Every learning experience undertaken in the classroom expresses the teacher's views. The way in which lessons are planned, concepts are explored and processes and skills are developed, and the way in which the teacher establishes the learning environment and engages the students, reflects the teacher's perceptions of learning. Teachers' beliefs about the learning process are crucial; therefore, a thorough understanding of how mathematics is best learned by students needs to be a high priority for all primary teachers. The purpose of this chapter is to build on your previous knowledge from educational psychology and stimulate your thinking about how to help all children learn mathematics.

2.1 How can we support the diverse learners in our classrooms?

LEARNING OUTCOME 2.1 Understanding the 5 strategies that will help all students learn mathematics.

When starting a school year, teachers look forward to meeting their new class of students. Each year brings a new mix of children with varying personalities, strengths and needs. Students come from a wide variety of sociocultural situations. They may be Indigenous, speak English as a second language or be

recent arrivals. In the classroom we will usually have boys and girls and there may be students with special learning needs or disabilities. In Australia, we might be teaching in a rural or isolated setting that can influence our students' approach to learning and impact upon the learning environment.

The students in our classrooms are diverse in many additional ways. They vary in their cognitive, physical and social development and abilities. They come from different cultures and family structures. They have different background experiences, interests, levels of motivation and styles of learning. Ultimately, every child in every class is unique. As teachers, our role is to help each and every one of these children learn mathematics and develop to their maximum potential. The Australian Curriculum (ACARA v8.1, Student Diversity) acknowledges student diversity and states that, 'all students are entitled to rigorous, relevant and engaging learning programs . . . that [address] their individual learning needs.'

As a teacher of primary mathematics, the task of providing for all children might appear to be overwhelming but it can also be viewed as an exciting challenge. In the next chapter, we will examine how you can adapt lessons to meet the needs of individual students. There are many strategies you can use to support all students without having to individualise instruction for each one. In the following sections, we describe 5 of these strategies.

Creating a positive learning environment

Creating a positive learning environment is about maximising students' learning potential. It is about being aware of the cognitive, emotional and physical needs of individual students and the whole group. A teacher who establishes 'an inclusive and caring atmosphere of trust and belonging, [where] active engagement with mathematics is valued, communication skills [are] fostered and co-operative and collaborative efforts [are] encouraged' (Australian Association of Mathematics Teachers 2006, p. 4) is well on the way to helping children learn mathematics with understanding. The teacher is largely responsible for creating an appropriate environment.

Here are some recommendations.

- Make sure the classroom arrangement is safe and comfortable and that it supports the learning experiences. When the teacher explains or demonstrates to the whole class, all students need to be able to see. When working in small groups, desks need to be arranged in clusters rather than rows to encourage communication. If students are frequently using concrete materials, or other supplies, they need to be organised, packaged and available for easy access or distribution. A room arrangement allowing both teacher and students the ability to move around the room easily and safely is important to achieve.
- Make sure the classroom atmosphere is intellectually stimulating for learning mathematics. The teacher needs to encourage academic risk-taking and help children feel safe when problem solving, reasoning and communicating mathematically. Misunderstandings, incorrect answers and conceptual errors are naturally a part of learning as children construct their mathematical knowledge. Children who are comfortable about giving their intuitive responses and are not overly concerned with right or wrong answers are more likely to observe patterns and relationships, make conjectures, engage in discussions and take risks when doing mathematics.
- Communicate to students that there will be variability in their readiness to learn mathematics. Help students to understand that everyone learns things at different rates due to prior knowledge, context and experiences and that there will be differences in students' conceptual understanding and proficiencies. Learning mathematics is a long-term process. Sometimes progress can be slow but at other times learning will move ahead in moments of insight, where students will think and/or say, 'I've got it!', 'Now I understand!' or excitedly think 'Ahah!'.
- Reward students for critical thinking and creative problem solving so that they learn to value and implement those processes. Students who have a deep understanding of mathematical concepts and who use a problem-solving framework in assessments such as NAPLAN (National Assessment Program Literacy and Numeracy) will perform better than those who attempt to use clues and routine strategies (Perso 2011).

Table 2.1 shows how shifts in the classroom learning environment and professional practice can better support students.

TABLE 2.1 Shifts in classroom environment and professional practice

Desirable outcome (shift towards)	Undesirable outcome (shift away)
The teacher arouses curiosity, challenges students' thinking and actively engages them in learning.	Students are busy in memorising procedures and finding answers automatically.
The teacher initiates purposeful mathematical dialogue with and among students.	Classrooms are quiet places where students work individually.
The teacher facilitates learning by negotiating mathematical meaning and modelling mathematical thinking and reasoning.	Mathematics is treated as acquiring concepts in isolation and learning procedures.
The teacher promotes, expects and supports creative thinking and mathematical risk-taking in finding and explaining solutions.	The teacher is the sole authority for the 'right' mathematical method and for supplying the right answers to students.
The teacher strategically intervenes and provides appropriate assistance.	The teacher works at the front and helps students only when called upon.

Source: Adapted from AAMT (2006, p. 4).

Avoiding negative experiences that increase anxiety

Many children and some adults experience degrees of maths anxiety, apprehension and fear of mathematics which are expressed as negative attitudes toward doing mathematics. Mathematics anxiety leads to poorer performance, lower achievement in timed assessments, misunderstandings, and is displayed as lack of confidence when doing mathematics, avoidance of maths related activities and lower levels of motivation (see figure 2.1). Research comparing international results of achievement against attitude and confidence has indicated that maths anxiety is a widespread issue (Lee 2009 cited in Beilock & Willingham 2014; Go8, 2009) and that it can start early, continue throughout primary and secondary school, and into University level courses (Maloney & Beilock 2012). Furthermore, maths anxiety can have implications for the next generation when teachers pass these feelings onto their students in the early years of their education, influencing the maths achievement of both both girls and boys (Beilock & Willingham 2014). Practices where the learners are passive and the mathematics being learned is not connected to the real world can further emphasise prior negative experiences.

According to Attard (2011), by their final year of primary school, students are aware of what quality mathematics teaching and learning looks like. Whilst their engagement in mathematics is influenced by many factors, 'students showed preference for tasks that require active learning, elements of choice, challenge and the option of self-directed activities' (p. 375). Australian students (aged 11 to 12 years) in the Attard study were strongly influenced by effective teaching strategies and the relationship between teacher and students. The following qualities of a good mathematics teacher identified by upper primary students resonate with the AAMT standards for excellence as shown in table 2.1.

A good mathematics teacher:
- is passionate about teaching mathematics
- responds to students' individual needs
- gives clear expectations
- uses scaffolding rather than providing answers
- encourages positive attitudes towards mathematics
- shows an awareness of each students' prior knowledge. (Attard 2011, p. 375).

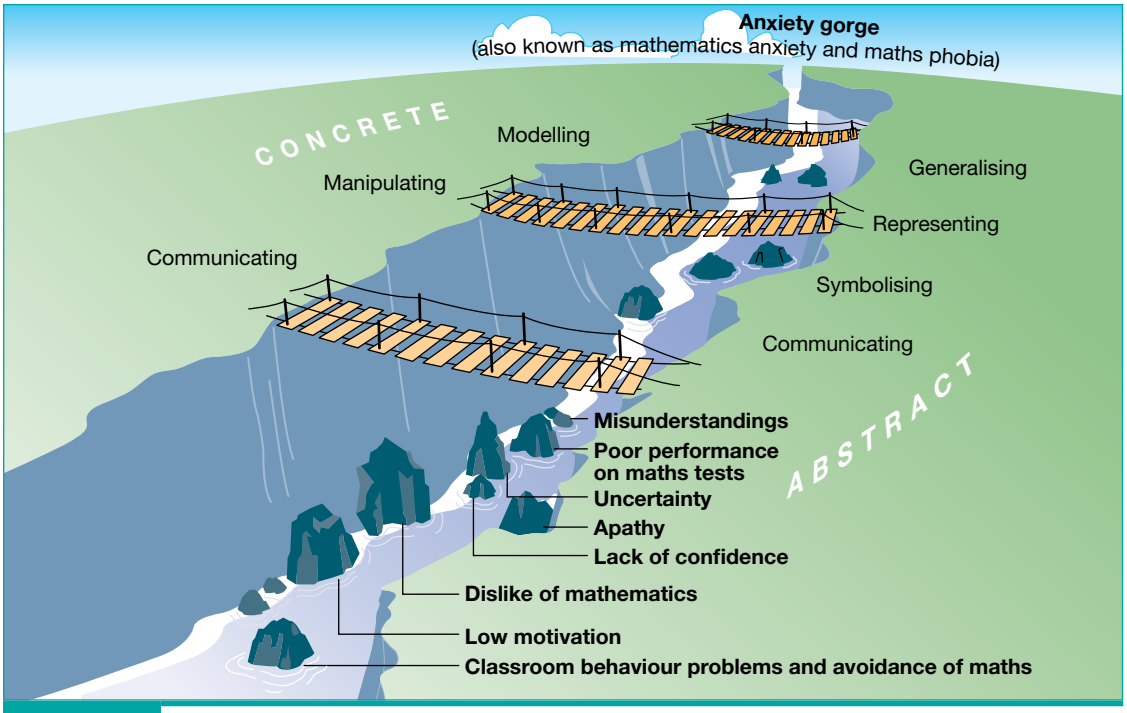


FIGURE 2.1 Bridges linking meaning to mathematics

These qualities of good teachers in combination with effective teaching and assessment strategies and a positive learning environment can help students maintain a positive attitude towards mathematics learning and a willingness to continue their studies of mathematics. In contrast, students who experience mathematics anxiety tend to take less mathematics during secondary school, thereby blocking their access to many careers.

What can be done about mathematics anxiety? Here are some suggestions for ways that can help students cope with this problem.

- *Emphasise meaning and understanding rather than memorisation*. Children attempting to memorise mathematics without understanding are likely to fall into the 'anxiety gorge' in figure 2.1. Helping students make connections between the concrete (models and manipulatives), the visual (pictorial representations and diagrams) and the abstract (generalisations and symbolic representations) facilitates understanding, promotes success at learning and helps relieve mathematics anxiety.
- *Model problem-solving strategies rather than presenting finished solutions*. Help students realise that using incorrect strategies and taking unnecessary steps are a natural part of developing problem-solving skills. Focusing on the process rather than the answer helps reduce the anxiety associated with getting 'wrong' answers.
- *Show a positive attitude towards mathematics*. If the teacher experiences maths anxiety, they need to make every attempt to overcome it. Students' attitudes are greatly influenced by the attitudes of their teachers. If the teachers enjoy teaching mathematics and share their interest and enthusiasm for the subject then they will tend to produce students who like mathematics.
- *Give students mathematical experiences that they will enjoy and that will interest and challenge them while allowing them to be successful*. Successful experiences in learning mathematics result in self-confidence. Similarly, mathematics that is relevant to students' own lives and makes connections to their world will help them maintain interest and enthusiasm.
- *Show a positive attitude towards students at all times*. Encourage, support and praise all students, and help them appreciate the power, usefulness and importance of mathematics.

- *Encourage students to explain their feelings about mathematics*. What do they like and why do they like it? This self-reflective (or metacognitive) diagnosis can help the teacher detect symptoms of mathematics anxiety. Rather than expressing those feelings out loud in class, giving time to students for keeping a mathematics journal can be very helpful. Just ten minutes a day with some prompting questions, such as 'What I enjoyed in maths today . . .'; 'What I found challenging in my learning today . . .'; 'Why I would do . . . differently, if I had a chance again'.
- *Be careful not to overemphasise speed or competition for assessment items or learning experiences*. Some children may enjoy the challenge of being compared, but others are uncomfortable with it — for these students, timed races cause apprehension and fear of mathematics. Use diagnostic techniques to identify students who are experiencing particular difficulty or need special help, and provide this help promptly and discreetly to get them back on track.

Establishing clear expectations

Clear expectations for student behaviour and achievement are necessary in a well-run primary classroom. Students tend to do what they think the teacher means rather than what the teacher says. Teachers need to think through what their expectations are of the students and be confident that their behaviour is in accordance with those expectations (Good & Brophy 2008).

Ways to establish clear expectations include the following.

- Make it clear that you respect and value student ideas and ways of thinking; and make it clear that you expect your students to respect and value each other.
- Expect academic rigour from your students. A mathematics class motto for your students could be: 'Do only what makes sense to you'. This motto encourages students to question, reflect and seek explanations and meaning. It also paves the way for constructing knowledge that is truly theirs. Another class motto might be: 'Maybe one answer — certainly many paths'. This motto will help your students realise that the process is just as important as the answer. Ask students to explain their thinking so you can see what stages and strategies they took to get to their solution(s).
- Encourage children to reflect on their learning — not just on what they have learned, but also on how they learned it (their thought process). Metacognition is an important part of learning. Individual reflection or interaction with others (both teachers and peers) encourages students to communicate and explain their thinking.

Treating all students as equally likely to have aptitude for mathematics

Do not let your words or actions suggest that some students are more likely than others to excel in mathematics. Doing so sends a message to other students that you expect them not to succeed. Make it clear that you expect students to succeed in mathematics regardless of their gender, social, cultural or economic background. Expecting high achievement outcomes from everyone sends a powerful message that mathematics is for everyone. Research consistently confirms that teacher expectations greatly affect student performance.

A complex assortment of social forces produces or influences inequalities related to mathematics. For example, parents of young children may indicate that they expect their sons to be better at mathematics than their daughters. School counsellors may subtly discourage students from particular socio-economic backgrounds from studying mathematics or pursuing careers where mathematics is important. Over time however, research has shown that teachers can do a great deal to help all students succeed in mathematics. To be an effective teacher, high expectations need to be upheld for all students and students need to be challenged equally. The language, culture and community background of students should be considered as assets, not as liabilities. In the development of assessment, there needs to be the flexibility to allow the teacher to identify and build on students' strengths. If weaknesses or gaps in students' knowledge are identified then support and encouragement need to be given without delay.

Research suggests that teachers continue to treat girls and boys differently in the mathematics classroom. Despite more than 30 years of extensive research on gender and mathematics learning, at least 20 years of intervention programs at many levels of education, curricula change and improved understanding of male–female learning styles and assessment preferences in many countries, there continues to be unequal learning experiences and outcomes for female students. This gender inequity continues to affect female participation in higher level mathematics courses and career options. According to Forgasz (2008) 'there are signs that the gender gap may be increasing' and that 'gendered patterns from the past are still evident in the context of contemporary mathematics education in Australia' (p. 13).

In terms of what happens in the classroom, it has been thought that teachers tend to attribute boys' failure at mathematics to a lack of motivation, whereas they tend to attribute girls' failure to a lack of talent or ability. However, according to research by Wilson et al. (2014), primary teachers attribute students' success and failure to 8 factors rather than gender, race and socioeconomic status as previously thought. If this is the case then ability, effort, luck, difficulty of the task, age and grade, out of school contexts, teaching and prior knowledge are all important attributions of success and failure in mathematics learning tasks.

Gender differences often arise with respect to learned helplessness — the belief that the individual cannot control outcomes and is destined to fail without the existence of a strong safety net. Learned helplessness includes feelings of incompetence, lack of motivation and low self-esteem. It usually develops from what is perceived as failure or lack of success in learning, and it is often associated with mathematics. Students feel there is little sense in trying because the opportunity for success is beyond their control. Both boys and girls experience learned helplessness, but girls are particularly susceptible to this syndrome. According to Yates (2009), patterns of learned helplessness can be identified efficiently and readily in order to help all students learn mathematics with more confidence and enjoyment.

The Australian Curriculum is based on equity ideals. That is, 'all students can learn and that every student matters' (National Curriculum Board 2009, p. 8). Actions that can be taken to address inequities include the following.

- Dispelling myths (such as 'mathematicians work in complete isolation' or 'only white males do mathematics') that discourage women and some minorities from pursuing careers in mathematics (Mewborn & Cross 2007).
- Having equally high expectations for all students and clearly communicating those expectations to both students and their parents.
- Engaging both boys and girls in solving difficult problems, raising questions and communicating their mathematical thinking — that is, making sure boys and girls participate equally in class discussion and receive equal shares of teacher time and attention.
- Making relevant connections between mathematics and students' lives (Coates 2007).
- Calling attention to role models of both genders and from the widest range of racial, cultural and ethnic backgrounds in both mathematics and science; also, helping students increase their awareness of career opportunities for people with strong mathematics backgrounds.
- Communicating to parents the importance of encouraging all their children — both girls and boys — to aspire to success in mathematics.
- Discussing learned helplessness with students experiencing problems and developing ways to prevent learned helplessness or to remedy it.
- Using a variety of methods to assess student performance and achievement (see later chapters for a range of assessment formats, interviews and portfolios).

Helping students improve their ability to retain mathematical knowledge and skills

Retention reflects the degree to which students can hold onto and use what they have learned. For example, if students can read a clock in class but have forgotten how to do so by the time they get home, we would say that their retention of this skill is very limited. Clearly, retention of knowledge and skills is an important aspect of learning.

Forgetting is a problem in all disciplines, but the nature of mathematics in particular makes it an issue because new concepts build on previous understandings. A gap of time during school holidays, a weekend, a day or even shorter periods can make the retention of skills and specific knowledge decrease dramatically. For instance, the knowledge that lets students answer questions like the following is quickly forgotten if it is not used regularly: What is a prime number? What is the commutative property of multiplication? How many millilitres in a litre? What is Euler's rule for 3D polyhedra?

Skills, too, are quickly lost if not used regularly. For example, students may have trouble (1) using a geoliner, (2) adding fractions with unlike denominators or (3) solving the area of an irregular 2D shape. Therefore, assessment of mathematical skills and knowledge can often report very changeable levels of performance and achievement depending on the time, day or feelings of the student.

Performance when problem solving, in contrast, is more stable over time and less susceptible to decline. One reason accounting for this is that problem solving is a complex behaviour supported by several higher-order thinking processes. Strategies for problem solving take time to develop and often improve with time but once established, they are retained longer than rote skills.

Retention is an important goal in mathematics education. It is essential to recognise its importance and plan teaching and learning to maximise it. Research suggests several ways for teachers to help children improve their retention.

- *Meaningful learning is the best way to increase retention*. Mathematical learning (knowledge, skills and processes) that have been developed with meaning and connected to previous understanding is retained longer.
- *The procedure used when a concept was learned can aid long-term retention*. For example, physically measuring the diameter and circumference of many different circles, observing patterns and recording them helps students remember that the ratio of the diameter and circumference is constant, more so than simply being told a formula by the teacher. Knowledge gained from exploration is more likely to be retained.
- *Establishing connections aids long-term retention*. Connections help children see how mathematical ideas are related to each other and to the real world. Mathematical topics should not be taught in isolation, but in conjunction with authentic problem solving and with applications in meaningful, real-world contexts. Research documents the value of establishing connections, not only to gain better understanding but also to promote retention.
- *Periodically reviewing key ideas helps to anchor knowledge and can contribute substantially to retention*. The spiral development of high-quality mathematics programs reflects the importance of regular review of mathematical topics for children at every age. These reviews may be explicit or implicit. Some teachers spend the first few minutes of each mathematics lesson explicitly reviewing previously learned concepts; others implicitly incorporate review of previous topics in discussions of new topics. In either case, reviews help remove rustiness, reinforce and refresh knowledge in ways that improve learning and contribute to higher achievement and greater retention.

2.2 Meaningful connections between procedural and conceptual knowledge

LEARNING OUTCOME 2.2 Articulating the differences between procedural knowledge and conceptual knowledge.

The overriding goal for mathematics education is for students to become mathematically proficient. According to the Australian Curriculum (ACARA v8.1, Key Ideas), mathematical proficiency incorporates 4 important strands: (1) Understanding, (2) Fluency, (3) Problem Solving and (4) Reasoning. In this section, we will elaborate on conceptual understanding and procedural fluency. In mathematics education, the relative importance of teaching students procedural knowledge and emphasising fluency versus developing conceptual knowledge and emphasising understanding has long been debated, but

the debate involves a false dichotomy. Clearly, not only are both procedures and concepts necessary for expertise in mathematics — they are intertwined.

As a teacher, there is a need to recognise what constitutes fluency and understanding and the importance of helping students make meaningful connections between them. Both types of knowledge are essential in order for students to be able to learn mathematics with understanding.

Procedural knowledge or fluency is reflected in skillful use of mathematical rules or algorithms. A student with procedural knowledge can successfully and efficiently use a rule or complete a process, such as a sequence of actions. For example, a student who is fluent in 2-digit by 3-digit multiplication can perform the steps in the long-multiplication algorithm quickly and accurately. However, the procedure that they learned might not have been the best way of doing the problem without errors or misconceptions. It simply was the preferred method of the time and best understood by the teacher (or parent).

Conceptual knowledge or understanding involves awareness of what mathematical concepts mean at a deeper level. For example, the meaning of multiplication includes equal addition or equal groups of an amount and there are many historical methods that build deeper understanding than the algorithm traditionally taught or in current practice. Students who understand concepts can link ideas in networks of connected meanings, incorporate new information into those networks and see relationships among different pieces of information (Hiebert & Grouws 2007).

Procedural knowledge alone helps students answer specific questions, but it lacks important connections. Conceptual knowledge requires that the learner actively think about relationships and make connections, while also making adjustments to fit the new learning into existing mental structures. Research shows that understanding knowledge won't diminish skills but may even help students recall and use skills (Hiebert & Grouws 2007). For example, in response to the question 'What is a square?' fluency might lead a student to respond routinely: 'A square is a shape with 4 equal sides and 4 right angles'. Procedural knowledge, however, wouldn't help the student understand other relationships and properties — for example, that a square is also a rectangle, a parallelogram, a regular quadrilateral, has 4 lines of reflective symmetry, and diagonals that bisect each other. In the classroom 2.1 requires that students demonstrate conceptual knowledge — that is, a deeper understanding of 4-sided shapes. The learning task could be expanded to ask 'What other properties of these shapes are you aware of?' and/or 'What purposes do these shapes serve in our everyday 2-dimensional and 3-dimensional world?'

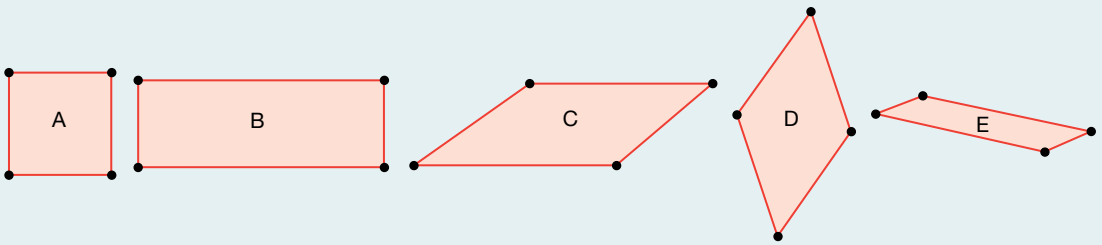
IN THE CLASSROOM 2.1

Different kinds of 4-sided shapes

Objective: Classify 4-sided shapes according to key features.

Year level: 3–4

- Study the following 2D shapes and look for similarities and differences.



- Consider properties of the shapes, such as side length, angles, diagonals, symmetry and pairs of parallel or perpendicular sides. Name and classify the shapes as square, rectangle, parallelogram or rhombus.
- Decide which quadrilaterals are most regular and discuss why.

There is a negative consequence of teaching for fluency without understanding. If teachers indicate that the most important thing in learning mathematics is to get the right answer, then students will have no motivation to think mathematically. For example, the addition algorithm for 23 + 49 could be memorised as: 'Add the 3 and the 9 to get 12. Write down the 2 and carry the 1, add 4 + 2 + 1 = 7. The answer is 72.' Rote learning (memorisation without meaning) has no place in school mathematics. This example highlights the lack of place value understanding and the absence of a variety of strategies including mental computation and estimation that give rise to efficient and meaningful methods, whilst also gaining the correct answers. One of the ever-present dangers associated with teaching by strict procedures is that there will be no understanding gained. Research suggests that students with highly developed rules for manipulating symbols are reluctant to connect these rules to other representations that might help them understand the mathematical meaning of the algorithms. In addition, a student who learns a procedure without meaning will have difficulty knowing when to use it, remembering how to do it, applying it in new situations and judging if the results are reasonable.

As a teacher of mathematics, it is important to help students establish connections and see relationships between understanding and fluency. They may not be able to do it automatically. Keep in mind, however, that research suggests, teaching and learning which focuses on developing understanding can also yield efficient skills (Hiebert & Grouws 2007). Finally, one way to focus on understanding is to focus on students' ability to communicate their ideas to the teacher and each other. The real challenge is to construct learning experiences that help students build connections between mathematical ideas.

The aims of the *Australian Curriculum: Mathematics* incorporate all of the ideas presented above, stressing the importance of both procedural and conceptual knowledge when ensuring that students:

- are confident, creative users and communicators of mathematics
- develop an increasingly sophisticated understanding of mathematical concepts, fluency with processes, and are able to pose and solve problems, and reason (in all content strands)
- recognise connections between the areas of mathematics and other disciplines and appreciate mathematics (ACARA v8.1, Aims).

2.3 How do children learn mathematics?

LEARNING OUTCOME 2.3 Describing behaviourist and constructivist approaches to learning.

Mathematical ideas have evolved from cultures throughout the world over thousands of years and mathematics continues to change at an ever-expanding rate with the aid of digital technologies. Hence, understanding, fluency, problem solving and reasoning are essential capabilities for mathematical thinking and for current and future demands (ACARA v8.1, Rationale).

The premise that mathematics needs to be learned with understanding is based on research that has been accumulating for many years and reflects advances in mathematics education, psychology and other areas of science. At the early childhood level children gain a great deal of intuitive understanding from their environment, through play and from social experiences. In the early primary years, teachers need to be aware that children's mathematical thinking might surpass their ability to record their learning except in visual representations. Later, once children have learned to read, they build their learning in mathematics alongside reading and writing and are more able to record their thinking in written and abstract representations. Mathematics, which is both hierarchical and interconnected, becomes more abstract as the ideas build. Evidence from educational psychology suggests that deep learning in mathematics takes time and need to build on concepts that are automatic (Hattie & Yates 2014). Ultimately, to learn more abstract mathematical concepts, children need to have developed enough both physically and psychologically to handle the abstraction.

Early in the twentieth century, the educational reformer, John Dewey, asserted that learning comes from experience and active involvement by the learner. Although much has been discovered since then about how children learn mathematics, the importance of meaningful experience remains unchallenged. Later, Jean Piaget argued that learners actively construct their own knowledge. This view of learning, known as

constructivism, suggests that rather than simply accepting new information, students interpret what they see, hear or do in relation to what they already know. Students learn mathematics with understanding by actively building new knowledge from their personal experiences and prior knowledge. Learning experiences developed by teachers, for students, are ideally based on how they believe children best learn. There are currently 2 prevailing theories of learning, behaviourism and constructivism. *Behaviourism* focuses on observable behaviours and is based on the idea that learning means producing a particular response to a particular stimulus. From this perspective, students learn specific skills (behaviours) by observing teachers demonstrating those skills in relation to specific stimuli (e.g. a mathematics problem) and then practising the skill. For example, a teacher demonstrates how to find the mean of a set of numbers and students learn to produce that type of response (finding the mean) when given that type of stimulus (a set of data). Behaviourism focuses on behaviours but excludes consideration of any thought processes students may use to arrive at their answers. *Constructivism* has a different focus which is about the thinking students do. From this perspective, learning depends not only on what the teacher does but also on the students themselves — how they integrate new ideas with their experiences and with what they already know. Both views of learning hold implications for teaching mathematics. Both are key to a further understanding of how children learn mathematics.

Building behaviour

The emphasis of behaviourism is on external actions and observable behaviours — on stimuli and responses. The main advocates of the behaviourist perspective over the years include Edward Thorndike, B. F. Skinner and Robert Gagné. Today, few learning theorists argue for an exclusively behaviourist approach to mathematics learning. It is true that behaviourism has had a significant impact on mathematics planning as teachers consider behaviourist psychology when developing learning goals and planning lessons. Nevertheless, strict adherence to a behaviourist approach to mathematics learning in primary school is inappropriate because of its lack of consideration of the child's own thinking. Therefore teachers, when helping children learn mathematics, can draw on behaviourist ideas, but need to do so wisely and with an awareness of certain limitations. Keeping this point in mind, we will examine behaviourism a bit more closely.

A major tenet of behaviourism is that behaviour can be shaped through reinforcement — that is, through positive and negative stimuli or rewards and punishments. Teachers can use reinforcement (e.g. feedback on a student's performance, praise or criticism in class) to encourage students to practise desired behaviours (e.g. learning complex written computations) but feedback should be used wisely. It is well documented that meaningful practice has great value and helps to consolidate mathematics learning, but excessive practice, premature practice or practice without sufficient understanding is associated with negative effects. Such practice often leads to a fear or dislike of mathematics and an attitude that mathematics does not need to make sense. Educators know, however, that making sense of mathematics is a major goal of mathematics learning.

A behaviourist approach can be useful in helping children learn a fixed set of skills in a fixed order — that is, to help children acquire fluency. Many behaviourists believe that mathematics is sequenced in a linear fashion, where one idea builds on another and where proficiency in one skill is used to develop proficiency in the next skill. For example, the concept of multiplication as repeated addition suggests that students need to master addition before moving to the concept of multiplication. However there is more than one meaning of multiplication, not just repeated addition. Whilst the hierarchical view of mathematics learning makes it a popular candidate for a behaviourist approach, there are deeper issues to consider with respect to the connected and interrelated nature of maths.

When planning a lesson from a behaviourist perspective, the teacher would first attempt to accurately state the objectives, or goals, of the learning experience. Such statements give the teacher direction in planning lessons and give the students clear expectations — both valuable outcomes that are consistently supported by research. Then, once an objective has been clearly stated, the teacher would identify as many prerequisites as possible for achieving that goal. These prerequisites are the building blocks for planning the learning experiences. For example, the objective: use the formula $A = \frac{1}{2}\,ba$ to find the area of a triangle, has been stated by the teacher. Please note, this is quite different to the content descriptor: 'Establish the

formulae for rectangles, triangles and parallelograms and use these in problem solving' (ACARA v8.1). Some prerequisite questions would be as follows.

- What is area?
- What is a triangle?
- What do b and a represent?
- What is a triangle's base?
- What is a triangle's altitude?
- What do the symbols next to each other in the formula mean?

Clear answers to these prerequisite questions are helpful in order to reach the objective; however, the following questions could also be considered prerequisites.

- How do you multiply by a fraction?
- How do you multiply 2 whole numbers?
- What are the units of measurement in this case?
- Why are we working out this problem?

This example illustrates the difficulty in constructing a complete set of prerequisites for any objective in mathematics, no matter how simple the objective may seem. Furthermore, this approach does not take into consideration the students for which the lesson is being designed. The whole perspective of their prior learning and/or misconceptions that the students might have is not being considered. Of course, the teacher needs to consider prerequisites when preparing lessons, but should be guided by common sense, rather than by zeal to state every prerequisite.

From the behaviourist perspective, clarifying the goals for learning focuses the teaching on the desired outcomes. What behaviour does the teacher want the students to exhibit by the end of the learning experience? Hence, the process of forming behaviourally oriented objectives may be useful, but it ignores a more important goal — learning mathematics with meaning. Students may be able to demonstrate a desired behaviour without understanding what that behaviour means. They might also learn other concepts and processes not originally planned as a result of the learning experience undertaken.

If the objective of the lesson is to use the formula correctly to find the area of a triangle, the focus of the lesson is on lower-level cognitive outcomes, where the students are only expected to correctly identify and use the base and altitude of the triangle to calculate the area. If, on the other hand, the objective of the lesson is to explain how the area of the triangle is half the area of a specific rectangle or parallelogram, the students are directed toward a higher cognitive level of understanding, problem solving and reasoning about the area of the triangle. From this higher-level perspective, the question for the lesson includes connections between triangles with base and altitude, to rectangles and parallelograms with the same dimensions of length and breadth or altitude as shown in rectangle *BEFC* in figure 2.2). The focus of the lesson therefore shifts to encompass spatial relationships not just number and algebraic manipulation with a geometry application.

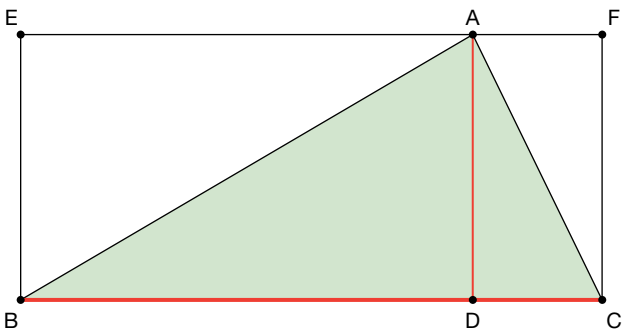


FIGURE 2.2 The area of triangle BAC is half the area of rectangle BEFC. The length of the rectangle equals the base of the triangle and the width of the rectangle equals the altitude of the triangle.

This example shows how emphasising behavioural outcomes for a lesson may result in lower-level cognitive procedures. Unfortunately, many mathematics programs include a hefty proportion of such outcomes, probably because they can be easily measured. The result is programs where students are 'shown' algorithms and mathematical relationships are 'illustrated' on the textbook pages. Unfortunately students' conceptual understanding and important connections are sadly underemphasised or omitted.

The attractive features of the behaviourist approach are that it provides clear guidelines, allows for short-term progress and lends itself well to curricula, which focuses on accountability. Knowing what outcomes will be assessed in standardised assessments gives teachers specific direction for designing their lessons; however, a real and constant danger in using a behaviourist approach is that it leads to a focus on simple, short-term objectives that are easily measured, easily forgotten and unconnected. These simple objectives can lead to short-term proficiency of specific skills, rather than to higher-level understandings and the development of connections that make knowledge meaningful and useful. The emphasis on short-term objectives often results in a de-emphasis of long-term goals and higher-level cognitive processes such as problem solving and reasoning.

Nevertheless, careful use of a behaviourist approach can result in consistent learner involvement and can even promote higher-level thinking in mathematics if designed creatively. Lesson outcomes can be identified that encourage children to develop concepts and use critical thinking. This possibility is illustrated by the use of behaviourally oriented verbs in stating outcomes, verbs such as *explore, justify, represent, solve, construct, discuss, use, investigate, describe, develop* and *predict*. Hence, the behaviourist approach leads to some useful ideas in teaching mathematics:

- behaviour can be shaped by reinforcement, drill and practice
- students can be supported in learning specific skills in a fixed order
- clear statements of objectives help teachers design lessons directed at specific learning outcomes
- clear statements of objectives and learning outcomes give students a clear idea of expectations.

Although identifying specific outcomes is an important part of mathematics teaching and learning, the constructivist perspective helps teachers focus more directly on helping students understand and make sense of mathematics.

Constructing understanding

During the first half of the twentieth century, William Brownell advanced a notion of meaningful learning that was a forerunner of constructivism. Brownell conceived of mathematics as a closely-knit system of ideas, principles and processes — a structure that should be the cornerstone for learning mathematics. Connections among concepts should be established to make 'arithmetic less a challenge to the pupil's memory and more a challenge to his [or her] intelligence' (Brownell 1935, p. 32).

Research has consistently confirmed that isolated 'learning' is not retained. Mathematics can and should make sense to learners. If it does, it will have meaning to learners and will be understood as a discipline with order, structure and numerous relationships — and will be likely to be called on in a variety of problem-solving situations. Meaningful learning provides the basis for mathematical connections and is an integral part of the constructivist perspective.

In addition to Brownell, Jean Piaget, Jerome Bruner and Zoltan Dienes have contributed to the growth of constructivism. Figure 2.3 summarises their frameworks for the learning process. Many of the major recommendations for teaching mathematics advocated by the Australian Association of Mathematics Teachers (2006) and the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, NCTM 2000) are based on their theories of how children learn mathematics. These documents provide strong support for changing from the traditional behaviourist approach to a constructivist approach. In fact, research has shown that students learn mathematics well only when they *construct* their own mathematical understanding.

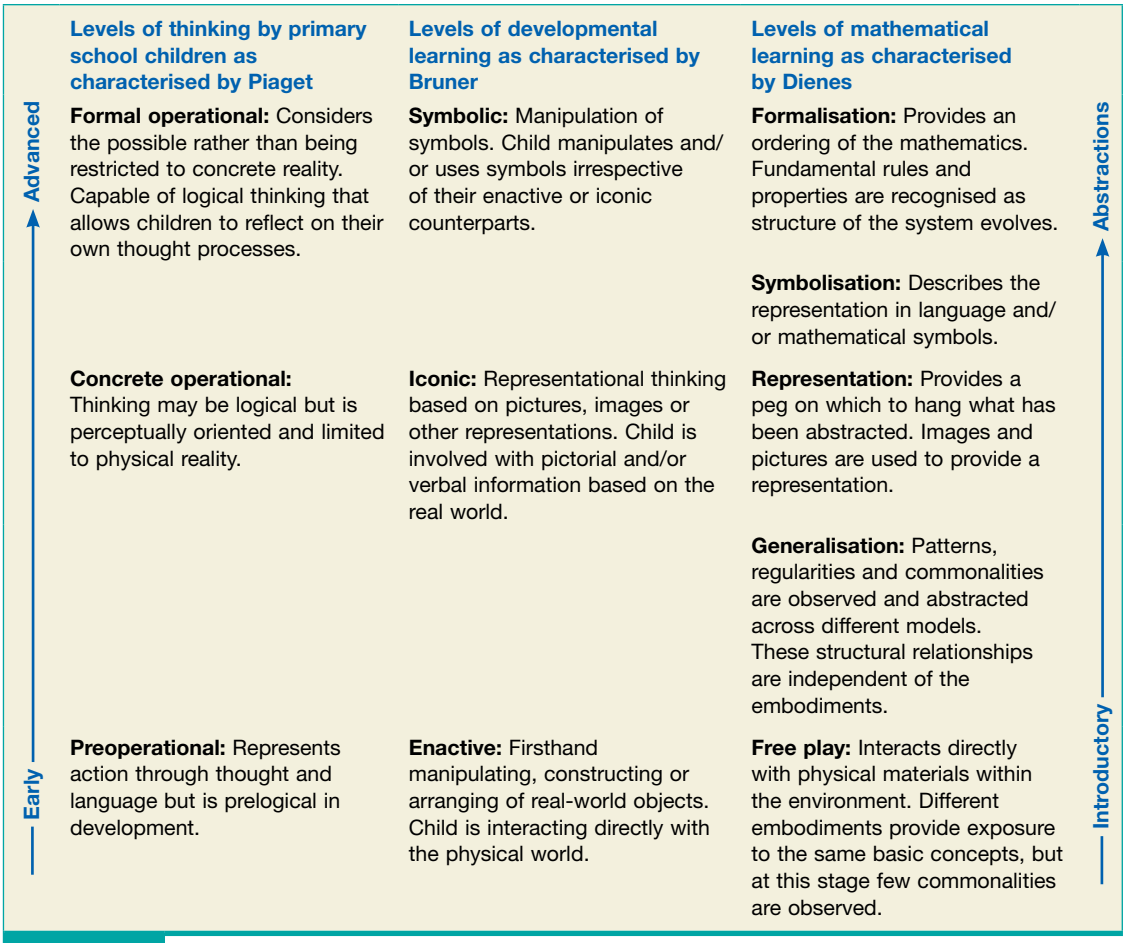


FIGURE 2.3 Frameworks of the learning process

What, then, does it really mean for students to construct their own mathematical understanding? It means different things to different people, but 3 basic tenets of constructivism help us answer this question.

1. *Knowledge is not passively received; rather, knowledge is actively created or invented (constructed) by students*. Piaget (1972) suggested that mathematics understanding is made (constructed) by children, not found like a rock or received from others as a gift.
2. *Students create (construct) new mathematical knowledge by reflecting on their physical and mental actions*. They observe relationships, recognise patterns and make generalisations and abstractions as they integrate new knowledge into their existing mental structure (Dienes 1960).
3. *Learning reflects a social process in which children engage in dialogue and discussion with themselves as well as with others (including teachers) as they develop intellectually* (Bruner 1986). This tenet suggests that students are involved not only in manipulating materials, discovering patterns, inventing their own methods or algorithms and generating different solutions, but also in sharing their observations, describing relationships they've found, explaining their procedures and justifying their processes and solutions.

To assist in *your* understanding of this new information, reflect upon something you have learned recently. You may want to think about what makes sense and where you are still unsure. Perhaps you will write a journal entry. Then discuss your examples and ideas with your peers or tutor. By actively

working through these ideas, reflecting on the process and discussing it with others, you will be more likely to add this new information to your personal knowledge bank than if you just memorised someone else's idea.

These tenets have significant implications for learning and teaching mathematics. They also suggest that, from the constructivist perspective, learning is a process that takes time and reflects a passage through several developmental stages. Research has established that each stage of children's cognitive development provides a window of opportunity for a range of learning activities in mathematics. At each stage, the lower limit of what children can learn is determined by the concepts and skills that they have already learned. The upper limit is determined by tasks that they can successfully complete with scaffolding or support from someone more skilled or knowledgeable. The Russian psychologist Lev Vygotsky referred to the child's *zone of proximal development* (ZPD) to describe this range of learning activities and experiences (Bay-Williams & Herrera 2007).

Research suggests that learning activities that fall within a child's zone of proximal development have a high probability of success, whereas activities outside the zone have much less likelihood of success. While we want to provide learning experiences that help students move to a higher level of thinking or to master a more advanced skill, we do not want the tasks to be so far beyond their reach that they cannot succeed and might subsequently give up. Vygotsky's notion of a child's zone of proximal development challenges the teacher to know his/her students well and to have a good understanding of the limits of their zone.

Learning is active and internally monitored; it is a process of acquiring, discovering and constructing meaning from experience. In this context, teaching mathematics with the use of concrete models and making connections with the children's experiences help them make sense of mathematics.

Whilst Piaget, Bruner and Dienes characterise children's levels of development somewhat differently (see figure 2.3), overall, their proposed frameworks are remarkably similar. A careful examination of these frameworks reveals 4 important observations about how children learn.

1. Several characteristic and identifiable stages of thinking exist, and children progress through these stages as they grow and mature.
2. Learners are actively involved in the learning process.
3. Learning proceeds from the concrete to the abstract.
4. Learners need opportunities to communicate their ideas with others.

2.4 How can we help children make sense of mathematics?

LEARNING OUTCOME 2.4 Making connections between the 4 recommendations and constructivist approaches to learning.

Teaching occurs only to the extent that learning occurs. Therefore, effective teaching of mathematics rests heavily on considerations about how children learn. The process of building bridges from the concrete to the abstract and helping children cross them is at the heart of good teaching — and it is a continual challenge.

A blend of learning theory, research, teaching experience and thinking about how children learn mathematics leads to practical recommendations for teaching mathematics. It is important to recognise personal experiences and prior knowledge in actively building new mathematical knowledge. What follows are 4 recommendations for helping children make sense of mathematics, based on the 4 previously listed observations, derived from the frameworks of Piaget, Bruner and Dienes. These recommendations provide a strong foundation for mathematics learning, and we extend and apply them throughout this text. No priority of importance is suggested by the order in which they are listed.

Recommendation 1: Teach to the developmental characteristics of students

As children grow, they progress through identifiable stages of development (see figure 2.3); they learn best when mathematical topics are appropriate for their developmental level. Topics should be presented in enjoyable and interesting ways that challenge children's thinking and allow them to add new concepts and skills to existing ones. Effective and efficient learning of mathematics depends on this type of teaching — it doesn't just 'happen'.

The learners in our classrooms have many diverse characteristics — each child is different from every other child. Nevertheless, there are common characteristics that are typical of children in a particular age group and it is useful for teachers to know about them. In this section, we focus on 3 types of development that influence mathematics teaching and learning: cognitive development, physical development and social development. *Cognitive development* has to do with how a child thinks and reasons. It also influences how the child learns new information. As educators, we probably pay more attention to our students' cognitive development than to any other development. The knowledge and skills we want children to learn are mainly cognitive in nature. *Physical development* has to do with the child's muscles and motor skills. If we want children to participate actively in mathematics, then a child's physical development must also be considered when planning lessons. *Social development* has to do with how children interact with others. It also helps describe their self-concept and how they feel about themselves as individuals and as members of a group. If the teacher designs lessons that require students to interact with them and with each other, then their social development and group dynamics need to be considered.

TABLE 2.2 **Characteristics of learners in the early years (ages 4–7)**

Cognitive characteristics (thinking and ways of learning)	**Suggestions for teachers**
• Piaget's preoperational stage • Centration — able to focus on only one idea or stimulus at a time • Irreversibility — unable to recognise the reversibility of changes to actions or objects • Begins to understand ideas beyond firsthand concrete experiences	• Use everyday experiences to connect mathematics concepts to children's life experiences • Break tasks into manageable parts • Encourage informal exploration, choice and experimentation with concrete materials • Use questions to stimulate thinking and challenge misconceptions • Begin using pictures and symbols to represent concrete actions
Physical characteristics (muscle and motor skills)	**Suggestions for teachers**
• Developing control of large and small muscles, fine-motor skills • Short attention span	• Provide short, hands-on activities with choice and opportunity for movement • Focus more on process than product • Provide materials that can be easily handled or manipulated
Social characteristics (self-concept and interpersonal skills)	**Suggestions for teachers**
• Egocentric — focuses more on self than others, talks at rather than with others • Developing sense of self and abilities • May have a best friend and exclude others • Learning to express oneself and make decisions	• Encourage friendship skills, sharing, taking turns and working with another person • Provide positive support, challenge and feedback • Allow children to make their own plans for manipulating things in the environment and solving problems

Source: Kostelnik, Soderman & Whiren (2004).

Children aged 4–7 are in the early years (Foundation to Year 2). Children in this age range are rapidly developing cognitively, physically and socially. They are active learners. With each year of growth, their cognitive ability to understand concrete, pictorial and abstract ideas increases. Their physical abilities to focus, control muscle movements and demonstrate motor skills also increase during these years. Socially, they are learning to understand and express themselves and to understand and interact with others. Table 2.2 summarises some key ideas to keep in mind when teaching mathematics to children in the early years (Kostelnik, Soderman & Whiren 2004).

Children aged 8–11 are in the primary years (Years 3–6). Children in this age range are becoming more independent learners. Their thinking is becoming increasingly sophisticated, as is their physical and social development. These children are also active learners who enjoy the challenges of working independently and with others. Table 2.3 summarises some key ideas to keep in mind when teaching mathematics to children in the primary years (Kostelnik, Soderman & Whiren 2004).

It is useful to consider the general developmental characteristics of children in primary school and how this may impact mathematics teaching. Keep in mind, however, that there is a big difference in development between children at the beginning of these age ranges and those at the end. Note that tables 2.2 and 2.3 include suggestions that teachers need to consider when planning experiences that support learners at particular developmental levels.

TABLE 2.3 Characteristics of learners in the primary years (ages 8–11)

Cognitive characteristics (thinking and ways of learning)	**Suggestions for teachers**
• Piaget's concrete operational stage • Able to decenter, can focus on part/whole • Understands reversibility of actions or objects • Able to classify and sequence by attribute • Can use logic and concrete objects to solve problems	• Use concrete materials and link to symbols • Provide opportunities to manipulate, classify and sequence both objects and numbers • Explore multiple representations and strategies • Have students regularly explain thinking
Physical characteristics (muscle and motor skills)	**Suggestions for teachers**
• Able to complete more complex physical skills	• Provide experiences that promote active, physical involvement • Provide opportunities for children to use real tools and materials, make models and diagrams and conduct experiments
Social characteristics (self-concept and interpersonal skills)	**Suggestions for teachers**
• Growing more independent • Small groups of friends have major influence • Hard to accept failure or criticism	• Provide opportunities for problem solving and projects • Provide opportunities for group work • Provide positive support, challenge and feedback

Source: Kostelnik, Soderman & Whiren (2004).

Recommendation 2: Actively involve students

Actively involving students in the learning process encourages them to make sense out of what they are doing and thereby develop greater understanding of mathematics. There is an ancient Chinese proverb that emphasises the importance of active involvement for students as they construct their own mathematical meaning: I hear and I forget; I see and I remember; I do and I understand.

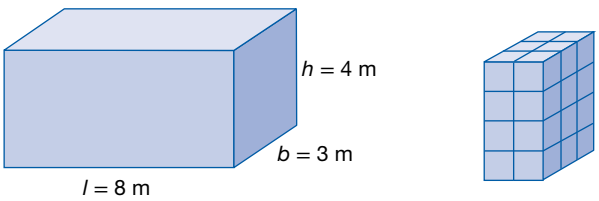
Active involvement sometimes requires physical activity but it always demands mental activity. One of the daily challenges of teaching is to do the things that encourage and reward active involvement. These include directly interacting with the children, offering them hands-on experience with concrete materials, having them interact with learning materials such as print and technology-based materials and having them reflect on what they have done, thought and experienced in class. Broadly speaking, we can classify such methods of active involvement into 2 categories: (1) exploration and sense making and (2) reflection and use of metacognition.

Encouraging student exploration and sense making

One way to actively involve students is to encourage physical involvement. For example, a group of students, Foundation to Year 6 (aged 5–11), have been asked to solve the handshake problem. If the group were to meet for the first time today and everyone was to shake hands with everyone else, how many handshakes would there be? First, establish the number of children in the group. For upper primary students, 25 is a good number; for younger students a group of 4 is recommended as a starting place. Next, clarify that each person shakes hands once with every other person. For example, Jackson shaking hands with Sarah is 1 handshake not 2. Allow students to solve this problem any way that makes sense to them.

There are a number of strategies for solving this non-routine problem. With the teacher available to help scaffold their method, a small group could demonstrate how many handshakes by acting it out! From there, individual or small groups of students might draw a diagram, write their data in a table, look for a pattern or create a number sentence and find a rule. Whatever method chosen, each learner will need to make sense of the process. At that this stage of development, students wouldn't jump to the mathematical equation of $(25 \times 24) \div 2$ through pure logic — that is, 25 people shake hands with 24 other people but they don't shake hands twice so divide by 2 (hence 300 handshakes). Whatever their age, the method(s) students use might surprise you! When the students have arrived at a solution that makes sense to each and all of them, the solution strategy they have explored should be discussed with others as it will help them remember the problem and the method. The stages of problem solving can be elaborated to see if everyone in the group did actually understand, do and check.

Another example would be a lesson on volume for Year 5 to 7 students (aged 11–13). A traditional approach might include giving students the formula for finding the volume of a rectangular prism: V = *length* × *base* × *height*. The teacher could show students drawings of various rectangular prisms and ask them to use the formula to compute the volume of each.



In the first example, Volume = $8 \times 3 \times 4 = 96$ m^3 and in the second example that has cubic units marked in, Volume = $2 \times 3 \times 4 = 24$ units^3.

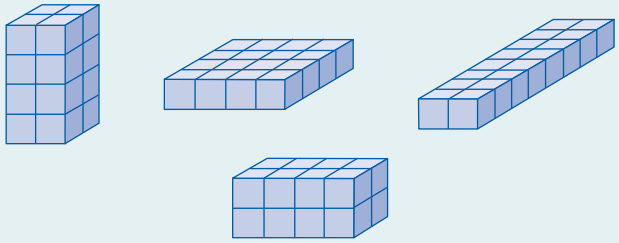
A more active, developmentally appropriate approach would help children understand volume through an activity such as In the classroom 2.2, where children are given a fixed number of cubes, say 16 blocks, and asked to build different buildings, boxes or prisms. As they build the structures, they begin to relate the dimensions of the boxes to the volume of a prism. Although finding the volume of rectangular prisms is the main objective for this activity, students learn that different arrangements can have the same volume. Furthermore, they come to understand why they need to multiply the dimensions of a box to find the volume, so the formula for the volume of a box evolves naturally from this active involvement.

IN THE CLASSROOM 2.2

Volume of prisms with blocks

Objective: Calculate volumes of rectangular prisms and draw different views of prisms (ACARA v8.1, Year 7)

Each of these boxes holds 16 blocks. Record the length, width, height and volume (number of blocks) in the table.



Length	Width	Height	Volume

- Using 24 blocks, estimate first, how many different boxes you can make. _____
- Make as many different boxes as you can. Record the length, width, height and volume in the table. How many different boxes did you make? _____

Length	Width	Height	Volume

- Imagine you have 4 piles of blocks. You must use all the blocks in each pile to make boxes.
  Pile A has 16 blocks. Pile B has 25 blocks.
  Pile C has 36 blocks. Pile D has 50 blocks.
  - Which pile of blocks would let you build the most different boxes? Why?
  - Which pile would let you build the fewest different boxes? Why?
- Build the boxes, fill in a table, like the one above for each pile and see if you are right. What did you find out?
- What patterns do you see in your tables? _____
  _____

An activity like this involves children in using models, making decisions and thinking about mathematics, rather than just methodically applying a formula. It also lets them physically experience important mathematical properties. For example, using 16 blocks they build a $4 \times (2 \times 2)$ box and they build a $(4 \times 2) \times 2$ box. Since both boxes have 16 blocks and the same dimensions, they are demonstrating that $(4 \times 2) \times 2 = 4 \times (2 \times 2)$. The boxes are the same, just oriented differently. Similarly, as they continue to build boxes with 16 blocks, they construct models that show $(2 \times 2) \times 4 = (2 \times 8) \times 1$ and, more important, they practise number facts and develop algebraic thinking. Comparing the teacher-directed approach with the student-centred exploratory approach, demonstrates that actively involving students in discovering the method (or formula) will be more likely to result in increased understanding and retention. It is easier to remember something we've figured out for ourselves than something we have been told how to do with no meaning and possibly no connection to previous learning.

Encouraging student reflection and the use of metacognition

Another way to actively involve students is mentally, that is by helping them develop their metacognitive skills. Metacognition refers to thinking about one's own thinking (i.e. cognition about cognition). It includes what one knows or believes about oneself as a learner and how one regulates and adjusts one's own behaviour. Children need to become aware of their own strengths, weaknesses and typical behaviours and of the repertoire of procedures and strategies they typically use to learn and do mathematics and, more specifically, solve problems.

In a sense, metacognition means looking over your own shoulder — observing what you do as you work and thinking about what you are thinking. Students who monitor their mathematical thinking search for understanding and strive to make sense of the mathematics being learned. Competent problem solvers are efficient at keeping track of what they know and of how successful or otherwise their attempt to solve a problem is proceeding. They continuously ask: What am I doing? Why am I doing it? How will it help me? Is there a better way to do this?

Research suggests that what students know or believe about themselves as mathematics learners not only greatly affects their performance but also influences their behaviour as they do mathematics (Bransford, Brown & Cocking 2000). Children who are skilful at metacognition can learn, for example, that practice improves their performance on particular tasks, that drawing a picture often helps them understand a problem or that word problems are difficult to decipher at times because of the language used. This type of metacognitive knowledge often helps students to control and adjust their behaviour. For instance, if Shauna knows that she frequently makes keystroking errors with her calculator, she would be more likely to estimate her answer first, work slowly when keying in and check the reasonableness of her answers.

The development of metacognitive skills requires that children observe what they know and what they do and reflect on what they observe. Encouraging students to think about their thinking is an important aspect of mathematics teaching and learning. Here are a few things the teacher can do to help students develop metacognitive awareness.

1. Help students be explicit about how they work when solving problems. For example, consider asking questions like: How did you know which was the relevant information? Why did you use that strategy? Did you check your solution? Teachers routinely consider such questions themselves when solving problems, but if they present complete, polished solutions to their students, it could inhibit students from discussing their own thinking strategies. There is value in helping children become aware of the behind-the-scenes thinking and strategising. When teachers share their own thinking, they provide a model for students.
2. Prepare students to accept and be aware of common problem-solving experiences by pointing out various aspects of problem solving — for example, some problems take a long time to solve and some problems can be solved in several different ways.
3. Encourage students to become more aware of their strengths and weaknesses and to think about their own mathematical thinking. For example, asking students to consider the following: What mathematics problems do you like best? Why? What mathematics problems are the most difficult for you?

Some educators recommend that teachers have students with special needs brainstorm how they learn best and then help them to use those strengths (Diezmann, Thornton & Watters 2003). This may include helping them use specific learning approaches or styles, helping them use specific methods for communicating their thinking and their approaches to problem solving, and encouraging them to rephrase tasks or problems in their own words.

Recommendation 3: Move learning from concrete to abstract

When thinking about how learning should move from the concrete and pictorial to the abstract, it is important to keep in mind that *concrete* is a relative term. To one child, joining 2 blocks and 4 blocks is

concrete, but 2 + 4 is not; another child may view 2 + 4 as concrete and $x + y$ as abstract. Symbols and formal representations of mathematical ideas follow naturally from the concrete level, often via pictorial representation but only after conceptualisation and meaningful understanding have been established. Without such understanding, children do not feel comfortable working with mathematical symbols and the mathematics does not make sense to them. As Bruner (1986) noted in his framework, children need opportunities to work with objects in the physical world before they will be ready to work with pictures and other representations. After working with pictures and other representations, children will be ready to work with symbols.

Using concrete materials and models

Concrete materials and models are critical throughout primary school for helping students learn mathematics. Mathematical ideas are abstract by their very nature, so any model that embodies them is imperfect and has limitations. However, even though the model is not the mathematics, models provide a context for the mathematical concept under consideration. Research shows that learning occurs best when students have a meaningful context for the mathematical knowledge and understand fundamental relationships associated with the knowledge. Helping children establish meaningful connections between the model (context) and the mathematics can be challenging but is well worthwhile.

Suppose the teacher is developing the concept of a circle. A plate might be used as a model to illustrate the concept of a 2-dimensional shape, but it would also illustrate other mathematical concepts such as area, boundary, circumference and possibly diameter. When a concept is first being formed, the learner has no way of knowing which attributes characterise it. For example, in this case,, the learner may think that irrelevant attributes matter (e.g. the design on the plate, its attractive finish or an imperfection) and may even fail to take in the relevant attributes of shape. The learner may end up not being at all clear about what characterises a circle.

The teacher could introduce a model such as a coin, but students might still focus on the interior rather than the boundary. In contrast, models such as a bike tyre and a ring would reinforce the roundness associated with a circle but would also make it clear that a circle is associated with the outer edge or boundary of the models (see figure 2.4). Research has shown that directing children's attention to the important attributes enhances learning, so it might help to take a piece of chalk or a water-soluble pen and trace around the outer part of the coin or plate to highlight the circle.

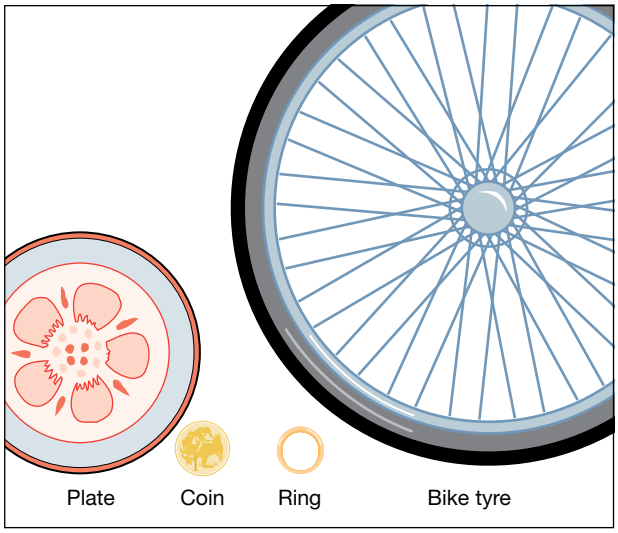


FIGURE 2.4 Some models of a circle

The use of perceptually different models, such as those shown in figure 2.4, is called *multiple embodiment*, or *multiembodiment*. Research has shown that transfer across contexts is difficult if children have experienced the concept from a single viewpoint rather than in multiple contexts. The more different the models or contexts, the more likely students are to focus only on the common characteristics and make the correct abstractions. The teacher cannot expect children to abstract correctly from a single model in mathematics. Multiembodiment helps students abstract or generalise appropriately. It also decreases the likelihood that children will associate a mathematical concept with just one particular model, which is a danger whenever a single model is used to illustrate the concept.

Mathematical learning depends heavily on abstraction and generalisation. Multiembodiment rests on the value of experiencing a mathematical concept in a variety of different physical contexts. Each embodiment has many different attributes. Children need to experience many different embodiments (along with models that do not embody the concept) before they can recognise which attributes all the embodiments share, allowing them to make the correct generalisations.

Students also need to recognise that even embodiments of the same kind have variable characteristics. For instance, consider again the plate shown in figure 2.4 as an embodiment used to develop the concept of a circle. To help children realise that they should ignore such features as size, design and imperfections, different kinds of plates could be shown. Pointing out features that can vary encourages children to focus their attention on a feature that does not vary — the roundness of the plate. Furthermore if shown a square or oval-shaped plate that illustrates the difference — that is, by using examples that are not circular (*non-examples*) the attribute of roundness will be made clearer. Multiple examples of a model can increase the likelihood that learners will focus on the mathematically significant attributes. Multiple examples will also help in overcoming misconceptions, which frequently occur.

Similarly, the teacher could use rings of different designs, styles and materials to show children that these attributes don't matter and help them recognise that in each case a ring models a circle. However, some rings don't model a circle, such as the 'adjustable ring' shown here.



This ring does not model a circle because it is not a closed figure. Non-examples play an important role in concept formulation. Students learn more when presented with a combination of examples and non-examples of a mathematical concept than when shown examples alone.

Making formal representations follows conceptualisation

Children need many experiences with concrete models and pictorial representations before they can work meaningfully with abstract representations and symbols. Initial experiences with models helps children to develop conceptual knowledge about the topic, as does visualisation and talking about the topic. Finally, the teacher can show children how symbols are used to record what has been expressed orally about the models or pictures.

How long should a model be used? That depends on both the student and the mathematical content. In general, too little time is spent with models. That is, students are rushed (or dragged) too quickly through first hand experiences with concrete materials and pictorial representations and then confronted with symbolic representations. Students need to feel comfortable with a model and both observe and talk about the key mathematical features it embodies. It is a good idea to keep concrete models available for students somewhere nearby and visible. They will know when they no longer need a model and will replace it with a faster method. They will also know that if some idea is not making sense thet the model or picture can support their thinking and use it as needed. At this stage, leaving the model doesn't mean that they will never use it again. The same model may be used at various levels throughout primary school to help children develop new or more sophisticated concepts.

Recommendation 4: Use communication to encourage understanding

The use of models, manipulatives and real-world examples provides many opportunities for thinking, talking and listening. The importance of communication in mathematics learning is demonstrated by the fact that communication is one of the main aims of the Australian Curriculum (ACARA v8.1). When children explain their approaches, talk about mathematics, make conjectures and defend their thinking orally, as well as in writing, their understanding deepens. Talking and writing about mathematics are essential in learning mathematics.

Using oral communication to encourage sense making

Lev Vygotsky, who originated the concept of the zone of proximal development (ZPD) is also well known for his ideas about the role of language in learning. Vygotsky believed that learning is a social experience — that is, interactions with others challenge learners to make sense of new ideas. Socio-constructivist theory supports the idea that teachers need to provide children with frequent and regular opportunities to talk about their thinking. Often, communicating our thoughts to others can clarify them for ourselves.

Students at all levels should talk about mathematics before being asked to communicate about mathematics symbolically. Just as speaking precedes writing for children, so should talking about mathematics precede writing about it using abstract representations. Both student-to-student communication and student-to-teacher communication are important in the learning process.

This kind of talking between students is natural and gives them many opportunities to use their own language to explain what they did, justify their thinking and share their methods. Teachers need to take every opportunity to stimulate and encourage the process of student-to-student communication, even if much of it takes place outside of immediate hearing.

In the following example Year 4 students have been exploring prime and composite numbers using square tiles as a model. They were asked to build squares or rectangles for the numbers 1 to 20 and describe what they have found and why they think the results are interesting or unusual. Students were challenged to describe any patterns they observed and predict theories about the numbers based on the model of tiles. The recording was left open to the students, but centimetre square dot paper was provided. These pictures were recorded by Rob for 1, 2, 3, and 4 tiles. The language used by Rob to explain his intuitive discovery to his teacher was similar to the following.

Rob: So every even number is composite, that is, you have more than 1 way of showing it.

Teacher: Every even number? What about 2?



This question stimulates additional thinking about the first 4 numbers and encourages Rob to justify his generalisation or to modify his theory. When the whole class was asked ‘Have you made any discoveries?’ their findings included the mathematical language: odd, even, prime, composite, square numbers, factors, rows, columns and areas. Questions are a vital element in the learning process. Students should ask each other questions, ask their teacher questions, and teachers should ask students open-ended questions. Such interactions give students opportunities to talk about their ideas, receive feedback on their thinking and hear other points of view. Thus in a socio-constructivist classroom students learn from one another as well as from teachers.

Using written communication to convey thinking

In writing about mathematics, students provide insight into what they are thinking and what they understand. Just like talking, putting our thoughts in writing helps us to think more deeply and clarify our thoughts. Young children who are still developing their writing skills can communicate in writing by drawing pictures or dictating their thoughts to a more skilled writer. Reflective writing (or journals) can be used frequently in mathematics for benefit of both the teacher and the students. The teacher can give prompts to help students get started. For example:

Today I learned about/found out that . . .
I was surprised about . . .
I discovered that . . .
Something I'd really like to know more about is . . .
I think the answer is . . . because . . .
Another way to do it is . . .
The thing I liked best was . . . because . . .
I still don't understand . . .

Additionally, after students have had experience with a concept through the use of models, pictorial representations and oral language, they can begin to learn the formal mathematical notation to communicate those ideas. Using conventional symbols to represent mathematical concepts is a valuable part of being able to communicate mathematically. In the example above, the teacher might show Rob the number sentence $1 \times 4 = 4$ and the sentence $2 \times 2 = 4$ which are both correct. The student can see in pictures and symbols that there is more than one arrangement for the number 4. Hence, it is a composite number and indeed special as it is a square number.

Talking and writing about mathematics makes it come alive, become more personal, and it keeps student interest high. For the teacher, reading students' writing carefully and listening with intent to what students say, as well as noticing what they do *not* say, helps the teacher tailor their learning experiences to all the unique individuals in their class.

MAKING CONNECTIONS

For many years, standards published in Australia (AAMT) and overseas (e.g. NCTM) have included recommendations for effective mathematics teaching that reflect what we know about how children best learn maths. They include making mathematics more meaningful and teaching mathematics through problem solving. As new curricula and teaching and learning strategies are introduced that reflect these recommendations, we must also consider how the diverse learners in our classrooms will need support to be successful learners of mathematics. Mathematics sometimes serves as a gatekeeper to advanced educational opportunities and careers therefore all students need to be successful at mathematics.

As mentioned at the beginning of this chapter, there are many ways in which the students in our classrooms are diverse. When we consider diversity among our students, we need to be aware of social, racial or ethnic background. The Australian Curriculum: Mathematics (ACARA v8.1, General Capabilities) states that intercultural understanding is gained when students value their own cultures and beliefs and those of others. Mathematics itself is rich in culture and history and there are many similarities in language, symbols and expression but there can be differences in reasoning and applications, including technology. It is important that students recognise similarities and differences, and create connections and cultural respect as they interact with people of diverse cultures.

The majority of children commence school with abundant intuitive mathematical understandings in many concepts, which have been gained in their everyday life. This is particularly true in Aboriginal and Torres Strait Islander communities where children are well grounded in, for example, concepts of space, number, pattern, time and distance. Through appropriate teaching and learning experiences it is possible for all students to explore and appreciate the knowledge of Aboriginal and Torres Strait Islander cultures.

In a similar way, students in primary school can be provided with teaching and learning experiences that recognise the contributions made by mathematicians from the Asia–Pacific region.

Another cultural characteristic of students, which impacts on their learning, is socioeconomic disadvantage. Lubienski (2007) researched how students from low- and high-socioeconomic backgrounds responded to a mathematics curriculum that was problem-centred. She found that students were more resistant to learning mathematics through problem solving and discussion if they were socioeconomically disadvantaged. They preferred to have the teacher explain how to do it or give the answer. They were sometimes confused by class discussions because they weren't sure which ideas were right or wrong. They also focused more on the real-world aspects of a problem and missed the mathematical idea being presented. As a result of her research Lubienski cautions us to be careful not to stereotype students from low socioeconomic backgrounds, but to be mindful that the way children have been raised may impact on their approaches to learning and their response to curriculum changes.

Lubienski provides 4 recommendations for teachers. These recommendations overlap with best practice principles already discussed in this chapter. First, we need to push for meaningful learning rather than rote memorisation. Students are less likely to forget content they have learned meaningfully. Second, teachers need to be sure students are learning what was intended from the problems presented in the curriculum. Teachers need to provide scaffolding so students do not rely on explicit teacher instruction. Third, teachers need to analyse achievement data and identify important topics that are causing difficulty for struggling students so that remediation can be provided. Finally, she suggests that teachers of students from disadvantaged backgrounds need to advocate that those students have access to the highest-quality teachers and curriculum. An implication for teacher change is that, teachers working with a large variety of students need to be open minded and learn all they can about the culture of socioeconomic groups to which they do not belong.

Finally, The Australian Curriculum: Mathematics (ACARA v8.1, Student Diversity) states that the diverse needs of students living with disabilities, students who are gifted or talented, and students with English as an additional language or dialect are to be considered when adjusting the curriculum, to provide additional support and time, and/or extension.

A GLANCE AT WHERE WE'VE BEEN

Children in classrooms today are increasingly more diverse. Teachers need to create a positive classroom climate to support their diverse learners. This includes avoiding negative experiences, establishing clear expectations, showing positive attitudes toward mathematics, treating aptitudes as equal and working to improve student retention.

Mathematics learning can and must have meaning. This statement is the cornerstone of all teaching. Understanding and fluency are essential elements of mathematics learning as are the processes of problem solving and reasoning. Mathematics learning is a slow process that requires years of development. Many individual differences exist amongst learners and the rate of learning varies greatly. Given these variables, the essential role of teachers is to help children construct mathematical knowledge that is meaningful to them at their stage of learning. Both behaviourist and constructivist learning theories can assist teachers to plan appropriate learning experiences, establish an inviting learning environment; and organise the classroom to ensure that all children are actively participating, experiencing, learning, abstracting and constructing mathematics that is meaningful and relevant to them.

Research and learning theory suggest that children move through distinct stages of development and that there is significant value in children being actively involved in constructing their own knowledge. Learning moves from concrete to abstract and communication plays an important role in facilitating the construction of new knowledge. In addition to hands-on experiences, children learn from describing, explaining, clarifying, making conjectures and reflecting on what they have done. They also learn from exploring, watching, listening, reading, applying ideas, giving and following directions, imitating and practising. All of these experiences and more contribute to learning mathematics. The responsibility of deciding the proper balance of learning experiences rests with teachers.

THINGS TO DO: FROM WHAT YOU'VE READ

1. Reflect upon diversity. What does it mean to you? What are the benefits and concerns of teaching in a classroom filled with diverse learners?
2. Select one of the methods for creating a classroom climate supportive of diverse learners. Explain how you plan to carry this out in your classroom.
3. Provide some mathematics examples to help distinguish between fluency and understanding.
4. Defend the statement in figure 2.2 that triangle *BAC* has an area that is half the area of rectangle *BEFC* using the different regions in the diagram.
5. Examine the learning frameworks in figure 2.3. Explain how the models proposed by Piaget, Bruner and Dienes are alike. How are they different?
6. Identify the 3 tenets on which constructivism is based.
7. Select table 2.2 or 2.3. Describe how this information will influence your mathematics teaching for junior or primary year students.
8. What is metacognition? Examine the activity described in In the classroom 2.1. Describe how metacognition might be used to think about your thinking during that activity.
9. Write about the recommendations for helping children make sense of mathematics making connections to the 3 tenets of constructivism.

THINGS TO DO: GOING BEYOND THIS TEXT

IN THE FIELD

1. *Three dimensional patterns with blocks.* Implement In the classroom 2.2 with a student in Year 5 using $2 \times 2 \times 2$ wooden or plastic cubes but make it completely open ended. Ask how many different structures (boxes, solids or prisms) can you make with 24 cubes? Ask them if their prism is the same or different in another position? Would it make sense to try this activity when blocks were not available? How can recording paper (cm dot or isometric paper help or hinder this activity? Why?
2. *Concrete materials.* Identify models and materials that are available for students to use in a maths classroom during a school visit. Did you see them being used? If so, what was being used and how?
3. *Equity in whole-class lessons*. Observe a mathematics class and be sensitive to the verbal and physical clues that suggest that there is differential treatment (boys/girls, students from different racial, ethnic, social backgrounds). If you observe evidence, of differential treatment, discuss what is happening and how you feel about the situation.
4. *Student attitudes.* Ask a child some questions to learn about his or her attitudes and perceptions about mathematics. There are various mathematics attitudinal surveys available online for students of different ages (Kaput Center for Research and Innovation 2011, Anderson 2007, Whitin 2007).

   Alternatively, use these or similar questions: If I say, 'Let's do some mathematics'. What would you do?; 'What school subjects do you like most? Are you good at them?'; Do you think knowing mathematics will help you when you grow up? Explain why; Do you think your teacher likes to help children learn mathematics? Explain why.

   In your report, summarise what you learned about the child's feelings about mathematics.

IN YOUR JOURNAL

5. Reflect on your own experiences of learning mathematics when you were in primary school. Describe how you learned mathematics.
6. Start a mathematics journal with 2 main columns. Firstly record activities that you undertake over a 24–48 hour period that involve mathematics. In the second column write down the mathematical

concepts and processes you were using when doing these activities. You can use the Australian Curriculum as a guide to consider the strands and content areas as well as the problem solving, reasoning and communicating that you did. Finally if you choose to write a summary at the end of each day about your feelings; what areas were you confident and competent?

7 Consider the 2 learning theories, behaviourism and constructivism. Describe personal experiences that suggest the influence of these theories on your learning of mathematics.

WITH ADDITIONAL RESOURCES

8 Examine an article discussing gender/race/culture as related to learning mathematics. Describe significant issues raised and implications for you as a teacher.

9 Access and read the article 'Trends and factors concerning gender and mathematics in Australia' (Vale 2008). Choose 1 of the 3 sections: Achievement, Affect or Participation and discuss implications for mathematics teaching and learning.

10 Examine the book *International Perspectives on Gender and Mathematics Education* (Forgasz et al. 2010) and read one of the articles of most interest to you. There are 18 contributions in this book. Identify why the issue you have read about is significant for mathematics education.

11 Examine the book by Reimer and Reimer in the Children's literature connections and report how you might use this resource in a classroom. Do you think this personalisation helps promote more interest in learning mathematics? Do you think helping children learn more about women's contributions in mathematics is an important aspect of teaching and learning? Explain.

WITH TECHNOLOGY

12 Spend some time with the Australian Association of Mathematics Teachers: Top drawer for teachers (http://topdrawer.aamt.edu.au) and go into one of the topics in more depth. You will find much more under each category Big ideas, misunderstandings, good teaching, assessment, activities, downloads and acknowledgements. Use this resource to help you plan a unit of work (for example, 5 or 10 sequential lessons) for a particular year level.

13 The website maths300 (http://maths300.com/about.htm) has been developed over time to inspire an approach to learning maths that works with the 4 recommendations discussed in this chapter. These exemplary lessons have been developed by teachers for teachers and build in exploration, communication and technology. Choose one of the free lessons in the sample tour (e.g. Temperature graphs (for the topic of data representation) and read and analyse the information given. Discuss the benefits of the lesson ideas and why the students would be working with a socio-constructivist approach.

14 Visit the Australian Council for Educational Research (ACER) website (www.acer.edu.au) and browse the publications, current articles and other useful links. When you are ready put 'student diversity' into the search engine and notice what comes up. Report on something related to students with special needs, Indigenous perspectives or equity and mathematics that you find on this site.

15 Access the Dr. Math website (http://mathforum.org/dr.math/) and review the questions that primary teachers and students send for Dr. Math to answer. Algebra and geometry feature highly in the questions that students ask. Why do you think this is so? What would you ask Dr Math and why?

CHILDREN'S LITERATURE CONNECTIONS

Disney: Donald in Mathmagic Land.
Can be streamed from the internet. This classic movie animation opens students (of all ages) to the history, beauty and complexity of mathematics.

McKinney, S. & Hinton, K. V. (2010). *Mathematics in the K–8 Classroom and Library*. ABC-CLIO, LLC: Greenwood Publishing.
Also available as an eBook. This reference includes 45 books with suggestions for teacher's use in the strands of Number, Algebra, Geometry, Measurement, Data & Probability, and across the curriculum.

Perl, T. (1978). *Math Equals: Biographies of Women Mathematicians + Related Activities*. Menlo Park, CA: Addison-Wesley Publishers.
Each chapter presents a short biography of women of diverse cultures and from different historical periods. The activities are great for middle year students or pre-service teachers.

Reimer, L. & Reimer, W. (1997). *Mathematicians Are People, Too: Stories from the Lives of Great Mathematicians* (2 volumes). New York: Pearson.
Share moments of mathematical discovery experienced by famous mathematicians with your students. Use these stories to demonstrate that both males and females are mathematicians, that mathematicians come from a variety of cultures and that the work of mathematicians is applied to many diverse fields. Use this book with children in Years 3–7 to experience the stories of great mathematics, including Pythagoras, Galileo, Pascal, Albert Einstein and Ada Lovelace.

Scieszka, J. & Smith, L. (1995). *Math Curse*. New York: Viking.
This book shows us that maths is everywhere. With humour and great illustrations, topics are presented in this book, such as fractions, counting, dividing, Fibonacci numbers, data, shape, problem solving and measurement, to name just a few. This book lets children appreciate mathematics in a new way. This book can be used at all levels of mathematical learning.

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ACKNOWLEDGEMENTS

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Tables 2.2 and 2.3: © *Helping Children Learn Mathematics* 9th Ed., Reys, Lindquist, Lambdin, Smith, John Wiley & Sons, 2009, p. 26

CHAPTER 3

Planning and teaching

LEARNING OUTCOMES

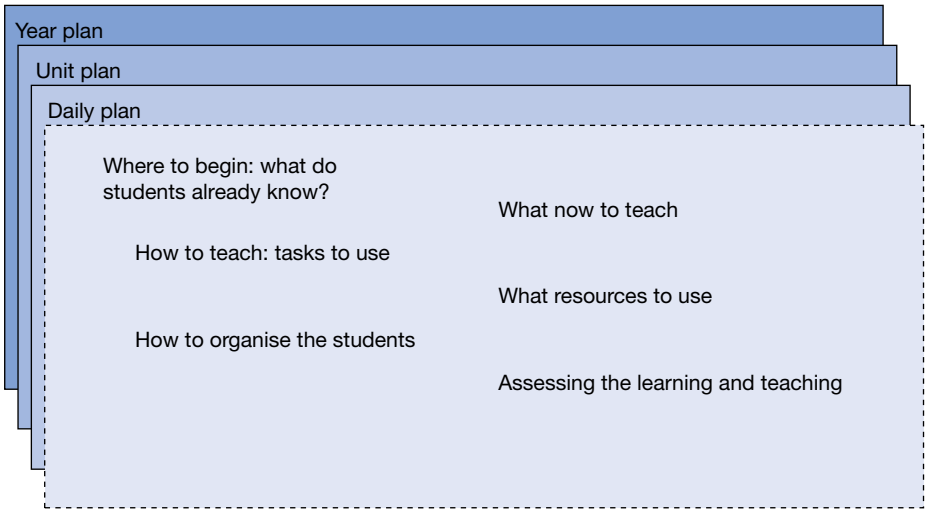
3.1 Identifying what questions a primary mathematics teacher should answer before beginning to plan.
3.2 Understanding why mathematics lessons must be carefully planned.
3.3 Recognising what levels of planning the teacher must create.
3.4 Identifying the 3 types of lessons used to teach mathematics, and the purpose of each.
3.5 Establishing how teachers can meet the needs of all students.
3.6 Understanding how teachers can integrate planning with assessment and analysis.



‘Tell me and I forget. Teach me and I remember. Involve me and I learn.’

Benjamin Franklin

Chapter 3 concept map



Introduction

The teacher's role in the classroom involves 4 important elements: planning, teaching, assessing and analysing. In this chapter, we focus on the planning and teaching elements. First, we discuss what constitutes strategic planning and then explain the important components that are a part of this process including addressing questions such as: what, how, when, where and why of planning. We then discuss levels of planning (for the year, for units, for lessons) and a range of ways to differentiate to meet the needs of all students. Recognising and appreciating the importance of planning evidence and its usefulness for informing your teaching is also interconnected with these discussions. This chapter also briefly touches on assessment and analysis (covered in detail in a later chapter). In coming chapters, we explain how to help children learn specific mathematics concepts and skills, and we elaborate on each of these teaching functions. This chapter then, serves as an organiser for much of what you will learn throughout the rest of this text.

The chapter makes general recommendations for planning for mathematics teaching in today's global and technology-rich society and provides contextualised suggestions for teaching mathematics in that context. Like every other chapter in this text, this chapter reflects our acceptance of the idea that children actively construct their own new understandings as they build on what they already know.

3.1 Effective planning and preparation for teaching: using strategic questions to inform teaching practice

LEARNING OUTCOME 3.1 Identifying what questions a primary mathematics teacher should answer before beginning to plan.

Effective teaching of mathematics that encourages successful mathematics learners requires of the teacher a sound knowledge of the content, 'the what', and knowledge of how to teach 'the what' in ways that support all learners. Knowing how to plan for these critical aspects is vital to children's learning and future education. Each year that a teacher teaches a child, that year provides a step up from what was learned in previous years (ACARA 2011). However, so that your children can achieve this growth,

several important questions must be asked and answered by you as their teacher — before you write lesson plans and before instruction begins. We discuss these questions and answers in the following sections.

What mathematics content knowledge and pedagogical content knowledge do I know and need?

As a teacher of mathematics, you will be required to teach concepts that you already know about and some that require you to investigate further because you are unsure. This will require you to explore available resources to assist you with building and expanding your content knowledge. This process is not unusual and indeed is expected. That is, if you are not sure about something investigate and learn about it! This process is critical to demonstrating the *Australian Professional Standards for Teachers*, (AITSL 2014) standards, 'know the content and how to teach it' and as a proficient teacher, "apply knowledge and understanding of effective teaching strategies to support students"... numeracy achievement'. It is further reiterated in the AAMT *Standards for Excellence in Teaching Mathematics in Australian Schools* (2006, p. 2), Domain 1: Professional knowledge, states that:

> Excellent teachers of mathematics have a strong knowledge base to draw on in all aspects of their professional work, including their decision making, planning and interactions. Their knowledge base includes knowledge of students, how mathematics is learned, what affects students' opportunities to learn mathematics and how the learning of mathematics can be enhanced. It also includes sound knowledge and appreciation of mathematics appropriate to the grade level and/or mathematics subjects they teach.

The important role that strong knowledge plays in all aspects of teaching is emphasised in the Department of Education and Training Queensland (2015) *How to Teach Mathematics* resource for teachers. This resource provides extensive discussion of the role of a strong knowledge base in mathematics teaching.

Findings from international and national studies of content knowledge (CK) and pedagogical knowledge (PCK) of mathematics teachers considers both as critical professional knowledges for teachers, requiring specific attention to both (Baumert et al. 2010; Krauss et al. 2008; Van Driel & Berry 2012). These studies indicated that mathematics teachers with in-depth content knowledge, exhibited a stronger cognitive connectedness to the pedagogical content knowledge required to teach lessons. As teachers need to work with students' prior knowledge and existing conceptions and identify their misconceptions and errors, addressing these aspects requires strong pedagogical content knowledge. This knowledge has been identified as allowing for a deepening of understanding of students' understanding of mathematics, knowledge of the mathematics curriculum and the selection of instructional tasks. Krauss et al. (2008) identified that knowledge that focused on knowledge of mathematics tasks was also critical. They found that suitably selected mathematics tasks provided the foundation for students' constructions of mathematical knowledge and represented critical learning opportunities; knowledge about the potential of mathematics tasks is an important component of mathematics pedagogical content knowledge.

There are many ways you can improve your content knowledge and pedagogical content knowledge doing so will not only make you more skilled but will also increase your confidence! Educators (Loewenberg Ball, Thames, & Phelps 2008) have emphasised how important it is for you to work through lesson problems and activities before presenting them to students because by working through the lesson and taking the role of the learner, you deepen your understanding of the mathematics and you experience firsthand what your students will be doing. These same educators suggest that teachers identify the mathematical goals of each lesson and what is most important for all students to learn, in order to avoid teaching in ways that do not meet those goals or that even conflict with them. This process, too, requires that teachers understand the mathematical content at a deep level. Even if the

lesson topic is familiar to you, you need to review the mathematics you will be teaching in order to experience it in the same way the students will and so that you can focus on students' conceptual understanding.

How will I differentiate my teaching to ensure that all students are learning?

The term, differentiation, has been identified as referring to different aspects of schooling (Emanuelsson & Sahlstrom 2006; Ollerton 2014). For example, organisational differentiation relates to ability grouping and pedagogical differentiation refers to the actions carried out in classrooms in different ways for and by students and teachers (Emanuelsson & Sahlstrom 2006). Pedagogical differentiation is the focus here.

The intention of pedagogical differentiation includes challenging all students by providing them with tasks at different levels of difficulty. This provision means that your selection of tasks, the pace of the work and student interest need to be taken into consideration (Maes 2010; Ollerton 2014). As a consequence, individual student differences are provided for. When differentiation is considered in this sense, flexibility on the part of the teacher and student is required to support further student learning and participation in mathematics.

Differentiation requires instructional clarity. In your role as a teacher, you will draw on what you know about students, their readiness to learn, their interests and preferred learning styles to present learning at different levels of difficulty to ensure that all students are challenged at their readiness levels (Maes 2010). Readiness in this sense refers to students who are the same age but differ in their readiness to learn, their learning styles, their life experiences and circumstances. These differences are shown to influence what students know, what they need to learn, the pace at which they learn and the learning support needed from teachers and other. Teachers can present information for students orally, visually, and or through demonstration. The intention of the teacher is to attend to individual learners as well as the whole class. In doing so, the goal of the teacher is to meet students at their beginning points and move them along a continuum of growth and learning as far as possible, with no end point or ceiling.

Your actions as a teacher are critical to providing differentiated tasks (Sullivan, Mousley & Zevenbergen 2006). These actions require clear mathematics content descriptions and learning intentions that contribute to student achievement and support them with inquiring and engaging in learning about mathematics. For example, open-ended tasks work to provide students with opportunities for extending their mathematical thinking, exploring a range of options and making decisions and generalisations as they work through the tasks. Consideration is required of how the tasks are sequenced and how the students engage with a succession of tasks to ensure they have the necessary experience to complete the task (Sullivan et al. 2006). These considerations include the direction of the teaching and learning, the activities undertaken by the teacher and students and predictions of the students' thinking and understanding that emerges through engaging in the task (Sullivan et al. 2006).

In the last chapter, we discussed some of the cognitive, physical and social developmental characteristics of students in primary schools, and these overall trends provide a good starting point as you begin planning and differentiating mathematics experiences for your students. Remember, however, that children do not all progress through developmental stages at exactly the same time and that they may, therefore, vary greatly in their readiness for learning. Thus, some students in Year 1 will understand addition and develop fluency with basic facts before some Year 3 students. Likewise, some students in Years 4 or 5 may have difficulty visualising the blocks shown in In the classroom 2.2; they may need to 'build' these boxes before the task becomes meaningful. Children at the concrete operations stage must root their ideas in concrete objects; they have little ability to manipulate abstract relationships. These children would find it difficult (if not impossible) to learn how to find the volume of a cube by looking at a 2-dimensional drawing.

What do my students already know?

As a teacher, you must identify and build on what your students already know. Instructional time is precious and should not be wasted on lessons that are too easy or difficult for your students. Of course, if students in your classroom are at different academic levels or if they have learning problems or disabilities, providing equitable treatment may mean adapting instruction to meet their particular needs. To meet the needs of all of the children in your class, you first need to assess them. Paper-and-pencil tests are useful tools in assessment, especially if you go beyond looking at final scores and analyse how children reached their answers. Also, you can learn a lot about your students simply by observing and talking with them; this can reveal much that would not be noticeable from tests. You should use the results of assessment to help select instructional materials and approaches, group students for instruction, adapt materials and decide what needs to be taught or retaught to individuals and to the class as a whole.

After assessing your students, you must organise the mathematics so you teach it in a way that is appropriate and understandable to them. Mathematics includes both conceptual and procedural knowledge; thus, you face 2 challenges — helping students develop both types of knowledge and helping them understand the relations between them. In no other discipline is previous knowledge more critical — for example, it would clearly be pointless to ask students to estimate a distance in kilometres if they didn't already have some sense of the length of a kilometre.

The importance of children's previous knowledge is clear from the way that quality mathematics programs are organised, both to provide continuous development and to help students understand the basic structure of mathematics. Scope-and-sequence charts give an overview of how particular topics are developed across year levels. A careful examination of such a chart reveals how the sequence of activities related to a particular topic is organised in a *spiral approach*, giving students repeated opportunities to develop and broaden concepts. The spiral approach incorporates and builds on earlier learning to help guide the learner through the increasingly intricate study of a topic. For example, angle measurement is informally introduced in primary years and returned to many times in later years, each time at a higher level of sophistication. Figure 3.1 shows how the spiral approach uses previous experience to develop the concept of 'angle'.

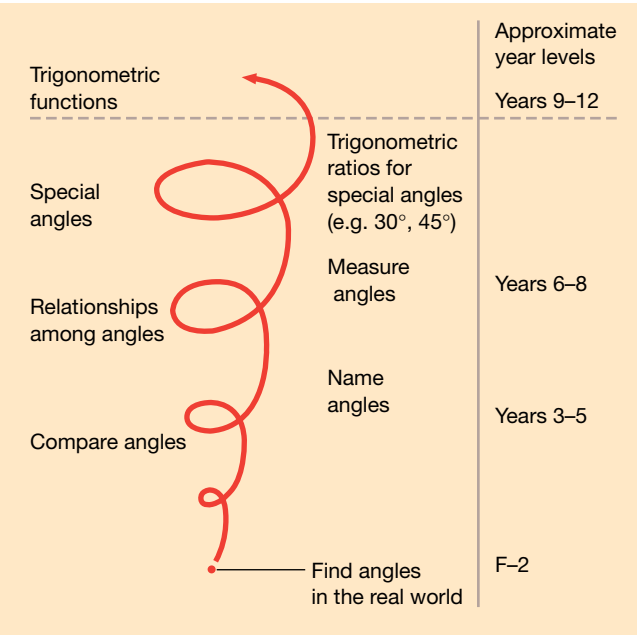


FIGURE 3.1 Spiral approach towards developing the 'angle' concept

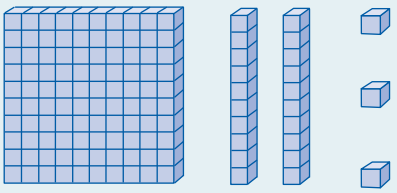
The description we just gave of the spiral approach represents an ideal; the reality, however, may be somewhat different. In theory the spiral mathematics curriculum provides for continuous growth and development, but in practice many topics are revisited without appreciable change in the level of treatment from one year to the next. Consequently, much valuable time may be spent each year treading water, merely reviewing topics rather than building on prior learning or introducing new topics.

The spiral approach holds profound implications for learning and teaching. For example, when planning a lesson, you must consider what prerequisites students must have to succeed in the lesson and you must check to see if students have them. Often, you will find students who have never learned, incorrectly learned or simply forgotten such prerequisites. Detecting these weaknesses early and quickly will let you plan the right kind and quantity of review, so you can properly prepare students for later lessons without wasting time. In the classroom 3.1 describes a way of gathering information about students' growing understanding of place value. Early in Year 2, students may be able to do no more than read 2- and 3-digit numbers but as the year progresses, their understanding of place value will grow and they will gain the ability to explain the digits by representing them with base-ten blocks. Maintaining a record of their growth is important in order to identify gaps or misunderstandings.

Besides knowing what students learned in earlier years, teachers must look ahead to see what their students will need to know tomorrow, next month and next year. Year 3 teachers must know what their students learned in kindergarten, Year 1 and Year 2, as well as what their students will have to know in Year 4 and beyond. This broad perspective helps teachers appreciate the importance of their role. It also helps them detect gaps in their students' mathematics learning, so they can fill in any gaps as quickly and effectively as possible.

IN THE CLASSROOM 3.1

Place value



Objective: Identify students' growing levels of understanding about place value.

Year level: 4

Materials: Base-ten blocks



- At the beginning of the school year, interview children individually to assess their understanding of place value. For each student, record the results on a card with the student's name. Then, quarterly, interview the children again to see how their understanding of place value has grown. Be sure to record your observations on the cards to maintain an ongoing record of the development of each student's understanding and of any misunderstandings that may become evident in the interviews.
- Conduct the interviews at a table with a piece of paper that you can write the numbers on in large print and with the base-ten blocks in a tub for the children to use.
- With each student:
  1. Write a 2-digit number on the paper, such as 48. Ask the student to read the number to you.
  2. Point to an individual digit in the number and ask the student to explain what it means. For example, point at the 4 and ask 'What does this 4 mean?' Repeat this question for each digit.
  3. Ask the student to use the base-ten blocks to represent the number and ask them to explain why the blocks represent the number.
  4. Repeat the process with a 3-digit number (such as 137) and with a 4-digit number.

Of course, a gap for 1 student may not be a gap for another. If an individual student has a gap in understanding that most students do not have, you will need to work with that student to fill in the gap. It is not appropriate to force the whole class to review something that only a couple of students do not understand. As a teacher, you will need to make many instructional decisions directed at helping individual students while continuing to challenge all your students, extending their experiences and their levels of understandings, as illustrated by the spiral approach in figure 3.1. In addition, you should maintain records of how individual children's understandings develop (e.g. as indicated by In the classroom 3.1) and pass this information on to teachers in the following year. This will help those teachers continue to extend the learning of individual students.

What kinds of tasks will I give my students?

What are the appropriate kinds of tasks for your students? This is another key question you must answer before you begin planning lessons. Suitable tasks might include inquiry projects, open-ended questions, constructions and applications (ACARA 2011). The tasks you select must be engaging and worthwhile to students (Roche, Clarke, Sullivan, & Cheeseman 2013).

> For worthwhile learning in mathematics, students need mathematically appropriate, engaging and challenging tasks. At the same time, the decisions which the teacher makes (in planning, and 'on the run') can make a considerable difference in how the task plays out, the level of persistence shown by students, and the resulting learning, cognitively and affectively (p. 32).

Expanding on this idea, Roche et al. (2013) describe a series of student actions whilst working on suitably engaging and challenging tasks including: concentration, self-application, belief that they can succeed, making an effort to learn and taking some risks.

Focus on the mathematics

It is extremely important that mathematics lessons maintain a focus on the important mathematical topics. Thus, one of your top priorities when you start planning lessons will be to decide on the mathematics objectives of each lesson. Certainly, you can improve your lessons and make them more engaging for students by using children's literature as a springboard, by involving students in hands-on explorations and by setting problems in real-world contexts. However, your first question when planning a lesson should be, 'What mathematics do I want the children to learn from this?' It is not appropriate to use manipulative materials just as toys, to engage students in talking and writing only about their feelings and not about the mathematics, or to do lessons that are just 'fun for fun's sake'.

As a teacher, you will be responsible and held accountable for helping children meet national and state standards. You can do this only if your mathematics lessons involve students in tasks that are mathematically rich. When children are genuinely engaged in solving mathematical problems that make sense to them, the learning they take away from that experience is likely to be deep and lasting. Important mathematical processes such as those described by the proficiencies of *The Australian Curriculum: Mathematics* (2015) are recommended by the *NCTM Principles and Standards for School Mathematics* (2000) and discussed in a later chapter. Further insights can also be gained by consulting national documents and web resources such as Australian Association for Maths Teachers (2016), *How to Teach Mathematics* (Department of Education and Training 2015) the NCTM's *Curriculum Focal Points for Prekindergarten Through Grade 8 Mathematics (NCTM)*. The NCTM material may be found in appendices A and B of this text.

Provide necessary practice

Part of deciding about appropriate tasks is considering how to provide necessary practice. Students need practice in order to acquire fluency with a mathematical idea or procedure. Just as we must practise while learning to walk or drive a car, we must practise basic addition facts or how to make patterns.

The choice is not if, but when. Research has long indicated that drill and practice should follow, not precede, the development of meaning. During instruction, the teacher's task is to ensure meaningful attention by students to what is to be learned using whatever methods are available. Practice, also referred to as rote learning (Mayer 2002) or drill, is seen as one means to this end but not the only means that dictates teaching and learning. In addition, drill and practice should not consume as much time as developmental work that helps children to understand the idea or procedure. Whenever possible, practice should be meaningful — it should be much more focused on reinforcing conceptual understandings than on going over rote procedures. Finally, practice should help children apply their knowledge in new contexts.

Many textbook series provide practice on the content presented, lesson by lesson. Your decision whether to assign this practice material (in addition to all the other practice that students get — for instance, in the course of problem solving) must be based on your students' needs. You have to strike a balance between 2 conditions: one is that children must understand the mathematics before practising it; the other is that practising a skill that has already been mastered is boring and thus can be harmful. Therefore, you should include practice on previously learned topics as a small but regular component of each lesson. Keep in mind that completing a meaningful task will often provide just as much practice as completing textbook exercises, if not more.

How will I encourage my students to talk, what kinds of questions will I ask and how will I group my students?

You can help your students learn mathematics better by encouraging them to communicate their thoughts to you and to each other, by asking them the right questions at the right times, and by grouping students in ways that are appropriate to their strengths and needs and to the mathematics being taught.

Talking

You can use classroom talk, or discourse, to make certain that the classroom discussion helps students make sense of the mathematics you are teaching. Classroom discourse also helps children who are learning English develop and practise their language skills. Just listening to others discuss concepts can help clarify those ideas for some students. You will be the organiser, cheerleader and conductor of the entire process of discourse, making appropriate tools available and finding ways to help children assume responsibility for their own learning. One teacher describes her efforts to improve and increase her students' discourse:

> After some experimentation, I realised that if I rephrased questions to start with 'How,' 'What,' or 'Why,' students needed to give a longer response than just 'yes' or 'no.' I also discovered that I could give a request such as 'Tell me about . . .' rather than ask a specific question. Frequent use of the think-pair-share strategy also encouraged discussion by allowing time for students to compose their thoughts and by helping shy students become more confident in responding in front of the class. The length and depth of student responses gradually increased. In addition, I began to insist that students explain their reasoning and justify their responses for even factual textbook questions. At first, many students seemed to feel that their response was incorrect whenever I asked for an explanation. With reassurance that I was just curious about their thinking, they began to share more often and soon began to justify their responses without being prompted (Cady 2006 p. 461).

A flow of ideas — not only from teacher to student but also from student to teacher and from student to student helps every child learn mathematics and is a natural corollary of interactive teaching (Ewing 2007). Here are some ways you can help create effective classroom talk.

- Provide students with opportunities for 3-way interaction between teachers and other students. Through this process active student engagement in learning emerges.
- Provide students opportunities to develop a deeper understanding of mathematics that is transferable beyond the classroom. Here the focus is on providing quality learning experiences rather than the

press for a fast pace and learning isolated rules and procedures to cover the required quantity of work before it is tested.

- Encourage and support students with developing mastery of the language of mathematics. The mathematics classroom that provides contexts in which students can safely express their thinking as they develop and apply mathematical understandings will enhance the teaching and learning of mathematics. It provides a stable social audience and an environment where reasons, justification and so on are shaped.
- Create and sustain a community of learners of mathematics — to which the students and teacher belong. It is a bridge or territory shared by teacher and student. When they use language, they are not transmitting information, they are consciously engaging in simultaneous understanding. It is through this process of active, responsive understanding that meaning — here mathematical meaning — is realised.
- Explicitly teach mathematics vocabulary. Rubenstein (2007) notes that some vocabulary may be confusing for students. For example, words like *round* or *square* have multiple meanings in mathematics. Words like *product* or *reflection* have different meanings in mathematics than when used in everyday language. Learning to understand and use correct terminology will enhance students' abilities to communicate mathematical ideas effectively. It is important to develop the concept or mathematical idea before introducing the vocabulary. Then, students already have the knowledge and the focus is on communicating it effectively (Burns 2007).

Questioning

Good questions are vital in facilitating learning. Questions that are aimed at checking children's knowledge of a fact or their ability to perform a skill are relatively low level and are the easiest to create during a lesson. Higher-level questions require children to analyse or synthesise information, such as when we ask students to explain why a procedure works or how they decided an answer is correct. Research has shown that in mathematics lessons, teachers tend to use questions of the first type much more frequently than those of the second type (Gough 2006; Zevenbergen et al. 2001). Children's ability to answer lower-level questions may correlate with high scores on some achievement tests. However, such questions give children the wrong picture of mathematics by implying that mathematics involves only simple answers and that there is only one correct answer per question. So, plan on being a teacher who asks plenty of higher-level questions that encourage children to do more talking about mathematics — talking that explores why a procedure works, what might happen if something were changed in the problem and how mathematics could be applied in real-life situations.

To be this type of teacher, you will need to consider the types of question you want to ask and write them in your lesson plans. Try to prepare a range of questions, but focus on questions that require children to think rather than merely supply a fact from memory or perform a learned procedure. If you ask only lower-level questions, students may come to believe that the main goal in mathematics is to memorise facts rather than to understand and make sense of mathematical ideas. Remember, however, that creating higher-level questions can be challenging — you'll have much more success if you develop such questions before the lesson begins.

The NCTM Standards (2000) includes 5 categories of higher-level questions teachers should ask.

1. *Questions that help students work together to make sense of mathematics:* 'What do others think about what Janine said?' and 'Can you convince the rest of us that that makes sense?'
2. *Questions that help students rely more on themselves to determine whether something is mathematically correct:* 'Why is that true?' and 'How did you reach that conclusion?'
3. *Questions that seek to help students learn to reason mathematically:* 'How could you prove that?' and 'What assumptions are you making?'
4. *Questions that help students learn to conjecture, invent and solve problems:* 'What would happen if . . .' and 'Do you see a pattern?'
5. *Questions that relate to helping students connect mathematics, its ideas and its applications:* 'How does this relate to . . .' and 'Have we ever solved a problem like this one before?'

Differentiated student grouping

Teachers can further encourage student participation in learning by altering the ways that students work. These ways include assigning students to groups based on similar-readiness, mixed-readiness, similar-interests, mixed-interests, similar-learning styles or mixed-learning styles (Rubenstein et al. 2015; Wilkinson & Penney 2014). Teachers can work with students who are advanced on a particular topic to further their thinking and understanding. They can work with students who need additional instruction and guided assistance with their learning and create mixed-readiness teams of students who work to solve a problem, defend their team's approach and explain the reasoning behind the solutions (Tomlinson 2000). Differentiation in this sense moves beyond covering the content or creating activities. It provides an alternative approach where students of varying backgrounds, experiences, interests and readiness levels are likely to experience appropriate challenge and make sense of powerful ideas.

Through these various groupings, a class can be kept as a unit or a learning community. In doing so, the teacher can respond to the different learning needs of students (Great Schools Reform 2014; NCTM 2006; Rubenstein et al. 2015). The idea is not to bring all students to the same achievement level. Rather, the intent is to enable students to progress fully in their achievement, through a combination of shared topics for all students and differentiated activities for students according to their achievement level. In this way, students are more likely to learn mathematics effectively in and with their class, as collaborators with their peers and with their teachers, feeling confident about their progress.

How will I group students? Many teachers use whole-class instruction too extensively. You should keep in mind that small-group work can help children develop their reasoning abilities through active involvement, can ensure that children work on content focused on their particular needs and can at the same time help them learn to work together to solve problems. Having individuals work alone can also serve important purposes. Here are some guidelines to help you decide when to use which grouping pattern.

1. Use large-group instruction:
   - if the topic can be presented to all students at approximately the same point in time (i.e. if all students have the prerequisites for understanding the initial presentation)
   - if students need continuous guidance or modelling from the teacher to master the topic.
2. Use small-group instruction:
   - if students can profit from student-to-student interaction with less teacher guidance
   - if you are trying to encourage exploration and communication about mathematics
   - if you are trying to help students benefit from the acquisition of cooperative learning skills.
3. Use individual instruction:
   - if students can follow a sequence or conduct an activity on their own
   - if the focus is individual practice to attain fluency.

Most effective teachers use a combination of grouping patterns (Tomlinson & Germundson 2007). For example, a teacher might develop new material with the whole class and then have children work individually or in small groups for one or more activities designed to help them consolidate the new learning. Communication with one another is especially valuable because the students help each other master mathematical ideas. The teacher moves from group to group, providing assistance by asking thought-provoking questions as needed. The whole class is then brought together again for a teacher-mediated discussion so that children can hear ideas from students in other groups. The teacher ends the discussion by providing closure on the lesson for the day (Cheeseman 2009). This method of integrating whole-class and small-group work has proved more effective than whole-class instruction alone for developing higher-level thinking skills (Cheeseman 2009).

A variation on small-group learning involves just 2 children working together on a task. This is particularly appropriate for young children who may not be developmentally ready to work in larger groups. The process of working together promotes the learning of both children. Educators have pointed out that small group work is not a guarantee that students will engage in learning. You are tasked with directing the teaching and learning, the activities undertaken by the teacher and students and predictions of the students' thinking and understanding that emerges through engaging in the task as they work in groups (Sullivan et al. 2006).

Before you place children in groups, carefully consider their individual abilities and personalities. For example, you might think that placing low, middle and high achievers in separate groups would better meet their needs, but research has found that homogeneous (similar ability) groups do not help increase overall student achievement and may promote inequity — that is, this type of grouping has not been found to narrow achievement gaps. In fact, low-achieving students actually perform worse when placed in homogeneous groups than they do in heterogeneous (mixed ability) groups. Middle- and high-achieving students do show some gain when placed in homogeneous groups; this may be helpful at times, but grouping them heterogeneously holds even more promise (Tomlinson & Germundson 2007).

Small-group learning may require more planning and class time than whole-class instruction, but this is more than outweighed by the positive results: increased mathematical communication, social support for learning mathematics, opportunities for all students to experience success in mathematics, increased likelihood that children will see more than one approach to solving a problem and opportunities to deal with mathematics through discussion of meaningful problems (Fitzgerald & Bouck 1993).

What materials will my students and I use?

Planning for instruction also involves selecting materials to enhance or support the text or curriculum. Teachers have varying levels of freedom when selecting materials to support their lesson objectives. Some can start with the textbook lessons and supplement them creatively, finding ways to adapt the textbook to the learners in their classroom. Others may be required by the school to use particular curriculum materials or textbooks and not to make more than the minimum adaptations needed for the students in their class. Either way, you will need to make the best use of the available materials, given your goals for the lesson and the children in your class. Whatever curriculum materials you use, sometimes you will want to locate additional resources. Educational publishers and the internet provide lots of options. You can also consider resources such as children's literature, manipulatives and technology.

Textbook lessons

Beginning teachers often wonder how much to depend on a textbook for lesson planning. To what extent can you use textbook lessons 'as is'? How and why should you adapt them for your individual use? Obviously, answers to these questions depend on the textbook, on how closely its goals and methods align with your school and system requirements, and on your own teaching preferences. It certainly does not make sense to develop your own original lesson plans if you have a textbook that provides an appropriate outline for each lesson. If you determine that the lessons are developmentally appropriate for your students, that they engage the students in genuine problem solving and that they promote conceptual development, there is no reason not to adapt them as seems necessary. Teachers' guides to textbooks can give you a variety of ideas for teaching lessons using the textbook. Guides also often provide suggestions for follow-up activities, such as written work and games. Many guides offer suggestions for remediation, enrichment and alternative tasks for children with learning disabilities and other special needs.

If you are required to teach from a textbook with a more traditional approach, you can considerably enhance the lessons by embedding them in a motivating theme or problem-solving context. Such a context will give you and your students an organising framework for teaching and learning and will help tie the lesson (or several lessons) together. Contexts for lessons may be taken from real-life situations (e.g. from current events or from special days such as holidays noted on a calendar) or may be developed from integrative themes tied to other subjects such as science, history or children's literature. To get a sense of what a context is, look at table 3.1, which shows some examples of how you can use contexts to enhance mathematics lessons. Make sure you give students open-ended tasks that allow for multiple solution paths or multiple solutions. This allows students at different levels to participate and learn where they are. It also lets students select the approach that best fits their learning preferences. For example, one student may prefer to draw a picture to solve a problem while another needs to use blocks. It is relatively easy to start with traditional textbook tasks and make them more open ended (Kabiri & Smith 2003), as illustrated in figure 3.2. A useful reference for teachers is Sullivan and Lilburn's

Open-Ended Maths Activities (2004). Notice that not only does the open-ended problem in figure 3.2 encourage critical thinking more than the traditional problem does, but it also gives students the opportunity to practise using the formula for volume many times rather than just once.

TABLE 3.1 **Using contexts in mathematics lessons**

This is a context	This is not a context
• Sorting M&Ms for party favours gives a reason for using this manipulative. • Planning a schedule for a class field trip involves telling time. • Ordering carpet for classrooms in the school is a context. We need to know the area of the room to find out how much to order. • Measuring classroom objects to decide how to fit them into storage containers gives purpose to a lesson.	• M&Ms (alone) are a manipulative, not a context. • Time is a mathematics topic, not a context. • Area of rectangles is a mathematics concept, not a context. • Measuring classroom objects is a mathematics procedure, not a context.

You may also need to supplement lessons in order to increase the focus on problem solving and conceptual development. For example, if your text does not provide enough meaningful experiences or hands-on practice with manipulatives for the students in your class, you may need to find other ways of providing such experiences and practice. Ultimately, you will be the one who best knows your students and their needs. One teacher describes how he made time for more problem solving while teaching from a structured mathematics curriculum (Buyea 2007). Treat your textbook as one of many available resources to help you teach mathematics to your students. Adapting textbook lessons can be challenging but also very rewarding. If you have the freedom to do so, another option is to select curriculum materials that are developed by state or national authorities.

Traditional problem: A rectangle measures 12 centimetres wide by 14 centimetres long. What is its area?

Open-ended problem: You have 72 metres of fencing to build a dog run in your backyard for your pet. Each fence piece is 1 metre in length, so each side of the dog run will be in whole metres. You have decided to use a rectangular structure that will stand by itself under some trees (that is, it is not along any fence or building). Using any method you like, find all possible rectangular dog runs you could build.

FIGURE 3.2 A measurement problem shown

Standards-based curricula

Since the 1990s and the development of the Australian Education Council's *National Statement on Mathematics for Australian Schools* (1991) and the associated *Mathematics — A Curriculum Profile for Australian Schools* (1994), most Australian states have adopted an outcomes-based approach to education and drawn up their curriculum documents. New South Wales retained the format of a syllabus, prefacing content descriptions with an 'outcomes' statement, a finer-grained list of 'key ideas' and lists of 'indicators' — behaviours that demonstrated student learning. In his critique of an outcomes-based education (OBE), Dr Kevin Donnelly, Senior Research Fellow at the Australian Catholic University, claimed that many of those in the field of education felt that OBE did not 'represent "world's best" curriculum and that it [failed] to successfully support teachers in their work' (p. 1).

Donnelly (p. 3) continues:

> OBE curriculum documents are not syllabuses or work programs as such and, when compared to a syllabus approach, give greater weight to formative, criterion-based assessment, in opposition to summative assessment and high-risk tests, and adopt a constructivist, developmental approach to education. Unlike a syllabus, where subject knowledge forms a critical part of the curriculum, it is also the case that OBE places greater emphasis on dispositions and attitudes.

Subsequently, various states and territories have reconceptualised their curriculum documents in terms of a standards-based education. Standards-based statements tend to be closer to syllabus statements in describing clearly the knowledge and skills that students will have acquired at each year level.

> One of the defining characteristics of both a syllabus and a standards approach is the central importance of the disciplines and the belief that generic skills and competencies can only be taught within such a context. (Donnelly 2007, p. 9).

The new Australian Curriculum (ACARA 2015) has been written in terms of year-by-year achievement standards, supplemented by detailed descriptions of content to be taught during each year.

In this chapter, we provide sample lessons based on the content descriptions of the Australian Curriculum. These lessons challenge students to understand the mathematics, to use higher-level thinking and to learn mathematics through problem solving. Many traditional programs report that they have incorporated the NCTM's Standards in their texts, but this often means they have incorporated the content standards without fully implementing the instructional approaches recommended by the NCTM. These particular programs encourage children to use higher-level thinking and to be actively involved in their mathematics learning.

Teachers find it challenging to teach mathematics in the ways recommended for developing student understanding. Often, they must significantly change their approach to mathematics instruction. At first they may need to spend more time preparing and planning for lessons. Sometimes teachers seeking to give their students additional support in these challenging lessons end up eliminating essential components. If you develop these lessons, you must be careful to include classroom discourse, allow students to learn the mathematics concepts through problem solving and apply the mathematics.

You should also realise that new ways of teaching and learning mathematics are not only challenging for teachers, but may also be confusing for parents. So, it is important to keep communication lines open with parents. Some schools invite parents for mathematics nights where teachers can explain the rationale for new instructional methods. Mathematics nights can also give parents opportunities to experience directly these new types of mathematics lesson while giving teachers opportunities to suggest ways for parents to help their children at home. Some teachers send children home with letters for parents at the start of each unit so that parents know what is happening in the mathematics class. Remember that parents and teachers are partners in educating children. Anything you can do to help parents understand your goals should help you reach those goals.

Children's literature

Teachers often use children's books to enhance lessons in language arts, social studies, science, music, art, dance and drama, but it is also easy, effective and appropriate to use children's literature in mathematics lessons. 'Children's books can spark students' imaginations in ways that exercises in textbooks or workbooks often don't' (Burns 2007, p. 45). Stories can provide children with a common starting point from which to share and discuss mathematical ideas. Mathematics and language skills develop hand in hand as children talk about problems and read and write about mathematical ideas. In addition, you can incorporate multicultural and Indigenous children's literature in your mathematics lessons as a way of making cultural connections for your students.

Use of children's literature in mathematics lessons can enhance learning in many ways. Children's literature:

- provides a context or storyline that can launch or develop mathematical concepts
- provides illustrations that clearly represent mathematical concepts
- provides quality illustrations that are motivating to the reader
- is a source of problems or a basis for generating problems
- provides styles and formats that can motivate the class to write problems or help them to design your own book (Thiessen 2004).

Many reference books for teachers can help you identify stories that work well for integrating children's literature with lessons on particular mathematics topics. The Children's literature connections at the end of each chapter in this text includes numerous such suggestions.

Manipulatives

Effective teachers use manipulatives when appropriate to provide concrete experiences that help children make sense of mathematics and build their mathematical thinking. The use of manipulatives is also an effective way to meet your students' diverse learning needs. Manipulatives let teachers illustrate mathematical ideas in multiple ways using a variety of different models and representations. Manipulatives support problem solving and also increase the active engagement of the children. They can test and verify their ideas, eventually forming mental models from the concrete models. The time children take to become familiar with the materials is time well spent — during this process, children construct knowledge as they make connections between the models and the mathematical ideas.

All through this text, we stress the importance of having children use formal and informal manipulatives to model mathematical ideas. Commonly used materials in primary school include counters or tiles, interlocking cubes, pattern blocks, attribute blocks, tangrams, base-ten blocks, fraction models, geoboards, measuring instruments, spinners and dice and play money. Informal materials include leaves, buttons, matchsticks, toys etc. (Appendix C shows patterns for many of the better-known manipulatives.)

While some teachers or parents may be concerned that children will become overly reliant on manipulatives, research has shown that children will progress from random guessing to using tools such as manipulatives to using their own mental strategies. It is also suggested that children need opportunities to explore different tools and strategies and discuss how and why they are useful (Jacobs & Kusiak 2006).

Roberts (2007) reminds us that having students use manipulatives and models does not guarantee they will be able to construct the desired mathematical understanding. The teacher will need to provide careful planning and delivery of the task and encourage thoughtful reflection by the students. When selecting manipulatives, the teacher should ask the following questions.

- Does the manipulative or model clearly and accurately represent the concept?
- Do the materials lend themselves to an efficient guided-discovery activity?
- Does the activity include a structure or recording procedure that will facilitate students' ability to construct the desired knowledge?

Practical suggestions for incorporating manipulatives into your instruction are provided by Burns (2007).

- Talk with children about how manipulatives will help them learn. These conversations are essential from time to time.
- Set ground rules. Manipulatives are not toys or games; the children should be expected to stay on task, completing assigned problems or activities.
- Set up a system for storage of the manipulatives. Children should be able to reach and remove things easily themselves without your assistance.
- Provide time for free exploration. Students will be more willing to complete the assigned tasks once they have satisfied their own curiosity.
- Display charts listing the names of manipulatives the children may use. This communicates that you value manipulatives and helps students learn the names and spellings.
- Send home a letter explaining why the children will be using manipulatives. You might also have children take home materials to use with their families.

Remember that young children, whose fine-motor skills aren't fully developed, may have difficulty if manipulatives are too small or too difficult to connect. To help children who have special needs, you might use materials with built-in organisation (such as an abacus) or materials that fit together (such as connecting cubes), rather than separate objects (such as base-ten blocks). Or you can use containers that keep counters separated, such as egg cartons or mats with clearly separated sections. Finally, package

manipulatives into sets for 1 student, 2 students or a small group for easy distribution and collection. Remember to have the children pass out and collect the materials as much as possible. You don't have to do it all!

Information and communication technology (ICT)

The increased availability of ICTs in today's homes and classrooms is having a profound effect on teaching and learning. There are several compelling reasons for their implementation and maintenance as part of the teaching and learning process. One, ICT allows the teacher to transform the ways of learning in classrooms. Second, ICTs have increased the options available to meet specific learning needs of all students. Third, ICTs can be used for demonstrating ideas, focusing questioning, assigning tasks for individual learners and groups and to create multi-modal texts across the curriculum The Australian Curriculum: Technologies (ACARA 2015b) aims to develop the knowledge, understanding and skills to ensure that, individually and collaboratively, students:

- investigate, design, plan, manage, create and evaluate solutions
- are creative, innovative and enterprising when using traditional, contemporary and emerging technologies, and understand how technologies have developed over time
- make informed and ethical decisions about the role, impact and use of technologies in the economy, environment and society for a sustainable future
- engage confidently with and responsibly select and manipulate appropriate technologies — materials, data, systems, components, tools and equipment — when designing and creating solutions
- critique, analyse and evaluate problems, needs or opportunities to identify and create solutions.

ICTs have become essential tools in our larger society and are one of the most common forms of technology used in primary classrooms. Many students today have access to desktop, laptop and handheld devices such as ipads. However, just the availability of computers is not enough to impact on instruction. Government (both state and national) expenditure on digital technology infrastructure in Australian schools has established a situation where innovative teaching practices can be developed with schools having access to sufficient numbers of up-to-date computers using a standard suite of software and web-based resources such as Scootle (Education Services Australia 2016).

There are 6 types of computer software available. Students use each type in a slightly different way and the different types also offer different benefits to children. One educator describes each software type in terms of its potential for developing problem-solving and higher-level thinking skills (Crown 2003).

- *Drill-and-practice software* provides practice for a procedural skill that has been previously taught. The computer presents a problem to the student, waits for an answer and indicates whether the student is right or wrong. Some drill-and-practice programs keep records and provide a score or they may even adjust the difficulty level of problems based on student responses (e.g. number and type of errors). The primary focus of drill-and- practice software is to strengthen automatic recall.
- *Tutorial software* provides instruction on new skills or processes. It may introduce information, present examples and provide practice. Depending on whether the student grasps the concept easily or has difficulty, the program will move to a more difficult lesson or provide additional instruction. Some tutorials are quite effective in presenting new information; however, they give students little opportunity to think creatively. Also, tutorial software might encourage children to focus on procedures or rules prematurely, before they have solidified their conceptual knowledge of the topic.
- *Simulation software* allows students to experience events and explore environments that might be too expensive, messy, dangerous or time-consuming to experience and explore in reality in the classroom. For example, through computer simulation, students can run a business or go on an expedition, or they can conduct a probability experiment (such as flipping coins or drawing randomly from a collection) with many more trials than would be feasible to conduct by hand. Simulation software has the potential to support problem solving and critical thinking. However, teachers have to carefully plan related discussions and activities to help students gain these benefits.

- *Educational game software* engages students in activities that address specific educational skills or may aid in the development of logical thinking or problem-solving skills. Games usually include randomised events, offer an opportunity to 'win', and present some obstacles to winning that the student must overcome. Often these games provide drill-and-practice opportunities. In addition, they can help children improve their higher-level thinking skills by analysing game situations and developing strategies. However, games do not typically develop key mathematical content knowledge.
- *Problem-solving software* is designed to aid in the development of higher-order problem-solving strategies. Problem-solving programs are similar to simulation programs, where students are placed in a situation in which they can manipulate variables and receive feedback; however, these programs do not necessarily model real-life situations. One type of problem-solving software involves programming. Students write programs in a computer language such as Logo to instruct the computer to operate in a particular way; in the process, they develop logical thinking and problem-solving skills. Problem-solving software can improve students' performance on problem-solving tasks but may not develop their mathematics skills or concepts unless the teacher can help them look beyond the task itself and focus on the mathematics.
- *Tool software* can enhance both teaching and learning and may be the most effective type of software for developing mathematics through problem solving. Tools help efficiently and effortlessly with graphing, visualising and computing. Useful classroom tool software includes word processors, databases, spreadsheets, graphing programs, hypermedia, dynamic geometry software, computer microworlds and applets (electronic versions of manipulatives, also called *virtual manipulatives*).

Today, many software programs of the types just described are available on the internet, which gives teachers and students access to people, information and resources that have never before been so accessible. The AAMT *Position Statement on Digital Learning in Mathematics* (2014) recommends:

1. All students have ready access to appropriate technology and the associated virtual learning environments and communities that support and extend their mathematics learning experiences.
2. Teachers at all levels are actively involved in exploring ways to take full advantage of the potential of technology for mathematics learning within the total curriculum. Professional learning should focus on new pedagogies for teaching mathematics.
3. Students will be assessed in their understanding of mathematics both with, and without, digital technologies; and in their awareness of when to choose to use digital technologies.
4. Education authorities make available to teachers, professional development opportunities to support the development of knowledge and skills necessary for the successful use of technologies in classrooms.

This statement strongly supports the stance of the Australian Curriculum, Reporting and Assessment Authority which states:

> All young Australians should develop capacity for action and a critical appreciation of the processes through which technologies are developed and how technologies can contribute to societies. Students need opportunities to consider the use and impact of technological solutions on equity, ethics, and personal and social values. In creating solutions, as well as responding to the designed world, students consider desirable sustainable patterns of living, and contribute to preferred futures for themselves and others.

The Australian Curriculum (ACARA 2015c) content description for Year 2 'Shape' in the Measurement and Geometry strand states that students are expected to 'describe and draw 2-dimensional shapes, with and without digital technologies'. It also explicitly describes the use of digital technologies for Year 3 in the Number and place value sub-strand, and in the Statistics and Probability strand. From Year 4, the use of digital technologies is encouraged in all strands of the mathematics curriculum.

Calculators not only let students compute quickly and accurately, but they also let them examine patterns, solve problems, develop mathematical concepts and perform other higher-level tasks. Research has shown that students who used calculators in a variety of ways had higher achievement than students who used only paper-and- pencil methods. They were better at mental computation, had better attitudes toward mathematics and did not become overly reliant on the calculators. Their problem-solving abilities were also enhanced (Sutton & Krueger 2002). (See later chapters for more about calculators and how they can be used to develop higher-level thinking.)

Today, more and more students have access in school to technology that goes beyond calculators and desktop computers, including laptop computers or data collection devices; classroom networks; interactive whiteboards; digital cameras; video cameras; calculator-based laboratory systems; and CDs, videos, MP3s and DVDs. As a teacher, you must not only learn to be comfortable using these technologies but must also consider how they might be integrated into your mathematics lessons. Do not bring in technology simply to impress or motivate your students. Remember that technology will improve student achievement only if it is used to develop the higher-level thinking skills involved in collecting and analysing data, investigating patterns and solving problems.

Finally, teachers must ensure equitable access to technology. Make sure not to use technology only as a reward for early finishers; rather, plan opportunities for all children to use technology during the lesson. If there is not enough technology available for all to use it at the same time, design a schedule that gives each student an opportunity to use it on a regular basis.

3.2 Planning for effective teaching

LEARNING OUTCOME 3.2 Understanding why mathematics lessons must be carefully planned.

Planning lies at the heart of good teaching (Cheeseman 2009). Children learn best from lessons that are interesting and carefully organised, directed by thoughtful questions and enriched by activities and materials that give them the opportunity to develop ideas about mathematics. Careful development of ideas, based on clear explanations, careful questioning and effective use of manipulatives and technology, is particularly important in helping children learn mathematics. Teachers make many of the decisions related to these factors during the preparations just discussed. Once those decisions have been made, it is time to organise them in plans for the year as a whole, for units and for individual lessons.

Teachers plan in a variety of ways. Some just list the objectives they want children to attain or at least take a step toward attaining. Some jot down key questions they want to ask. Some lay out materials for children to use or run off worksheets for them. And some read the comments in the teachers' guide to the textbook. All of these approaches require teachers to think through what they plan to do and how they will do it.

Few experienced teachers take the time to write out a complete, detailed plan for every mathematics lesson they teach. Some do it occasionally, when they know that the idea they want to teach must be developed with particular care or when the lesson is one they haven't taught before. Experienced teachers often have detailed mental plans, although they may not write more than a page of sketchy notes. They may have taught a similar lesson many times before and therefore have a good sense of appropriate sequencing, timing and questioning as well as potential pitfalls. Beginning teachers, however, lack that experience and so it is particularly important for them to write detailed lesson plans. Careful planning helps make initial teaching experiences good ones, for both the children and the teacher. In fact, research shows that both experienced and pre-service teachers benefit from thinking deeply about the lesson tasks, teacher activity, students' responses to them and possible interventions before teaching the lesson (O'Donnell & Taylor 2006/2007).

The importance of planning

Good plans give you the security of knowing what you will do and say, of having interesting activities and materials ready for the children's use, and of anticipating what the children might do. A good,

detailed lesson plan gives you a way to judge how well the lesson went. Even though you might not be able to follow your plan precisely, it will help you evaluate your teaching and assess your students' learning. A written plan can also be shared with other teachers who can provide helpful comments and suggestions. Moreover, writing detailed plans helps you learn how to plan in your head.

3.3 Levels of planning

LEARNING OUTCOME 3.3 Recognising what levels of planning the teacher must create.

Planning for your mathematics class is done at 3 levels. You start at the broadest level with a plan that sets out your goals and objectives for the year. Then, moving down a level, you plan how to achieve those goals and objectives by organising the mathematics content into units. Finally, at the narrowest level, you plan daily lessons for teaching specific parts of that content. Each level of planning is important in ensuring that you meet all your year-level requirements in a logical fashion.

Planning for the year

Before the school year begins, you need to consider what you want to have the children in your class accomplish during that year. Of course, you do not need to develop these goals on your own. Most schools prepare scope-and-sequence charts or curriculum guides, or they rely on those provided with the textbook series they use. The Australian Curriculum describes the achievement standards for each year level together with content descriptions and elaborations of these. These are developmentally paced to provide students with deep knowledge and skills in mathematics, and to make clear 'connections across strands and the sequential development of concepts from Foundation to Year 10' (ACARA 2015c). The NCTM's *Principles and Standards for School Mathematics* (2000) indicates, in broad terms, the mathematical content and processes appropriate for students at each of the year-level bands (pre-K–2, 3–5, 6–8 and 9–12). The NCTM's *Curriculum Focal Points for Prekindergarten Through Grade 8 Mathematics* (2006) will also be a useful reference. (See appendices A and B for these content expectations.) You should be familiar with the Australian Curriculum Standards, your state education requirements and your school's scope-and-sequence guide or curriculum before you do any planning. Such materials are designed to ensure that children are taught the desired range of content across year levels. You should also check with your principal and other teachers to determine whether any changes have been made to meet the needs of the children in your school and to learn how much flexibility you have in making changes for your class.

After determining the goals of mathematics teaching for the year and the order in which topics are to be taught, you need to consider the approximate amount of time you want to spend on each phase of the curriculum, based on the relative importance of each. Doing this helps you be sure that you can fit in all the mathematics content you want to include. The goals for the year can be organised by textbook chapters or units. The Australian Curriculum and support documents, your state education requirements and your school's curriculum guide or the textbook's planning chart can help you with this and with the decision about how much time to assign to teaching each topic. Prepare a brief outline listing each unit, its objectives and the estimated number of days and dates you will teach it.

Planning for units

Once you have outlined your year, you can begin planning your units. Begin by outlining, in sequence, the topics that are to be taught and how much time you will spend on each. You can then decide on supplemental materials, instructional strategies and assessments. Next, outline what you want to accomplish each week — that will make the process of developing daily lesson plans much easier. Some schools require teachers to maintain a lesson plan book in which they note the objectives (and sometimes other details as well) for each day's lessons for a week. Even if this is not required, it is a good idea because it keeps you aware of progress toward meeting your goals for the year, it gives you a guide to follow

each day and provides a platform for developing an evolving planning resource. You cannot expect to follow this guide exactly, but as you plan each week or unit, you can review the progress of the children in your class and then vary or pace the content to be taught to meet their individual needs. Alternatively, you can plan to teach different content to small groups or individuals.

Planning for daily lessons

Finally, once you have outlined the units, you are ready to write daily lesson plans. Putting your lesson plan into writing will help you clarify many of your ideas, give you a 'road map' to follow while teaching and give you a record that you can use to evaluate each lesson and to plan subsequent lessons. Consider the questions discussed at the beginning of this chapter and use the process of writing the plan to help you think deeply about the lesson, the steps involved in teaching it and the time you will need for each step. Then, if you diverge from the plan, you will be able to pinpoint where you diverged and return later to pick up where you left off. Your plan should be concise yet complete. Since it is just for your use, you may reach the point where you can write just short sentences or phrases to cue your memory. Use a format that lets you refer easily to the plan while you are teaching.

3.4 Planning different types of lessons

LEARNING OUTCOME 3.4 Identifying the 3 types of lessons used to teach mathematics, and the purpose of each.

When writing daily lesson plans, you must also consider the format of the lesson. The most familiar format for mathematics lessons, review-teach-practise, may also be one of the least effective, especially when it is used to introduce new content. In this lesson format, the teacher begins by reviewing homework or problems worked during a previous lesson; then the teacher briefly explains a new concept or demonstrates a new skill, using sample problems; finally, the teacher assigns exercises (of the same type as the sample problems) for students to practise. Typically, for the remainder of the lesson, students practise solving similar problems on their own while the teacher assists individual students.

According to the report, *Teaching Mathematics: Using research-informed strategies* (Sullivan 2011), having a planning model assists with information teaching and incorporating differentiation. The task of the teacher is to pose challenging problems that students then think about; individual students present ideas for solutions to the class, which then discusses these ideas. The teacher uses the discussion to highlight the key mathematical concepts and then the students practise similar problems.

Review-teach-practise lessons in the United States and Germany focus on skill acquisition, on teaching students to do something; in Japan, the typical lesson focus is on understanding mathematical concepts. In US and German classrooms, students spend more than twice as much time practising routine procedures as students in Japanese classrooms (although many Japanese students practise skills in paid tutoring sessions after school). Japanese students spend most of their class time inventing new solutions and thinking about mathematical concepts. Analysis of TIMSS data suggests that what occurs in lessons is critical to students' learning. The specific topics taught and how these topics are presented and developed shape what students learn and are able to do. In the United States, lessons often consist of episodic encounters between students and curricular content, rather than ensuring that students stay focused on the content, as is done in other countries. Topics and concepts are presented in a fragmented and disjointed manner, and underlying themes and principles are either not identified or merely stated but not developed.

Instead of the traditional review-teach-practise format, we suggest using 3 different lesson formats: (1) investigative lessons, (2) direct instruction lessons and (3) explorations. These formats can provide structure for most of the situations you will encounter during mathematics lessons. You will probably choose to use different types of lesson on different days, depending on your goals and your students' previous experiences with the content. However, no matter which format you are using, be sure to keep in mind the overarching goals of ensuring student understanding and maintaining active student involvement.

You will probably design many lessons that use aspects of more than one of these formats. In other words, the 3 lesson formats should not be seen as entirely distinct. Nevertheless, it is useful to outline the general characteristics of each form, as we do in the sections below.

No matter what lesson format you select, the beginning of a lesson plan usually shows basic information, such as the lesson objective(s), how you will assess students, what adaptations you might make (i.e. gearing up and gearing down) and what materials you will need. The lesson itself generally consists of 3 phases: (1) introduce (how you will begin the lesson); (2) investigate, instruct or explore (depending on the lesson format) and (3) reflect/ summarise (how you will bring the lesson to a close). In the lesson plan, the lesson outline details these phases of the lesson. Figure 3.3 shows a generalised lesson plan covering all 3 formats; in this figure, you can see how the lesson outline differs among the formats. The descriptions below should help you learn how to write lesson plans using each format; we also provide a sample lesson plan using each format, which you could try out with children.

The kind of teaching envisioned by standards-based curricula is probably very different from the kind of teaching you experienced when you were in primary school. The emphasis is shifting from a teacher-directed classroom to one in which students are actively involved in learning. Though still in control, of course, the teacher involves the students through discussions and activities, usually by incorporating much small-group work. Direct instruction and discussions with the whole class have not disappeared, but children learn much more effectively when they work with materials to solve problems and talk about their results with each other.

Generalised lesson plan for all 3 formats

Introductory information

Year level _____ **Date** _____

Topic/Title _____

Content description(s) (Australian Curriculum)

Objective(s) Determine what you want your students to know or do. Align with national and state standards.

Assessment What types of information will you collect to help you determine how well individual children achieved the objective(s) of your lesson and where they had trouble? Assessment information may come from a variety of sources — for example, classroom observations, written class work (either individual or group) or homework. Your dual goals in planning for assessment are to be able to make appropriate follow-up instructional decisions and to be able to report on students' progress (e.g. by assigning grades or writing a narrative report). Be sure to describe the criteria you will use to decide who 'got it' and how you might translate your assessment to a grade or comment.

Gearing down What will you do if your lesson, as planned, turns out to be too advanced for some or all students? How can the lesson be modified (on the spot) for less able students?

Gearing up What will you do if your lesson, as planned, turns out to be too easy for some or all students? How can the lesson be modified (on the spot) for more able students?

Materials List all materials needed:

- mathematics manipulatives, technology (and how many of each)
- books, including page numbers (and how many of each)
- other supplies (scissors, markers, posters etc., and how many of each)
- handouts (attach a copy of each)
- references (include bibliographic information for books or resources used in planning the lesson)

Lesson outline (include approximate times for each segment of your lesson)

Introduce (setting the stage)

Describe briefly how you will begin the lesson. Your aim is to motivate students to get involved in the lesson activities and to ensure that all the children understand what they are to do. Include questions you might ask. Introduce the context you will use (e.g. real-life examples, manipulatives, problems or puzzles). Establish that students have background knowledge to engage in the lesson.

FIGURE 3.3 Generalised lesson plan *(continued)*

FIGURE 3.3 *(continued)*

Investigate (for investigative format)
This part of the lesson could consist of one continuous problem-solving activity (whole class, small group or individual) or it could be divided into several segments, each involving different activities or groupings. For each segment, give a step-by-step description of what you (the teacher) will be doing and what the students will be doing. In particular, provide examples of the types of question you might ask and of any products that the students will create. If your lesson is divided into segments, give an approximate time for each segment; also, describe how you plan to make transitions from each segment to the next.

OR

Instruct (for direct instruction format)
Whole-class teacher instruction and modelling What concepts, definitions, formulas or algorithms will you explain and model? List all the examples you will use and order them appropriately. Include directions for the activities that will follow. *Activities* Activities may be done by individuals or small groups. Students work with you on problems or exercises that follow rather closely from what they learned during your whole-class instruction and modelling. Then, when they are ready to practise independently from you, circulate, assisting individual students, answering questions and, for students who finish early, providing tasks that let them extend their learning. Make a list of the questions you will ask as you circulate. Include a copy of any written materials that students will work on, such as a worksheet. Include several simpler problems that you could use in helping students who have trouble. Include several more challenging problems that you could assign to early finishers.

OR

Explore (for exploration format)
For each exploration, give a step-by-step description of what you will be doing and what the students will be doing. In particular, give examples of the types of question you might ask and of any products that the students will create. If you plan to rotate students through several activities, include the approximate time for each and describe how you plan to make transitions from each segment to the next.

Reflect/summarise (includes homework setting)
Describe how you will bring the lesson to a close. Find a way that motivates the students to reflect on their activities and what they learned. Include questions you might ask yourself and/or your students to help you determine whether the lesson was a success.

Regardless of the lesson format, the teacher orchestrates the classroom discourse by 'deciding when to provide information, when to clarify an issue, when to model, when to lead, and when to let students struggle with a difficulty' (NCTM 1991, p. 35). You learn to make these decisions through planning and through experience. No one mode of teaching can be considered best. You should learn many teaching modes and use them as appropriate. The NCTM Standards (NCTM 2000) urges teachers to consider the goal, the task and the students (i.e. to ask what will best help them learn).

Investigative lessons

Investigative, or problem-based, lessons are most appropriate for helping children develop problem-solving skills, learn new concepts or apply and deepen their understanding of previously learned topics. In an investigative lesson, students try to solve a problem or conduct an investigation on their own. The problem or investigation may have been brought up by the teacher or by one or more of the students; however, the lesson itself revolves around ideas that the students generate through their own investigations. Obviously, the teacher is responsible for guiding the lesson but the students are expected to develop their own approaches, strategies and solutions.

In the *introduce phase* of an investigative lesson, you make sure that the students are motivated to do the lesson and that the problem at hand is explained. The introduction might come from a student presenting a problem that he or she has been struggling with and asking classmates to work on it too. Alternatively, you might provide motivation by reading a children's story or a newspaper clipping, playing a game, presenting a puzzle or using some other context that involves the problem to be worked on.

Once all the students understand the challenge, the *investigate phase* of the lesson begins. In this phase, the students work on the problem. This might involve learning stations, individual work, paired problem solving or small groups, and students might use models, manipulatives, computers, calculators or other tools. In an investigative lesson, the problem should truly challenge the students, so it is important that you do not explain up front how to approach the problem. The students should not simply mimic strategies or skills that you have just shown them. Instead, they should decide for themselves how to get started and what to try. Your role as teacher is to let them go. You should circulate around the room, listening in on conversations, observing what individuals or groups are doing and occasionally interjecting questions or comments to guide students as they develop their approaches.

During the *reflect phase* of the lesson, the class talks together about its findings. Using information you gathered from observing the children at work, you orchestrate a discussion in which various groups or individuals report what they tried and what they discovered. It is important to be neither too negative nor too positive about student ideas and responses. Encourage students to challenge each other's ideas. They need practice in learning how to judge when mathematical ideas are valid and when they are not. Also be sure not to finalise the solution to the problem prematurely.

Students learn a lot from listening to each other and from hearing about alternative approaches. Your job is to encourage the students to share their ideas while maintaining control of the discussion and trying to guide it in ways that advance your curricular agenda. During the final portion of the reflection phase, it may be important to help students be explicit in stating what they have learned from the investigation. This is time for stating generalisations or rules that have been formulated. It may be tempting to cut short this phase, but do not do so. Talking together and sharing ideas are critical to meaningful learning of mathematics.

Figure 3.4 shows an investigative-format lesson plan for Year 2, based on measurement outcomes in the Australian Curriculum. In the lesson outline portion of this plan, the teacher details how he or she wants to conduct the lesson. Read the outline to get a sense of what the teacher intends to do. Notice that the introductory phase of the lesson includes background information and discussion that students need before they start their investigation. Now look at the outline of the investigate phase and note that the teacher has considered how the students will be organised, moving the children back and forth between small-group and whole-group discussion as needed. Finally, the outline of the reflection phase includes additional questions to be asked at the end of the lesson.

Direct instruction lessons

In direct instruction lessons the teacher plays a more central role in directing the instruction than in investigative lessons. Direct instruction lessons are appropriate when the teacher wants to communicate specific knowledge, to introduce new vocabulary or to teach specific procedures. In a direct instruction lesson, the teacher exercises more control than in an investigative lesson and the lesson generally has a tighter focus. The 3 phases of a plan for a direct instruction lesson are usually (1) introduce, (2) teach and (3) reflect/summarise (see figure 3.5); each phase may be subdivided into smaller segments of various types, depending on the content of the lesson.

As with investigative lessons, in the *introduce phase* of the lesson, the teacher provides a motivating introduction and introduces the students to the problem. Led by the teacher, the class talks together about how to approach the problem and what previously learned concepts and skills might be useful. In the *teach phase*, the teacher models for the students how to proceed but also elicits ideas from the students while helping them move toward the concept or skill that is the goal of the lesson. Guided practice is included during the teach phase — students actively practise using the concept or skill, with guidance and feedback from the teacher. Afterward, the students have a chance to work with the ideas presented earlier in the lesson and to make them their own, working independently from the teacher. The task during this phase should be closely related to the original task but not so similar that the students are simply copying what the teacher has just shown them. Finally, during the *reflect/summarise phase*, the teacher and students reflect and summarise on what was learned.

Sample investigative lesson plan

Introductory information

Year level: 2 **Date:** 15 March

Topic: Measurement: using informal non-standard units

Content description (Australian Curriculum): Compare and order several shapes based on length, area, volume and capacity using appropriate uniform informal units (ACARA 2015).

Lesson objectives:
The students will:
1. estimate length with informal non-standard units (handspans and cubits)
2. measure length with informal non-standard units (handspans and cubits)
3. develop fluency with iterating units to measure length.

Assessment (practical)
Objective 1: Provide reasonable estimates using informal units for a selection of objects other than those already measured.
Objective 2: Measure with informal units to a reasonable accuracy a selection of objects other than those already measured.
Gearing down Do the first handspan and cubit measure together. Assign partners if needed.
Gearing up Encourage students to measure other objects but with different informal non-standard units (such as paces or tiny steps).

Materials
Activity sheets (each one directing students to different locations in the classroom to measure objects — for example, door width, window width, desks, noticeboards, cupboards).

Lesson outline

Introduce (15 minutes)
Who remembers measuring length? Show me what you might do? How did you measure it?

Today we are going to use 2 other ways of measuring length using our hands and arms.

Handspan: With your fingers spread wide [***demonstrate***] the distance from the tip of your thumb to the tip of your little finger. Everyone show their handspan. Compare with your neighbour; are they the same? This is why handspans are called 'non-standard units'.

Cubit: The distance between your elbow and the tip of your longest finger [*demonstrate*]. Everyone show their cubit. Compare with your neighbour; are they the same? This is why a cubit is called a 'non-standard unit'.

Handspans and cubits let us make approximate, not exact, measurements. Suggest when such approximate measures could be used.

Demonstrate [*have some students demonstrate*] how to measure the length of the board with handspans and then with cubits. Show how to use free hand (or other) to mark the place of one unit before moving.

Investigate (15 minutes)
Assign students to groups of 4 and send each group to one of the locations, providing appropriate activity sheet(s). The length of each object (limit to 3 or 4 objects) listed on the sheet is to be estimated first in handspans and cubits and then measured using these units. Both the estimate and the measure are to be recorded as individual results and then a group result is also decided.

Reflect (30 minutes)
Collect group results from each group, recording on transparency, IWB (interactive whiteboard) or computer.

Why are there differences? How did you decide on the group result?

Discuss how to handle part units. Decide on a rule.

If you were to measure the board with cubits, rather than handspans, would you need more or fewer? Why? Have students estimate how high the door might be in both handspans and cubits. Have them justify their estimation.

Have students estimate and record the teacher's height in handspans and cubits. Discuss justification with regard to how they accounted for the observed variations. Discuss (and carry out) how to check their estimates.

Homework/research What other informal, non-standard units might the students' families or friends use?

FIGURE 3.4 Sample investigative lesson plan

Sample direct instruction lesson plan

Introductory information
Year level: 3 **Date:** 15 May
Topic: Multiplication and division: division ties to multiplication

Content description (Australian Curriculum): Recall multiplication facts of 2, 3, 5 and 10 and related division facts (ACARA 2015).

Lesson objectives
The students will:
1. model division number stories with arrays using counters and record using diagrams and numbers
2. identify multiplication and division fact families for the numbers 2 and 3
3. develop fluent recall of multiplication and division facts for the families of 2 and 3.

Assessment
Answer pen-and-paper questions from a worksheet.

Gearing down Using a number line, have students skip count in 2s and 3s (starting from 2 and 3).
Gearing up Challenge students to write and solve multiplication and division stories and share with a partner or within a group.
Materials Activity sheets with division stories represented in various ways, counters, overhead transparencies or interactive whiteboard.

Lesson outline

Introduce (20 minutes)
Have students form groups of 2, 3, 4, 5 or 10.
Count the numbers of groups each time.
Record on overhead or interactive whiteboard using diagrams of ordered groups and number statements (arrays were introduced in a previous lesson).

Problem You have 12 lollies (counters) to be shared between 3 people. How many lollies will each person receive? Give each student 12 counters to model the question.
Discuss how students went about answering the question.
Record responses on the overhead transparency or interactive whiteboard as an array and in numbers.

Teach (15 minutes)
Next to the diagram of the 4×3 array, write the problem as a table:

Children	Counters	Counters in all
3	?	12

How many counters for each child? Write 3 different number statements:

$12 \div 3 = 4$ $\quad \frac{12}{4} = 3$ $\quad 4\overline{)12}\ 3$

Label the following parts of each division representation:
- the *quotient* (the answer in each)
- the *dividend* (the total before sharing)
- the *divisor* (the number of equal parts that the dividend is being split into).

Problem There are 15 counters. Each child receives 4 counters. How many children are there?

Model the array, having the students form the array with their counters. There are 3 counters ***remaining***.
This is called the ***remainder***; the quantity left over when a set of objects is shared equally.
Have the students fill in the multiplication division table on the overhead transparency or interactive whiteboard.
Have volunteers write the problem in the 3 different representations.

Have students attempt other similar problems, modelling the array, completing the table and writing the 3 number representations — for example:
- *21 puppies are placed equally in 3 pens. How many puppies in each pen?*
- *17 markers are shared equally between 5 children. How many markers does each child receive?*

Provide an activity sheet with similar problems for the students to attempt, either individually or in groups, before discussing their solutions.

FIGURE 3.5 Sample direct instruction lesson plan *(continued)*

FIGURE 3.5 *(continued)*

Reflect/summarise (15 minutes)
Have students as a whole class talk about the strategies they used to provide the answers to the problems. Point out the relationships between multiplication and division, made visible through the array model.

Put up on the board a division statement (e.g. 10 ÷ 2 = 5) and ask students to suggest a story to match the number statement.

Homework/research Students have another activity sheet with further word questions about divisions and some number statements to change into word questions.

Figure 3.5 shows a direct instruction lesson for Year 3 based on outcomes from *The Australian Curriculum: Mathematics*. Notice the role of the teacher during the teaching phase of the lesson. The teacher provides important information, models the procedure, guides the students to practise the procedure and gives them opportunities to practise. During the reflect/summarise phase, students not only have an opportunity to reflect on their learning but also share the strategies they used.

Explorations

Sometimes teachers want to provide a series of learning activities, or *explorations*, for students to work on either independently or in small groups around the room. These activities are often called learning centres or work stations; they are commonly used in early childhood classrooms with very young children who are not ready to participate in more formal lessons, but they can also be appropriate for older primary and middle school students. Explorations should be structured so the children can complete them without teacher assistance and allow for more individual choice, while working with assigned materials, than in an investigative lesson. Explorations are not just free time. They provide opportunities for students to work independently on important mathematical tasks. Students can work at their own pace and make decisions about their learning. The learning activities can all be related to a particular topic and provide various levels of difficulty or complexity. They can also provide a variety of opportunities to practise a skill and may encourage the development of conceptual knowledge or of problem-solving skills. In addition, explorations can provide remedial or enrichment activities for students who need them. A set of explorations can be made available on particular days of the week, over the course of a unit, or over a period of days in place of whole-class lessons.

Explorations don't necessarily resemble a typical whole-class lesson although you still need to identify the student objectives and carefully plan the activities. You need to consider how the activities will be managed and think about how you can package the necessary materials for easy access and cleaning.

Explorations let you plan to meet the needs of students with varied learning styles and interests. Some students may prefer to work with technology, while others may want to use manipulatives. Planning a variety of activities will give students choices. Also, by considering how you can make particular activities less difficult or more challenging, you can plan to meet individual needs.

Consider your class size and how many children you want to work with each activity at the same time. You might plan for 5 or 6 stations and have groups of children rotate through all of them, spending a predetermined amount of time at each location. Or you could provide several sets of materials for each activity so several groups or individuals can be working on the same activity at the same time. One educator calls these types of experience *menus* and suggests providing a menu of activities related to a particular unit or topic (Burns 2007). You can require that all students complete certain explorations on the menu, while others could be optional. Once again, this is also an effective way for you to meet students' individual learning needs. You could assign students who are struggling to complete particular activities, while assigning those who are more advanced to complete more challenging activities. Alternatively, the students could decide in what order they would like to complete the activities and how much time they will spend on them. Once students are familiar with the routine of completing these

types of activity and are able to work fairly independently, explorations can also give you time to assess individual students or small groups.

An exploration generally involves 3 phases: (1) introduce (2) explore and (3) reflect (see figure 3.6). The *introduce phase* of an exploration will probably involve telling the students about the available activities, what you expect from them for each activity, how they should manage materials, how they will move from one activity to another and what products they will create and you will check. You can give older students written instructions for each activity but young children will need you to model the activities and provide oral instructions. Often, you will do best to gradually introduce the activities involved in an exploration to the whole class over several class periods. Then, when students are sent to work independently, they have previous experience with the activities and are more likely to be successful when working on their own.

During the *explore phase*, students work independently for a specified amount of time. Be sure you have planned the step-by-step procedures for each activity. Be sure students understand what to do when they have completed one activity and are ready for another. Also consider your role. Will you be circulating? What questions will you ask about particular activities? You might carry a clipboard and make notes about student learning as you observe or keep a list so you can check products that particular groups have created and can mark groups off whether the products look satisfactory.

In the *reflect phase*, find time to talk with the students, letting them share strategies or products they created. You may want to instruct everyone to complete one particular activity and make that the focus of the discussion for that day.

Figure 3.6 shows a Foundation year exploration plan based on Number and Algebra outcomes from the Australian Curriculum. Notice that in the *introduce phase*, the teacher provides instructions so the children can work independently while completing their activities. In the explore phase, the teacher lists organisational details. In the *reflection phase*, the teacher uses questions to get the children to talk about their experiences.

Sample exploration lesson plan

Introductory information

Year level: Foundation **Date:** 2 June
Topic: Patterns and Algebra: making patterns
Content description (Australian Curriculum): Sort and classify familiar objects and explain the basis for these classifications. Copy, continue and create patterns with objects and drawings (ACARA 2015).

Lesson objectives:
The students will:
1. use concrete objects to create linear, repeating patterns
2. distinguish between a pattern and a non-pattern
3. develop fluency in continuing patterns.

Assessment
Objective 1: Create patterns on 'place mats' that are completely correct.
Objective 2: Correctly identify examples of 'placemats' with patterns and non-patterns.

Gearing down Use 2 different shapes to create a repeated pattern (Station 1)

Gearing up Use 3 or more objects to create a pattern and label a pattern numerically (e.g. 1, 2 . . . or 1, 2, 3 . . .)

Materials
'Placemats' made from coloured card with squares marked along each side
Station 1: Orange and yellow stickers
Station 2: Rubber stamps and stamp pads
Station 3: Markers and fruit stencils
Formal and informal manipulatives for students to build further patterns once set work is completed.

FIGURE 3.6 Sample exploration lesson plan *(continued)*

FIGURE 3.6 *(continued)*

Lesson outline

Introduce (15 minutes)
Clapping/stamping pattern (e.g. clap–clap–stamp) for class to copy. Change pattern for class to copy. Have class listen to a non-pattern of clapping and stamping. Do they recognise it?

Today we are going to decorate placemats for the class/school picnic we are having next week. We are going to do this by placing one object in each square, arranging the objects to make a pattern. How do you know it is a pattern? [***repeated elements***].

Using the materials from each station, demonstrate how to decorate the placemats. Show one pattern and one non-pattern and ensure students can distinguish them. Suggest how patterns can be described (e.g. yellow–orange, yellow–orange or 1–2, 1–2).

Explore (15 minutes)
The class is organised into workstations in groups of 3 or 4 to decorate their individual placemats. These are to be checked by the teacher. Students are given counters or blocks to create other patterns.

While circulating, the teacher shows 4 or 5 prepared 'placemats' and asks students which show patterns and which do not.

Recorded observations: Who is making patterns? Who is not? Who needs further guidance? What kinds of pattern are students making? How does each student describe the pattern?

Reflect (15 minutes)
Students move back to classroom centre and show their 'placemats', describing their patterns.

Repeat clapping–stamping pattern, but make it more complex. Ask students to describe the pattern, using numbers and/or letters.

3.5 Meeting the needs of all students

LEARNING OUTCOME 3.5 Establishing how teachers can meet the needs of all students.

Once you have created a lesson plan, one more step remains before you are ready to teach it. Be sure to review the special needs of your students and make adaptations to your plan so all students can have the opportunity for success and challenge. Equity of educational opportunity is one of the fundamental principles of teaching.

The AAMT *Standards for Excellence in Teaching Mathematics in Australian Schools* (2006) articulates this in the 3 domains: Professional knowledge, Professional attitudes and Professional practice. The Australian Curriculum is written with the expectation that 'schools will ensure that all students benefit from access to the power of mathematical reasoning . . .' (ACARA 2015c).

Throughout this text, we discuss ways of promoting equity. It is important to note that providing equal opportunities to learn does not necessarily mean providing exactly the same instruction to every child. *Equity* implies fairness and justice. What is fair and just for one student may be different from what is fair and just for another. Providing equitable instruction means adapting instruction appropriately for the needs of each student.

Teachers are often concerned that planning to meet the needs of each student in their classroom will be an overwhelming task. However, rather than create a different lesson plan for each student, you should create a single lesson plan and then consider how you can adapt it to make it more accessible for the different learners in your class. To help in planning, think of 3 students from your class: (1) a student who is ready for the lesson, (2) a student who will require help with the lesson and (3) a student who is ready to work beyond the lesson. Keep the different needs of these 3 students in mind while you plan. In addition, if you have any students who are learning English, keep their needs in mind too. Next, think of some supportive or enrichment strategies that you can use to meet students' individual needs while helping all the children meet the mathematical goal of the lesson. In this text, we

refer to adaptations for students who need help with a lesson as strategies for *gearing down* the lesson and adaptations for students who are ready to work beyond the lesson as strategies for *gearing up* the lesson. Be careful not to go too far and 'lose the integrity of the mathematics content and pedagogy or set expectations too low for students' (Brodesky et al. 2004, p. 150).

Some educators believe that a teacher must ask 3 questions while planning to individualise instruction for students who need extra help (Karp & Howell 2004, p. 119).

1. What organisational, behavioural and cognitive skills are necessary for students with special needs to derive meaning from this activity?
2. Which students have important weaknesses in any of these skills?
3. How can I provide additional support in these areas of weakness so that students with special needs can focus on the conceptual task in the activity?

It is useful to begin collecting ideas and approaches for adapting lessons. Currently, much is being written about how teachers can provide equitable instruction for students from different cultures, students who are English-language learners, Indigenous students and students with identified special needs. Knowing some of these recommendations will give you a head start at becoming an effective teacher.

Teaching students from other cultures

As Australia becomes more culturally diverse, the task of making mathematics relevant for all students both becomes more challenging and offers more opportunities. One educator (Spicer 2005, p. 10) suggests the following.

- Integrating the mathematics of world cultures into K–12 instruction can be a key step in increasing the engagement and mathematical empowerment of students, regardless of their backgrounds:
- Classroom engagement can increase when students see mathematical connections to their cultural roots.
- Students can gain a new respect for the cultures of others when they understand how mathematics was developed by people from many countries and backgrounds.
- Tying mathematics to other aspects of life, including art, literature, technology, and games, helps students see its relevance. (Excerpted from Spicer, J (2004). 'Mathematics of world cultures — A world of possibilities', *ENC Focus,* (12) 32. Reproduced with permission of Eisenhower National Clearinghouse.)

It is important to plan lessons and activities that take into account the realities of students' daily lives as well as to ensure that students' cultural backgrounds are reflected in the mathematics curriculum and materials. Doing this allows teachers to model interest in learning about cultures different from their own. The Making connections sections throughout this text can help you learn more about mathematics in various cultures. Even if you do not have students from those cultures in your classes, they are interesting topics for all children to explore.

Students' cultures may also affect their learning preferences. While being careful not to stereotype students from a particular culture, you may benefit from learning more about the general learning preferences of children from that culture. Keep in mind, too, that ethnicity is not the only cultural factor to be considered. Cultural differences can also be recognised among students from families of low-, middle- and high-socioeconomic status (SES) (Payne 1996). For example, students from low-SES families may lack important background knowledge or may not have mastered certain cognitive strategies; of course, such deficits can be identified and remedied.

You should also understand cultural norms and how they will show themselves in your classroom. For example, you should be aware that some students from other countries may need help adapting to new teaching and testing styles (Abrams & Ferguson 2004/2005). Students whose previous teaching focused on memorisation and repetition may become frustrated in a classroom where the teacher uses a more open-ended approach. In addition, some behaviours considered acceptable in the students' home culture may not be as well received in their new location. For example, in some cultures students do not look

their teachers in the eye when speaking with them. That behaviour may be viewed as disrespectful if we do not understand the reason it occurs. With support, students can maintain their own cultural identity and still learn to function successfully in class in the new culture. Teachers must communicate instructions and expectations clearly and help students adjust to the new culture. Be aware that some of your students may feel and act in ways that could make it difficult for them to succeed in class without your help and understanding: 'In many countries, students are not to speak unless the teacher asks them a question directly. To volunteer answers might be considered boastful or conceited. Many students will not question what the teacher says, even if they know it to be wrong' (Miller & Endo 2004, p. 789).

Teaching English-language learners

Students who are learning English have special needs in our classrooms. According to one expert, there are 5 basic principles of teaching that teachers must adhere to in order to meet the needs of English-language learners (Herrell 2000, p. xiv).

- *Give students comprehensible input (i.e. language they will understand*). This may involve using clear and concise language and avoiding slang. Emphasise and repeat key vocabulary and concepts and accompany these key points with explanatory visuals, models or physical actions.
- *Give students opportunities to increase verbal interaction during class activities (i.e. to talk more with you and with the other children).* This will help English-language learners practise their language skills. Be aware, however, that some children may need to listen to and make sense of a new language for an extended period of time before they will try to speak it in class. Be sure to respect this silent period. Also, match questions to the child's proficiency level. For children who are not yet using the language, allow them to respond with actions (e.g. by demonstrating a skill). For children who are able to speak only in single-word sentences, phrase your questions so one-word answers will suffice or provide choices of answers from which they may select. Respond to the student's intent, not to their pronunciation, grammar or choice of words. In addition, you may need to wait longer for English-language learners to respond to questions, giving them time to translate the question, think about it and then translate the answer. Finally, remember that not all cultures value verbal communication as much as ours does.
- *Teach in a way that contextualises language*. A manipulative or graphic organiser may help students connect old and new knowledge and make your instruction more meaningful. Approach new ideas by beginning with students' current knowledge and by connecting to their lives to provide context. If you can, use both English words and words from students' native languages.
- *Use teaching strategies and groupings that reduce the anxiety of the students*. Look for ways to give English-language learners additional support. Involve as many of their senses as possible. Give students ways to respond besides writing or speaking in English. Allow them to interact with a partner or in a small group, as this is less stressful than having to speak in front of the entire class. Often, working with a peer who speaks the same language can make a student more comfortable.
- *Assign activities in the classroom that offer students opportunities* for active involvement. Using manipulatives and everyday objects can bring meaning to a concept. Some students will be more proficient demonstrating with objects than explaining with words.

Teaching students with identified special needs

Students with identified special needs are those students who have been tested, have been admitted into special education services and have an individual education plan. These students can include gifted children, children with physical handicaps and children with learning disabilities. There are 2 common myths about teaching students with learning disabilities (Karp & Howell 2004, p. 119).

- Myth 1: 'Students with special needs are vastly different from the regular school population and must be spoon-fed information or they will not be able to learn it.' If you teach based on this belief, the result may be passive students who rely on others to tell them how to approach a problem. Since the

goal is to develop independent learners, you must find ways to provide support to these students while encouraging them to develop self-reliance and confidence.

- Myth 2: 'Students with special needs are just like all other children in the class, and 'good teaching' is good teaching for all students.' In order to be successful, students with identified special needs require different learning conditions and depend on different learning strategies than the other students in the class. They often possess a significant deficit in one or more areas, and these deficits can be barriers to their learning (see figure 3.7).

Memory: visual memory, verbal/auditory memory, working memory

Self-regulation: excitement/relaxation, attention, inhibition of impulses

Visual processing: visual memory, visual discrimination, visual/spatial organisation, visual–motor coordination

Language processing: expressive language, vocabulary development, receptive language, auditory processing

Related academic skills: reading, writing, study skills

Motor skills: writing legibly, aligning columns, working with small manipulatives, using one-to-one correspondence, writing numerals

FIGURE 3.7 Potential barriers for students with special needs

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To teach children with special needs of any kind (gifted, physically handicapped or learning disabled), you will need to make adaptations in the curriculum and in your teaching. Guidance is provided in the Australian Curriculum with a focus on student diversity and the General Capabilities (ACARA 2015) Figure 3.8 shows 9 ways to make such adaptations (Ebeling, Deschenes & Sprague 1994). To decide which adaptations are appropriate for which students, you must be familiar with 3 things: (1) individual students and their special needs, (2) the activity and its subject-content demands (what is required to complete the assignment successfully) and (3) the options available for making adaptations. The adaptation categories in figure 3.8 can be useful when planning for all kinds of learner in your classroom.

9 types of adaptation

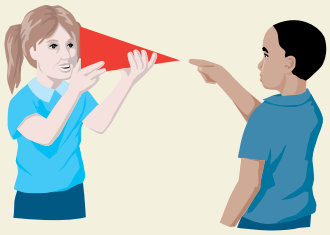
Size Adapt the number of items that the student is expected to learn or complete.	**Time** Adapt the time allowed for learning, task completion or testing.	**Level of support** Increase the amount of personal assistance with a specific learner.
For example: Reduce the number of social studies terms a learner must complete at any one time.	***For example:*** Individualise a timeline for completing a task; pace learning differently (increase or decrease) for some learners.	***For example:*** Assign peer buddies, teaching assistants, peer tutors or cross-age tutors.
Input Adapt the way instruction is delivered to the learner.	**Difficulty** Adapt the skill level, the problem type or the rules on how the learner may approach the work.	**Output** Adapt how the learner can respond to teaching.
For example: Use different visual aids; plan more concrete examples; provide hands-on activities; place students in cooperative groups.	***For example:*** Allow the use of a calculator to figure out mathematics problems; simplify task directions; change rules to accommodate learner needs.	***For example:*** Instead of answering questions in writing, allow a verbal response; use a journal for some students; allow students to show knowledge with hands-on materials.

FIGURE 3.8 9 types of curriculum and instruction adaptation *(continued)*

FIGURE 3.8 *(continued)*

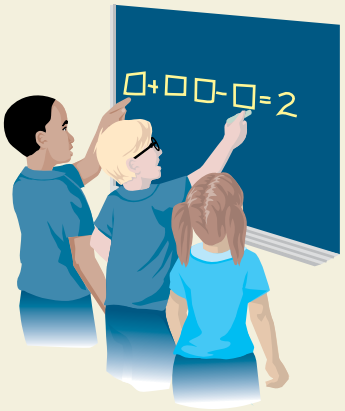
Participation
Adapt the extent to which a learner is actively involved in the task.

For example:
In geography, have a student hold the globe while others point out locations.



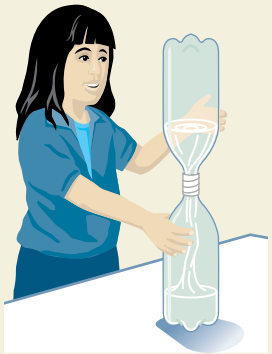
Alternative goals
Adapt the goals or outcome expectations while using the same materials.

For example:
In social studies, expect one student to be able to locate just the states while others learn to locate capitals as well.



Substitute curriculum
Provide different teaching and materials to meet a learner's individual goals.

For example:
During a language test, one student is learning computer skills in the computer lab.



Source: Reproduced with permission from Ebeling et al. (1994).

3.6 Assessment and analysis in planning

LEARNING OUTCOME 3.6 Understanding how teachers can integrate planning with assessment and analysis.

Assessment should be an integral aspect of mathematics teaching. You need to ascertain whether you taught what you think you have taught and whether each child has learned what you think he or she has learned.

All the assessment information teachers collect is useful as they plan lessons. They will know more about the achievement and progress of each child as well as what his or her individual needs are.

Many teachers also find it helpful to keep an evaluative record of the effectiveness of their lessons. They jot down notes in their plan books or their teachers' guides about the things that went well and the things that didn't during each lesson (see figure 3.9). They keep records of activities tried, articles read, ideas they want to try and other anecdotal records. These notes help them to plan the following year. You may think you will remember what happened, but you will probably forget.

Likewise, the AAMT (2006, p. 4) suggests that excellent teachers of mathematics:

- regularly assess and report student learning outcomes, both cognitive and affective, with respect to skills, content, processes and attitudes
- use a range of assessment strategies that are fair, inclusive and appropriate to both the students and the learning context
- maintain ongoing, informative records of students' learning outcomes that are used to map student progress and to plan future learning experiences
- provide constructive, purposeful and timely feedback to students and their parents, and to school authorities as required.

These teaching practices are based on 3 foundational assessment principles. Within the *Standards for Excellence in Teaching Mathematics in Australian Schools* (AAMT 2006) it is suggested that a student's learning of mathematics should be assessed in ways that:

- are appropriate
- are fair and inclusive
- inform learning and action.

The Australian Curriculum provides content descriptions along with elaborations for teachers to gauge student understandings.



FIGURE 3.9 A sampling of stick-on notes from a Year 5 teacher's textbook

MAKING CONNECTIONS

Effective primary mathematics teachers must understand the mathematics being taught; be aware of their students' developmental characteristics; consider what their students already know; think about the kinds of task they will give their students in order to maintain a focus on the mathematics, provide necessary practice and take advantage of strategic moments; think about how they will encourage students to talk, what kinds of questions they will ask and how they will group students; and consider the kinds of

materials they will use in class, including curriculum materials, children's literature, manipulatives and technology.

A GLANCE AT WHERE WE'VE BEEN

Planning lies at the heart of good teaching. Planning helps ensure that all essential content is included, permits scheduling the work in feasible periods of time and in a sensible sequence, helps control the pace of a lesson, aids in holding children's attention, helps avoid unnecessary repetition while ensuring necessary review and practice, and helps the teacher feel confident. Planning must be done for the year, the unit and the lesson. Lesson plans should include clearly stated objectives, procedures, time allotments and assessment. The review-teach-practise lesson format, though widely used, is not very effective. Different types of lesson (investigative, direct instruction and exploration) can be more effective when used appropriately. When planning, teachers must also consider equity — how to meet the needs of all their students, including students from other cultures, English-language learners and students with identified special needs. Finally, assessment and analysis are integral aspects of each lesson, not only for judging student progress but also for judging the effectiveness of the lesson.

THINGS TO DO: FROM WHAT YOU'VE READ

1. From the questions to ask before planning, select one and discuss why it is important to you.
2. Discuss one way you plan to supplement your mathematics textbook materials when you teach.
3. Name a specific special need that students in your class may have. How can you adapt lessons for those students?
4. Why is lesson planning important? Why is it important to plan for the year and unit in addition to making daily lesson plans?
5. What must you consider as you plan a lesson? What should you include in a daily lesson plan?
6. Briefly describe the most important differences among the 3 different types of lesson described in this chapter. When and why would you choose to use each type?
7. Use the lesson plan outline to write a lesson plan for a lesson that you might teach.

THINGS TO DO: GOING BEYOND THIS TEXT

IN THE FIELD

1. Interview several Year 2 and 3 students using the directions from In the classroom 3.1. Were all the children able to use the base-ten blocks to explain the place values? Describe the different levels of understanding that you found. Did you find that any of the children had misunderstandings about place value? Compare your results with those of others in your class and then compile a summary of the children's different understandings.
2. *Analysing classroom interactions and communication.* Observe in a classroom, keeping a record that lets you answer the following questions: What proportion of time is the teacher talking to or at students? What proportion of time are students talking with teachers? What proportion of the time are students talking with other students and the teachers about the tasks assigned? Identify if the talk is about conceptual ideas or simply social talk. In your opinion, are the interactions and communication equitable classroom discourse about significant mathematics? Why?
3. Locate an article that discusses differentiating lessons to suit all learners in the classroom. Talk over these suggestions with a classroom teacher and write a summary of the ideas you want to try in your classroom.
4. *Equity for students with special needs*. Observe in a mathematics class where students identified for special education services are included in the room. What types of adaptation are made for them?

How is technology used in this classroom to support students' with their learning? If you were this teacher, what else would you do for these children?

IN YOUR JOURNAL

5 Reflect on your skill and confidence related to lesson planning. What areas are strengths? What areas do you hope to improve? What do you plan to do to increase your skill and confidence?

6 Describe a lesson you think would lend itself to working with children in cooperative learning groups. How would cooperative learning groups support children in this lesson?

WITH ADDITIONAL RESOURCES

7 Select a topic to teach. Locate resources such as literature, manipulatives or technology to enhance your teaching of the topic.

8 Select a content topic. Compare the way it is taught in a standards-based text versus a more traditional text.

9 Select a content topic. Follow its development through the year levels of the Australian Curriculum. Trace what is review and what is new development of mathematical content.

10 Examine a textbook and the supplemental materials that accompany it to see what assistance is provided for adapting instruction for various levels of students. Are suggestions for modifying lessons included? Are there extra-challenge ideas? Are there review activities for students who need extra practice? How would you use these materials if your class included students with widely varying levels of understanding and skill?

WITH TECHNOLOGY

11 Investigate the *How to Teach Mathematics* website on Scootle (Department of Education and Training 2015) and examine one maths topic area, e.g., fractions. From the example selected, write a sequence of 2 lesson plans that focus on that topic. You can use the planning template provided on that site and ensure that you include the content descriptions from the Australian Curriculum.

12 Collect 5 resources with ideas for incorporating different technologies in teaching mathematics in the primary year levels. For each, describe the technology. Write a lesson plan that describes how you would incorporate one of the ideas in a particular class.

CHILDREN'S LITERATURE CONNECTIONS

The Children's Literature Web Guide www.acs.ucalgary.ca/~dkbrown/index.html
This site collects an enormous number of links to children's literature sites, from authors and publishers to discussion boards and teaching ideas. Well-organised and easy to navigate, it's one of the largest such sites on the Internet.

Carol Hurst's Children's Literature Website www.carolhurst.com
Carol Hurst, a teacher, storyteller, and librarian, has been consulting and writing about children's literature and teaching for many years. Find reviews, magazine articles, book excerpts, and many suggestions on how to use children's literature in the classroom.

Fairrosa Cyber Library of Children's Literature http://fairrosa.com
A personal and comprehensive site of children's literature sites on the Web curated by a children's librarian who is well-known in online children's literature circles.

Internet Public Library Youth Division: Reading Zone www.ipl.org/cgi-bin/youth/youth.out.pl?sub=rzn0000
Provides a range of children's literature sites suitable for children and adults. Particularly strong in sites with children's work and online stories.

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ACKNOWLEDGEMENTS

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CHAPTER 4

Enhancing learning and teaching through assessment and feedback

LEARNING OUTCOMES

4.1 Understanding practices that can enhance learning and teaching.

4.2 Evaluating methods and strategies for gathering information for what students have learned and what they think and feel about their learning.

4.3 Investigating how learning, activities, feedback, teaching and evidence connect.

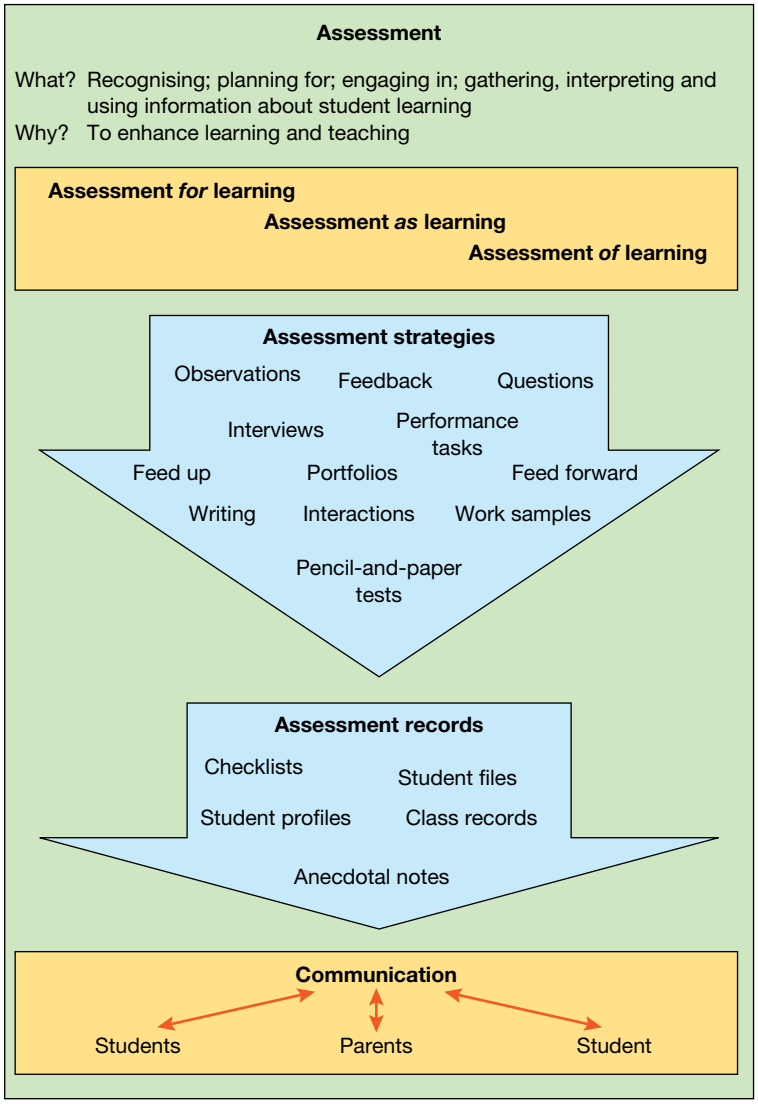
4.4 Using evidence to inform student progress and teaching.



‘Nobody else can make anybody else learn anything. If you are a gardener . . . you don’t make the flowers grow. Your job . . . is to provide the optimum conditions . . . to allow it to grow itself.’

Sir Ken Robinson

Chapter 4 concept map



Introduction

How do educators know what children know? What can we do to find out what knowledge and understandings children bring with them to our classrooms? How do we know what children have learned in our classroom? How do we develop opportunities that create future learning affordances? Exams? Quizzes? Marks and grades? Homework? Projects? Standardised tests? Teacher–student interviews? According to the Australian Association of Mathematics (AAMT 2013, para. 1), 'Whatever is worth teaching is worth evaluating'. Callingham (2010, p. 41), takes this further by outlining the importance of assessment in terms of learning and teaching:

> Assessment is arguably the most powerful element in teaching and learning. Quality assessment can provide information to students, teachers, parents and systems in effective and useful ways. To be helpful, however, it must be broad ranging, collecting a variety of information using a range of tasks before, during and after a teaching sequence.

Let us revisit the quotation at the start of this chapter. What is it that educators can do to provide optimum conditions for students to learn? These are the learning affordances generated by the experiences within our classrooms and they are the opportunities we create that enhance learning and teaching. This would bring to mind the title of this chapter and that may assist in answering the question at the start of this paragraph — assessment and feedback. Indeed, under the domain of Professional Practice of the *Australian Professional Standards for Teachers* provided by the Australian Institute for Teaching and School Leadership [AITSL] (2014), Standard 5 is *Assess, provide feedback and report on student learning*. The inclusion of a specific standard that addresses assessment, feedback, and reporting indicates the importance of these in a teacher's practice. Teachers who are highly accomplished are able to:

> Develop and apply a comprehensive range of assessment strategies to diagnose learning needs . . . provide targeted feedback based on timely judgements of each student's current needs in order to progress learning . . . use data from internal and external student assessments for evaluating learning and teaching, identifying interventions and modifying teaching practice . . . construct accurate, informative and timely reports to students and parents/carers about student learning and achievement [AITSL, 2014].

Assessment can be viewed as assessment *for* learning, assessment *as* learning, and assessment *of* learning (Callingham, 2010). However, assessment strategies and tools may be used to address more than one type of assessment — it is when the strategy or tool is used, how it is used, and what follows its use that determines this. Another crucial element in enhancing learning and teaching is feedback. In their seminal paper on feedback, Hattie and Timperley (2007) described it as existing on a continuum informed by the learning context as part of the teaching process. This feedback provides information to students regarding where they are going, how they are going, and where they will go next — often referred to as feed up, feedback, and feed forward.

4.1 Enhancing learning and teaching

LEARNING OUTCOME 4.1 Understanding practices that can enhance learning and teaching.

Assessment *for* learning

Much of your effort and time as a teacher will be focused on classroom assessment — assessment *for* learning. This focus will be on information about student achievement that you will need, moment by moment, day by day, as:

> the value of assessment for learning lies in the ways in which it can contribute to the main aim of education, which is to develop in students the capacities of independent, effective and responsible learning. (Black, 2015, p. 175).

Phases of assessment

The process of assessment for learning can be viewed as linking teaching and learning to assessment through 4 phases: (1) planning, (2) gathering, (3) interpreting and (4) using. Let's consider an actual classroom example (from Moskal 2000) as we explore these phases.

While planning for an upcoming unit on decimals, Ms Lee decided it would be important to gather information about what her students already understood about place value, ordering and density of decimals. So, she posed the following task in class: 'Write all the numbers between 3.4 and 3.5 on the board.' One of the students, Dana, wrote: 3.41 3.42 3.43 3.44 3.45 3.46 3.47 3.48 3.49. Nakisha disagreed, asserting there were more numbers between 3.4 and 3.5. Ms Lee asked the class who they thought was correct. She asked them to use paper and pencil to take a stance and explain their reasoning.

Figure 4.1 shows the response of 3 students from Ms Lee's class. Jim thought Nakisha was right. To support his claim, he offered examples of more numbers between 3.4 and 3.5. Afwandi also agreed; but instead of offering specific examples of other numbers, she simply claimed that there are infinitely many numbers between 3.4 and 3.5. Juan offered an example of how to find additional numbers between any pair of decimals.

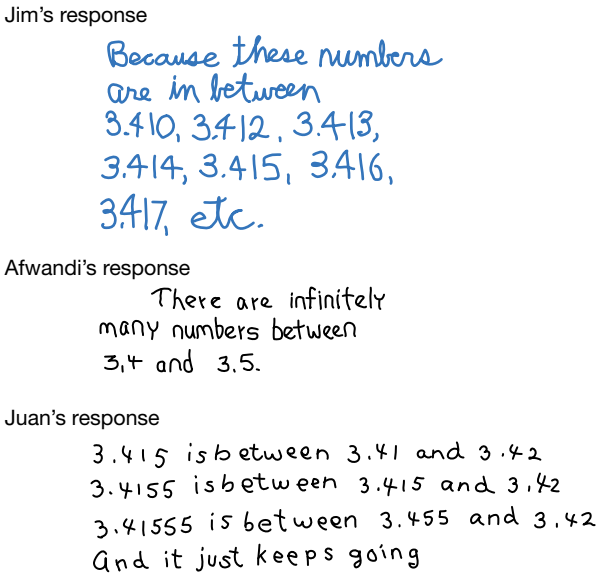


FIGURE 4.1 Responses from three of Ms Lee's students

Source: Moskal (2000, p. 192).

Ms Lee reviewed all of her students' papers. Juan appeared to have some rudimentary knowledge of the concept of infinity. Jim's response concerned her because he seemed to be (incorrectly) claiming that 3.410 was missing from Dana's list. Dana had listed 3.41, but maybe Jim didn't realise that the trailing zero on 3.410 didn't change the value of the number. It wasn't clear from Afwandi's response what she understood about the density of decimal numbers. It seemed possible that her answer about 'infinitely many numbers' was merely a phrase she had heard before and echoed without really understanding.

Ms Lee decided more instruction was needed, particularly regarding zeros in decimal numerals. She decided to try — by posing another task — to help her students think about zeros in decimals at the same time that she would gather more information about their thinking. So she assigned this task: 'Identify the numbers below that are equal: 02.3 20.3 20.03 20.030 2.3. Explain how you know which numbers are equal' (Moskal 2000, p. 193).

Ms Lee's approach to teaching about decimals provides an example of the planning, gathering, interpreting and using phases of the assessment process. She began by *planning* a task (write all the numbers between 3.4 and 3.5). After her students had completed the task, she *gathered* their responses and *interpreted* them. Finally, she *used* the information she had gathered to plan another assignment for her students. Of course, the events in each phase of this model may vary, depending on the type of assessment you use, the mathematical topic and your students' responses. However, making instructional decisions is one of the primary ways that teachers use information gained through the classroom assessment process. For example, Ms Lee might also decide to talk with Afwandi or to engage her in a task-based interview to gain more insights into her understanding of 'infinitely many'. Note that it was Jim's response, and similar responses from other students, that convinced Ms Lee to assign another task to help students reconsider the role that zero plays in decimals.

Now that we have examined a specific example, let's think, more generally, about the 4 phases of classroom assessment.

When you are *planning an assessment*, it is important to think about the purpose of the assessment, the methods you will use to collect data, how you will interpret the evidence you collect (e.g. how you will award points for students' written problem-solving work) and how you will summarise and use your findings. Will the outcome of your assessment be to change your lesson plans for the next week? Or will you use the information when explaining to parents what their children's strengths and weaknesses are? How you answer these questions may affect the sorts of data you decide to collect.

When you are *gathering evidence*, think about the variety of tasks or activities you might use and try to choose appropriately. For example, if you want to find out what your students understand about decimals, you could have them complete some worksheets on addition and subtraction of decimals.

However, that task might not provide you with as much information as having them show you how they could use base-ten blocks to represent an addition or subtraction problem with decimals or having them write about using base-ten blocks and draw pictures to accompany their writing.

When you are *interpreting evidence*, it is important to think in advance about the criteria you will use to judge the adequacy of responses. You should think about how you *hope* your students respond to a task or question and what sorts of difficulties or misconceptions they *might* have. What sorts of responses will you consider exemplary, adequate or flawed? What will you consider adequate evidence of understanding?

Finally, when *using results* from assessments, think about how you will communicate your judgements and how they will affect future instructional decisions. For example, how well will a numeric or letter grade help your students understand where they went wrong and what they need to do differently? Can you supplement the grade with written comments? Can you find time for brief individual conferences with some of your students? How will you use what you've learned about your class, as a whole, in planning for future instruction?

Assessment *as* learning

Assessment *as* learning is the experience itself and is 'informal, undertaken as part of the teacher's "normal" activity' (Callingham, 2010, p. 40). Callingham also describes assessment *as* learning as incorporating interactions between the teacher and student and feedback. Although interactions and feedback contribute to and are strongly related to the value of assessment *as* learning, they are also valuable within assessment *for* learning and assessment *of* learning as they generate learning affordances. However, the distinguishing feature within assessment *as* learning is the temporal immediacy.

Interactions

In a mathematics classroom, or any classroom, interactions occur between the teacher and students and between the students. These interactions can be considered as discourse, described by Björklund Boistrup (2015) as:

> more than the entirety of what is communicated and the way it is communicated. It is also present in what is not communicated, or what is communicated through gestures, attitudes, presentations, patterns of actions and the rooms and furniture. (p. 133).

In her research in classrooms in Sweden, Björklund Boistrup (2015) identified 4 construed discourses. Each of these were described in terms of verbal and non-verbal communication, feedback, and the evidence (or not) of interest in answers, underlying mathematical processes, and reasoning.

1. Do it quick and do it right, where the aim is for students to quickly offer the teacher the correct answer. The feedback is focused on the correctness of that answer.
2. Anything goes, where students offerings are encouraged through tools such as open questions, though structure for the use of resources during the activity are lacking. Feedback is limited to approval for offering answers, with minimal critique of the student answers (with incorrect answers not queried or corrected) or suggestions on how to use the resources.

3. Openness with mathematics, where tools such as open questions are used to encourage students to provide alternative mathematical processes for an identified incorrect answer. Feedback is from the teacher to the students, often through the use of questions and at a fast pace.
4. Reasoning takes time, where emphasis is on challenging students, developing new learning, and student reflection. Feedback occurs in both directions between the teacher and students, at a slower pace, with pauses for thinking and longer discussions.

Each of these construed discourses generates different learning affordances. These learning affordances are created from the activity, which itself includes feedback that may focus on just the answer, the mathematical processes, use of resources, or reasoning. Being aware of the discourse in your mathematics experiences can enable you to generate the depth of discussion needed for your students to develop mathematical understandings. As Björklund Boistrup (2015) states, it can enable you to identify the opportunities you are providing for learning and the types of feedback you use.

Feed up, feedback, and feed forward

Although the term 'feedback' was used in the section on interactions, there has been much work on considering what feedback is. In her work, Björklund Boistrup (2015) includes further details regarding the components of feedback that are likely to be used with the different construed discourses. As noted earlier, Hattie and Timperley (2007) provided the seminal description of feed up, feedback, and feed forward. Fisher and Frey (2011) describe each of these 3 components of feedback as essential.

1. Feed up sets out the requirements and resources for the task and outlines what needs to be accomplished in that task.
2. Feedback scaffolds students' learning by providing timely indications of whether their understandings and learnings are appropriate in terms of the requirements for the task.
3. Feed forward informs the planning of what experiences or learning should occur after the existing task.

As a teacher, you need to use each of these components to create optimal conditions for your students. Considering these components together with Björklund Boistrup's (2015) construed discourses above, it can be seen that not including feedback or providing inappropriate feedback can greatly reduce the learning affordances for your students.

4.2 Gathering information on student learning

LEARNING OUTCOME 4.2 Evaluating methods and strategies for gathering information for what students have learned and what they think and feel about their learning.

Assessment *of* learning

Assessment *of* learning is primarily designed to document what students know and are able to do. For example, tests are sometimes given (or other sorts of evidence collected) to decide:

- in which class to place students
- whether to move a student from one group to another
- what grade to record on a student's report card.

When assessments have serious consequences such as these, we call them 'high-stakes assessments'. In the situations just described, high-stakes decisions about the students themselves are being made as a result of the assessments.

However, more frequently these days it is not just the individual students who are affected by the consequences of high-stakes assessments. For example, tests may be administered in order to document the achievement of a group of students or to compare one group of students with another. Individual teachers, schools or regions may be held accountable for their students' test scores, taken as a group. Many schools, regions and states now use student test results to make funding decisions.

In Australia, the National Assessment Program runs the National Assessment Program — Literacy and Numeracy [NAPLAN] (National Assessment Program [NAP], 2013) which incorporates nationwide tests

for literacy and numeracy. These tests are completed by students in Years 3, 5, 7 and 9, and replaced state tests that differed from one another and were administered in different states to students of different ages (Lowrie & Diezmann 2009). A contemporary interest regarding national and international standards and accountability, along with development of an Australian national curriculum, necessitated an implementation of ways to assess the performance of Australian students against recognised national minimum standards. This was enabled through the introduction of NAPLAN, although this also changed the landscape and introduced 'a new era of testing and assessment in Australia' (Lowrie, Greenlees, & Logan, 2012, p. 145).

As a form of summative assessment, a national assessment program of this kind can provide data for 'macro-level (systemic) analytical studies and micro-level (individual school and/or class) analytical studies' (Norton 2009, p. 26). The data can be used as evidence of state, district, school or even class-level mathematics achievement, with an aim to make comparisons. However, the same data could be used at a school, class or individual level to diagnose students' weaknesses or strengths and refine teaching programs. In this latter context, summative assessment can be viewed as formative assessment (Lokan, Doig & Underwood 2000). That is, standardised tests, generally viewed as assessment *of* learning, can also be used as assessment *for* learning.

Other forms of standardised testing in recent use in Australia have focused on the early years of schooling; for example, the *Count Me in Too* program by the New South Wales Department of Education and Training (2009) is an online resource that is available for educators. Likewise, the Victorian Department of Education and Training (2014) also provides a *Mathematics Online Interview*. The task-based interviews that are central to these programs are used primarily as diagnostic assessment. Student learning is mapped by a thorough process; in some cases, it is mapped to a developmental continuum. The resultant profile of a student's knowledge, understandings and skills, while providing a record of achievement, can also serve as information to inform teaching. Hence, diagnostic tests as assessment tools are both summative and formative in nature — making them assessment *of* learning and assessment *for* learning. As they do not enable teacher feedback during the experience, they are not generally viewed as assessment *as* learning. However, students within the assessment process may self-assess by identifying areas they do not know or understand.

It should now be clear that there are not necessarily clear distinctions between formative and summative assessment. The context, purpose and ultimate use of the related information is what determines its nature and its focus as assessment *for* learning or assessment *of* learning (or both). Thus, in upcoming sections, the range of assessment strategies outlined can be used for formative assessment or for summative assessment, depending on how the information gathered is used.

Making teaching and learning decisions

As a teacher, you will be making many teaching and learning decisions. As with any decision, the better informed you are, the wiser are the choices you usually make. How assessment information is used in making decisions has shifted from past practices (for example, see table 4.1).

One shift in assessment practices is a move toward gathering assessment data continuously rather than periodically or at the end of a chapter, when it is too late to modify your instruction. Another trend is using a wide variety of sources of information (rather than primarily tests). The third change is related to keeping the needs and progress of students in mind when making long-range teaching plans (rather than following a set plan of study regardless of how well students are doing).

Monitoring student progress

Several shifts in monitoring student progress are also called for by the National Council for Teachers of Mathematics [NCTM] (1995) Assessment Standards (table 4.1). These include assessing a wide range of student capacities rather than focusing just on testing students' factual knowledge and skills, providing more extensive and elaborate feedback to students on their progress, using a wider variety of methods for gathering information (rather than relying primarily on tests and quizzes) and involving students more in the assessment process.

TABLE 4.1 **Major shifts in assessment practices**

Towards	Away from
Shifts in assessing to make teaching and learning decisions	
Integrating assessment with instruction	Depending on scheduled testing
Using evidence from a variety of assessment formats and contexts	Relying on any one source of information
Using evidence of every student's progress toward long-range planning goals	Planning primarily for content coverage
Shifts in assessing to monitor students' progress	
Assessing progress toward mathematical power	Assessing knowledge of specific facts and isolated skills
Communicating with students about performance in a continuous, comprehensive manner	Simply indicating right or wrong answers
Using multiple and complex assessment tools	Primary reliance on answers to brief questions on quizzes and tests
Students learning to assess their own progress	Teachers and external agencies as the sole judges of progress
Shifts in assessing to evaluate students' achievement	
Comparing students' performance with performance criteria	Comparing student with student
Assessing progress toward mathematical power	Assessing knowledge of specific facts and isolated skills
Certification based on balanced, multiple sources of information	Relying on only a few, narrowly conceived sources of evidence
Profiles of achievement based on public criteria	Single letter grades based on variable or nonpublic criteria

Source: Adapted with permission from *Assessment Standards for School Mathematics*, Copyright 1995 by the National Council of Teachers of Mathematics. All rights reserved.

Evaluating student achievement

Teachers today are also doing things differently in terms of grading and certifying students' achievement (table 4.1). One change is judging students' performance against stated criteria (standards or expectations) rather than by comparing students with one another. The second and third shifts were mentioned earlier: assessing a broad range of components of mathematical capacities rather than isolated skills and knowledge, and using a variety of sources for gathering assessment information. Finally, there are trends toward making achievement criteria more open and public.

The number of shifts listed in this section is an indication of the need for change in many of our assessment practices. This change will come through time and open minds. This chapter is an opportunity to begin asking yourself questions about how you will assess students in order to make decisions, monitor progress and evaluate achievement.

4.3 Ways to assess students' learning and dispositions

LEARNING OUTCOME 4.3 Investigating how learning, activities, feedback, teaching and evidence connect.

There are many different ways to gather information about the learning, dispositions and interests of students. What type of information to collect and how to obtain it depends on the purpose for which it will be used. You first need to ask yourself what you want to assess.

For example, if you want to assess the principle discussed earlier in the text — that mathematics learning should make sense to students — then you want to observe students and ask them questions that elicit their level of understanding, to find out whether they have attached meaning to what they are doing. If you want your students to be persistent and willing to approach problems in a variety of ways,

then you need to assess these characteristics through problems of a challenging nature. If you want your students to be able to communicate well and to work well with others, as discussed in an earlier chapter, then you should make those characteristics part of your assessment.

Once you have engaged your students in activities likely to elicit the behaviours that you want, you need some sort of guidelines or framework to use in looking at the data you collect. To see how well your students can communicate about the notions of area and perimeter, you might ask them to write a letter to a younger child comparing and contrasting these 2 ideas and providing an example of a real-world problem using each measure. But how will you judge which student letters are better than others? You may choose to use a rubric, or scoring guide.

4 Exemplary work	Task solved correctly and efficiently, or the student may even have gone beyond expectations of the task (e.g. problem solving in more than one way, or the student has extended the problem and completed a more difficult version). Communication of processes and thinking is clear and effective.
3 Good work	Task was completed correctly but possibly inefficiently, or not communicated clearly, or with some errors so minor that the teacher is confident of the student's understanding. Any errors or communication difficulties should be able to be corrected with minimal feedback from teacher.
2 Marginal work	Task partly completed or completed with major errors. There may be either a lack of evidence of understanding or evidence of some misunderstandings. Some further teaching will be required.
1 Needs more instruction	No progress or only minimal progress. Work may include evidence of faulty reasoning that the problem was not understood or that inappropriate methods were used. Teacher intervention is essential.

FIGURE 4.2 General holistic scoring guide for mathematics problem solving

A *rubric*, or *scoring guide*, is a rating scale that can be designed or adapted for use with a certain class of students or a particular task. Generally, a scoring guide is used to assign anywhere from 0 to 10 points to student work to provide a rating of performance. This is different from awarding points when grading a quiz or test, where you generally count the number of correct answers to arrive at a score. Instead, with a rubric you look at performance on the task and rate that performance along a continuum. As Stenmark and Bush (2001, p. 118) explain:

> *Scoring* is comparing students work to criteria or rubrics that describe what we expect the work to be. *Grading* is the result of accumulating scores and other information about a student's work for the purpose of summarising and communicating to others.

Rubrics, or scoring guides, can be either holistic or analytic. Holistic criteria assign a single score based on the overall quality of the student's work. Holistic scoring guides may have just 2 or 3 levels, or more, and should include general descriptors of the achievement necessary to attain them. For example, you might award points for a student's problem-solving work as shown in the scoring guide in figure 4.2. On the other hand, when it seems important to score various dimensions or traits of student work separately, analytic scoring rather than holistic scoring is appropriate. (See figure 4.10 for an example of an analytic scoring guide for assessing open-ended questions.)

One very straightforward way of constructing a scoring guide is to think about sorting students' work into 3 piles:

1. those who do not understand the outcome (concept or process)
2. those who are developing understanding (but are not quite there)
3. those who understand the idea and can apply and communicate it.

For any specific task, you can develop *performance indicators* to help judge which student work to put in which categories. Ultimately, these performance indicators provide descriptions of how well the students are making sense of the mathematics.

For example, you might design a task for students to show what they know about the meaning of decimals by using base-ten blocks to build specified decimal numbers (e.g. 3 tenths, or 1 and 5 hundredths) and to demonstrate decimal relationships (e.g. build 2 tenths and build 22 hundredths and explain which one is larger). Each flat block could represent a whole number, each rod a tenth, and each little cube a hundredth. In this example the student has displayed 2.46.

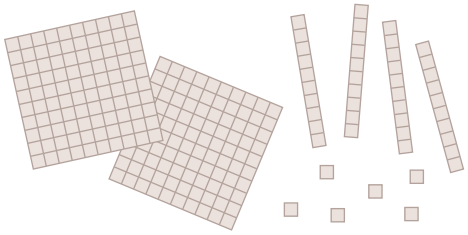


Figure 4.3 shows a 3-category rubric, or scoring guide, with performance indicators for this task. This type of scoring guide is often called an annotated holistic rubric.

Sample performance indicators for a student who:

Does not understand	Is developing understanding	Understands
• When told to use 1 flat to represent a whole, cannot use blocks to show how many tenths would be equivalent to that 1 whole • When told to use 1 flat to represent a whole, cannot use blocks to show how many hundredths would be equivalent to that 1 whole • Has difficulty understanding that the same blocks can be used to represent different powers of 10 in different problems (may think a flat must represent 100 if earlier experiences assigned a value of 1 to each small cube)	• When told to use 1 flat to represent a whole, can state which block would represent 1 tenth (rod) and which block would represent 1 hundredth (little cube) • Can use blocks to show why 10 tenths and 100 hundredths are each equivalent to 1 whole • May interchange value of blocks when building decimals • May have difficulty building decimals with blocks, especially when the decimal includes zeros (e.g. 3.04) • May have difficulty stating what number is represented by a given collection of blocks (e.g. 5 flats, 2 rods and 3 little cubes)	• Understands that the same blocks can be used to represent different powers of 10 in different problems • When told which block represents 1, can state what number is represented by any collection of blocks • Can build, order and compare decimals using blocks to illustrate • Can build alternative models for the same decimal (e.g. can represent 3.45 either with 3 flats, 4 rods and 5 little cubes or with 3 flats and 45 little cubes)

FIGURE 4.3 Scoring guide with performance indicators for base-ten block task on decimals

Observation

Observing students as they work is a commonly used assessment method in the classroom. However, many teachers never go one important step further: to make notes about their observations or think carefully about what they want to look for and why. Watching and listening may seem easy and commonplace, but it takes practice and planning to hone your observation skills. For example, it is helpful to plan what you will observe on a given day.

Your students are solving problems involving addition facts. You may want to observe and record which students are using physical materials, which are doing most of the problems mentally, which are using thinking strategies to determine basic facts (which will be discussed later in the text), and which are relying on memorised facts. At times, you may plan to observe only one student in a cooperative group setting.

- Does Kenn jump right in or wait for others to begin?
- Does he know his doubles facts? Does he know the facts that make 10?
- Is he able to use known facts to determine unknown facts? If so, what strategies does he use? What sense does he make of the mathematics? Does he accept or challenge the ideas of others?

From such observations, you can gain insight into a student's attitude and disposition toward mathematics. This knowledge, in turn, can help you plan ways to encourage strengths and work on weaknesses.

Your notes will be useful anecdotal records for individual assessment, for future planning and for reporting to parents. Your observations can also help you decide what to do immediately in the classroom, while you are leading a discussion or presenting a new concept.

Some teachers find it helpful to set aside a page in a notebook for each student or to use individual index cards in a file. However, it can be clumsy to page through the notebook or fumble through the cards when you are walking around the room. To make observation notes easier, some teachers tape an index card for each student on a single page of heavy cardboard, with the cards overlapping as shown in figure 4.4.

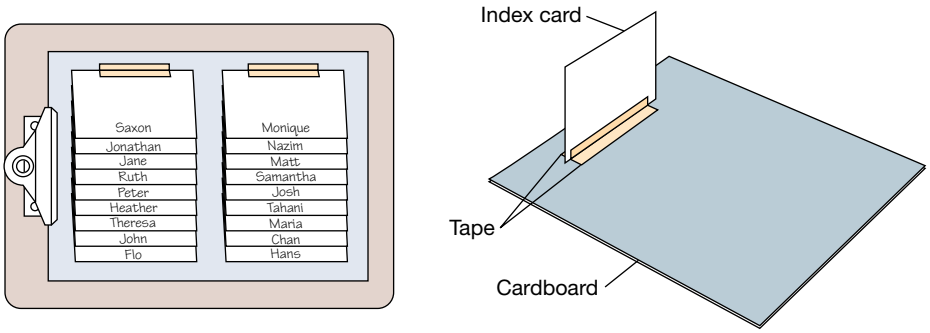


FIGURE 4.4 Flip cards for recording classroom observations. Cards can be arranged alphabetically or by classroom seat assignments, whatever will help you find the right card quickly.

When you want to write notes about a particular individual, you just flip to his or her card. When a card is full, you can file it away in that student's file and replace it with a fresh card. Another handy idea is to carry a page of computer labels or adhesive notes on your clipboard as you circulate around. Make notes about individual students on separate labels or notes (putting name and date on each). Later, you can paste these notes into the individual students' folders or pages in your observation notebook. If you want to remember to collect data on certain students on a given day, write their names on labels or adhesive notes in advance as a reminder. Whichever form or process you use, it is extremely important to remember to date your observations.

You can read more about assessing via classroom observation in Star and Strickland (2008) and AAMT (n.d.).

Questioning

In the previous 2 chapters, we discussed the value of asking questions of students to help direct their mathematical thinking. Good questioning techniques can also complement and enlighten your assessment observations. Asking good questions is an art that needs to be developed and practised. When you teach through questioning, you actively involve students and know more about what they are thinking. In planning your lessons, you should think of questions that will help you gauge whether students are doing the following:

- making sense of the mathematics
- approaching a problem in different ways
- clearly explaining their thinking
- showing evidence of generalising.

Limit the number of questions that can be answered yes or no or with one-word responses. For example, instead of asking 'Should we add, subtract, multiply or divide to solve this problem?' ask,

'What could we do to understand this problem better?' or 'What sort of picture can we draw to help decide what operation to use?' As a teacher, you should also work to develop more high-level and open-response questioning, asking students to explain or defend their answers or to describe another way to solve the problem. An effective framework to use for developing higher-level questions is Bloom's Taxonomy (as cited in Forehand, 2005). Many resources exist on the web that provide example questions for each of the levels of Bloom's Taxonomy or one of the revised versions (for example, see the website by Churches in the reference section). Remember that students need time to think about their answers, so practise incorporating pauses after questions — although this might seem easy, it can be difficult to not jump in and provide prompts or even answers.

Finally, recognise that you can involve many more of your students in thinking deeply about mathematics if you think of ways to get everyone involved in asking and answering questions (Reinhart 2000). For example, pose a question to the class and ask everyone to 'think-pair-share'. The students should first think about their own answers, then pair up with a partner to talk about their ideas, and finally be ready to offer suggestions to the entire class. (For even more opportunities for small-group discussion, you can have the pairs join up for discussion in groups of 4 or 6 before you bring the whole class together to talk about the question.) With this questioning model, everyone is encouraged to think and talk about the question. The conversation in the classroom no longer 'ping-pongs' from teacher to individual student and back.

Some classic and recent references on questioning include Burns (1985); Truxaw and DeFranco (2008) and Mewborn and Huberty (1999).

Interviewing

Interviewing is a powerful way to learn about a student's thinking and to give her or him some special attention. Key factors in a successful interview are establishing rapport with the child, actively listening to responses, accepting responses without judging and encouraging the child to talk and explain.

For example, at the beginning of the school year, Ms Morihara conducted short interviews with her new Year 4 students to assess their understanding of odd and even numbers. Here is what Tonya said during that interview:

Ms Morihara: Tonya, is 9 an odd or even number?

Tonya: It's odd . . . unless you go by 3s, then it's even.

Ms Morihara: Oh, that's interesting. Can you show me what you mean with tiles? [Ms Morihara gave Tonya 9 tiles to explain her thinking. Tonya stacked the tiles into 4 groups of 2 and showed her the 1 that was left over.]

Tonya: See, if we divide evenly by 2s, there is 1 left over so it is odd.

[Tonya then divided the 9 tiles into 3 groups of 3.]

Tonya: But, if we divide it into groups of 3, there aren't any left over, so it is even. That is why 9 is both odd and even.

Ms Morihara: Thank you very much, Tonya, for explaining your thinking to me.

A short interview can provide a wealth of information regarding a student's understanding. Tonya is doing some interesting thinking about number relationships, particularly what it means to separate various numbers of tiles into even-sized piles. She seems to recognise that there may be different numbers of tiles left over depending on what sized piles the tiles are sorted into, and she is interested in focusing not just on the number of piles but also on the number of leftovers. Although she is confused about the definitions of 'odd' and 'even', she seems on the verge of understanding the more complex notions of 'factor' and 'multiple'. If you had a chance to probe Tonya's thinking even further, what questions would you ask her? How would you attempt to change her misconceptions?

Although you will not have time to ask in-depth questions of each of your students each day, it can be enlightening to choose a few children to interview each week until you have had a chance to talk with each

individual. Teachers are often surprised at how students value private time with them and how much they learn in a short time. You may be able to squeeze in an interview or 2 each day during whole-class silent reading time, workstation time (when other students are circulating from station to station) or seat-work time.

Before interviewing a student, you need a basic plan of what concepts or skills to assess, what materials you need, what questions to ask, and when and how you will record the information. You may want to have alternative paths to take if the interview proceeds in different ways.

The interview itself generally has 3 parts: initiation, questioning/hypothesis formulation and questioning/hypothesis testing (Long & Ben-Hur 1991). During the initiation phase, the interviewer puts the student at ease by chatting informally, asking nonthreatening personal interest questions, showing the child the materials that will be used and ensuring that he or she is familiar with them, and explaining the purpose of the interview.

Next, begin posing tasks, making certain to rephrase questions in the student's own language when necessary and encourage the student to explain and elaborate.

It is the interviewer's responsibility to ask questions in a variety of ways in order to formulate hypotheses about how the child is thinking about the task and what the child does and does not understand. It is critical to remain nonjudgemental about student responses, even though students often look for subtle clues about the correctness of their answers. Some noncommittal phrases that are helpful in prompting children to explain their thinking further (without indicating whether previous responses have been right or wrong) include the following.

- I am interested in knowing more about your thinking. Talk to me about it.
- Pretend you are the teacher and I am your student. Please help me understand.
- Can you explain that in a different way?
- I like it when you take the time to explain your thinking.
- I think I understand now, but what if . . .?

As the second phase of the interview, begin to formulate hypotheses about what the child knows and where the child has trouble.

The third phase is the time to ask questions specifically designed to test hypotheses, to see if the difficulties really are what they seem to be. A big temptation for many teachers is to slide into teaching during an interview. It is important to resist the temptation to fix errors or misconceptions on the spot. If there is time, teaching may be appropriate, but re-teaching or tutoring is often better reserved for another day. The focus of a classroom interview should be to figure out what a student knows, not to help him or her on the spot. In the classroom 4.1 is a plan for interviewing Year 3 students about their understanding of place value.

Additional sources of information on interviews include the *Count Me in Too* online program provided by the New South Wales Department of Education and Training (2009) and the *Mathematics Online Interview* provided by the Victorian Department of Education and Training (2014).

IN THE CLASSROOM 4.1

Primary interview on place value

Objective: Determine students' understanding of place value.

Year level: 1–3

Materials: Unifix cubes, paper and pencil, base-ten blocks

Tasks:

1. *Show me 24 Unifix cubes.* Observe whether child counts accurately (says numbers in proper order, uses one-to-one correspondence and has a method for keeping track of which cubes have been counted).
2. *Now let's suppose we want 34 cubes. Can you show me that?* Does child count out 10 more or count over from the beginning?



3. *Please write the number 34 here.*
4. *Let's put the 34 cubes into groups of 10. How many tens do you think we can make? Do you think you'll have any cubes left over? Count out the 34 cubes into piles to make sure.*
5. *Let's look at 34, the number you wrote. Show me with the cubes what the 3 means. Now show me what the 4 means.*
6. *Now let's make 4 piles of 10 cubes each and have 5 left over. Do you know how many cubes we have here all together? How can you tell?* Does student know to count the piles by 10 and add the leftovers? Or does he or she still count all?
7. (a) If the child has trouble with the previous questions, check further by asking him or her again to count out several different numbers of cubes (e.g. 17, 26 or 30) and to predict how many tens and how many left over. Ask the reverse question: *For example, if we have 2 tens and 7 left over, how many cubes would that be?*
   (b) If the child has no trouble with the questions above, probe for understanding of place value of larger numbers, such as 123 or 347, using base-ten blocks and asking how many piles of 100 could be made and how many piles of 10 and how many ones. Also ask the reverse question: *If we have 3 flats and 6 sticks and 2 units, how many cubes would that be altogether?* As an extension check to see how many different ways the child can show 136.

Interview reminders: Listen and watch carefully! Let them do it! Don't teach! Be flexible. Ask for more examples, if needed.

Ask: How would you show a friend? How would you explain it to a little kid?

Performance tasks

In a class that is alive with problem solving and investigations, many valuable opportunities arise to observe students working on performance tasks. In fact, the only way to assess some skills is through performance tasks. For example, if one objective is for children to know how to measure with a ruler, then they need to be assessed doing such measuring. Questions on written tests that present a picture of a ruler perfectly lined up next to a picture of some other object to be measured do not reveal much about the actual skill of measuring. From this task you can judge whether the child can read measurements from the ruler, but you have no idea whether he or she would actually be able to use a ruler effectively.

Performance tasks generally mirror the real world, are open-ended and require time for grappling with a problem. It is often helpful to pair children when observing performance of such tasks so that you can hear their conversation as they work. In planning, you should list some of the areas you want to observe as the pairs work on appropriate tasks.

As an example of a performance task, look back at In the classroom 2.2 (chapter 2). In that activity, children are asked to build different boxes (right prisms) with some blocks. As they build boxes of various dimensions and various volumes, the formula for the volume of a box evolves naturally. The children are involved in using models, making decisions and thinking mathematically, rather than just applying a formula. Figure 4.5 lists 3 areas you might plan to observe and examples of the notes you might make while observing the performance task from In the classroom 2.2.

As indicated in the previous chapter, there should be an emphasis on worthwhile mathematical tasks. You may want to begin with simple tasks and then build up to longer, more complicated ones. Often, you will be surprised at the tenacity of very young children on a task that is engaging them; you should not rule out richer tasks for this age. Badger (1992) and Moon (1993) are classics that provide interesting reading about performance assessment in the primary mathematics classroom. The use of performance tasks is demonstrated in Paek and Foster (2012), who found connections between the use of complex performance tasks and subsequent student mathematical achievement.

Self-assessment and peer assessment

Self-assessment is an activity that engages many people. For example, actors, athletes and musicians often study videotapes of their performances in order to figure out how to improve. In a similar way, when

mathematics students engage in activities that promote self-awareness and self-evaluation, they can eliminate weaknesses and become better problem solvers. Students are often the best assessors of their own work and feelings. When students evaluate their own work, the responsibility for learning is theirs.

	Jason	Amy	Lisa	Tyler	Alana	Kristin	Tara	Drew
Strategy								
Used trial and error								
Organised by length (or height or width)								
Found and used patterns								
Made other comparisons								
Used a combination of strategies								
Result								
Found few or all combinations								
Found the connection to factors of the number								
Found/did not find the formula								
Able/unable to describe the procedure								
Attitude								
Worked well with others								
Curious, explored other shapes, other numbers								
Enthusiastic								
Remained engaged								

FIGURE 4.5 Observation guide for block-building activity (In the classroom 2.2)

You can begin the self-assessment process by having students validate their own thinking or their answers to selected exercises. For example, in a nonthreatening way, you could ask Liam to show you how he arrived at the answers to each of the problems below:

$$\begin{array}{r} 28 \\ \times\ 46 \\ \hline 168 \\ 112 \\ \hline 280 \end{array} \qquad \begin{array}{r} 24 \\ \times\ 39 \\ \hline 216 \\ 72 \\ \hline 936 \end{array}$$

It is important to ask questions about correct answers as well as incorrect ones, and to ask about standard approaches as well as nonstandard, so that children do not think they are questioned only when something is wrong. Asking only about incorrect answers does not help build their self-esteem or establish the feeling that they have control of the mathematics.

It is helpful for students to learn to ask themselves a variety of mental self-assessment questions while they are engaged in problem solving. You can help students develop the habit of self-assessment by prompting them with questions such as 'What are you doing now?', 'Why are you doing that?' and 'How will it help you find the solution?' Eventually, these questions will become second nature to the

students and you will no longer need to prompt them. Being able to think about one's own thinking and to monitor one's own problem-solving efforts (*metacognition*) is one of the goals of problem-solving instruction.

Engaging students in self-assessment of their problem-solving efforts after they have finished solving problems is another useful way to improve their monitoring abilities. For example, you might ask students to write a sentence or 2 in response to prompts like those in figure 4.6 after they have finished a problem-solving activity.

Think about the problem you just worked on. Then answer these questions by circling what you think:

1. How sure are you that your answer is right?

ABSOLUTELY SURE	PRETTY SURE	SORT OF SURE	NOT SO SURE	I KNOW I GOT IT WRONG

Why are you sure (or not sure) about your answer?

2. How hard was this problem for you?

VERY, VERY HARD	PRETTY HARD	SORT OF HARD	NOT SO HARD	REALLY EASY

Why was this problem at this difficulty level for you?

3. Have you ever solved a problem like this one before?

YES, MANY TIMES	YES, ONCE OR TWICE	NOT SURE	DON'T THINK SO	NO, DEFINITELY NOT

If so, describe the other problems and say why they were like this one. If not, what was different about this problem from the others you've seen?

FIGURE 4.6 Self-assessment questions for problem solving

Students can also analyse each others' strategies for solving problems. As they listen to and discuss how another student or group of students solved a problem, they begin to see different ways to proceed and to make judgements about which way makes the most sense to them, which seems easier or different and which leads to stumbling blocks. Not only will they learn from such discussions and self-assessments, but you will also learn a lot about the students.

Sometimes knowing about students' attitudes, beliefs and feelings about mathematics and mathematical tasks helps you, as a teacher, know how to design lessons more effectively. No one is better at assessing how a student feels about a given task than the student who is doing it. You may want to give a simple attitude inventory (see figure 4.7). How would you plan a lesson that involves problem solving if you find from the survey that most of your students have a negative attitude toward problem solving? How could this information help you in grouping students in the lesson?

You might assign students to write mathematics journals or letters to themselves or to others. One teacher had her students write letters to themselves about their strengths and weaknesses at the beginning of each grading period. Months later, they read back over their letters to see if they had made progress in any of the areas they had previously self-identified as weaknesses (see figure 4.8). This information can be shared with parents during conference time. You may learn more from children by asking them questions such as the following.

- What are you really good at in maths? What things do you need to improve on?
- If you were telling a friend about your favourite thing you do in maths class, what would you tell him or her? What other things about maths class might you tell your friend? Why?
- If you and a friend got different answers to the same problem, what would you do?
- How do you know when you have solved a problem correctly?

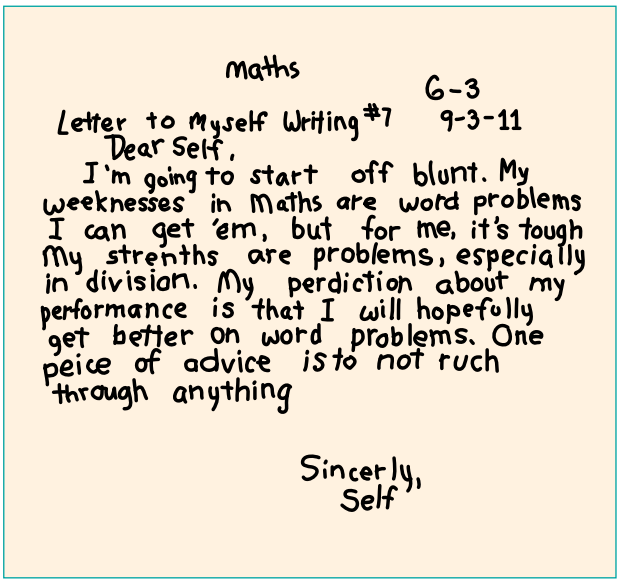
Attitude inventory items

Pretend your class has been given some mathematics story problems to solve. Mark true or false depending on how the statement describes you. There are no right or wrong answers for this part.

_____ **1.** I will put down any answer just to finish a problem.
_____ **2.** It is no fun to try to solve problems.
_____ **3.** I will try almost any problem.
_____ **4.** When I do not get the right answer right away, I give up.
_____ **5.** I like to try hard problems.
_____ **6.** My ideas about how to solve problems are not as good as other students' ideas.
_____ **7.** I can only do problems everyone else can do.
_____ **8.** I will not stop working on a problem until I get an answer.
_____ **9.** I am sure I can solve most problems.
_____ **10.** I will work a long time on a problem.
_____ **11.** I am better than many students at solving problems.
_____ **12.** I need someone to help me work on problems.
_____ **13.** I can solve most hard problems.
_____ **14.** There are some problems I will just not try.
_____ **15.** I do not like to try problems that are hard to understand.
_____ **16.** I will keep working on a problem until I get it right.
_____ **17.** I like to try to solve problems.
_____ **18.** I give up on problems right away.
_____ **19.** Most problems are too hard for me to solve.
_____ **20.** I am a good problem solver.

FIGURE 4.7 A problem-solving attitude survey for primary students

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Maths G-3

Letter to Myself Writing #7 9-3-11

Dear Self,

I'm going to start off blunt. My weeknesses in Maths are word problems I can get 'em, but for me, it's tough My strenths are problems, especially in division. My perdiction about my performance is that I will hopefully get better on word problems. One peice of advice is to not ruch through anything

Sincerly,
Self

FIGURE 4.8 Sample student self-assessment 'letter to myself' written at the beginning of the year. Students can compare this letter with a letter written at the end of the year to show their growth over the school year.

Work samples

Work samples can include written assignments, projects and other student work that you collect and evaluate. Figure 4.9 reproduces the work of 4 children on the following problem.

> A carpenter makes only 3-legged stools and 4-legged tables. At the end of one day, he had used 31 legs. How many stools and tables did he make?

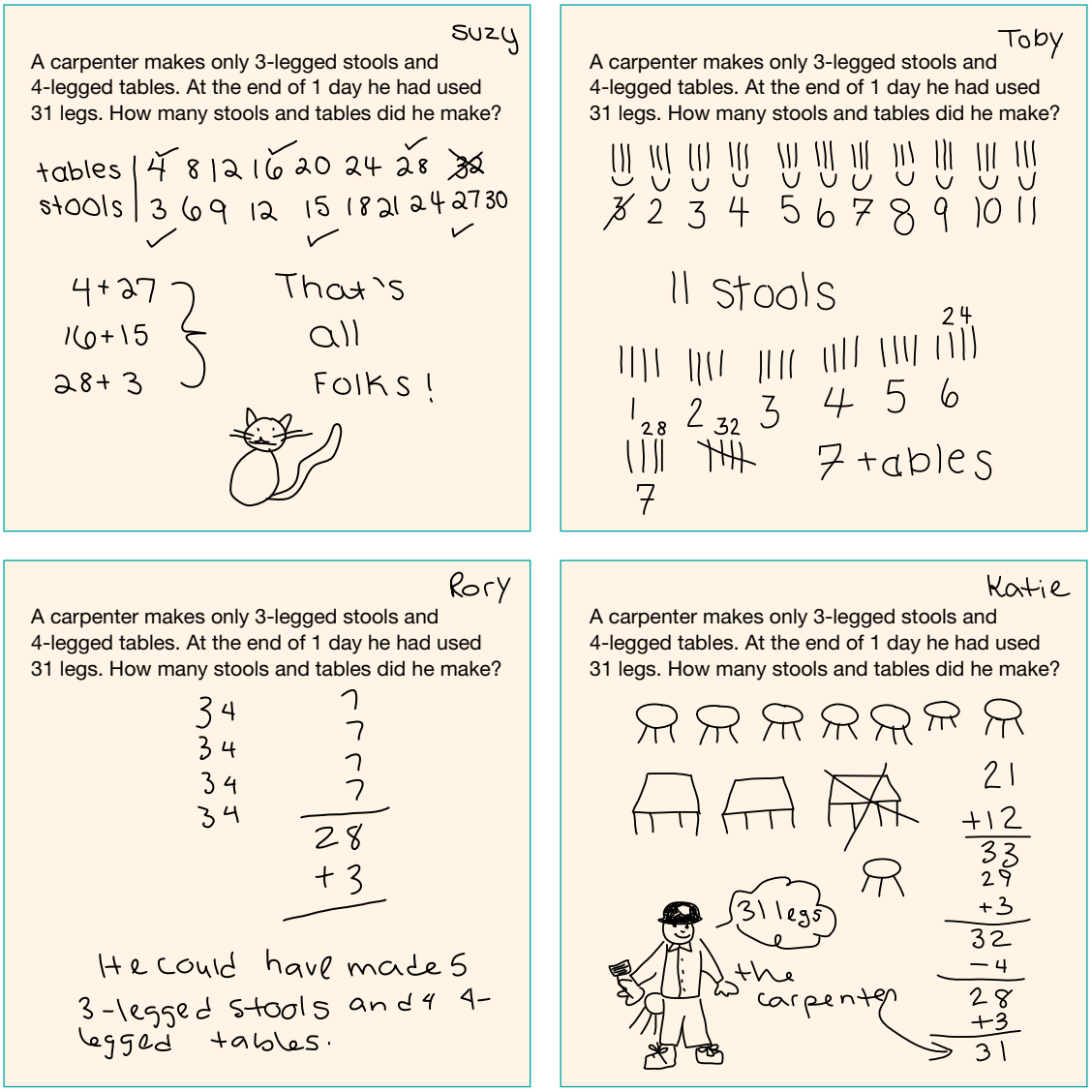


FIGURE 4.9 Samples of children's problem-solving work solving on 3 measures

Figure 4.10 provides a typical analytic scoring scale for assessing children's work in problem solving on 3 measures (understanding the problem, planning a solution and getting an answer).

Table 4.2 shows 1 teacher's scores for each of the 4 children on the 3 measures in figure 4.10. There are, in fact, many different schemes for scoring problem solving and each involves judgement. For example, try scoring the papers in figure 4.9 using the same analytic scoring scale to see whether you agree with the scorer in table 4.2. (You may well disagree, but you should be able to justify your position.)

Analytic scoring scale

Understanding the problem
0: Complete misunderstanding of the problem
1: Part of the problem misunderstood or misinterpreted
2: Complete understanding of the problem

Planning a solution
0: No attempt or totally inappropriate plan
1: Partially correct plan based on part of the problem being interpreted correctly
2: Plan could have led to a correct solution if implemented properly

Getting an answer
0: No answer, or wrong answer based on an inappropriate plan
1: Copying error, computational error, partial answer for a problem with multiple answers
2: Correct answer and correct label for the answer

FIGURE 4.10 A scale for scoring problem solving

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TABLE 4.2 **Scoring for children's work in figure 4.9 using the scale in figure 4.10**

	Understanding the problem	Planning a solution	Getting an answer	Total
Suzy	2	2	1	5
Toby	1	1	0	2
Katie	2	2	1	5
Rory	2	2	1	5

Perhaps more important than scoring children's work is analysing it to see what you can learn about them. Looking again at the work of the 4 students in figure 4.9, for example, you might conclude that only Suzy searched for more than one answer to the problem.

Portfolios

Portfolios have long been used to evaluate works of art, but in recent years this technique has also been used for assessment in mathematics education and for other school subjects. Generally, portfolios are purposeful collections of work. The items chosen for inclusion in a portfolio display the student's effort, progress over time and level of accomplishment. Usually the student plays a key role in deciding what examples will be included. He or she may also help determine criteria for judging merit and evidence of student self-reflection (Paulson, Paulson & Meyer 1991). A student's portfolio might include special problem-solving tasks, writings, investigations, projects and reports. These samples could be presented through a variety of media, including paper and pencil, CDs or DVDs and CD-ROMs. Some teachers keep both classroom working portfolios, in which students keep the majority of their work, and assessment portfolios, which contain selected samples of their work for purposes of making judgements about students' learning, attitudes and dispositions in mathematics.

If your students keep portfolios, you will need to answer the following questions.

- What is the purpose of the portfolio? Who will be the audience? (The students themselves? You as teacher? Parents?)
- What will be included and who selects? (Problem-solving work? Tests and quizzes? Writing about mathematics? Only best efforts or a collection of representative work?) See figure 4.11 for a list of the items one teacher chose to require.
- Where will the portfolios be kept? (In notebooks? In folders? In the classroom? In student desks or lockers or at home?)

- Will you grade the portfolios? If so, how? (Will the portfolio include already-graded items and, if so, is a student graded twice on the same material? Is the portfolio graded on the basis of completeness, comprehensiveness or insight? Can a well-compiled portfolio that contains many poor papers receive a high grade? Can a messy, incomplete portfolio belonging to an excellent mathematics student — and containing all high-standard assessment items —receive a low grade?)
- How often will you give feedback? (Monthly? Quarterly? At the end of each grading period?)

Minimum requirements for inclusion in your portfolio at the end of each grading period:
1. 3 homework assignments
2. 3 writing samples
3. 2 quizzes
4. 2 examples of class notes
5. 1 example of special project work
6. 1 example of cooperative group work.
Your portfolio is not limited to the above. At the end of each grading period, you will choose 5 pieces of work to remain in your portfolio to represent the grading period's work.

FIGURE 4.11 One teacher's specifications for portfolio contents

These are only representative questions; there are many more to be considered. Portfolios can be a rich source of information for you and your students.

A particular benefit of portfolios is their value as a self-assessment tool for students. It is important for students to date the entries so that they (and others) can see their growth over time. It is also helpful if they describe each task and reflect on it. Figure 4.12 provides some advice for getting started with mathematics learning portfolios.

Getting started with portfolios
Start small and simple.
- Portfolios are a lot of work — especially if you have many students.

Developing a rationale is important.
- Why do you want your students to keep portfolios?
- What will you say to students and parents about your goals?

Portfolios need to be accessible.
- Establish a central place in your classroom for keeping portfolios.

Class time allocated for portfolio work is time well spent.
- Students need guidance in labelling and choosing.
- Sharing samples of student work in class is important.
- Students need guidance in being reflective.

A table of contents is essential.
The criteria you identify for judging students' portfolios reflect your goals and values.
- What categories of mathematical tasks will you require be included?
- Will you require examples of both draft and revised work?
- Will you require written reflections?
- Will you put grades on the portfolios? If so, how?

FIGURE 4.12 Tips for getting started with portfolios

Writing

The NCTM (2000) Standards call for more emphasis on communication in mathematics. As stated in figure 4.13, numeracy and literacy go hand in hand. Student writings as a form of communication can provide another source for assessment. You may want students to keep a journal or to add writing to other assignments.

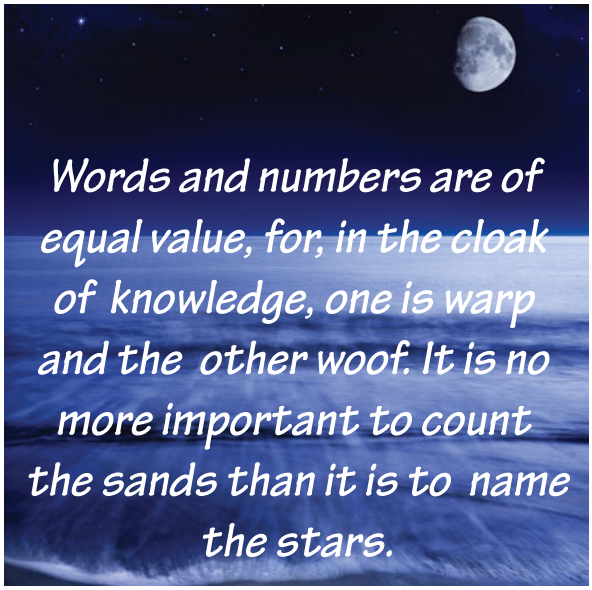


FIGURE 4.13 An idea to engage students in writing about numbers

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A simple way to begin is to ask students to write about what they did or did not understand for one assignment, how they felt about an activity, what they learned today in class, or what they like about mathematics. Your creativity in providing suggestions will help spark theirs. For example, you might ask them to write a letter to a friend about mathematics class or to write a poem about triangles. As writing becomes a part of mathematics class, you can use it to assess children's knowledge of and attitudes toward mathematics. Figure 4.14 provides some sample ideas for writing prompts.

Describe an activity our class might do to help us understand how big one million really is.

Let's play teacher! You want to teach some Year 1 students how to subtract 2-digit numbers. Give examples of the problems and materials you will use. Outline what you will say and what you will have the children do.

Audrey claims that her teacher could improve the average test score in the class by 10 points if she just added 10 points to everyone's score. Kelly doesn't believe that would work. What do you think? Explain your thinking.

Which decimal is closer to one-half, 0.307 or 0.32? How do you know? Explain how you could show or prove it.

What connections are there between fractions and decimals? Explain and use examples in your explanation.

FIGURE 4.14 Sample writing prompts for having children explain their understanding of mathematical concepts

Writing is an important aspect of mathematical communication and the broader goals of developing students' literacy and numeracy. Within the proficiency strands of *The Australian Curriculum: Mathematics* (ACARA 2015), in particular Understanding, Problem Solving and Reasoning, writing is a valuable learning and assessment tool. Bicer, Capraro, and Capraro (2013) reported on the positive impact that writing had on students' mathematical problem solving. As one of the 4 proficiency strands in the Australian Curriculum, this provides an effective strategy for teachers to use. The other 3 proficiency strands, Understanding, Reasoning, and Fluency, could also benefit from the use of journals as it provides children with the 'opportunity to share their ideas by using their own words' (Bicer et al. 2013, p. 363). Specific examples of using journal writing include using number names and units (Understanding); referring to the date, day of the week, and seasons (Fluency); outlining plans for solutions (Problem solving); and, comparing and contrasting strategies or plans for solving a problem (Reasoning).

Teacher-designed paper-and-pencil tests

Tests can inform and guide your teaching, rather than simply determine grades. Students learn from tests, but too often the lessons they learn may not be those intended. When too much emphasis is placed on right/wrong answers from written tests, students may believe that it is not important to be able to show why procedures work, or to explain how they solved a problem, or to be able to solve word problems (because a good grade may be obtained without ever doing these things). As some educators have observed, students often decide what is valued by what gets tested — 'or what you test is what you get' (Wilson & Kenney 2003).

Carefully constructed and correctly analysed tests can tell us a lot about students. For example, the children's papers in figure 4.15 are for a simple test on subtraction of 2-digit numbers, but they reveal a lot about the students. Notice how many of them missed the problems with zeros. If the first or seventh item had not been on the test, the teacher might not have realised that Jim regroups — and then ignores — even when regrouping is not necessary. (Try analysing the kinds of errors the other students made and what characterises each student.)



FIGURE 4.15 Samples of children's papers for a written test of subtraction

Most paper-and-pencil tests do not give you the opportunity to learn how a student arrived at an answer. One way to gain more insight is to ask the students to explain in writing what they did. If children are accustomed to explaining orally in class, then this task will be easier for them and you will receive more meaningful explanations. Ask for explanations on a few items and accept them in the children's language. Otherwise, they will soon learn to parrot explanations that are meaningless to them.

There is no reason why paper-and-pencil tests cannot include estimation items and require manipulatives and calculators. Tests can also permit the use of textbooks or notebooks, ask thought-provoking questions and require students to connect new learning to previous learning.

They need not be races against time in which the students regurgitate all they remember from the week or about the procedure last taught.

Thoughtful, well-constructed tests are one way — often a very efficient way — to gather information. Alone, they do not give a complete assessment of students' knowledge, but they can add 1 more piece to the puzzle.

Standardised achievement tests

At some point you will be asked to administer standardised national and possibly international tests. The results of these tests may be used to take a look at your students' achievements in general. For example, if your students did poorly in measurement, then you need to rethink what you are doing in that area. If student test results come back late in the year, you might ask to see the results from the previous year for students in your class. Sometimes this information can provide valuable insight into what areas of study need extra work this year.

Remember that norm-referenced tests are designed to spread children across the whole normal distribution. There will always be 50% of the students below average and 50% above average. Thus some items on such tests are not intended for everyone at a year level to answer correctly. The fact that an item on multiplying fractions appears on a Year 4 test does not necessarily mean that this is a recommended skill for all students at this level. Refer to the mathematics scope and sequence tables (ACARA 2012) for expectations for your year level.

As mentioned earlier in this chapter, standardised tests are changing. Some allow the use of calculators; some ask questions in an open format rather than multiple-choice format; some ask for explanations; and some include problem-solving situations. Be sure your students are familiar with the format of whatever standardised tests they are required to take. In addition, you should strive to help your students learn in ways that are meaningful to them and that help build their confidence and the attitude that they can do mathematics. These are some of the best ways to ensure that your students will be able to show what they know on standardised achievement tests.

4.4 Keeping records and communicating about assessments

LEARNING OUTCOME 4.4 Using evidence to inform student progress and teaching.

It is important to keep both informal and formal records of students' learning and their disposition toward mathematics. Certain types of recordkeeping may be required by your school system, but you can always keep additional records of your own.

A *word of caution:* Do not become burdened with a multitude of records, but keep enough that you can reflect on your students' progress and can justify any major decisions about them or what to teach. The type of record you keep will also depend on the ways you report information to students, parents and the school administration. Make the most of technology — the ease of taking photographs and recording students' thoughts on their activities can lessen the burden. How powerful would it be to play a child's recording for a parent to listen to?

Recording the information

Teachers are aware of many things about their students. For example, you may know that Treena is always willing to answer in mathematics class, that Joshua does not work well with Katrina, that Yong knows how to compute but seems to lack understanding about when and why certain operations are used, and that Cary is often absent on mathematics test days. It might be that you need to jog your memory about Rachel before her parents come for the next parent–teacher interview. Or maybe you are thinking about changing the composition of cooperative groups and wondering if Kerry and Richard worked well together in the past.

Several techniques for recording information are described here. You may find others or modify these to suit your needs.

Checklists

There are a variety of ways to record individualised information about students' understandings, attitudes or content achievement. They may be simply lists that are marked in different ways, rating scales or annotated checklists (see figures 4.16 through 4.19). The beauty of checklists is that they can be adapted to your situation and, can help you think about your goals and the needs of each child.

MATHS CLASS PARTICIPATION

NAME	DATE: March	Centre A	Centre B	Centre C	Group leader	Group recorder	Group reporter	Bulletin board	Materials
Atkins, Willie		2	4	6	5	13	10	✓	✓
Bero, Rommel		9	10	11	10	5			March
Connel, Brenda		2	3	5			10		✓
Coroi, Troy		11	10	9	5		13	✓	March
Cosby, Kim		11			5		13		

FIGURE 4.16 Checklist of student participation: useful for tracking which children need to be encouraged to participate and in which activities or roles. For example, Brenda has worked at mathematics centres and has been responsible for materials. She may need encouragement to become a group leader or group recorder.

CLASS OBSERVATION
Class Mathematics Week Feb. 6-10/13-17

NAME	COMMENTS	ACTION NEEDED	ACTION TAKEN
Atkins, Willie	Persisted with problem solving until successful		
Bero, Rommel	Needs a challenge	*	Yes
Connel, Brenda	Showed others how to use scales accurately		
Coroi, Troy	Needs base-ten blocks to do subtraction	*	Yes
Cosby, Kim	No homework for 3 days	*	Called

FIGURE 4.17 Annotated class observation checklist: provides teacher with reminders of points to discuss when communicating with parents and students

You cannot spend all your time keeping records, so select a few significant aspects of students' learning and attitudes and target a few children each day. You may want to observe and keep records about children working in cooperative groups one week and the next week focus on individual children's understanding of a new topic.

STUDENT DISPOSITIONS						
CLASS: Year 3 Maths MONTH: October	Helen	Art	Quoh	Beverly	Iza	Jim
Confidence Is sure of answer Knows how to proceed	✓ ✓			✓ ✓		
Flexibility Will change direction Tries several ways	If ✓ prodded		✓ ✓		✓	
Perseverance Stays with task Enjoys involved problems	✓	✓		✓		
Curiosity Wants to find out why Challenges		✓	✓	✓		✓
Sharing Works well with others Shows leadership		✓		✓		✓

FIGURE 4.18 Checklist of students' dispositions. The checklist is helpful in deciding which students might need special activities or encouragement, or as information to inform conferences with parents or students.

Student profile in Mathematics

Date Feb.

Student Jeremy Gelder Teacher Matt Bell Year 2

Content areas

- Number sense & numeration: Still sees numbers as ones - needs to develop place value
- Estimation: Good feel for numbers – size of quantities
- Concepts of operations: Good problem solver
- Computation
- Geometry & spatial sense: Geometry he loves
- Measurement
- Statistics & probability
- Fractions & decimals
- Patterns & relationships: Better at spatial patterns

Math power

- Problem solving: Likes to think in creative ways
- Reasoning
- Communication: Explains well
- Connections
- Concepts: Developing
- Procedures: Doesn't like to follow procedures – invents his own

Dispositions

- Confidence: Growing
- Flexibility
- Perseverance: On many tasks
- Curiosity: Great
- Reflection

Other comments

FIGURE 4.19 Sample student profile

If you use a checklist, you will want to keep it handy so that you can make quick entries as children are engaged in tasks. If you wait until the end of the day, you may forget some of the day's gems or be too preoccupied with other tasks to jot them down.

If you are recording information that is meant only for you, keep the checklist away from the eyes of your students. Be especially sensitive to children's feelings. A public checklist that shows the progress of each class member's skill attainment may be a great boost for those at the top but detrimental to those at the bottom, who often need the most encouragement.

Student files

Many teachers keep a record of students' learning in the form of a file of work samples. If children are keeping their own portfolios, those may suffice, and you will not need to keep additional samples of their work in your own student files. In either case, you may want to keep a summary profile for each child, like the one shown in figure 4.19. Sometimes the school's format for reporting to parents requires that you keep such a record.

Class records

Sometimes the only class record is the grade book. Although it may be necessary to keep such a record, there are limitations. Alternatively, you could modify it with shorthand entries to tell you more than attendance; grades on assignments, quizzes and tests; and cumulative grades.

You could also supplement the grade book with a cumulative checklist made from your own records. You can determine which items to include by reviewing the usefulness and frequency of your entries in these earlier checklists. The cumulative checklist will then give you a picture of the class as a whole and help you plan.

Communicating the information

Most teachers have 3 main audiences to whom the information they have gathered will be communicated: students, parents or guardians, and administration. Each group will receive different amounts of information in different ways.

To students

Much of your communication of information will be done orally or through actions, but you will also be commenting on work samples, portfolios, tests, journals or other forms of assessment materials. You might also give a letter or number grade. All of these communications influence children's feelings about the value of different aspects of mathematics, their expectations of what they can accomplish and their sense of their own worth.

Be positive and fair. Use information from your checklists or from student portfolios when letting each student know whether he or she is meeting expectations. Look for ways to demonstrate to each that he or she can do what is expected. Finally, remember that it is important that children grow in their ability to self-assess. One of your goals should be to help children become independent learners. If they always have to rely on you to validate their work and their thinking, they will not reach this goal.

To parents or guardians

You will be reporting to parents or guardians both in written and oral forms. The format of these reports is often determined by the school system, but you are responsible for the quality. It is important from the beginning that you keep records and use them to illustrate or justify your comments. Phone calls can be useful for immediate feedback, not only when there is a problem but also to applaud good behaviour or outstanding achievement. Parents and children are often surprised and gratified when a teacher calls home to report something good. Some teachers try to make good calls home to all parents early in the year. This practice sets up good relationships with parents and establishes the expectation that a teacher phone call need not necessarily mean bad news. Some use regular newsletters about

classroom activities and goals or letters to parents to tell them about special events or request their help. Many also use email and digital communication to maintain contact. (Figure 4.20 shows a letter one teacher sent to parents to enlist their assistance in developing and maintaining their children's mathematics portfolios.)

At parent–teacher interviews, you have an opportunity to explain goals and to enlist parents' aid in helping their child meet them. It is helpful if you have collected a wide variety of samples to share with the parents as you talk about their child's progress. Parent–teacher interviews are also times for gathering information. Parents often provide insights about their children that may not be evident to you in the classroom. The information you glean from them should be included in notes in your student files.

To the school administration

When you begin teaching, you will need to find out what types of records and grades are required in your school and how they are used.

- What sort of report card format is used? A/B/C letter grades? S/I/N (satisfactory, improving, needs improvement) progress indicators? Numeric grades? Narrative reports?
- Does the administration expect written reports on each child?
- Is there an official checklist?
- Is the class grade record sufficient?
- Is information on students passed to the next teacher, kept in a permanent file or used to make tracking decisions?
- Is the information used in teacher evaluations?

The type and use of records vary from school to school, so it is necessary to find out about your own situation. When you know how the records are to be used, you will be better able to provide information that is suitable for particular uses.

Dear Parents,

Your child will be developing a 'portfolio' in maths class this year. One of the primary purposes of this project is to allow students the opportunity to reflect on their work, including exemplary work, work that is considered a 'personal best' or work that they feel is in some way memorable. Through this project, students can begin to develop critical self-evaluative tools that will be extremely valuable in the years ahead.

A second purpose of the portfolio project is to allow students to share some of their work with you. Please take some time this week to discuss your child's portfolio with him or her. Your child has been given written guidelines for the portfolio project. Please review the guidelines together.

Please help your child this week as he or she finalises selections for the portfolio. There is no set number of pieces required, but a selection of 5 or 6 pieces is suggested. Help your child select pieces that best meet the guidelines and that are *most* representative of the major maths content areas that we have covered during this grading period. The attached 'News Bulletin' from this grading period may provide you with a better idea of the topics your child has studied and the activities he or she has participated in. Students have been asked to provide written accounts about *why* they chose particular pieces for their portfolios, so be sure to talk with your child about why he or she is choosing each piece for inclusion in the portfolio. (If you wish, you too may include some written comments about the pieces selected or the selection process.)

After you have read the 'News Bulletin' and have helped your child with his or her portfolio selections, please sign and detach the form below. Your child is to return the signed form, along with his or her portfolio, by Friday, November 15.

Thank you for helping to make your child's mathematical experience a memorable one!

Sincerely,

I have read and discussed the 'News Bulletin' with my child and have helped him or her select choice pieces for his or her portfolio.

FIGURE 4.20 Letter to parents about mathematics portfolios

SPOTLIGHT ON EARLY CHILDHOOD

Unveiling hidden mathematics knowledge

Young children constantly explore their world by actively doing things and looking at things. As they make meanings from their experiences they will often ask questions and talk about what they are doing, what they see and what they like or dislike. They will also often talk about what they are thinking or even about imaginary things and stories going on in their head. Many will also draw pictures if appropriate materials are available (particularly if parents or teachers encourage or emphasise drawing as a play-oriented learning activity). Children will naturally use drawing as a way of recording and making meanings from their explorations or for creating new ideas.

The drawings in figure 4.21 provide a rich example of how a child's mathematics knowledge might be hidden and not assessed unless one talks to the child and asks for an explanation or story to go with the pictures. Here, a 5-year-old boy, Charlie, has drawn pictures of bugs and insects, something that commonly fascinates children. What is not clear by looking at these drawings is how much they reveal of Charlie's mathematics knowledge. It is not immediately evident that they are about how Charlie has noticed mathematical patterns in his environment. When asked to explain his pictures, Charlie pointed to the repeating images in the first picture and said:

> A pair of wings, a pair of wings, a pair of wings, a pair of wings, a pair of wings.

When asked about the other picture, he said it was about how each caterpillar has 5 legs. He then proceeded to point to each individual caterpillar and its legs and count:

> 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5.

He persisted with his counting until the legs on all 10 caterpillars were pointed to. Only by interacting with Charlie was it revealed that he was, without prompting, doing mathematics by identifying patterns and using numbers to describe patterns. In addition, as Charlie completed his counting, it could be seen that his one-to-one counting skills were still developing (not all the caterpillars have exactly 5 legs).



FIGURE 4.21 A 5-year-old child's drawings of insects and bugs

Children's drawings can be very useful in determining their thinking. However, to assess what children are learning it is essential to interact actively with them by talking and asking questions. When a child draws a picture, upon the request of a teacher or spontaneously, it is not the case that 'a picture paints a thousand words' unless the child tells you what the picture means. That is, assessing a young child's learning necessarily requires diverse assessment methods. It is also the case that children do not separate their world into the discipline areas by which a school curriculum is organised. They experience their world in an integrated way and therefore report their learning in holistic contexts.

MAKING CONNECTIONS

Your goal as a mathematics teacher must be to help *all* students develop a rich understanding and appreciation of mathematics, problem-solving capacities and high levels of procedural skills. In this regard it is vital that you develop teaching and assessment practices that are inclusive of students' interests, cultural and home backgrounds, language capacities and other individual and group characteristics.

Curriculum movements in Australia and elsewhere have shifted from merely reporting final achievements to assessment that is 'educative, formative, an integrated part of teaching and learning, and aimed at improving student performance' (Shalicky 2007, p. 23). One way to address these broader goals is to make connections to students' personal interests in designing learning tasks that are motivating, personally meaningful and contextually rich. Shalicky (2007) reported on the use of an open-ended mathematical inquiry, with a focus chosen by an individual student that integrated teaching, learning and assessment. For example, Anita, a Year 6 student, posed the question 'How does measurement apply to pigeons and their lifestyle?' Her interest in breeding and training pigeons started at age 7 when her grandad gave her 4 pigeons. Anita's investigation and final report provided the teacher with opportunity to assess Anita's measurement knowledge, understandings and application skills, as well as how she used mathematical thinking to reason and interpret within a real-world context. The task therefore was valuable to the assessment of numeracy — not just mathematics learning — since it provided for the 3 key components of numeracy as outlined by the Australian Association of Mathematics Teachers (2009): mathematical, contextual and strategic — in summarising the work of Willis and Hogan (Hogan 2000).

Numeracy is integrally linked to literacy, making it essential that your mathematics assessment practices take into account the literacy demands of all teaching, learning and assessment activities. Perso (2009, p. 16) outlines how children are disadvantaged and will have difficulty succeeding with numeracy tests such as NAPLAN if they are not explicitly taught how to crack the many different codes in mathematics. Examples of potential challenges include:

- terms particular to mathematics (e.g. number sentence)
- mathematical terms with meanings that differ to their use in other contexts (e.g. complete, dense)
- symbolic representations of terms (e.g. '×' for all 3 of 'times', 'multiply' and 'lots of').

Socio-economic status (SES) is a strong predictor of student academic achievement in many countries (Perry & McConney 2010). Various international studies have shown that individual student SES as well as a school's aggregated socio-economic profile are associated with achievement on standardised assessment tests, including local, national and international tests (Organisation for Economic Co-operation and Development 2005; Sirin 2005). As a teacher you need to be aware of these facts, even though you do not have direct influence over your school's socio-economic profile or related government funding initiatives. You do, however, have much control over your classroom practices. According to the AAMT *Standards for Excellence in Teaching Mathematics in Australian Schools* (2006), to be an excellent teacher of mathematics you need to regularly assess and report student learning, use a range of assessment strategies, maintain ongoing informative records of student learning outcomes that are used to plan future learning experiences, and provide 'constructive, purposeful and timely feedback to students and their parents, and to school authorities' (AAMT 2006, p. 4). These points reflect those provided in Standard 5 of the Professional Practice domain of the *Australian Professional Standards for Teachers* provided by AITSL (2014).

A GLANCE AT WHERE WE'VE BEEN

Assessment is an integral part of teaching. Teachers should have a repertoire of assessment strategies that can be used to help students develop mathematical understandings and learning that meet the needs of the curriculum (AITSL, 2014).

Assessments *for* learning help teachers make informed decisions about the varied aspects of their day-to-day work: planning instruction, making teaching decisions, monitoring student progress and evaluating student achievement. The assessment process itself has 4 phases: (1) planning the assessment, (2) gathering data, (3) interpreting the data and (4) using the results for future lesson planning.

Assessment *as* learning incorporates interactions between the teacher and student, as well as feedback, particularly as occurs within the learning experience. Interactions involve the elements of the experience, not just verbal communication, and these can comprise the components of feed up, feedback, and feed forward.

Assessments *of* learning are primarily designed to document what students know and are able to do. Final exams, as well as state and national standardised tests, are familiar examples of assessments *of* learning.

There are many methods used to gather, analyse and present the information from assessments, whether the assessments are focused on student learning or on students' dispositions (attitudes and beliefs) toward mathematics. In the classroom, these include informal observations, interviewing, peer assessments, portfolios, writing and testing. Assessments, rubrics, or scoring guidelines also provide solid frameworks. Assessment information needs to be recorded in ways that make it easy to analyse and use. There are numerous techniques for recording and communicating information. Portfolios, in particular, can provide a broad and useful picture of student progress and show growth over time. Assessment can make a difference in how you help your students learn. You can use it in a positive way to encourage the children to become independent learners, to modify your instruction and to communicate with parents and others with an interest in students' achievement.

THINGS TO DO: FROM WHAT YOU'VE READ

1. Explain the difference between assessment of learning (summative assessment) and assessment for learning (formative assessment). Give 2 examples of each.
2. Identify and briefly describe each of the 4 phases of assessment.
3. A narrow view is that assessment in mathematics focuses on how well students can carry out procedures in a limited amount of time. Describe a broader view of mathematics assessment.
4. Try scoring the papers in figure 4.9 using the analytic scoring scheme shown in figure 4.10. Compare your scores with those shown in table 4.2. Be prepared to justify the scores you give.
5. Describe 3 ideas for getting your students involved in self-assessment.
6. List 3 ideas for keeping records of your observations of students as they work during class. Do you have to observe every child during each class session? If not, how will you be sure to give each student adequate attention?
7. What sorts of assessment records would be good to keep to assist with a parent–teacher interview? What other ways can you communicate with parents about their children's progress?

THINGS TO DO: GOING BEYOND THIS TEXT

IN THE FIELD

1. How might the use of scores impact on assessment? On your next visit to a classroom, ask the teacher to share several problem-solving questions. Working together with the teacher, analyse and score them using the analytic scale in figure 4.10.
2. Have a look at several of the work samples provided in the Mathematics Portfolios on the Australian Curriculum: Mathematics Foundation to Year 10 webpage (www.australiancurriculum.edu.au). If you were to work with these students in mathematics, what would you plan to occur next to assist their mathematical learning?

3 Work through a range of items on one of the NAPLAN sample tests. As you work, link the mathematical skills used to the Australian Curriculum: Mathematics. Ask a peer to work through the same questions and to also note down the mathematical skills they used. Compare your lists of mathematics that you used for each question – were they the same?
4 Ask 1 or more classroom teachers to describe the assessments they use in mathematics, the feedback used with each, and how they used the results of those assessments. What do you notice?

IN YOUR JOURNAL

5 Which of the recommendations for shifts in assessment (table 4.1) have you experienced in your own years as a student? Briefly describe when and how. Which of the shifts are most closely connected to the classroom? How are they connected?
6 Write a 'letter to self' similar to that shown in figure 4.8. In your letter you should reflect on your current strengths and weaknesses, as well as your future goals as a teacher of primary school mathematics.
7 Defend your position regarding this statement: Students' work on tests provides the most accurate assessment of their mathematical understanding.
8 Does student use of calculators interfere with assessing their understanding of mathematics? Explain your position on this issue.
9 Which assessment practices do you think provide the best evidence of children's understanding of mathematics? Explain why you like these particular practices.

WITH ADDITIONAL RESOURCES

10 Use a web-based library search to identify a teacher-authored journal article about assessment or grading. (Good journal options *include Australian Primary Mathematics Classroom, Australian Mathematics Teacher, Teaching Children Mathematics* or *Mathematics Teaching in the Middle School*). Write a brief summary of the article, followed by your own reflection about 2 good points in the article and 1 point that you disagree with or that you think could have been improved.
11 Analyse a test in a primary mathematics textbook. Would it make a difference if the students used calculators? What items would you need to change?
12 Examine several end-of-chapter tests from primary school textbooks. How well do they align with the content of the chapter? How well do they align with the Australian Curriculum's expectations for children's learning of these topics at this year level? To what extent do the tests emphasise vocabulary, concepts, procedures and processes (such as problem solving, connections, communication, reasoning and representation)?
13 Analyse several end-of-chapter tests from primary school textbooks to determine whether they require writing about mathematics. If not, what changes could you make so that they would do so? What other changes could you make to the tests so that they would be more authentic?
14 Examine a standardised test. Does it allow use of calculators? Identify examples of good and poor questions, and explain your opinions.

WITH TECHNOLOGY

15 Create 2 tasks for children that use a calculator – one that demonstrates the learning affordances offered through the use of the calculator and one that uses the calculator in a way that doesn't develop mathematical understandings. List the differences in how the calculator is used and the way the activity would be conducted with children.
16 What are the key points that need to be considered to ensure that an activity with a calculator is used in a way that extends mathematical understandings?
17 The National Assessment Program website (www.nap.edu.au) provides copies of NAPLAN test papers. Obtain copies of the Numeracy tests for a particular year for different year levels. Examine and discuss the literacy demands of the items and how they might be particularly challenging for some younger students.

18 Obtain a NAPLAN Numeracy test paper and for 3 of the items develop a classroom teaching and learning activity that focuses on helping students learn the mathematics concepts and/or skills that these items emphasise.

CHILDREN'S LITERATURE CONNECTIONS

Cohen, M. (2006). *First Grade Takes a Test*. New York: Star Bright Books.
Sammy, Jim and George can't find the correct answers to some of the questions. Classmate Anna Marie has no trouble finding the answers and she is eventually moved to a special class. The Year 1 class is sad without Anna Marie. Eventually, Anna Marie returns and tells the class she has missed them. However, is getting the highest score that important? This book prompts teachers to consider how children think when given a test. View the study guide available here www.spokenartsmedia.com/StudyGuides/First%20Grade%20Takes%20a%20Test.pdf

Finchler, J. (2000). *Testing Miss Malarkey*. New York: Scholastic.
This is the story of a class and how upset everyone is — including the teacher and principal — over the Instructional Performance Through Understanding test. See how things change as teacher and students prepare for the test. Consider how these suggestions for using the book www.thenedshow.com/assets/test_prep_lesson_malarkey-thenedshow.pdf or the study guide www.spokenartsmedia.com/StudyGuides/Testing%20Miss%20Malarkey.pdf can be used to enable your students to connect with the story.

Murphy, S. (2000). *Pepper's Journal*. New York: HarperCollins.
Joey and Lisa keep a journal on their new cat, Pepper, for one year. This book will help provide students with insights into how a journal is kept and may offer them a good introduction to journal writing. There are activities in the back of the book for teachers and parents. View the activities suggested on this page www.mathstart.net/peppers-journal.html. How can these be used to develop the mathematical ideas further?

Prelutsky, J. & Smith, L. (1998). *Dr. Seuss Hooray for Diffendoofer Day*! New York: Random House.
This is the story of a school with a very diverse faculty who teach in a wide variety of ways. Then, one day, the principal brings in a test for everyone to take. The students are worried, but the teacher reminds them that they know everything that is on the test, plus more. The principal tells them that all students from miles and miles around will be taking this test to see who is learning and what school is the best. If the school doesn't do well, the school will be torn down and they will have to attend a school that only teaches and behaves in one way for all students. At the end, the school does very well and everyone passes the test. Readers see that everyone learns differently and that people can learn by doing things differently. Select one of the activities available from the list here www.seussville.com/activities/HOORAY_TeachersGuide_0.pdf and use it as the base from which to create a mathematical experience.

Scieszka, J. & Smith, L. (1998). *Maths Curse*. London: Puffin Books.
This is the story of a child's mathematics-related thoughts and questions prompted by the class teacher saying, 'You know, you can think of almost everything as a maths problem'. The examples are humorous, relate to a typical school day and include concepts and problems related to measurement, numbers and geometry. A teacher could use the ordinary-day context of the story as a prompt to learn about students' attitudes to mathematics and their awareness of mathematics in daily life. View the suggested sessions available from this page www.readwritethink.org/classroom-resources/lesson-plans/solving-math-curse-reading-1123.html and write a summary of how you might use these in your classroom.

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CHAPTER 5

Processes of doing mathematics

LEARNING OUTCOMES

5.1 Explaining how teaching mathematics through problem solving is different from simply teaching students to solve problems.

5.2 Describing mathematical reasoning and how it assists children with making sense of mathematical knowledge and relationships.

5.3 Explaining how primary school children can be encouraged to communicate their mathematical thinking.

5.4 Identifying what connections are important for primary school children to make when learning mathematics.

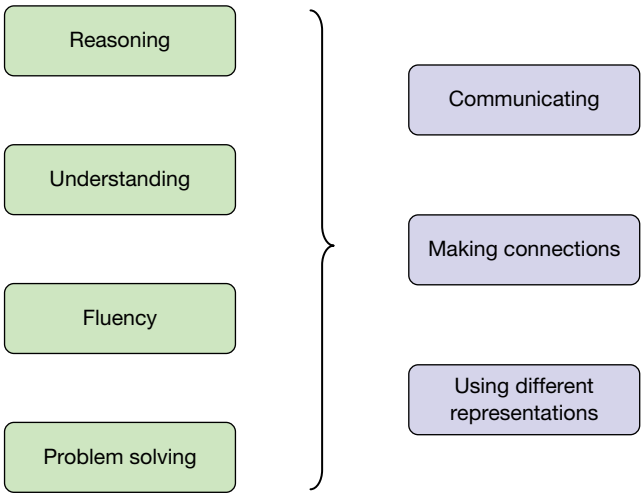
5.5 Explaining the processes of representation in primary school mathematics.



‘Too often we give children answers to remember rather than problems to solve.’

Roger Lewin

Chapter 5 concept map



Introduction

As we have seen in earlier chapters, school mathematics is moving in new directions. In many mathematics classrooms these days, it is rare for a lesson to be taught to rows of children studiously bent over worksheets, practising computations, rules and formulas. Instead, it is more common to observe behaviour where children work in small groups — while talking and using drawings or tiles or blocks to model a problem — and share their ideas, observations and problem-solving processes. They listen carefully to each other, and they challenge and question each other. The lesson is often consolidated through whole-class discussion.

In doing mathematics, students are actively involved in a wide variety of physical and mental actions — actions that can be described by verbs such as *exploring, investigating, patterning, experimenting, modelling, conjecturing* and *verifying*. Doing mathematics is learning mathematics.

As mentioned in chapter 1, the Australian Curriculum's 4 proficiency strands interact with the 3 content strands. The proficiency strands 'provide the language to build in the developmental aspects of the learning of mathematics and have been incorporated into the content descriptions of the 3 content strands' (Australian Curriculum Reporting and Assessment Authority, 2012). How students achieve understanding, fluency and the ability to solve problems and reason are summarised in table 5.1.

TABLE 5.1 Student attainment of mathematical proficiencies

Proficiency strand	Process of attainment
Understanding	• Connecting related ideas • Representing concepts in different ways • Identifying commonalities and differences between aspects of content • Describing mathematical thinking • Interpreting mathematical information
Fluency	• Calculating answers efficiently • Recognising robust ways of answering questions • Choosing appropriate methods and approximations • Recalling definitions • Regularly using facts • Manipulating expressions and solving equations

(continued)

TABLE 5.1 *(continued)*

Proficiency strand	Process of attainment
Problem solving	• Using mathematics to represent familiar and meaningful situations • Designing investigations • Planning approaches • Applying existing strategies to seek solutions • Verifying that answers are reasonable
Reasoning	• Explaining thought processes • Deducing and justifying strategies used and conclusions reached • Adapting the known to the unknown • Transferring learning from one context to another • Proving that something is true or false • Comparing and contrasting related ideas • Explaining choices made

Source: © Australian Curriculum, Assessment and Reporting Authority 2012 (p. 5).

The *Principles and Standards for School Mathematics* (NCTM 2000) identifies 5 processes of doing mathematics: (1) Problem Solving, (2) Reasoning and Proof, (3) Communication, (4) Connections, and (5) Representation. Teachers of mathematics in Australia also aim to incorporate these into their programs from the earliest years, giving them equal emphasis with the content strands (i.e. Number and Algebra, Measurement and Geometry, and Statistics and Probability). The Australian Curriculum and the NCTM Standards make it clear that doing mathematics means engaging in these fundamental processes, and that having students do so makes the learning a rich experience and the content meaningful.

5.1 Problem solving

LEARNING OUTCOME 5.1 Explaining how teaching mathematics through problem solving is different from simply teaching students to solve problems.

What comes to mind when you think of problem solving? Some people think of challenging situations they may have encountered in real life, such as when their car got stuck in mud and they were unable to get it out until they solved the problem by putting floor mats under the wheels to provide greater traction. Similarly, when you think of problem solving in the mathematics classroom, you may think of challenging situations involving numbers or shapes or patterns. For example, a state official may need to figure out how many distinct licence plates can be produced if each must be printed with a unique identifier consisting of exactly six characters. Perhaps the characters may be chosen only from among the 10 numeric digits (0–9) and the 26 letters of the alphabet. How many different licence plates are possible? That's a problem!

Alternatively, when *you* think of mathematical problem solving, you may remember the story problems or word problems that often came at the end of each chapter of the mathematics text when you were in primary school. After you had learned to perform certain computations (say, multiplication of fractions), your text may have provided problems in contexts that used those same skills. In many cases, these may not have been genuine problems for you because the techniques to use had been clearly outlined in the preceding pages.

The Australian Curriculum identifies problem solving as a proficiency to be developed as students learn mathematics. In problem solving:

> Students develop the ability to make choices, interpret, formulate, model and investigate problem situations, and communicate solutions effectively. Students formulate and solve problems when they use mathematics to represent unfamiliar or meaningful situations, when they design investigations and plan their approaches, when they apply their existing strategies to seek solutions, and when they verify that their answers are reasonable (ACARA 2012, p. 5).

Problem solving is a 'major means of developing mathematical knowledge' (NCTM 2000, p. 116). By general agreement, a problem is a situation in which a person wants something and does not know immediately what to do to get it. Problem solving is the foundation of all mathematical activity (Stacey, 2012). As such, problem solving should play a prominent role in the primary school mathematics curriculum.

For example, when students were confronted with a problem that piqued their interest, building a 1-metre staircase with centicubes; it quickly became obvious to the class that simply counting the number of centicubes needed to make a staircase 1 metre (100 centicubes) high was not practicable. By encouraging the students to deal with a simpler problem by drawing the staircase, counting the number of centicubes in some of the steps and then extrapolating to larger numbers, the students could begin to see a generalisable rule. During their practice examples, students talked to each other, helped each other with their understanding and were guided by their teacher when needed. The problem required the students to draw on their arithmetic knowledge of pairs of numbers to 10 (and then to 100) and to identify the relationship between the number of pairs and the number to which the pairs added. Students had to be able to apply arithmetic procedures efficiently and correctly in order to concentrate on the patterns in the problem. Explaining their thinking to the class and the teacher reveals the depth of their mathematical proficiency.

Consider another problem; one involving a game that can be played by students in the upper primary years. To investigate this particular problem, you will need to play the game with 1 or more opponents. You'll need 2 dice, 12 small counters or tiles or markers each and a piece of paper on which to make a game board for each person (see In the classroom 5.1).

IN THE CLASSROOM 5.1

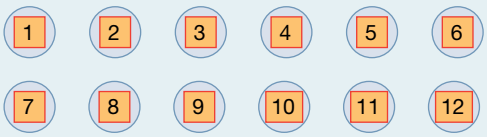
Rolling the dice

Objective: Use a dice game to investigate experimental and theoretical probabilities and to develop analytical reasoning skills.

Year level: 3–6

Materials: 12 counters, tiles or markers for each player; 1 pair of dice for each 4 players

Rolling the dice game board



Rules of the game: Up to 4 players can play together with 1 pair of dice. Each player has his or her own game board. You begin by placing all 12 of your counters on your game board. The game board has 12 spaces numbered 1–12. You may place as many counters as you choose (0–12) on each space. You may leave spaces blank. You may also put 1 or more than 1 counter on any space.

Players take turns rolling the dice. The first player rolls the 2 dice and finds their sum. (For example, if 2 and 3 are rolled, the sum is 5.) Each player may remove 1 counter from his or her 5 space. Even if there is more than 1 counter on that space, only 1 may be removed. If there are no counters on that space, no counters may be removed from any space. The next player rolls the 2 dice and finds their sum (e.g. 4 + 4 = 8). Each player now removes 1 counter from his or her 8 space, and so on. The goal of the game is to empty your board. The first player with no counters left on his or her board is the winner.

Analysing the game: When you have played several times, talk with each other about these questions.

- Which sums were rolled most often?
- Which sums were rolled, or not rolled, very often?
- Why do you think some sums came up more often than others?
- Can you prove which sums are most likely to occur?
- What do you think is a good strategy for placing your counters on the game board? Why?

Writing about the game: Write advice to a friend who is new to the game. Tell him or her your favourite strategy for placing the counters on the game board and explain why you believe this strategy is a good one.

Place your counters on the game board, putting as many or as few as you like on each of the 12 numbered spaces. (For example, you might choose to put all your counters on your favourite number, 4. Or you could put 1 counter on each space. Alternatively, you might put 2 counters on each of the even spaces and leave all the odd spaces empty. It's your decision.) Roll the dice. Say you get 3 and 5. Find the sum (3 + 5 = 8). If you have any counters on the 8 space, you may remove 1 of them. When anyone rolls, everyone plays, so your opponents should also be removing a counter from the 8 space on their boards, if they can. Roll the dice again, sum the results and remove another chip. The goal of the game is to be the first person to remove all your counters from your board. Try the entire game a couple of times before reading on.

Here's the problem: Find a good strategy for placing your counters at the start of this game so that you are more likely to be able to clear your board quickly. Which squares were good to avoid? Were there certain sums that rarely (or never) came up when you rolled the pair of dice? Why? Which squares were good to put counters on? Were there certain sums that came up rather often? Why? Can you explain why some sums were more likely than others?

A Year 4 class played this game several times, worked on the problem of finding a useful game strategy, talked about what they discovered and then wrote about their findings. Figure 5.1 provides details regarding what some of the children wrote.

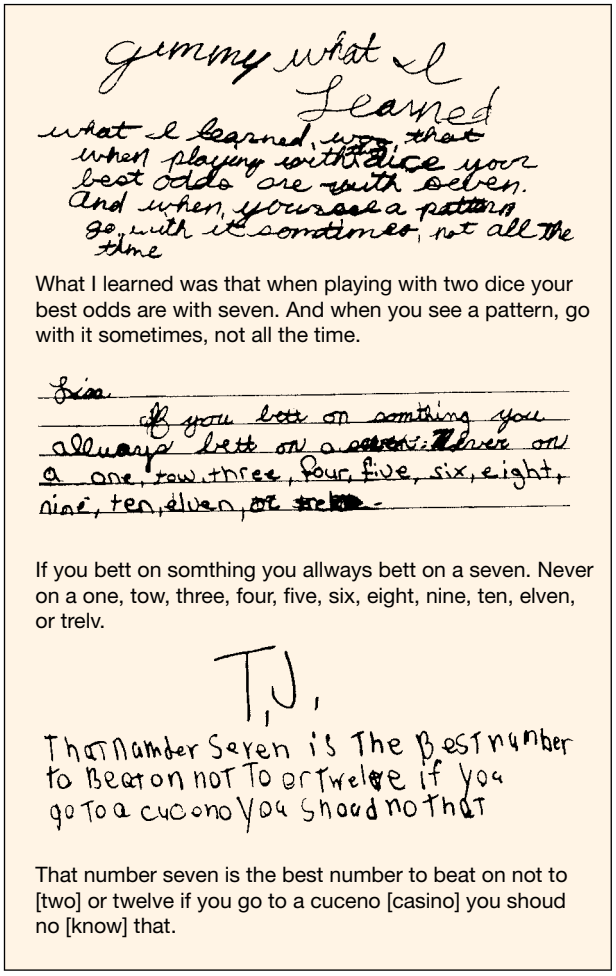


FIGURE 5.1 Year 4 students' writing about playing the dice game

In a classroom where mathematics is taught through problem solving, students given the challenge of the dice game might already have had previous experiences with making organised lists or making tables. Nevertheless, their teacher would *not* have provided them with advice about exactly how to solve the problem if he or she wanted this to be a true problem-solving challenge. A reasonable approach is to begin experimenting by playing a few games. It is helpful to keep a list of the sums obtained when the dice are rolled. If you don't keep a list, it may be hard to be sure which sums come up more often than others. Haphazard experimentation is not likely to produce a good, well-justified solution. Solving this problem requires students to think logically and to make some important decisions, particularly about how to systematically keep track of the sums that come up. (An even better way is keeping track of the pairs of dice that lead to those sums; writing down $5 + 2 = 7$ is more informative than just writing 7, since 7 could be obtained in several different ways.) A skill such as making an organised list or a table can be useful in a wide variety of problem-solving situations.

When you roll 2 dice, you are much more likely to roll a sum of 7 than a sum of 3, since there are only 2 ways to get 3 (1 + 2 and 2 + 1); however, there are many more ways to get 7. As this problem helps develop general problem-solving expertise, it also deepens students' understanding of probability (helping them to recognise that we can compare the likelihood of rolling various sums by seeing how many ways those sums can be formed).

The probability of rolling a 7 is actually $\frac{6}{36}$ whereas the probability of rolling a 3 is only $\frac{2}{36}$ (refer to figure 5.2 to see 2 ways of representing the 36 different possible sums: 1 + 1, 1 + 2, . . ., all the way up to 6 + 6). Students who work on this problem are learning about probability *through* problem solving. They are also developing and using a strategy for playing a game that can be used to solve other problems in mathematics as well as in other contexts.

What kinds of problem are appropriate for primary school students? In the early years, most school mathematics problems are related in some way to the children's own experiences because their world is relatively circumscribed and children relate best to concrete situations. By upper primary school, however, the universe of problem contexts should diversify. Increasingly, problems can grow out of situations in the world at large or from the investigation of mathematical ideas. Mathematics teaching in the upper year levels can take advantage of the increasing sophistication of students and their growing knowledge of such topics as probability, statistics and geometry.

Possible sums

2	3	4	5	6	7	8	9	10	11	12
1 + 1	1 + 2	1 + 3	1 + 4	1 + 5	1 + 6					
	2 + 1	2 + 2	2 + 3	2 + 4	2 + 5	2 + 6				
		3 + 1	3 + 2	3 + 3	3 + 4	3 + 5	3 + 6			
			4 + 1	4 + 2	4 + 3	4 + 4	4 + 5	4 + 6		
				5 + 1	5 + 2	5 + 3	5 + 4	5 + 5	5 + 6	
					6 + 1	6 + 2	6 + 3	6 + 4	6 + 5	6 + 6
1 way	2 ways	3 ways	4 ways	5 ways	6 ways	5 ways	4 ways	3 ways	2 ways	1 way

Ways to make each sum

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

FIGURE 5.2 Two tables show the number of ways to get each possible sum when rolling the dice.

Upper primary students can deal with messier and more complex problems than can children in earlier primary years, not only because they are more capable and confident in working with mathematical ideas than younger children but also because they can use technology to alleviate much of the drudgery that — until recently — often constrained school mathematics to considering problems with 'nice numbers'. Computers, calculators and electronic data-gathering devices such as calculator-based laboratories (CBLs) and calculator-based rangers (CBRs) provide simpler methods for gathering and analysing data that in years past might have been considered too much trouble. Similarly, classroom internet access makes it possible for students to look up facts and figures quickly and easily for use in posing and solving a wide variety of real-world problems. Graphing calculators and easy-to-use computer software enable students to move effortlessly between different representations of problem data. They can also compute with large quantities of data and with 'messy' numbers, both large and small, with relative ease. As a result, problems in the primary school can and should be responsive to student questions and interests.

This discussion about problem solving as one of the fundamental processes for school mathematics has described the general nature of problem solving and briefly considered the types of problem that may be appropriate for primary school students. However, the most important point is that problem solving can serve as a foundation for all mathematics teaching because problem solving also involves students in work with all the other fundamental processes of doing mathematics (reasoning, communicating, connecting and representing) and develops the proficiencies (understanding, fluency and reasoning) outlined in the Australian Curriculum. Problem solving is a way of teaching and learning. This means it involves more than the presentation of word problems; it involves the way you encourage children to approach mathematical learning. A situation is posed, as in a word problem, and then there is a search for a resolution. However, the situation that is posed often has a mathematical basis beyond the application of some procedures. Students or the teacher may pose the problem. In either case, using problems as a jumping-off point for mathematics teaching involves the teacher in posing questions that provoke student thought and also in encouraging students to pose their own questions. Using problem solving as a foundation for mathematics teaching requires students to engage in a search for a reasonable solution or solutions.

Researchers have documented changes in children's perceptions about mathematics when they are taught in a problem-rich environment (Bansilal 2012; Chazan, Sandow, Evitts, & Heinz 2011; Hunter 2014; Stacey 2012) reflecting the spirit of the Australian Curriculum (ACARA, 2011) and the NCTM Standards (2000; 2006). One example is students who were involved in the Cognitively Guided Instruction (CGI) program (Franke & Carey 1997). In the CGI model of teaching, the teacher poses a rich mathematical task. Students take time to work individually or in small groups to solve the problem and then share their approaches with each other. Students are encouraged to listen carefully to each other and to question each other about processes and strategies. The teacher's role is to choose appropriate tasks and to orchestrate the classroom discourse, using what he or she knows about the students' developmental level. A key aspect of CGI is the teacher's ability to analyse the children's thinking and to guide classroom problem solving accordingly. The CGI Year 1 students 'perceived of mathematics as a problem-solving endeavour in which many different strategies are considered viable and communicating mathematical thinking is an integral part of the task' (Franke & Carey 1997, p. 8). In a study of another problem-centred mathematics program, after 2 years in the program, Year 3 students scored significantly higher on standardised measures of computational proficiency and conceptual understanding and held stronger beliefs about the importance of finding their own or different ways to solve problems than those in 'textbook classes' (Wood & Sellers 1996).

Problem solving has been the focus of numerous articles, books, collections of materials and research studies (Moss, Hawes, Naqvi, & Caswell 2015; Mulligan 2015; Warshauer 2015). Because it is so fundamental to teaching and learning mathematics, we have devoted another entire chapter to it in this textbook. The next chapter focuses on helping you probe deeper into questions such as the following.

- What is involved in teaching mathematics through problem solving?
- What strategies for problem solving are helpful for primary students?
- How can problem solving be evaluated?

5.2 Reasoning and proof

LEARNING OUTCOME 5.2 Describing mathematical reasoning and how it assists children with making sense of mathematical knowledge and relationships.

From their earliest experiences with mathematical challenges and problems, children should understand that they are expected to supply reasons for their arguments. The question 'Why do you think so?' should be commonplace in the classroom. Teachers should not be the only ones asking 'Why?' Asking why comes naturally to small children. They should be encouraged to sustain their natural curiosity for justification as they share their mathematical ideas with each other. When children observe a pattern (e.g. whenever you add 2 odd numbers, the answer is even), they should be encouraged to ask why. A Year 2 student used square tiles to represent and sort various numbers (see figure 5.3) and then explained his reasoning as follows:

> All the even numbers are just rectangles because you can make them go 2 by 2. But all the odd numbers can't. Odd numbers make rectangles with chimneys. If you put 2 rectangles with chimneys together, the new one doesn't have a chimney. So 2 odds makes an even.

Ball and Bass (2003) offer another example of young children reasoning about odds and evens, and justifying their conjectures. If students are consistently expected to explore, question, conjecture and justify their ideas, they learn that mathematics should make sense rather than believing that mathematics is a set of arbitrary rules and formulas. This notion underlies the Australian Curriculum's Reasoning proficiency.

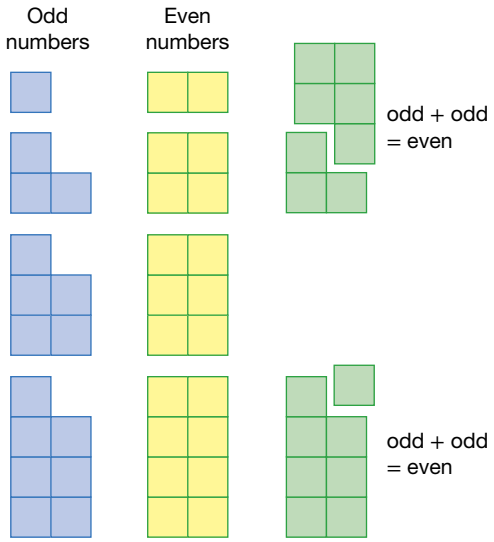


FIGURE 5.3 Pictures of odds and evens can help students justify why the sum of 2 odd numbers is always even.

Being able to reason mathematically is essential to making sense of mathematics and, ultimately, to justifying mathematical conjectures. Proof is often interpreted as a formal process, reserved for students in advanced mathematics. On the contrary, in the primary classroom, clear articulation of one's reasoning is the goal because this serves as an important precursor to formal proof. Young children have limited experience with what constitutes a reasonable argument. To help them develop their reasoning powers, both NCTM and ACARA advocate posing problems that encourage children to examine, explain and justify their own thinking — and the thinking of their peers. Children learn to reason by

being encouraged to detect fallacies and to critique their own thinking and the thinking of others (NCTM 2006; Stacey 2012).

Figure 5.4 shows 2 related problems used with a Year 3 class to engage the children in logical thinking and in critiquing one another's reasoning (McGivney-Burelle 2004/2005).

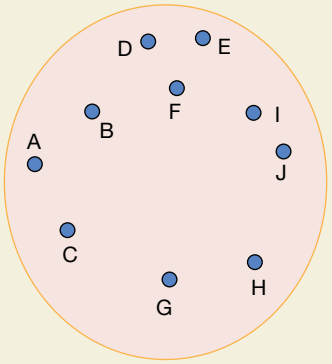
Playground problem: Version A

There are 10 children on a playground. Some of them are brothers and sisters. The oval represents the playground and the dots represent the children. The letters next to each dot represent the children's names.

Which of the children are brothers and sisters?

How many of the children are boys and how many of them are girls?

How many families are on the playground?



Playground problem: Version B

There are 10 children on a playground. Some of them are brothers and sisters. The oval represents the playground and the dots represent the children. The letters next to each dot represent the children's names.

To help us out, each child on the playground will point to his or her sister. On paper, we will draw a red arrow to represent this. For example, if there is a red arrow leaving from C and pointing to B, then we know that B is a sister of C.

Which of the children are brothers and sisters?

How many of the children are boys and how many of them are girls?

How many families are on the playground?

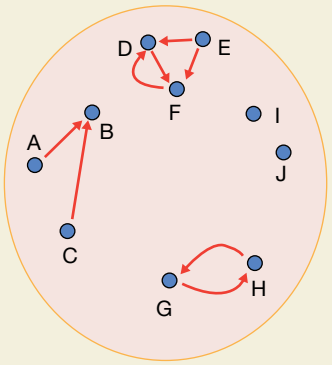


FIGURE 5.4 Two logical reasoning problems

Source: Adapted from Jean M. McGivney-Burelle's 'Connecting the dots: network problems that foster mathematical reasoning', Reprinted with permission from *Teaching Children Mathematics*, copyright 2004 by the National Council of Teachers of Mathematics.

As soon as their teacher introduced the Playground problem: Version A, children immediately raised their hands and began guessing how many boys and girls were on the playground. A variety of answers were proposed, but the most common suggestion was that there were probably an equal number of boys and girls on the playground. As one student explained, 'Well, it has to be fair!'. After further discussion, however, some children recognised that there was not enough information given in Version A to determine how many boys and girls would be on the playground. Eventually the children convinced each other that 'it could be anything that adds to 10. There could be 10 girls and no boys or 10 boys and no girls and anything in between' (McGivney-Burelle 2004/2005, p. 273).

Their teacher then posed Playground problem: Version B, which included additional information. She helped them understand the new situation by examining a simpler problem first — asking 2 children to come forward to act as brother and sister. Marcus pointed to Jessica (pretending she was his sister), while Jessica left her arms by her sides. When Rosie joined them, representing another sister, the students figured out that Marcus should point to both girls and the girls should point to each other. The children then broke into groups to discuss what could be concluded from the Playground problem network with red arrows showing sister relationships. After some thought, they noted the following.

- Arrows from both A and C to B mean B must be a sister to both A and C. Because there are no arrows pointing to A or C, they must be brothers. So B must be a girl, while A and C must be boys.
- Arrows from E to D and from E to F mean D and F must be sisters of E, so D and F are girls. No arrows pointing toward E mean that E must be a boy.
- Mutual arrows between G and H mean they are sisters to each other; therefore they are both girls.

What about I and J? The children suggested it was impossible to determine their gender and thought they must be 'only children'.

> 'But why couldn't I and J be brothers?' a student wondered aloud. 'I mean, since kids are only pointing to their sisters, couldn't it be that I and J are brothers and so there are no arrows between them?' After some discussion along these lines, the students concluded that it was impossible to infer the relationship between I and J. The most definitive answers that the class could come to were that there were '*at least* five girls (B, D, F, G, H), *at least* three boys (A, C, E), and *at least* four families (ABC, DEF, GH, with I and J from either the same family or different families).' Beyond that it was purely speculation (McGivney-Burelle 2004/2005, p. 275).

Reasoning and proof cannot be taught in a single unit or lesson; they must be consistent expectations throughout all units and lessons in all year levels. 'By developing ideas, exploring phenomena, justifying results, and using mathematical conjectures in all content areas and — with different expectations of sophistication — at all grade levels, students should see and expect that mathematics makes sense' (NCTM 2000, p. 56).

However, what must primary teachers consider in order to promote an environment where children are nurtured and encouraged to make sense of mathematical ideas and thus develop their abilities to reason mathematically, providing proofs of mathematical conjectures? Russell (1999) identifies 4 important points about active mathematical reasoning in primary school classrooms.

1. Reasoning is about making generalisations.
2. Reasoning leads to a web of generalisations.
3. Reasoning leads to mathematical memory built on relationships.
4. Learning through reasoning requires making mistakes and learning from them.

Reasoning is about making generalisations

Mathematics is much more than just finding answers to specific problems or computations. Indeed, in today's world, machines can do much of the computational drudgery. Reasoning mathematically involves observing patterns, thinking about them and justifying why they should be true in more than just individual instances. A simple example of mathematical reasoning occurs when a kindergartner, proud of their newfound ability to think about big numbers, eagerly challenges their mother by asking, 'Do you

know how much 2 trillion plus 3 trillion is? It's 5 trillion!' There is no doubt that this child has never seen a trillion of anything, nor are they able to write numbers in the trillions. However, they are familiar with the pattern in smaller cases (2 apples + 3 apples = 5 apples; 2 hundred + 3 hundred = 5 hundred), so they are able to generalise to adding trillions.

Here is another example of making generalisations — the story of Katie, a Year 3 student who was working with a partner to find factor pairs for 120 (Russell 1999). Katie and her partner believed at first that 3×42 might be such a factor pair (a result obtained by incorrectly counting squares on a rectangular array on graph paper); however, Katie reasoned that answer must be incorrect because she remembered that 6×20 was a valid factor pair for 120, and from previous experience in finding factor pairs, she had figured out that if you halve 1 factor, you should double the other factor to keep the product the same. This is a powerful and useful generalisation. Using this line of reasoning, Katie reported that the correct factor pair must be 3×40, not 3×42 because — thinking about 6×20 — if 6 is halved to get 3, then 20 should be doubled to get 40 to keep the product the same: $6 \times 20 = 3 \times 40$. When children go beyond specific instances of mathematical ideas to consider general cases, they are reasoning mathematically.

Reasoning leads to a web of generalisations

A second point about mathematical reasoning is that it 'leads to an interconnected web of mathematical knowledge within a mathematical domain' (Russell 1999, p. 1). Students should expect newly encountered mathematical ideas to fit with ideas they have already learned. Students have much more mathematical power if they have many ways to think about a number or fact or assertion. Year 1 students might understand that 'three-quarters' is between 'half a dollar' and '1 dollar'; however, they have a much more extensive understanding of the fact that three-quarters is between one-half and 1 when they also understand this relationship in terms of fractions ($\frac{1}{2}$, $\frac{3}{4}$, 1), in terms of equivalent fractions ($\frac{2}{4}$, $\frac{3}{4}$, $\frac{4}{4}$), in terms of decimals (0.5, 0.75 and 1.00), and by visualising pieces of a pie or portions of a collection of trading cards. When students incorrectly claim that 0.25 must be larger than 0.5 because 25 is more than 5, they apparently have not developed this robust web of connections for these ideas. They are unable to reason about the other meanings they may know for these 2 numbers and are thereby unable to recognise the contradiction in their thinking. Reasoning statements to support students can be found at the AAMT Top Drawer website (AAMT 2013).

Reasoning leads to mathematical memory built on relationships

A third point about mathematical reasoning is that the development of a web of mathematical understandings is the foundation of what *mathematical memory* (or mathematical sense), a capability that provides the basis for insight into mathematical problems (de Freitas & Ferrara 2015; Kroesbergen, Luit, Friso-Van den Bos, & Ven 2013; Raghubar, Barnes, & Hecht 2010). For example, consider the problem of finding the sum of the first 100 counting numbers: $1 + 2 + 3 + \ldots + 100$ (see Snapshot of a lesson at the beginning of this chapter). Some time during your years of studying mathematics, you probably encountered a formula that would allow you to calculate the sum of any arithmetic series (a list of numbers where the difference between consecutive numbers is always the same — here, the difference is always just 1). If you are like most people, you have long since forgotten that formula because it is not something that you use every day. On the other hand, if you have ever seen a geometric illustration of that formula, your web of mathematical understanding may be strong enough to help you reconstruct the formula with little trouble.

Picture the sum from 1 to 100 as a set of stairs. Picture another identical set of stairs. Turn the second set of stairs upside down on the first, and you have a rectangle (100 wide, 101 high). (See figure 5.5 for a picture of a related, simpler problem: a 10-by-11 rectangle built from 2 staircases from 1 to 10.) It's easy to find the area of the 100-by-101 rectangle ($100 \times 101 = 10100$). The sum you want is half of this (100×101 divided by 2, or 5050) because the rectangle is made of 2 staircases instead of just the 1 original staircase. So $1 + 2 + 3 + \ldots + 100 = 100(101)/2$.

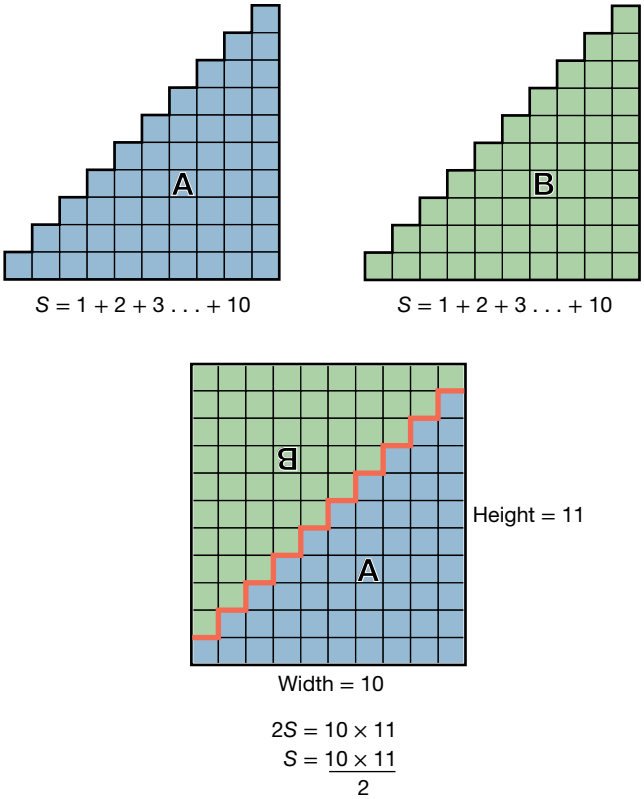


FIGURE 5.5 A 10-by-11 rectangle built with 2 staircases from 1 to 10 can help you remember the formula for the sum of a series of numbers.

Using the same idea, you can figure out how to find the sum, S, from 1 to any number, n. It's just $S = n(n + 1)/2$. A similar line of reasoning can be used to find the sums for various other arithmetic series (such as $1 + 3 + 5 + \ldots + 175$ or $12 + 15 + 18 + \ldots + 639$). In other words, if you can connect the idea of summing numbers in a series with the geometric illustration of the dual staircases, you'll never again need to worry about forgetting the formula for the sum of an arithmetic series.

Learning through reasoning requires making mistakes and learning from them

A final, important point about mathematical reasoning and proof is that one of the best ways to develop stronger reasoning and proof abilities is to study flawed or incorrect reasoning. There will be many times when your students think they've figured something out, but their reasoning just isn't quite right. That's just human nature. For example, a student might observe that you can make the problem $29 + 95$ easier to do mentally by adding 29 and 100, then subtracting 5 (obtaining the correct sum, 124). Can this same shortcut be used to make the problem 29×95 easier to do mentally (i.e. can you multiply 29 times 100, and then subtract 5)? It turns out that just isn't correct! Why? Rather than just tell your students that this doesn't work, it would be much better to help them investigate why the shortcut works for addition and why it does not work for multiplication. Some possible approaches would be to try adding and multiplying a wide variety of examples to try to figure out what is going on. Or it might help to represent the problems geometrically using a 29-by-95 array to represent the multiplication and comparing it with a 29-by-100 array may offer some insights; see figure 5.6.

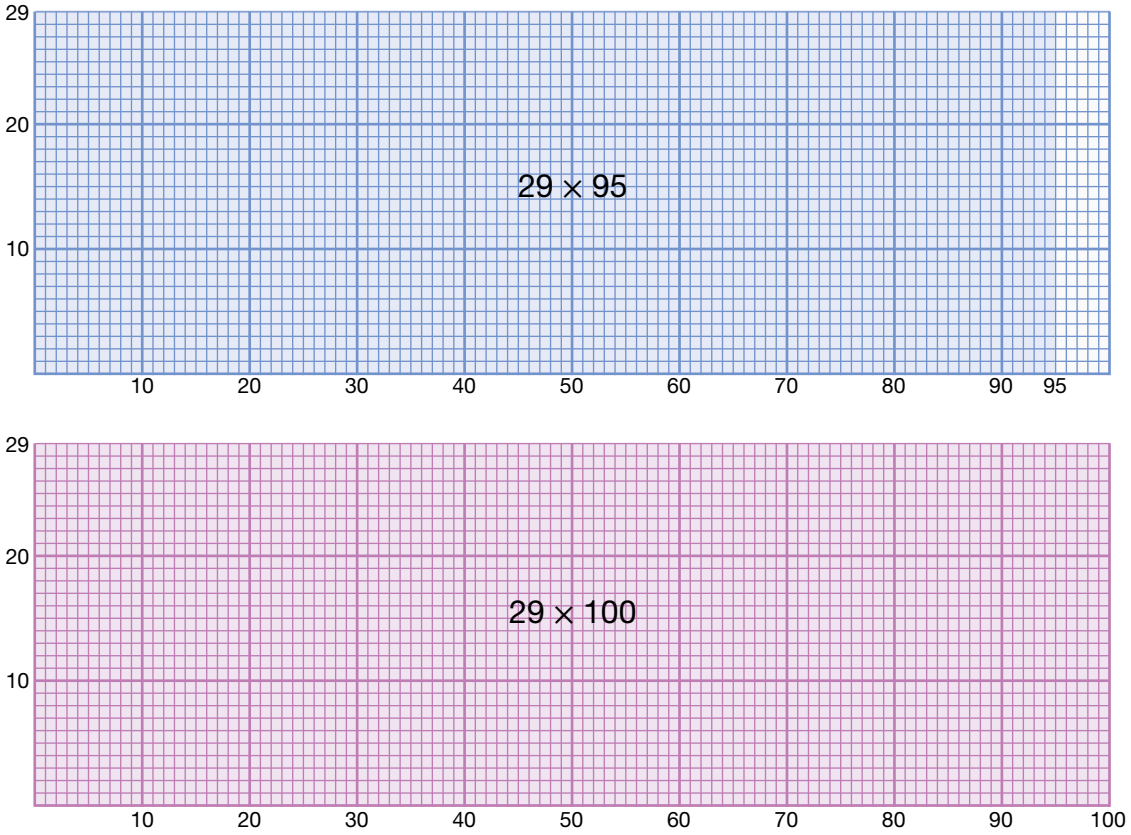


FIGURE 5.6 Visualising 29 × 95 and 29 × 100 helps students think about why 29 × 95 ≠ (29 × 100) − 5.

As a teacher, your role is to encourage students to examine their own thinking and the thinking of others as well as to help them uncover and understand flawed reasoning when it occurs.

5.3 Communication

LEARNING OUTCOME 5.3 Explaining how primary school children can be encouraged to communicate their mathematical thinking.

Because language is a powerful tool for organising thinking about mathematical ideas, it is extremely important for students to have many experiences with talking and writing about mathematics, describing and explaining their ideas. Conversely, it is also important that students often be on the receiving end of communications: hearing about, reading about and listening to the descriptions and explanations of others (Amos 2007; Chapin & O'Connor 2004; Lampert & Cobb 2003). Two-way communication about mathematical ideas helps students identify, clarify, organise, articulate and extend their thinking. Have you ever noticed how struggling to explain an idea helped you figure out what you were really thinking? Reflection and communication are intertwined. Sharing your ideas with others through talking or writing forces you to think more deeply about those ideas. Also, in thinking about your ideas, you often deepen your understanding and, consequently, are able to communicate even more clearly. Communication is obviously a process rather than an end in itself.

Students should be encouraged to communicate their mathematical thinking in a variety of modes: through pictures, gestures, graphs, charts and symbols, as well as through words (both spoken and written). Figures 5.3, 5.4, 5.5 and 5.6 clearly illustrate the power of visuals in communicating about mathematics. Such nonverbal communication is often useful in promoting learning.

Especially at first, students' efforts at communicating about mathematical ideas may be idiosyncratic (they may use symbols, expressions or notations that they make up on their own). Angela Andrews, a teacher who has written a fascinating book about the problem-solving experiences of her kindergartners, shares several examples of their naive uses of language and notation. For example, she reports that a student used the notation shown in figure 5.7 to represent the number of scoops of rice needed to fill a certain bottle at the rice table. When she questioned him about his writing, he confidently reported, 'Oh, that jar held $10\frac{1}{2}$ scoops, teacher.' His symbol for $10\frac{1}{2}$ was certainly not the standard representation, but it was effective in communicating his idea' (Andrews & Trafton 2002, p. 94).



FIGURE 5.7 Child's representation of 10

Source: Andrews & Trafton (2002, p. 94).

Over time, with more experience and practice, students learn to use conventional and more precise language to express their ideas. Indeed, mathematics as a language has a vocabulary, syntax and symbolism all its own (Morgan, Craig, Schuette, & Wagner 2014).

Sometimes words or phrases used in everyday conversation may be used in mathematics with different, more precise, meanings. The symbolism of mathematics (particularly equations and graphs) often helps clarify concepts and promote understanding. Throughout their primary school years, students should have daily opportunities for communicating about mathematical ideas. Gradually, they should be expected to incorporate more precise mathematical terms in their explanations.

Mathematics journals can be used to prompt students to write about mathematics. The regularity of writing in a journal can help students monitor their own understanding of mathematical concepts. Students can be encouraged to write about such questions as 'What am I puzzled about?' and 'What mistakes do I make and why?'. When teachers take time to respond to student journals on a regular basis, however briefly, the journals can become a regular chain of communication between student and teacher. Also, teachers can use students' writing diagnostically. When the writings of many students in a group or class reveal similar confusions or misconceptions, the teacher can more appropriately plan future lessons.

Other forms of writing in mathematics class include open-ended writing as follow-up to a lesson, having students write their own word problems, having them describe their solutions to problem-solving activities, having them describe a procedure or process, and writing about connections between and among ideas. Young children can be encouraged to use their own invented spelling. All students can benefit by accompanying words with pictures and symbols so they can express themselves as fully and completely as possible.

5.4 Connections

LEARNING OUTCOME 5.4 Identifying what connections are important for primary school children to make when learning mathematics.

Although mathematics is often represented as a list of topics or a collection of skills, this is a shallow view. Mathematics is actually a well-integrated domain of study. The ideas of school mathematics are richly connected. It is important for the primary school curriculum to provide children with ongoing opportunities to experience and appreciate the connectedness of the subject.

At least 3 types of connection are important in learning mathematics (Aguirre et al. 2013; Murata, Bofferding, Pothen, Taylor, & Wischnia 2012; Rogers 2014). First, ideas within mathematics itself are richly connected with one another. Students who learn about fractions, decimals and percentages in isolation from one another miss an important opportunity to see the connections among these ideas. For example, in figure 5.8, we can see why $\frac{1}{4}$, 2 tenths and 5 hundredths and 0.25 are actually all names for the same quantity.

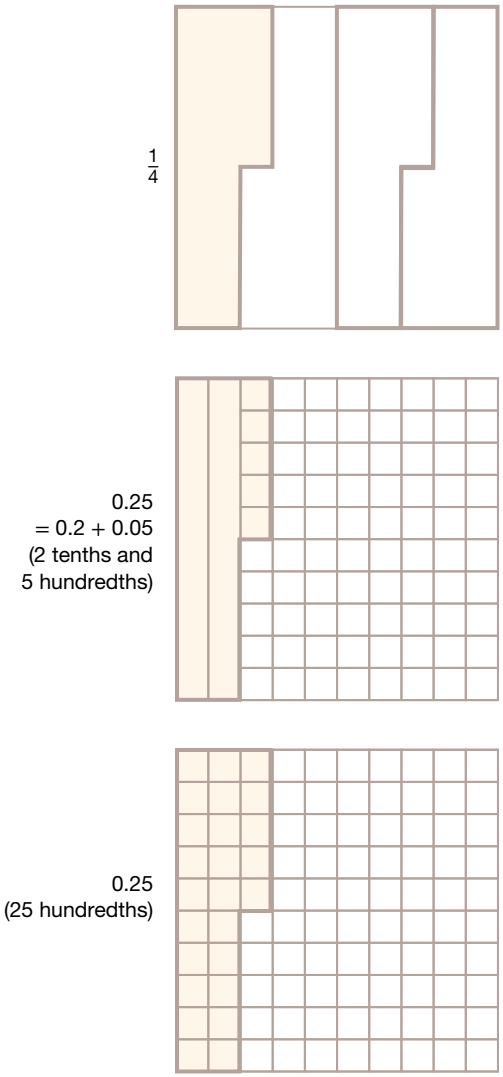


FIGURE 5.8 Some representations for 1 quarter ($\frac{1}{4}$), 2 tenths and 5 hundredths (0.25 = 0.2 + 0.05) and 25 hundredths (0.25) show that they are equivalent.

A second important type of connection is between the symbols and procedures of mathematics and the conceptual ideas that the symbolism represents. For example, why do we refer to 3^2 as '3 squared'? 3^2 is 3×3, or 9. A drawing of 9 dots, arranged in a 3-by-3 array, forms a square. Similarly, any array of x-by-x dots would form a square; thus students can recognise why x^2 is read as 'x squared'. The area of any geometric figure is generally reported in square units (e.g. square metres, square centimetres, square kilometres). Why? Because measuring area is actually just measuring how many 'squares' it would take to cover a surface. If the squares are 1 centimetre on each side, then you are measuring in

square centimetres. In fact, you can write 'cm^2' instead of 'square centimetres' for the same reason. So here we can *see* connections between number theory (the 'square numbers' 1, 4, 9, 16 etc. as shown in figure 5.9), algebraic language (x squared) and measurement (square centimetres).

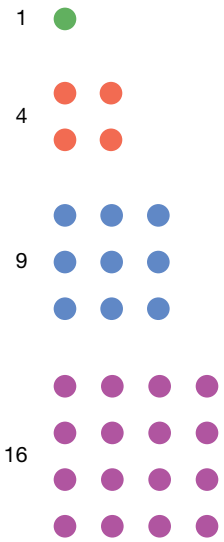


FIGURE 5.9 Arranging dots in square patterns connects the numbers 1, 4, 9 and 16 to their reference as square numbers.

A third type of connection is between mathematics and the real world or between mathematics and other school subjects. Engaging students in problems as they occur naturally in the classroom or in the lives of the children often provides a totally natural connection to mathematics. For example:

> Our class has 24 students. Today 4 desks are empty. Tony, Melissa and Emily brought their lunches from home. Everyone else who is here today wants to buy the school lunch. How many orders for lunch should be sent to the office?

As children encounter problems from real-world contexts where mathematics is a significant part of the solution, they come to recognise and value the utility and relevance of the subject. Classroom teaching should also provide many opportunities for children to experience mathematics as found in domains such as science, business, home economics, social studies, literature and art. Mathematical connections can be highlighted through integrated or thematic curricula as well as through mathematics lessons motivated by situations in children's literature. The famous poet Carl Sandburg (1960) wrote a poem entitled 'Arithmetic' that included images such as 'arithmetic is where numbers fly like pigeons in and out of your head' and 'arithmetic is numbers you squeeze from your head to your hand to your pencil to your paper till you get the answer'. After reading and discussing Sandburg's poem, children might write about their reactions or brainstorm other aspects of mathematics (with examples); they could even be prompted to write their own poems about mathematics after the style of Sandburg (focusing on their own views of arithmetic or their own thoughts about other aspects of mathematics).

5.5 Representations

LEARNING OUTCOME 5.5 Explaining the processes of representation in primary school mathematics.

When most people think of mathematics, they may think of numbers such as 2, 29 or 5280, or of numeric or symbolic expressions such as 5×2, $(a + b)(a - b)$ or $5798 \div 13$, or of equations such as $x^2 + y^2 = r^2$ or $2x + 7 = 13$. Alternatively, they may think of tables of numbers or graphs or geometric figures. All

are commonly used representations for mathematical ideas. Interestingly, it is often possible to use a variety of these different representations to illustrate the same mathematical ideas. Different representations for an idea can lead us to different ways of understanding and using that idea. This is the power of representation. Figure 5.10 identifies 5 ways that many mathematical ideas can be represented. Helping students become comfortable moving among these representations is an important goal of school mathematics (Barmby, Bolden, Raine, & Thompson 2013; Debrenti 2013, 2015; Moreira & Manrique 2014).

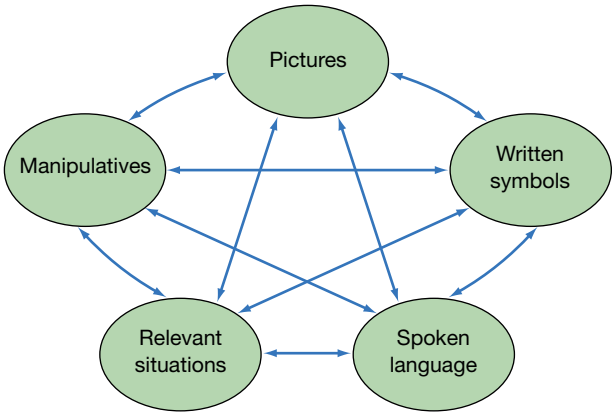


FIGURE 5.10 Five representations for mathematical ideas

Source: Adapted from Lesh and Landau (1983, p. 271).

The school mathematics curriculum has traditionally involved children in learning about a variety of representations for mathematical ideas; unfortunately, however, different representations have sometimes been learned in isolation from one another. Consider the following scenario (Clements 2004, p. 97):

Teacher: I will tell you a number and you write it. Please write for me the number *one-half*.

Student: [*Writes* $1\frac{1}{2}$]

Teacher: Please write for me the number *one and one-half*.

Student: [*Writes* $1\frac{1}{2}$] It is the same thing.

Teacher: [*Writes* $\frac{1}{2}$ *and points to it*] What would you call this?

Student: Half.

Even children who have a reasonable understanding of fractions as encountered in their everyday lives may not have made connections between those ideas and the language or symbolism traditionally used to represent them. 'The connections children make between language and written symbols may be different from the connections adults make' (Clements 2004, p. 97). Children often use the word *half*, not *one-half*, when they share in real life, for example, asking 'Can I have half your biscuit?' As a result, when the teacher asks them to write 'one-half,' they may write $1\frac{1}{2}$ — writing '1' to represent the 'one' in 'one-half' and writing '$\frac{1}{2}$' to represent the 'half'.

Similarly, students might learn about fractions in 1 chapter and decimals in another, but they also should be provided with opportunities to connect these 2 different representations for the same numbers. It is important to challenge students to connect, compare and contrast the utility and power of different representations.

The NCTM Standards (NCTM 2000) discusses 3 major goals for representation as a process in school mathematics: (1) creating and using representations to organise, record and communicate mathematical

ideas; (2) selecting, applying and translating among representations to solve problems; and (3) using representations to model and interpret physical, social and mathematical phenomena.

The Australian Curriculum notes that 'students build understanding . . . when they represent concepts in different ways' (ACARA 2011, p. 5).

Creating and using representations

Students can use several representations to illustrate the simpler problems given to them in order to find a useful general pattern that could be applied to more difficult problems. Although finding an answer by drawing pictures, counting and listing numbers is less efficient than using a formal rule, the pictures and lists of numbers were more meaningful to the students because they were just beginning to learn to see and articulate generalisable number patterns.

It is important for young children to have repeated opportunities both to invent their own ways of recording and communicating mathematical ideas and to work with conventional representations. The mathematical symbols and representations that are used every day (e.g. base-ten notation, equations, graphs of various types and traditional computational algorithms) have been polished and refined over many centuries. When students come to understand them in deep ways, they have a set of tools that expands their capacity to think mathematically.

Selecting, applying and translating among representations

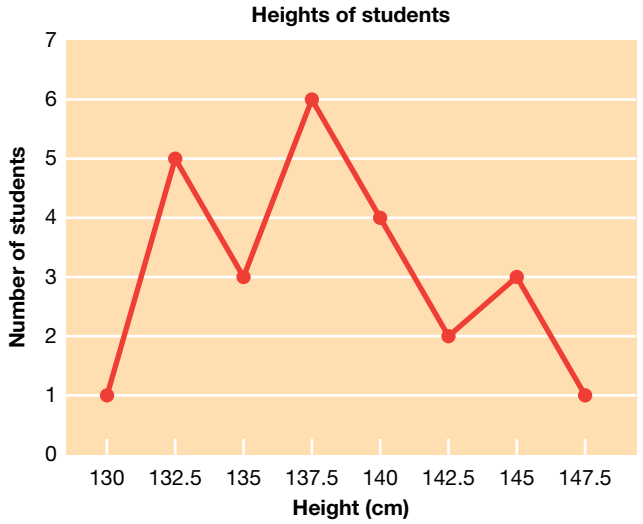
As mentioned earlier, mathematical ideas can often be represented in different ways. Each of these representations may be appropriate for different purposes. For example, a student who can think flexibly about numbers is probably able to think about the number 24 in many different ways: 2 tens and 4 ones, 1 ten and 14 ones, a little less than 25, double 12, the perimeter of a square with side 6, the area of rectangles with sides 2×12 or 4×6, and so on. Depending on the problem at hand, some of these representations may be more useful than others. Technology now offers students many opportunities for experiences with translating among representations. Data analysis software can help students easily compare and contrast various graphical representations. It is important that students consider the kinds of data and questions for which the graphical representation is appropriate.

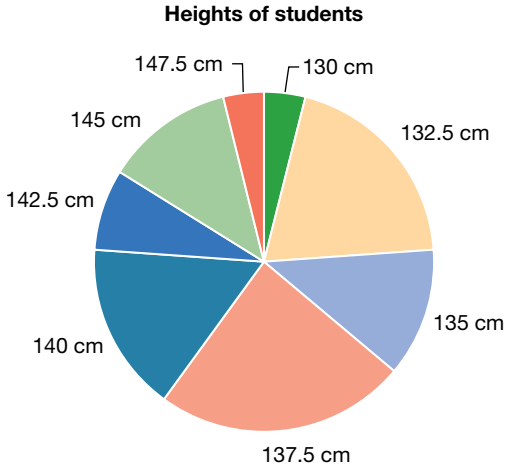
For example, figure 5.11 shows the results when a class was asked to describe the heights of the students in the class. The children recorded their heights in the appropriate rows in a spreadsheet and prepared 3 graphs: a column graph, a circle graph and a line graph. Which graph best represents the data collected? Children are used to being lined up by height. However, the circle graph does not order the heights as clearly as either the column or line graph. The line graph incorrectly gives the impression that there are children of heights between the measurement points. On the other hand, the column graph shows that the most common height of the children is 137.5 centimetres and that the second most common height is 132.5 centimetres.

Using representations to model and interpret phenomena

Much of mathematics involves simplifying problems — stripping away context and excess information to reduce the problem to symbols or representations that are easier to work with. This is mathematical modelling. For example, to solve the following word problem, you might reduce it to a picture or to a table of numbers:

> Alice is stacking soup cans for display on a shelf at the end of a supermarket aisle. She wants the display to look like a pyramid. On the top row she wants to put just 1 can, on the next-to-top row 3 cans, on the next row down 5 cans, and so on. Alice decides the display can be 6 rows high. How many cans should she start with on the bottom row?





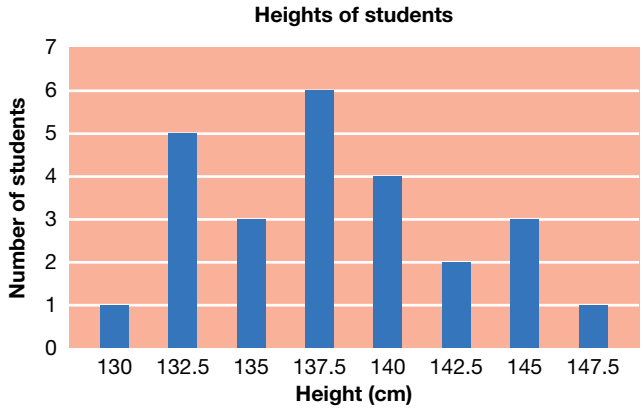


FIGURE 5.11 Which graph best represents the height of students in the class?

Once you've made a picture or a table of numbers, the fact that the problem is about stacking soup cans is no longer really important. You've modelled the problem and used the power of mathematics to solve it. Similarly, when students ask when solving word problems 'Do I add or do I subtract?' they are asking for advice about modelling the situation at hand. It is important to encourage students not to move too quickly and unthinkingly from real-world situations to abstract models. The best answer to the question about adding or subtracting is to ask 'What's going on in the problem?' or 'Can you draw a picture or can you act it out to help you decide which operation to use?'

It is also important to check back at the end of solving a problem to ensure that the solution obtained from the mathematical model fits the original situation. A classic example is a problem where students are asked to determine how many buses are needed to take a group of children on a field trip. The problem tells how many students need to be transported (e.g. 130) and how many can fit in each bus (perhaps 40). Many students correctly pick out the numbers in the problem and divide ($130 \sqrt{40}$); however, when the division doesn't come out evenly, they may offer decontextualised answers such as $3\frac{1}{4}$ or 3.25. They have forgotten to check back with the context of the problem to see that a realistic answer would be 4 buses or perhaps 3 buses and a large van.

To summarise, representations are ways of thinking about ideas (Barmby et al. 2013; Debrenti 2013, 2015; Moreira & Manrique 2014). Individuals develop their own idiosyncratic ways of thinking, but mathematics offers a broad repertoire of conventional representations that are helpful in problem solving and in communicating about mathematical ideas. One of the most important goals of mathematics teaching should be to help students build bridges from their own ways of thinking to the conventional, so that they come to understand, value and use these powerful mathematical tools. For a collection of readings all focused on the roles of representation in school mathematics, see Cuoco and Curcio (2001).

MAKING CONNECTIONS

As we have seen in this chapter, teaching mathematics involves students developing proficiencies involved in the process of doing mathematics as outlined in the Australian Curriculum. These also reflect processes identified by others, such as the NCTM's *Principles and Standards for School Mathematics* (2000). A classroom that focuses on engaging students through understanding, problem solving, fluency of recall and use of mathematical facts, and reasoning also encourages students to communicate their thinking, make connections between mathematical concepts and use a variety of representations. While there can be no doubt that all students benefit from these sorts of experiences, special attention may be necessary to ensure that students whose first language is not English are genuinely included in classroom conversations and problem-solving activities. These students are sometimes referred to as learners of English as a second language (ESL).

Some studies in Australia identify possible inequities in such a language-based approach to the teaching of mathematics (e.g. Zevenbergen, Sullivan & Mousley 2001). Students from non–English speaking backgrounds, Indigenous students or students whose family English is not that of the schoolroom might struggle with language demands as well as the cultural shift from teacher-focused instruction to student-focused learning. However, an environment that fosters both deep mathematical understanding together with rich language development enables students to participate more fully in the society of which they are a part.

Often, teachers respond to the difficulties of ESL students by stripping away much of the language associated with mathematics, leaving a procedural skeleton. While this may help these learners acquire proficiency with computation and symbol manipulation, it also narrows their mathematical education to arithmetic only, stripping much of the beauty and relevance from the mathematics curriculum and removing any focus on the processes and proficiencies discussed in this chapter.

By using a variety of helping strategies, teachers can support ESL students in their learning of both the procedures of mathematics and the language. Mixed-ability groups allow children to help each other with language challenges. By having students talk through word problems during whole-class sharing time before working individually, teachers can ensure that everyone understands the problem context. 'Study buddies' can help each other with language translation or with using concrete materials

or diagrams to show their thinking. Figure 5.12 provides several additional teacher tips that are useful for all students and particularly helpful when a class includes students with different language or cultural backgrounds. Such instruction can take account of diversity in children's ethnic and cultural heritage and primary languages whilst simultaneously supporting mathematics teaching that fosters deep understandings and proficiencies in the mathematical process.

1. Use tools to help you communicate with students.
   a. Use manipulatives, pictures, contexts, stories and drama.
   b. Write directions, key questions, terms, or ideas on chart paper, the chalkboard or the overhead projector.
   c. Demonstrate (show, don't just tell) what you want students to do.
   d. If you use handouts, have students read the directions. Have them repeat them in their own words before following them. Check for understanding.
2. Listen to students. Don't assume you know what they will say.
3. Watch for clues that students are understanding.
   a. Use eye contact.
   b. Watch their facial expressions and other body language.
   c. Notice if they are asking one another to explain what you are saying.
   d. Ask them to say back to you what you have explained (directions, new ideas etc.).
4. When students respond and you don't understand, try 1 of the following:
   a. Ask them to speak slower or use different words.
   b. Ask another student to explain what the first student said.
   c. Ask them to show what they mean by coming to the chalkboard, drawing a diagram or using concrete materials.
   d. If appropriate, promise to speak personally with them later, then do so.
   e. Identify what you think you do understand and say it back to the student: 'What I heard you say was that you have another method you'd like to use. But I'm not following your method. Please explain it again slower or show us at the board.'
5. Use lesson openings and closings to your advantage (and your students'!).
   a. Begin by asking them where you left off yesterday or ask key questions to review what they should have learned. Ask more than one student to say what was learned in a different way. Listen to how they phrase things. Ask other students for corrections or modifications.
   b. Include a summary at the end of your lesson. Have students tell what they have learned. Listen to how they say it. As above, ask for other ways to express the new ideas.
6. Use guided practice in your lessons, as appropriate, for example, for skill development. 'We'll do number 1 together.' Then develop it with them. 'Now, you do number 2.' Give them an item just like the first, monitor them, and see how they do. In this way you get evidence of their learning, and you see which students understand and can be resources for explaining to others. Continue alternating between examples done together and those done by students individually or in pairs.

FIGURE 5.12 Teacher tips for interactions in diverse classrooms

Source: Rubenstein (1997, p. 220). Reprinted with permission from *Multicultural and Gender Equity in the Mathematics Classroom: The Gift of Diversity*, copyright 1997 by the National Council of Teachers of Mathematics.

A GLANCE AT WHERE WE'VE BEEN

Learning and doing mathematics involves engaging in 5 key processes: problem solving, reasoning and proof, communication, connections and representations. These processes are inextricably linked with each other and with the mathematics content that students learn (number, algebra, geometry, data and measurement). Through these processes, students engage actively in making sense of the mathematics they are learning.

Problem solving requires that students be engaged in questions where the solution is not known and cannot be easily identified. Good problems give students an opportunity to extend what they know, building their understanding of mathematical ideas. An important role for teachers is to select challenging problems, mathematics tasks or opportunities that engage their students in the process of problem solving. Careful attention to establishing this environment will support students in enhancing their mathematical understanding.

Being able to reason mathematically is essential in order to make sense of mathematics and, ultimately, to justify or prove mathematical conjectures. To meet this challenge, children must be expected to supply reasons for their arguments from their earliest experiences with challenges and problems. Reasoning is about making generalisations and connecting new ideas with old ones to develop a web of mathematical understandings. Beginners' reasoning is often faulty, but that's okay. One of the best ways to help students develop stronger reasoning abilities is to encourage them to examine flawed arguments and to figure out where the thinking went wrong.

Communication provides a way for children to share their ideas with others. Their knowledge deepens as they help others understand their mathematical ideas and conjectures. Listening to the explanations of other children is equally important in the communication process. Two-way communication about mathematical ideas helps children identify, clarify, organise, articulate and extend their thinking. Children need to be encouraged to present their ideas in a variety of modes (e.g. pictures, gestures, graphs, charts and symbols) in addition to spoken and written words.

Connecting mathematical ideas helps children expand their understanding. At least 3 types of connection are important in learning mathematics: mathematics connected with other mathematical ideas, symbols connected to mathematical procedures and mathematics extended and connected to contexts outside of mathematics. Teachers need to emphasise mathematical connections to encourage their students to routinely seek connections and to enjoy that search.

A variety of representations can be used to express the same mathematical idea: pictures, manipulatives, written symbols, relevant situations and spoken language. The different representations lead to different ways of understanding and using that idea. Children should be encouraged to (1) create and use a variety of representations; (2) select, apply and translate among representations to solve problems; and (3) use representations to model and interpret physical, social and mathematical phenomena.

THINGS TO DO: FROM WHAT YOU'VE READ

1 Name the 5 key mathematical processes described in this chapter and identify their commonalities with those described as the proficiency strands in the Australian Curriculum.

2 (a) Solve this problem and clearly write up your solution: A programmer designed a new program that required the user to enter a 3-character password in order to be recognised as a legal user. The first character must be a W or K. The second character must be a number: 1, 2, 3, 4 or 5. The third character must be a symbol selected from $, % or *. How many possible passwords can be created for this program? What is the probability of guessing the password on the first try?

(b) Describe your work on this problem considering each of the following.

- Problem solving: Do you consider this task a problem? Why or why not?
- Reasoning and proof: Did your solution provide proof that your solution is correct?
- Communication: How did you communicate your solution? Did you use multiple ways to explain your solution? Listen or read another student's solution to the problem. Was it communicated in the manner you communicated your solution?
- Connections: Connect this problem with another mathematical idea. Connect this problem with another problem outside of mathematics.
- Representations: Show a solution to the problem in more than 1 way. Describe how you used different representations in these ways.

3 Make a list of as many different representations for the number 75 as you can. Compare your list with the lists of others. You might use equivalent expressions, models, words, pictures and so on. It is a sign of flexibility in thinking to be able to represent numbers and other mathematical ideas in many different ways.

4 What are the 3 distinct types of connection that are important in school mathematics? Give 2 examples of each type of connection.

THINGS TO DO: GOING BEYOND THIS TEXT

IN THE FIELD

1. *Mathematical processes*. Observe a primary school classroom while children are engaged in a maths lesson or investigation. Make a list of instances in which the children show evidence of using 1 or more of the 5 mathematical processes. What were the children doing or saying and what tasks were involved? What role did the teacher play?
2. Create a video snapshot of your explanation of 5 mathematical processes. Be innovative!!
3. Observe a classroom to identify how the teacher supports students with making connections to other mathematical ideas. Describe 3 examples.
4. '*Doing mathematics*?' Talk to 1 or more primary teachers about their vision of what it means to 'do mathematics'. Ask whether they connect mathematics to other areas of their curriculum and, if so, how. Write a reflection about how teachers' views of mathematics fit (or do not fit) with a view involving the 5 mathematical processes discussed in this chapter.

IN YOUR JOURNAL

5. Which of the mathematical processes and proficiencies are familiar from your own study of mathematics? Give some examples. Which of these have not been a part of your experiences in learning mathematics? Explain.
6. Select a problem from a mathematics problem-solving text. Solve the problem. As you solve the problem, keep track of what you are doing in your journal. After you have finished solving the problem and recording your activities, summarise your actions in solving the problem.
7. Explain in your journal how you would develop the 4 proficiencies of your students as you plan a unit of work.

WITH ADDITIONAL RESOURCES

8. Identify a journal article or book chapter about writing and communication in mathematics classes. (Consider *Australian Primary Mathematics Classroom (APMC)* or *Australian Mathematics Teacher (AMT)*, both publications of the Australian Association of Mathematics Teachers). Summarise the main ideas for developing proficiency in communication in the mathematics classroom.
9. Begin a resource folder for yourself that contains sections for each of the 5 mathematical processes. Collect and review recent articles in journals, such as *APMC* or *AMT* to locate ideas that support the 5 mathematical processes.

WITH TECHNOLOGY

10. Critique an online mathematics education website to identify the problem solving and reasoning strategies used to solve a problem e.g., https://www.youcubed.org/students-working-group-worthy-task/ http://topdrawer.aamt.edu.au/Reasoning
11. Explain how children are encouraged to incorporate these strategies in their learning
12. Collect some numerical data (e.g. height, arm span, hand span) about a group of children or adults (perhaps the students in your class). Record the data in a spreadsheet and create at least 3 different charts. Describe which chart best represents the data.

CHILDREN'S LITERATURE CONNECTIONS

The Maths Shed. Monsters in the movies http://www.mathematicshed.com/popular-culture-shed-2.html
This video story provides snapshots of monsters in movies. It serves as a hook to engage children in investigating and inquiry about the size and scale of monsters.

Khan Academy, Doodling in math: Spirals, Fibonacci, and being a plant [1 of 3] www.khanacademy.org/math/recreational-math/vi-hart/spirals-fibonacci/v/doodling-in-math-spirals-fibonacci-and-being-a-plant-1-of-3
This site provide some delightful example of how maths emerges from the art of doodling which then can lead to some interesting problems to solve.

AAMT Top Drawer Photo maths www.aamt.edu.au/digital-resources/
This is a suite of still images with potential for use in mathematics teaching and learning. Each resource is annotated to provide background and suggestions for using the resource in the classroom. Particular thanks to Matt Skoss and Jeff Baxter for their work on these projects.

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Figure 5.4: © Connecting the Dots: Network Problems That Foster Mathematical Reasoning, Jean McGivney-Burelle, *Teaching Children Mathematics*, January 2005, Volume 11, Issue 5, p. 272, National Council of Teachers of Mathematics
Figure 5.7: © Andrews, A. & Trafton, P. R. (2002). *Little Kids — Powerful Problem Solvers: Math Stories from a Kindergarten Classroom*. Portsmouth, NH: Heinemann
Figure 5.12: © Multicultural and Gender Equity in the Mathematics Classroom: the gift of diversity, J. Trentacosta / M. J. Kenney, 1997, Published by National Council of Teachers of Mathematics

CHAPTER 6

Helping children with problem solving

LEARNING OUTCOMES

6.1 Identifying the difference between solving problems and practice exercises.

6.2 Articulating teaching through problem solving and identifying which signposts guide this approach. Defining the types of problems used in this approach.

6.3 Naming and describing various strategies for problem solving which are used by primary students.

6.4 Explaining why looking back is such an important phase in the problem-solving process. Defining the questions students should learn to ask themselves when they are solving problems and reflecting on their solutions.

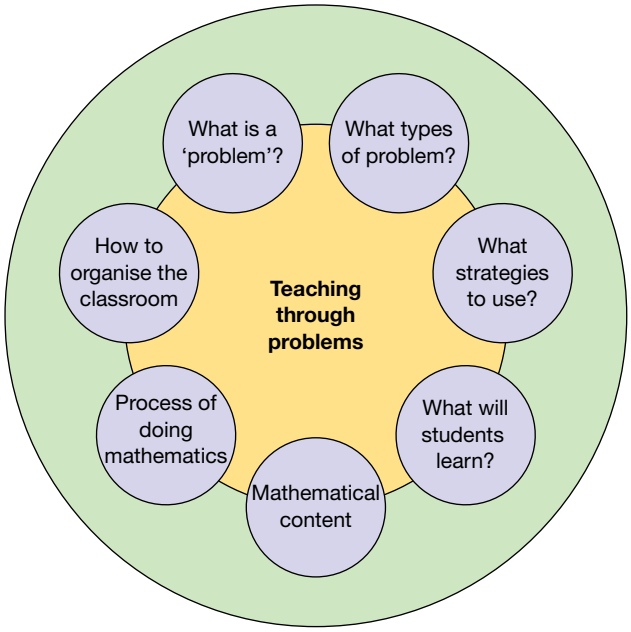
6.5 Identifying considerations for structuring problem-solving sessions to benefit a diverse range of primary school students.



'It is better to solve one problem five ways than to solve five problems one way.'

George Polya (mathematician)

Chapter 6 concept map



Introduction

The Australian Curriculum proficiency strands describe the 'actions in which student can engage when earning and using the content' (ACARA 2015, p. 4) and are incorporated into each of the 3 content strands. Problem Solving is 1 of the proficiency strands. Students solve problems when they:

- use mathematics to represent unfamiliar or recognisable situations
- design investigations
- plan approaches to investigations
- use their mathematical knowledge and skills to obtain solutions
- check their answers or solutions for logic and reasonableness. (ACARA 2015, p. 5)

Problem solving is interdependent with the other proficiency strands of understanding, fluency with facts and procedures, and reasoning. These are interwoven through the content strands, so students learn mathematical content together with a developing awareness of and expertise in the processes of doing mathematics. This means that teachers can use problems to introduce new topics, to form threads that connect topics within the mathematics curriculum and across other subject areas, and to ascertain whether students can apply what they have learnt to new situations (Midgett & Trafton 2001).

This chapter describes how primary school teachers can support children's mathematical growth through problem solving. For example, children can build new mathematical knowledge by having concepts introduced through problems set in familiar contexts (e.g. sports, games and everyday activities) or through problems involving a variety of representations (e.g. hundred charts, fraction bars and counting cubes). A strong mathematics program builds on the natural, informal problem-solving strategies that the child has encountered before entering school. As the opening quote by George Polya suggests, the more problem-solving strategies we become familiar with, the more appropriately we can handle unfamiliar problem situations. Having a selection of strategies to choose from makes for more efficient problem solvers. Many of the best problems for primary school children involve everyday situations — for example, Year 1 students will relate to problems such as 'How many more chairs will we need if we're having 5 visitors and 2 children are absent?' and 'How many biscuits will we need if everyone is to

have 2?' while Year 5 students might be more interested in problems such as 'Who has the higher batting average, Dave or Tamali?' or 'Which of these dice games is a fair game?'

No matter what type of problem is involved, children who are effective problem solvers plan ahead when given the problem. They ask themselves if what they are doing makes sense, adjust their problem-solving strategies when necessary and look back afterwards to reflect on the reasonableness of their solution and their approach.

6.1 What is a problem and what is problem solving?

LEARNING OUTCOME 6.1 Identifying the difference between solving problems and practice exercises.

A *problem* is something a person needs to figure out, something where the solution is not immediately obvious. Solving problems requires creative effort and higher-level thinking. If a child immediately sees how to get the answer to a problem, then it is not really a problem for that child.

Skill in solving problems comes through experiences with solving many problems of many different kinds. Children who have worked on many problems score higher on problem-solving tests than children who have worked on few. This chapter offers many suggestions for helping children become more proficient in mathematics through problem-solving experiences in the mathematics classroom.

As a teacher, you must be sure not to shield children from problem-solving challenges by assigning 'problems' that really are just exercises. For example, a page in a traditional children's textbook (after children have been taught how to add large numbers) might begin with exercises such as:

3194	5479	6754
5346	3477	8968
+8877	+6399	+7629

Next, there might be a story problem such as this one:

(A) 7809 people watched television on Monday.
9060 people watched on Tuesday.
9924 people watched on Wednesday.
How many people watched in the 3 days?

However, this story problem is merely an exercise with words around it, not a true problem. The only challenge is doing the computation. Sometimes tasks like this — which can be solved by applying a mathematical procedure in much the same way as it was learned — are called *routine problems* or exercises. Looked at this way, many traditional textbook word problems are routine problems. If the past week's work has been on addition, a textbook's routine word problems are typically solved by adding; if the topic has been division, then students can simply look for 2 numbers in the routine problems and divide. It is little wonder that children taught in this way flounder on tests or in the real world, where problems are not conveniently grouped by operation. By contrast, *nonroutine problems* — that is, true problems — generally require thinking because the mathematical procedures that children must use to solve them are not obvious. In the remainder of this chapter, when we refer to problems, we usually mean nonroutine problems.

Try the following nonroutine problem:

(B) Begin with the digits
1 2 3 4 5 6 7 8 9
Use each digit at least once, and form 3 4-digit numbers with the sum of 9636.

While solving this problem, children will get considerable practice in addition, but they will also have to try many possibilities. If they are thoughtful about applying previously learned mathematical ideas, they may be able to take a more efficient approach. For instance, the desired sum 9636 has a 6 in the

units place, so students who know something about sums of odd numbers and sums of even numbers might realise that the numbers being added cannot all have either 1 odd number in the ones place or 3 odd numbers in the ones place, since these choices would result in an odd sum. The digits in the ones place could be all even or could be 1 even and 2 odd, since these choices would result in an even sum. Any child who can add 4-digit numbers has the knowledge required to solve this problem, but it is still a genuine problem because the solution is not immediately apparent.

Whether a problem is truly a problem or merely an exercise certainly depends on the person trying to solve it. For example, figuring out how to share 75c equally among 3 people is probably not a problem for you, but it probably would be a good problem for a Year 2 student. What is a problem for Ann now may not be a problem for her in 3 weeks, and it may not be a problem now for Brendan. The problems that you select for children must truly be problems for them in order to give them appropriate experiences in problem solving. Children who think they should always be able to solve problems immediately and easily are likely to view as impossible any problem where the solution is not immediately apparent, and they are unlikely to persist in working toward a solution. Finding the right level of challenge for students is not easy, but you can do it by trying out a range of problems, providing time and encouraging students to explore many ways around obstacles.

You shouldn't underestimate children's abilities, as is illustrated by the following anecdote. Ms Chewning, a mathematics teacher, had been using a problem-solving approach to mathematics instruction with her Year 2 class. She read in a newspaper about a problem that 40% of the nation's Year 8 students had not been able to answer at even a basic level on a recent national mathematics test. She wondered how well her Year 2 students could do with the following problem:

> Jill needs to earn $45 for a class trip. She earns $2 each day on Mondays, Tuesdays and Wednesdays. She earns $3 each day on Thursdays, Fridays and Saturdays. She does not work on Sundays. How many weeks will it take her to earn $45?

Ms Chewning was delighted to find that:

> every single student attempted the problem, which was presented as optional. Such risk-takers they have become! Two students, using mental math only, presented me with the correct answer by the time I had completed writing the number story on the board. A total of 82% of the students, using a variety of strategies, successfully solved the problem in less than five minutes. Of the three students who struggled, two were right on track, making only minor computational errors, and the third achieved success after extensive trial and error. Needless to say, I was astounded. While I had expected them to be successful to some extent, I had not anticipated the speed and comfort with which they approached the task (University of Chicago School Mathematics Project 2001, pp. 204–5).

Results from national assessments have shown that many students have difficulty with problems that require analysis or thinking, though scores have been rising in recent years. In fact, findings from the 2007 National Assessment of Educational Progress (NAEP) — which is administered at Years 4, 8 and 12 — show that the percentage of students performing at or above the proficient level in both Year 4 and Year 8 has been increasing consistently since 1990 (Kloosterman et al. 2004; Kloosterman & Lester 2007; Lee, Grigg & Dion 2007). Year 4 students who perform at the proficient level are expected to consistently apply integrated procedural knowledge and conceptual understanding to problem-solving content strands.

American students have generally been found to be successful in solving routine 1-step problems like those found in most textbooks but to have more difficulty in solving multistep or nonroutine problems, particularly those that involve application of more than 1 arithmetic operation. Australian students display similar strengths and weaknesses to the US students. However, in the most recent Trends in International Mathematics and Science Study (TIMSS 2011), the key findings were that Year 4 students' scores had remained steady but in terms of the relative international position, Australia was outperformed by most of the Asian countries as well as England and the United States. For Year 8 mathematics,

Australia's overall results had improved from 2007 however, in relation to the scores from England and the United States, Australia was slightly outperformed — particularly in the areas of geometry and algebra (the full report can be found at www.timss.bc.edu/timss2011).

A report on the 1999 TIMSS video study noted that 'higher-level processes such as mathematical reasoning or making reference to mathematical relationships were hardly ever used for problems solved publicly in the Australian lessons'. It went on to recommend that 'Australian performance in mathematics may improve if time spent on short, repetitive problems is diverted to varied, more challenging problems with class discussion emphasising reasons and connections between ideas' (Stacey & Stephens 2008, p. 15).

6.2 Teaching mathematics through problem solving

LEARNING OUTCOME 6.2 Articulating teaching through problem solving and identifying which signposts guide this approach. Defining the types of problems used in this approach.

The primary goal of school mathematics teaching should be to ensure that students make sense of the mathematics they are learning. We emphasise this goal throughout this text. It is certainly important for students to learn — for example — how to subtract 3-digit numbers or how to add fractions or how to compute the area of a circle, but this is not enough. Students must also learn to make sense of mathematical concepts and procedures so that they will be able to use them flexibly and appropriately in unfamiliar situations both during their school years and afterwards. For example, figuring out what fraction is coloured (not white) in each of the quilt patterns shown in figure 6.1 is a motivating problem that could lead to class discussion, not only about the fraction answers but also about how students figured them out. Students could also be challenged to develop their own quilt patterns to illustrate designated fractions or to exchange as fraction challenges with one another. Using dot paper (see Appendix C) can help with block design. (See Westegaard 2008 about using quilt blocks to help construct understanding in mathematics.)

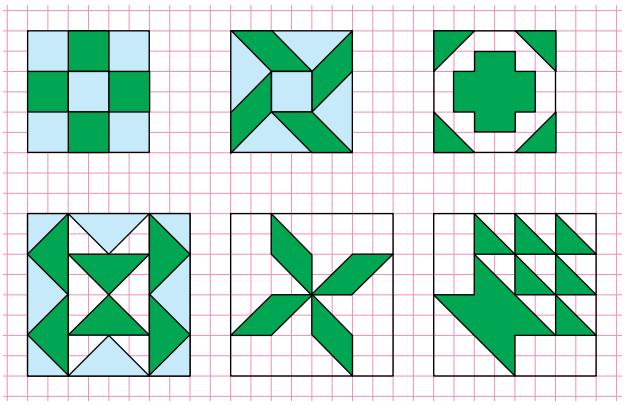


FIGURE 6.1 What fraction is coloured green in each quilt block?

It may seem efficient to teach by telling students exactly how to do things and then to have them practise until they are able to perform confidently, but this approach is actually not very effective. When children are told things, rather than figuring things out for themselves, they are less likely to remember them or to be able to apply the ideas in the future. Mathematical sense-making is best supported by a teaching approach in which students are confronted with problems, supported in their efforts to solve those problems and helped to discuss and consolidate the learning that results.

Here are some 'signposts' to guide you in teaching mathematics through problem solving (Hiebert 2003).

- *Signpost 1: Allow mathematics to be problematic for students*. This signpost means it is important to give students problems that challenge them to allow them to struggle and to help them examine and make sense of the approaches that they use. All students can benefit from problem-solving experiences. This teaching approach is probably unfamiliar to many teachers, and many may even find it questionable. They may think that good teachers should explain things clearly and completely so that students do not need to struggle at all. However, it is precisely through the experience of solving challenging problems that students grow in their mathematical understanding.

  Allowing mathematics to be problematic does not mean you must search for lots of extra problems to assign. Instead, it means allowing students to grapple with the everyday mathematical challenges that the school curriculum includes, rather than immediately stepping in to tell students what to do. Imagine, for example, that a group of primary students had no experience with subtracting multidigit numbers, though they understand the concept of subtraction. A problem-based approach to teaching multidigit subtraction would not have the teacher begin by showing children how to subtract when no regrouping is required (as in $579 - 342$) and later explaining how to regroup (as in $523 - 279$); rather, it would have the teacher encourage the students themselves to think about how subtraction is the same and different in the 2 cases and to figure out how to solve the second type of problem in a way that makes sense.
- *Signpost 2: Focus on the methods used to solve problems*. When students are challenged to solve an unfamiliar problem, they should be encouraged to talk with one another about their methods, to compare the methods and to think about their advantages and disadvantages. Given the problem $523 - 279$, some students might think about adding up to get from 279 to 523 ($279 + 1 = 280$, $280 + 20 = 300$, $300 + 223 = 523$, so the difference between 279 and 523 is $1 + 20 + 223$, or 244). Another student might think, '$523 - 279$ is hard to find, but I know it's the same as $524 - 280$ (adding 1 to each number) and that's the same as $544 - 300$ (adding 20 to each number). Aha! $544 - 300$ is easy to do! It's 244. So I know that $523 - 279$ is 244, too!' Still other students might choose to represent 523 with base-ten blocks and to think about trading in certain larger blocks for smaller blocks to make it possible to take away 279 blocks. This method most closely resembles the traditional paper-and-pencil approach.

  Learning begins when the students are challenged to make sense of a problem. Learning continues when they search for ways to solve it. Learning is extended when they share their approaches with one another. Students deepen their mathematical understanding by thinking about which approaches are easier and harder and by questioning each other about why various approaches make sense.
- *Signpost 3: Tell the right things at the right time*. Teachers who use a problem-solving approach to mathematics instruction must decide when to share information and when to let children figure things out for themselves. Speaking up too soon can eliminate the challenge and much of the learning, but it is also important not to leave students floundering when you can see they aren't making progress.

Here are some suggestions for telling the right things at the right time.

- Firstly, it is okay (indeed, important) to show students the written symbols of mathematics (e.g. notation for fractions, decimals, percentages, operations and equations) and to define technical language for them (e.g. the meaning of words such as *difference, quotient, quadrilateral* and *median*). Words and symbols are all just social conventions, so there is no way that students will 'discover' these things for themselves. However, the best time to tell students about these points is when they arise naturally during the course of students' mathematical endeavours.
- Secondly, it's also okay to tell students about alternative strategies for solving problems if these strategies have not been suggested by class members. The important thing is to encourage students to compare their own methods with those of their peers and with those that you might suggest.
- Finally, you should highlight the big mathematical ideas that come up during discussions of problem solving. In the course of solving a problem, children will uncover many important mathematical ideas. It is your job to guide the class in recognising and talking about these discoveries.

Factors for success in problem solving

Problem solving can be difficult to teach and to learn. Research on the characteristics of children who are successful or unsuccessful at solving problems, on the characteristics of problems, and on teaching strategies and classroom conditions that may help children be more successful at problem solving has led to some broad generalisations (McClain & Cobb 2001).

- Young children enter school able to solve many problems. Teaching should build on what children already know.
- Children can begin solving problems in the earliest year levels. They do *not* need to become skillful at computation before engaging in problem solving.
- Children's problem-solving abilities are related to their developmental level. They need to be given problems at appropriate levels of difficulty.
- Children should be taught a variety of problem-solving strategies to draw from as they meet a variety of problems. They should be encouraged to try solving a range of problems using the very same strategy and to try solving a single problem with more than 1 strategy. These efforts will help them recognise why some strategies are more appropriate for certain problems than for others. They can also be encouraged to adapt strategies to suit new types of problem.

Major factors that impact on students' problem-solving skills are knowledge, beliefs and attitude, control and socio-cultural factors. At all levels, teachers should be aware of the importance of all these factors when they teach problem solving.

- *Knowledge*. Students must learn to make connections between new problems and problems they have solved in the past. They must learn to recognise underlying structural similarities among problems and to choose the appropriate approach for solving each type of problem. (Do you see how this relates to the opening quote by George Polya?) That is, children should choose an approach based on a clear understanding of the problem rather than relying on surface features such as keywords — for instance, the fact that the statement of a problem includes the words *in all* doesn't always mean that addition is the right approach and *how many left* doesn't always indicate that subtraction is required.
- *Beliefs and effects*. Students' problem-solving abilities often correlate strongly with their attitudes, their level of self-confidence and their beliefs about themselves as problem solvers. Teachers must show students that they believe *all* students can be good problem solvers. Teachers must also encourage students to develop their own strategies for and approaches to problem solving. Teachers who believe there is only 1 way to solve a problem prevent students from truly experiencing what it means to be a problem solver and to do mathematics.
- *Control*. It is extremely important for students to learn to monitor and control their own thinking about problem solving. Research indicates that good problem solvers often spend a considerable amount of time up front making sure they understand a problem and at the end, looking back to see what they did, analysing how their solution might be modified or improved and thinking about how the problem is similar to and different from other problems. By contrast, weaker problem solvers tend to be impulsive, often jumping right in and crunching numbers with little regard for what they mean, without stopping to think about what approach might be most productive. Teachers must give students tasks and activities that encourage them to monitor, reflect on and control their own thinking.
- *Socio-cultural factors*. The atmosphere of the classroom should encourage students to use and further develop the problem-solving strategies that they have already developed naturally through experiences outside the classroom. Furthermore, the classroom climate itself (with its opportunities for discussion, collaboration, sharing and mutual encouragement among students) plays an important role in helping students become more skillful problem solvers. To use problems effectively, teachers need to consider the time involved, planning aids, needed resources, the role of technology and how to manage the class.

Choosing appropriate problems

Teaching through problem solving requires planning and coordinating the problems that you assign so that students have the chance to use a variety of problem-solving strategies and to analyse, write about and discuss their solutions. When you teach mathematics, you may be expected to use a textbook, so you need to consider how to use it most effectively. Begin by thinking about the big mathematical ideas that are involved in each chapter or unit. Then look at the problems provided in the text and think about how you can use them to prompt your students to get actively involved with the big mathematical ideas. Try to identify additional problems you could assign to accomplish this goal. Remember, the *signposts* we discussed as you select problems to assign. The problems you assign do not need to be extremely clever or original. Most good problems are quite simple. Sometimes an effective approach is to assign a problem that involves exactly the mathematical ideas you want to emphasise *before* students get to those ideas in the textbook. For example, if your students will soon be learning about multiplication, consider assigning a problem like this: How many coloured pencils would be needed if there are 18 children and each child needs 3 pencils? Their approaches will give you insight into what they already know and the strategies they already have. Trying to solve the problem will help them begin to think about the important idea of equal-sized groups.

The following list describes a wide variety of problem types, with examples of each type:

- Problems that ask students to represent a mathematical idea in various ways.

  (C) Think about the number 10 separated into 2 parts. Draw a picture to show ways that 10 things could be put in 2 parts. Make up a story to go along with your picture. Write a number sentence to go along with your picture.

  (D) Make up a story and draw a picture about marbles for this number sentence: 18 ÷ 6 = 3. (Figure 6.2 shows the responses of 2 Year 3 students to this problem.)

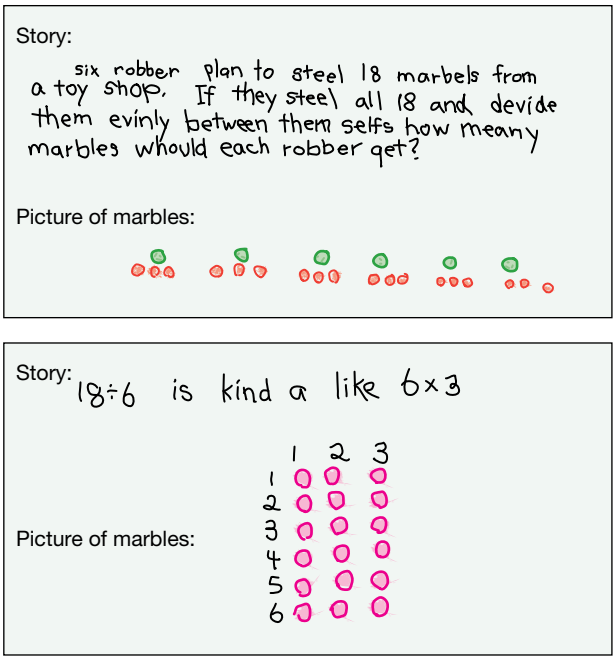


FIGURE 6.2 Stories about 18 ÷ 6 = 3 written by 2 Year 3 students

- Problems that ask students to investigate a numeric or geometric concept:

  (E) Begin with a square that measures 6 cm by 6 cm. The area of the square is 36 cm^2. How many different shapes can you think of that have this same area? For each shape, explain your thinking.

For example, how many different rectangles can you find with an area of 36 cm^2? How many different triangles? Can you think of trapeziums or parallelograms with an area of 36 cm^2? What other shapes can you think of with that area? (Tayeh & Britton 2005).

- Problems that ask students to estimate or to decide on the degree of accuracy required or to apply mathematics to practical situations (e.g. buying or measuring or building something):

  (F) Eric wondered, 'How many bran flakes are in an entire cereal box?' His younger sister predicted there were a million bran flakes, but his older sister said 'there is no way a million bran flakes would fit in that box'. How can Eric estimate the number of flakes in a 450 g box? How many boxes would be needed to fit 1 million bran flakes?

  (G) Erica is helping her father build a pen for her rabbit. She finds 4 pieces of timber in the garage that they can use for corner stakes and 36 m of chicken-wire fencing. She recognises that she can use these materials to make a variety of rectangular-shaped pens. Name 3 different sizes of rectangular pens they could build. What are the dimensions of the rectangular pen with the largest area?

  (H) Which is the better buy: a 180 g jar of jam for $1.79 or a 270 g jar for $2.79?

- Problems that ask students to conceptualise very large or very small numbers:

  (I) Have you lived 1 million hours?

  (J) How thick is a piece of paper?

- Problems that ask students to use logic, to reason, to strategise, to test conjectures or to gauge the reasonableness of information:

  (K) 3 children guessed how many jelly beans were in a jar. Their guesses were 80, 75 and 76. One child missed by 1. Another missed by 4. The other child guessed correctly. How many jelly beans were in the jar?

  (L) You and a friend have 2 dice to play a game. You can choose whether you will add or multiply the 2 numbers on the dice after you throw them. If you choose to multiply, you will multiply on all your turns and your friend will add on all turns. If you choose to add, your friend will multiply. You get to go first. The winner will be the first person whose dice give an answer of 12. Would you choose to add or multiply? Explain your thinking (Glanfield, Stenmark & Bush 2001).

- Problems that ask students to perform multiple steps or use more than 1 strategy:

  (M) Ellie has $10.00 in her pocket. She spends $5.50 for a movie ticket. The theatre offers a popcorn and drink special for $3.79. If Ellie buys the popcorn and drink, does she have enough money for a $1.45 chocolate bar, too?

  (N) How many days have you come to school so far this school year?

In addition to giving problems like these, you should sometimes give open-ended problems — that is, problems that can have more than 1 correct answer. With open-ended problems, the answer depends on the approach taken (of course, the answer must be reasonable). Students approach open-ended problems in very different ways, so such problems are ideal for ensuring that students at all levels can experience some measure of success. Open-ended problems are especially appropriate for cooperative group work that is followed with a class discussion in which the mathematical ideas and planning skills involved are explored and students can clarify their thinking and validate their decisions. Traditional textbook problems are rarely open-ended, but you can often make them open-ended by modifying them in minor ways (Kabiri & Smith 2003; Sullivan & Lilburn 2004).

(O) Figure 6.3 shows a traditional problem and its transformation into an open-ended problem based on buying from a restaurant menu. Such problems are familiar to most students and teachers.

An open-ended problem such as the one in problem (O) (figure 6.3) is much more effective when constraints ensure that students must do mathematics at a high level to solve the problem and that the focus stays on the mathematics. Of course, it is also important that the constraints seem reasonable or realistic, not arbitrary. In a problem that involves buying things from a catalogue, the constraints might

be that the students have to buy between 5 and 10 items, that no item can cost less than $1 or more than $50 and that the total cost of the items must be between $175 and $200. If the teacher does not impose appropriate constraints on the problem, students are likely to spend most of their time simply debating what to buy, with very little time spent estimating or computing how to spend their money. This can be fun for the children; however, it involves little or no mathematics.

MENU

Side dishes	
Green salad	$11.65
Coleslaw	$10.80
Baked potato	$10.50
Rice salad	$13.50
Mains	
Steak and chips	$29.50
BBQ pork ribs	$34.00
Grilled fish	$31.00
Grilled chicken	$25.95
Desserts	
Apple tart	$11.50
Sticky date pudding	$12.95
Ice-cream and fruit salad	$10.00

Traditional form: Robert, Beccy, CJ, Lauren and Ty have dinner at their favourite restaurant. Ty orders baked potato, grilled chicken and ice-cream and fruit salad. CJ orders steak and chips with a green salad, but no dessert. How much do these 2 meals cost altogether?

Open-ended form: Lauren has $65 to spend, but wants to go to the movies after dinner with her friends. A student concession ticket is $10. What meals could she have? Show the variations and how you worked out the cost of each meal.

FIGURE 6.3 A menu problem in 2 forms: traditional and open-ended

Source: Adapted from Kabiri and Smith (2003, p. 187).

In the classroom 6.1 shows an open-ended problem (P) adapted from a problem-oriented primary curriculum, *Investigations in Number, Data, and Space*. The children cut out 7 rectangles and are instructed to arrange them in order from biggest to smallest. A Year 3 teacher reports having used this problem with her class of bilingual Year 3 students, with good success in encouraging them to think about length and area. The problem allowed for multiple entry points and a variety of approaches for that diverse group of students. Her account of her students' conversations about their thinking is fascinating (Dwyer 2003).

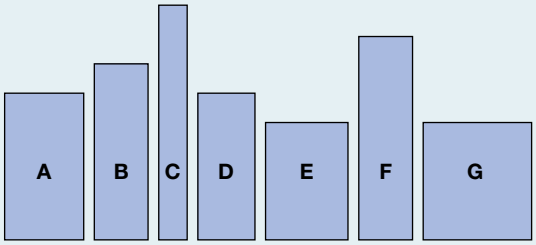
IN THE CLASSROOM 6.1

Which rectangle is biggest?

Objective: Order rectangles, define 'biggest' in different ways and find the number of tiles that cover rectangles.

Year level: 2–3

Materials: Colour tiles (1 tub per 6–8 students). For each student: scissors, glue, some black paper and a worksheet that pictures 7–10 rectangles of various sizes and shapes (e.g. skinny, fat and a few with the same length or width). Choose rectangle sizes so that colour tiles will fit on top perfectly, since part 2 of the activity involves covering the rectangles with colour tiles to find areas. Do not arrange the rectangles in any particular order by size. Label rectangles with letters, A, B, C etc.



Part 1: Ordering rectangles

A. Paired work
   Have the students work with partners to arrange the rectangles from smallest to biggest on their paper, and to glue them in place after they have decided. (The teacher should observe what students are saying and doing. Different orderings and disagreements are expected. The question is purposely ambiguous. What is important is to observe how students justify their reasoning.)
B. Class discussion
   Bring the group together, with their papers. Ask one pair to show how they ordered the rectangles. (For example, a common ordering is to arrange the rectangles 'standing up' (with their longer sides vertical) and to order them from tallest to shortest, with the wider one first whenever 2 have the same height.) Ask the students to explain why they chose their own particular order. Ask pairs who ordered the rectangles differently to show their papers and explain their thinking. For example, a pair may have chosen to arrange the rectangles 'lying down,' or they may have tried to order by area rather than by height. Encourage students to talk about the different ways they decided what was 'big' and what was 'small'.

Part 2: Covering rectangles

While all the rectangles are still visible, tell the children to imagine they are chocolate bars. Which would have the most chocolate? Which would have the least? How can you figure it out? Students may suggest different approaches. One possibility is covering them with colour tiles. Each colour tile can represent 1 square of chocolate.

A. Paired work
   Distribute a new sheet of paper showing all the original rectangles. Have students work in pairs to find how many colour tiles it will take to cover each rectangle. They should write the corresponding number inside each rectangle.
B. Class discussion
   Bring students together to compare findings. Were they surprised about any results? Were any rectangles covered by the same number of tiles? Encourage students to discuss whether they think the rectangles are the same size or different sizes. (If they think the rectangles are different, which is 'bigger' and why?) Were there any rectangles that were considered 'big' in the first activity (ordering rectangles) but not as 'big' in the second activity (covering rectangles)? (Note: Students may need numerous experiences before understanding that what we consider 'big' may change, depending on the attribute we are considering – for example, ordering by height rather than by area.)

Source: Reys, R. E., Lindquist, M. M., Lambdin, D. V. & Smith, N. L. (2009). Activity adapted from Akers et al. (1996, pp. 52–4).

Problems (Q) and (R), shown in figures 6.4 and 6.5, represent a different sort of open-ended problem. In both problems, the numeric answer is not at all difficult to determine (everyone should get the same answer). The challenge is to see how many different and interesting ways there are to find that answer.

(Q) Figure 6.4(a) shows that a pentomino is an arrangement of 5 square tiles where each tile has at least 1 side adjoining a side of another tile. Using graph paper and scissors, how many different pentominos can you make? How can you know that you have identified all the possible pentominos? Figure 6.4(b) shows 1 pentomino that will fold into a cube and 1 that will not. How many of the possible pentominos will fold into an open cube?

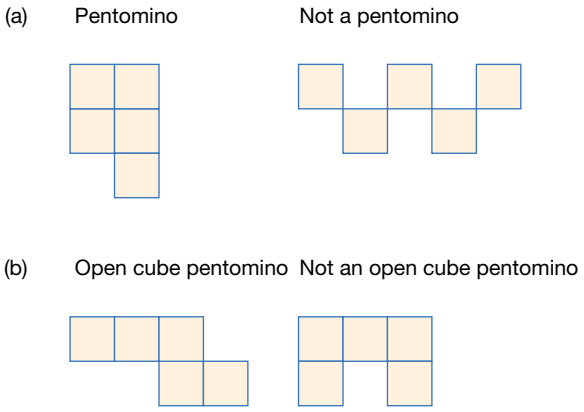


FIGURE 6.4 How many box pentominos can you make?

(R) Figure 6.5 shows marbles arranged in a pattern. How many marbles are there? Find the answer by counting or computing in as many different ways as you can.

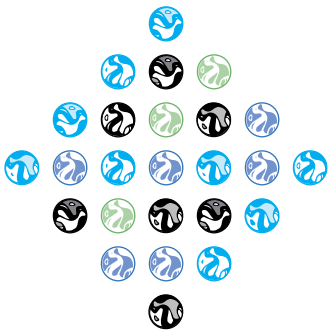


FIGURE 6.5 How many different ways can you find the number of marbles in the picture?

Finding problems

Where can you find problems to challenge your students? Most textbooks include a wide range of problems and we have seen that textbook problems in traditional form can often be converted easily to open-ended problems. In addition, there are many resources for problems that will stimulate and challenge your students. To find problems, a teacher might:

- investigate articles and books referenced in this chapter
- investigate a variety of websites
- write problems yourself (possibly using ideas from newspapers or from events in your community)
- use situations that arise spontaneously, particularly children's questions or conjectures. For example, after reading a story about a robbery, children might wonder, 'If a bank robber stole a million dollars, how heavy and bulky would the money be? Could he run down the street with it in his pocket? In a shopping bag?' For another example, read Phyllis Whitin's account of her Year 4 students' surprise when, while working on the pentomino problem (Q), they noticed that some of the pentominos had a perimeter of 10 units while others had a perimeter of 12 units — even though all the pentomino shapes were made from exactly 5 square tiles. This led Whitin's students to make and explore a variety of conjectures as they wrestled with this unexpected observation (Whitin 2004).
- attend problem-solving sessions at professional meetings and swap problems with other teachers

- have children write problems to share with each other
- use children's literature as a context for solving mathematical problems embedded in or related to the story (Ameis 2002; Bay-Williams & Martinie 2004; Bresser 2004; Burns & Sheffield 2004a, 2004b; Whitin & Whitin 2004).

It is never too soon to start a problem file, with problems categorised so you can locate them readily. Categorise them by mathematical content, by strategies or by how you are going to use the idea. Laminating the cards permits them to be used repeatedly by students for individual or small-group problem solving.

Having students pose problems

Encouraging students to write, share and solve their own problems is a good way to help them develop their problem-solving skills. By posing problems, students learn how problems are structured, they develop critical thinking and reasoning abilities and they learn to express their ideas clearly.

It is often helpful to begin by having students modify familiar problems. For example, Year 3 students may have read the classic story *The Doorbell Rang* (Hutchins 1986) and considered problems like this one:

> Suppose Mama baked 12 biscuits. If she has 2 children (or 4 or 6 or 12) and each child gets the same number of biscuits, how many biscuits does each child get?

This problem can be rewritten in various ways. A simple modification would be to change the numbers. Some changes in number don't really affect the difficulty of the problem (e.g. 18 biscuits and 6 children is hardly more difficult than 12 biscuits and 2 children). However, other changes in number might make a big difference (e.g. 12 biscuits shared by 8 children or 12 biscuits shared by 5 children are harder because they involve remainders or fractions). This could be turned into a more open-ended problem. By leaving the number of children unspecified, there can be more than 1 answer:

> Mama baked 18 biscuits. She can share them equally among her children without breaking any biscuits. How many different numbers of children can she have?

Another interesting way to reformulate a problem is to exchange the known and unknown information — here, for example, making the number of biscuits unknown:

> Mama baked a lot of biscuits. If she has 2 children (or 4 or 6 or 12) and each child got 3, how many biscuits did Mama bake?"

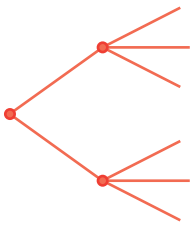
Here are 4 principles for helping students as they learn to pose problems (Moses, Bjork & Goldenberg 1990).

- Focus students' attention on the various kinds of information in problems: the information a problem gives them (the *known*), the information they are supposed to find (the *unknown*) and the *restrictions* that are placed on the answer. Encourage students to ask 'what if' questions. For example, what if you make the known information different? What if you switch what is known in the problem with what is unknown? What if you change the restrictions?
- Begin with mathematical topics or concepts that are familiar.
- Encourage students to use *ambiguity* (what they are not sure about or what they want to know) as they work toward composing new questions and problems.
- Teach students about the idea of *domain* (the numbers they are allowed to use in a particular problem). Extending or restricting the domain of a problem is an interesting way to change it. For example, the problem 'name 3 numbers whose product is 24' is very different depending on whether you consider a domain of all whole numbers, only even whole numbers, all integers (both positive and negative) or perhaps even fractions and decimals.

The teacher plays a key role in establishing a classroom environment where students are encouraged to think deeply about how problems can be changed or rewritten. The teacher can model an inquiring mind by frequently asking 'what if?' when discussing problems and by encouraging students to make conjectures and to reformulate problems.

Having students write their own problems (rather than modify problems already at hand) is also useful. Generally, this is best done after students have had considerable experience with modifying familiar problems. When you ask students to compose problems on their own, it is often important that you specify certain goals or constraints; otherwise, the assignment may become nothing more than an exercise in creativity. (Students may write fantastic stories with no discernible mathematics content or pose problems that are so complicated that no solution is possible.) You could specify goals or constraints, for example, by assigning students to write a word problem that matches a given mathematical expression or a given figure:

- Write a word problem for $250 \div 5 = 50$
- Write a comparison word problem for $12 - 8 = 4$
- (If you are not familiar with 'comparison' subtraction problems, see chapter 9, where they are defined, along with 'separation' subtraction problems and 'part-whole' subtraction problems.)
- Write a multiplication word problem for the tree diagram shown below (e.g. 2 kinds of ice-cream and 3 possible toppings for each gives 6 different types of sundae).



Situations or information in magazine advertisements, newspaper articles, books of world records, sales flyers and so on can also be effective in prompting students to pose problems. An approach used by a Year 6 teacher was to challenge her students to write and illustrate problems that involved multiplication or division and that included extraneous information. Students responded with creative problems such as this:

> At midnight, the wind over Cairns started to increase as Cyclone Lucas came closer. Every 10 minutes from then on, the wind doubled. At 12:30, the wind speed was 120 kilometres per hour and five trees were uprooted. At what speed was the wind blowing at midnight?



Students need help learning to write problems, and it is easy to integrate such lessons with language arts teaching. You could use a writing workshop approach, consisting of stages such as brainstorming and pre-writing, writing, several rounds of peer critiquing followed by rewriting and, finally, editing and publication. Student problems may be published on bulletin boards, on cards (to be made available for other students to solve) or in a class book of problems. The Year 6 teacher mentioned above obtained a small classroom grant to produce enough copies of her class's illustrated book of problems to distribute one to each Year 6 class in the school region. Her students proudly went on a field trip that took them from school to school delivering their books to Year 6 classrooms, where they shared problem-solving experiences with other students their own age. A Year 3 teacher engaged her students in taking 'the mathematician's chair' as they challenged their peers with problems they had written themselves (Hildebrand, Ludeman & Mullin 1999). The teacher assessed their writing by considering 3 aspects of each problem, on a scale of 1–4: the attributes of the problem, the structure of the problem, and the student's use of language conventions.

Using calculators and computers

The calculator's potential for helping children become more proficient at problem solving has been recognised since handheld calculators became widely available more than 35 years ago. Of course, if the focus of a lesson is on practising paper-and-pencil computation or mental computation, students should not be permitted to use calculators. However, when the focus is on problem solving or concept development, then calculators can often be an important aid, as we discuss in more detail in a later chapter. You should consider having students use calculators whenever:

- they let children solve more complex problems or problems with realistic, rather than contrived data
- they eliminate tedious and time-consuming computations and help reduce children's anxiety about being able to do computations correctly
- their special functions can help children explore mathematical objects, concepts and operations.

Because calculators present answers in decimal form, what comes up when students use a calculator while solving problems may surprise and challenge them. For example, consider this problem: 'If 1 bus can transport 34 students, how many buses will be required to take 489 students?' A student working on this problem keyed in 489 ÷ 34 and was surprised when her calculator showed 3.456138. How should this answer be interpreted? In this case, the calculator's answer presents an excellent opportunity for talking about division, remainders and fraction–decimal equivalents.

An educator reports the story of a Year 5 student named Jonathan who enjoyed using calculators to explore mathematical problems (Battista 2003). Jonathan overheard a Year 3 student working on finding pairs of whole numbers whose product was 9. Jonathan knew quite well that 9×1, 1×9 and 3×3 were the answers to the question. However, it prompted him to think about a related problem: Are there more numbers that multiply to make 9 if you can use decimals or fractions? Jonathan was able to come up with quite a few number pairs mentally ($4\frac{1}{2} \times 2$, 6×1.5, $8 \times 1\frac{1}{8}$, $4 \times 2\frac{1}{4}$ etc.). He spent the next 30 minutes brainstorming a long list of other possibilities and checking them with a calculator. Eventually, an adult asked Jonathan, 'How many pairs do you think there are?'

> Jonathan: 'Billions.' [*Short pause*.] 'There's 1 for every number. Like a googolplex times 9 googolplexths [*sic*]' (To Jonathan, a googolplex was an unimaginably large number; he had encountered the term in his reading.) (Battista 2003, p. 273)

Jonathan tends to use his calculator as a tool that helps him explore challenging mathematical problems — problems that he certainly could not easily explore without a calculator.

A popular type of problem-solving activity involving calculators is the 'broken key' activity. The children are assigned a series of problems and are told they may use a calculator but must pretend that certain keys on the calculator are broken. (The specified 'broken keys' can be number keys, operation keys or a combination of the 2. It may be helpful to give the children a small piece of masking tape to stick on the 'broken keys' as a reminder.) Suppose 1 of the problems is: 'Compute 5×39 when the '3' key is broken'. The children may suggest various ways to work around the broken key — for example, $40 - 1 + 40 - 1 + 40 - 1 + 40 - 1 + 40 - 1$ or $(5 \times 29) + (5 \times 10)$ or $(5 \times 40) - 5$. The teacher should ensure that students talk about and compare their approaches and the various operations they used. (You can read about how Year 4 and 5 students tackled the problem of multiplying 88×8 without using the '8' key or the '×' key in Ellis, Yeh and Stump 2007.)

Computers can also be important problem-solving tools. Like calculators, computers allow students to work on problems with realistic data. However, computers also let students experience problems of different types, such as problems involving graphics and graphing. For example, students could collect data from classmates on their favourite types of music, enter the data into a spreadsheet and then generate circle graphs or column graphs to represent the data. Students could also try to determine whether musical preferences differ among children of different year levels or between boys and girls — they could separate the data by year level or gender and draw a column graph that compares the preferences of 1 group with those of the other.

Many fine software programs provide a variety of problem-solving experiences. Some, such as *Numbers Undercover*, involve computation. Others, such as *The Factory Deluxe*, address spatial visualisation. Still

others, such as *Math Shop*, provide direct experiences with problem solving. *The Cruncher* teaches spreadsheet skills for solving such real-life problems as how many weeks of pocket money equal a new iPod.

Logo and BASIC, classic primary-level programming languages still available to students in some schools, can also provide rich problem-solving experiences. Working with these computer languages encourages students to make conjectures about what will happen, see what actually occurs and then try something else. Other tools, such as Geometer's Sketchpad, let students create geometric figures, make conjectures and explore relationships. For example, students can explore the sum of the angles of triangles, quadrilaterals and other polygons and try to predict the sum for 10-sided polygons. *Tinker Plots* allows students to investigate data collected.

In recent years, the internet has become an extremely rich resource for mathematics education. On the internet, teachers and students can find a wealth of challenging and interesting mathematics problems, search for data for real-world problem solving and seek answers to mathematical queries and conundrums.

6.3 Strategies for problem solving

LEARNING OUTCOME 6.3 Naming and describing various strategies for problem solving which are used by primary students.

The Hungarian-born mathematician George Polya proposed a now-classic 4-stage model of problem solving (Polya, 1973):

1. *understand* the problem
2. *devise* a plan for solving it
3. *carry* out your plan
4. *look* back to examine your solution.

This model forms the basis for the problem-solving approach used in problem-solving lessons in most primary school mathematics textbooks. However, Polya's model can be less than helpful if taken too literally. Except for simple problems, it is rarely possible to go through the steps in lockstep sequence. Moreover, the steps are not discrete, and it is not always necessary to perform every step. For instance, while trying to understand a problem, students may move into the planning stage without realising they have done so. Or simply understanding a problem may enable students to see a solution without any planning. In addition, going through the steps does not always help students find a solution. Many children become trapped in an endless process of reading, thinking, rereading — and rereading and rereading — until they give up.

Children need specific strategies that will help them move through the steps in a productive way. Polya himself delineated many of these strategies (or heuristics), and many textbooks provide lists of the strategies geared to various year levels. It is important, however, to distinguish between Polya's model itself and these strategies. Polya's 4-stage model provides a general picture of how to move through the process of solving a problem, whereas strategies are tools that may be useful for helping students interpret and describe mathematical situations or for helping them move forward at various points in the problem-solving process (Lesh & Zawojewski 2007).

In this section, we discuss 7 strategies that can help children solve a wide variety of problems:

1. act it out
2. make a drawing or diagram
3. look for a pattern
4. construct a table
5. guess and check
6. work backward
7. solve a similar but simpler problem.

Many textbooks introduce problem-solving strategies in a systematic way. If you are using a textbook that does not cover some of the strategies we discuss here, you can probably devise a plan for fitting them in by referring to the textbook's scope and sequence.

For additional guidance in helping children with problem-solving strategies, you can look at supplemental texts (e.g. Charles, Lester & Lambdin 2005). Of course, you should not limit students to using only the strategies that you have discussed in class. Rather, you should always encourage students to generate their own ideas about how to approach a new situation. If you see some students successfully using a strategy that you haven't discussed, you should encourage them to share their ideas with the rest of the class. You might also help the children think of a label for that new strategy so it can be referenced easily in future class discussions. You could list problem-solving strategies on a bulletin board for quick reference by students. For example, Jessica might discover that a good way to get started on a problem involving large numbers is to solve a similar but simpler problem with smaller numbers. When she shares her idea during whole-class discussion, you might suggest naming this the 'make-it-simpler strategy'; this would give you the option of beginning future problem-solving discussions by asking whether the make-it-simpler strategy might be useful in attacking the problem at hand.

The following discussions of specific problem-solving strategies include illustrative problems, covering a range of mathematical topics and year levels that could be used to develop each strategy. Most problems can be solved with any of a variety of strategies. Even so, for a given problem it is often the case that some strategies are more effective than others. For this reason, children should have a repertoire of strategies that they feel comfortable using. Sometimes students will have to use more than 1 strategy to solve a problem. Being able to draw on a repertoire that includes a wide variety of strategies allows students to attack many different types of problem. Moreover, when 1 strategy fails, the children have others to turn to, and this can help them develop confidence in their ability to find a path to a solution.

As you read, do stop and try to solve the problems!

Act it out

Acting out a problem helps children visualise what is involved in the problem. In using this strategy, either the children themselves perform the actions described in the problem or they manipulate objects. When teaching the strategy, you should stress that the objects used do not have to be the real thing — for example, real money is obviously not needed to act out a problem involving coins, only something labelled '5c' or '20c'. Children are adept at pretending, so they will probably suggest substitute objects themselves, but make sure they focus their attention on the actions, not on the objects.

In the early year levels, you can develop the act-it-out strategy using simple real-life problems:

(1) There are 6 children standing at the teacher's desk. Another 5 children join them. Now how many children are at the teacher's desk?

In later year levels, when the problems are more challenging, it is probably unrealistic for students actually to act them out. However, they might find it valuable to act out simpler variations of the problem to see if they can identify patterns of actions. Alternatively, they simply *think* about the actions involved and keep notes of what would happen at each step of acting out the problem if the problem situation involves multiple steps.

(2) There are 24 children in a class. Each child gives an apple to each of the other children in the class. How many apples are exchanged? Thinking about acting this out requires thinking about how many apples each child will need to bring to class. We can envision the first child giving an apple to each other child — that's 23, and the second child doing the same — that's 23 more, and so on.)

(3) A man buys a horse for $60, sells it for $70, buys it back for $80, and sells it for $90. How much does the man make or lose in the horse-trading business?

Make a drawing or diagram

Within the past week or so you have probably used the drawing or diagramming strategy to help solve a real-life problem. Perhaps you had to find someone's house by following a complicated set of directions,

so you drew a sketch of the route. Or maybe you wanted to rearrange a room and drew diagrams to see how the furniture could be placed. This strategy of making a drawing or diagram lets you depict the relationships among the different pieces of information in a problem in a way that makes those more apparent.

When teaching this strategy, stress to the children that there is no need to draw detailed pictures. Rather, encourage the children to draw only what is essential to represent the problem. For example, if the problem is about children in a classroom, stick figures (or even just circles) can represent the children and a square can represent the classroom — it is not necessary to draw the children's clothing or the walls of the classroom.

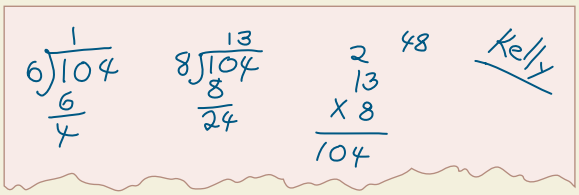
Here are 2 problems where students could apply the drawing or diagramming strategy:

(4) Aunt Katrina wants to sew 6 cloth patches on her blanket, in a straight line and spaced evenly apart (from each other and from the edges of the blanket). Each patch is 8 cm in diameter and the blanket is 104 cm long. How far apart should the patches be sewn? (Hartweg 2004/2005). See figure 6.6 for 2 Year 4 students' attempts at this problem.

(5) A snail is at the bottom of a jar that is 15 cm high. Each day the snail crawls up 5 cm, but each night it slides back down 3 cm. How many days will it take the snail to reach the top of the jar? (*Note:* The answer is not 7 days. Draw a picture to see why not!)

Aunt Katrina wants to sew 6 cloth patches on her blanket, in a straight line and evenly spaced apart. Each patch is 8 cm in diameter and the blanket is 104 cm long. How far apart should the patches be sewn? (Hartweg 2004/2005).

A Year 4 student named Kelly attempted to solve this problem without drawing a picture. She understood that she needed to divide up the blanket distance to figure out how far apart to space the patches. Her work shows that she tried dividing 104 by 6 and also by 8.

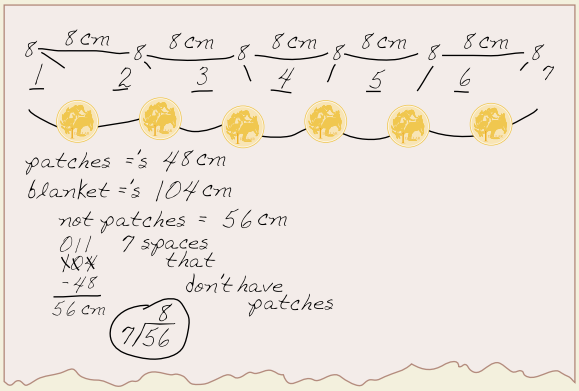


Kelly explained her thinking as follows:

> 'I knew dividing by 6 wouldn't work because there was some left over, so I divided by 8 and got 13. Then I checked my work by multiplying 13 by 8 and got 104.'

When pressed about why she chose 6 and 8 as divisors, she replied that there were 6 patches with a diameter of 8. She believed that her answer was 13, but was unable to explain what the 13 represented. Without a drawing to help her see relationships, she was unable to solve the problem.

Lizzie was a classmate of Kelly's. Her solution is shown below.



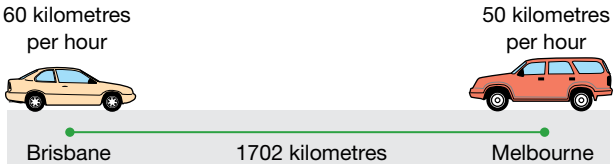
Lizzie taped dollar coins onto a piece of paper to represent the 6 patches and determined that the patches took up 48 cm because they were each 8 cm across ('patches = 48 cm'). She subtracted this amount from 104 to determine that the empty spaces totalled 56 cm. ('not patches = 56 cm'). She numbered the '7 spaces that don't have patches' on the blanket (before, between and after the 6 patches), and then divided the 56 by 7 to arrive at 8 cm for each of these spaces.

FIGURE 6.6 Year 4 students attempt the cloth patch problem.

Source: Adapted from Hartweg (2004/2005, pp. 280, 282) Reprinted with permission from *Teaching Children Mathematics*, copyright 2004/5 by the National Council of Teachers of Mathematics.

At times you can use this strategy as the basis for an activity, by presenting a picture or diagram for which the children have to make up a problem. For instance, you could show children the following picture, which could prompt them to pose the problem of how far each car goes before they crash or how long it takes them to crash:

(6)



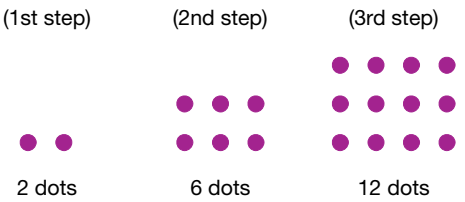
Look for a pattern

Recognising patterns, understanding the concepts of variable and function, and building on an understanding of the number system to describe relationships and formulate generalisations are some of the goals articulated in the Number and Algebra content strand of the Australian Curriculum (ACARA 2015, p. 6) Recognising, describing, extending and generalising patterns are important components of algebraic thinking. In many early learning activities, children have to identify a pattern in pictures or numbers. In problem solving, children look for patterns in more active ways — for example, by constructing a table that might help them see a pattern.

Here is a problem that appeared on the 1992 National Assessment of Educational Progress (NAEP) for mathematics.

(7) A pattern of dots is shown below. At each step, more dots are added to the pattern. The number of dots added at each step is more than the number added in the previous step. The pattern continues indefinitely.

Marcy has to determine the number of dots in the 20th step, but she does not want to draw all 20 pictures and then count the dots. Explain or show how she could do this and give the answer that Marcy should get for the number of dots.



Here is another problem where finding a pattern is useful.

(8) In a town of 90 000 people, 1 person starts a rumour by telling it to 3 other people. If each person who hears the rumour tells it to 3 new people every 15 minutes, how long would it take to spread the rumour to every person in town?

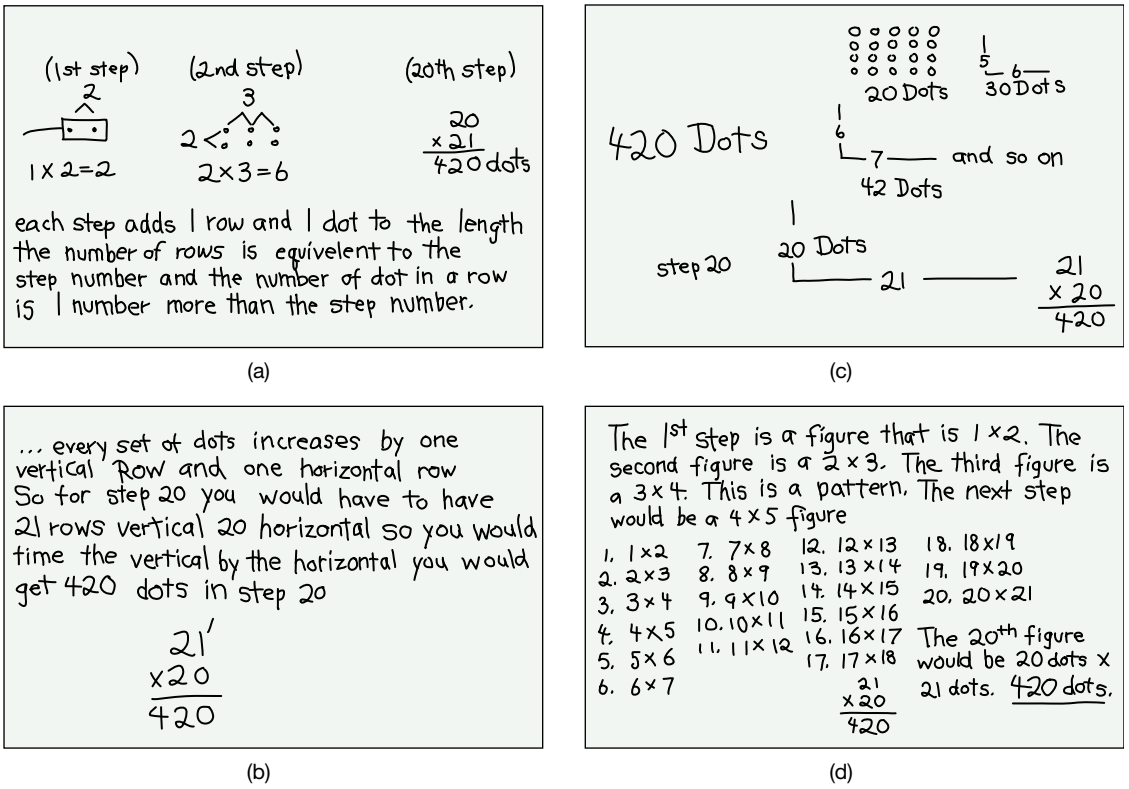


FIGURE 6.7 Examples of strategies used for finding the correct number of Marcy's dots

Source: Stylianou et al. (2000, p. 138). Reprinted with permission from *Mathematics Teaching in the Middle School*, copyright 2000 by the National Council of Teachers of Mathematics.

Construct a table

Organising information into a table often helps children discover a pattern and identify missing information. Constructing a table is an efficient way to classify and order large amounts of information; also, it provides a record of what has been tried so that children need not retrace nonproductive paths or repeatedly do the same computations. The following 2 problems lend themselves to the strategy of constructing a table:

(9) Can you make change for 25 cents using exactly 9 coins? 17 coins? 8 coins? How many different ways can you make change for 25 cents?

(10) Your teacher agrees to let you have 1 minute of recess on the first day of school, 2 minutes on the second day, 4 minutes on the third day and so on. How long will your recess be at the end of 2 weeks?

The mathematical idea involved in problem (10) can be stated in terms of other situations, and such reformulations can alter the difficulty level of the problem. Reformulation can also give children practice in recognising similarities in the structure of different problems — an ability that appears to be closely allied to good problem-solving skills. Here is a reformulation of problem (10):

(11) Suppose someone offers you a job for 15 days. They offer you your choice of how you will be paid. You can start at 1c a day, get 2c the next day, 4c the next day, and continue doubling the amount every day. Or you can start at $1 the first day, get $2 the next day, $3 the next day and continue adding $1 to the amount every day. Which would you choose? Why?

Textbooks frequently teach part of the table-construction strategy by having students read a table or complete a table that is already structured. However, it is vital that children also learn how to construct a table from scratch. They need to determine for themselves what form the table should have (e.g. how many rows and columns are needed), what the columns or rows should be labelled and so on. To teach children these skills, you can present problems that require children to collect information and then organise it into a table in order to report it. A spreadsheet can be helpful with this task, as it would be with problems (10) and (11). Table 6.1 shows the beginning of a spreadsheet that children could use to solve problem (11).

TABLE 6.1 **The beginning of a spreadsheet to solve problem (11)**

Day number	Double the amount each day		Add $1 each day	
	Day's pay	Total paid	Day's pay	Total paid
1	0.01	0.01	1.00	1.00
2	0.02	0.03	2.00	3.00
3	0.04	0.07	3.00	6.00
4	0.08	0.15	4.00	10.00
5	0.16	0.31	5.00	15.00

Guess and check

For years, children have been discouraged from guessing. Of course, random guessing is not good problem solving, but guessing can be a useful strategy if students incorporate what they know into their guesses — that is, if their guesses are educated guesses rather than wild guesses. Educated guesses are based on careful attention to pertinent aspects of the problem and on knowledge gained from previous work on similar problems. *The guess-and-check strategy involves making repeated educated guesses, using what has been learned from earlier guesses to make subsequent guesses better*. Too often, children just check a guess and, on finding that it is wrong, make another guess that may be even more off the mark. However, when this happens, instead of saying 'you're only guessing' in a derisive tone, you must help your students learn how to refine their guesses efficiently. Consider the following problem:

(12) Suppose it costs 27c to mail a postcard and 42c for a letter. Ian wrote to 12 friends and spent $3.99 for postage. How many letters and how many postcards did he send?

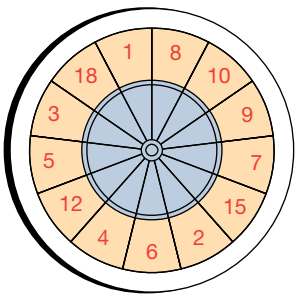
Suppose Robert begins by making a guess of 6 letters and 6 postcards — total postage $4.14 (too much). It is your job not to let him just make another guess randomly; instead, you must help him understand that his next guess should be fewer letters than 6. Why? Maybe he guesses 4 letters, so there must be 8 postcards — making the total postage $3.84 (too little). Now you must make sure he sees that there have to be more letters than 4 but still fewer than 6. That is, the only possibility is 5, and when he tries 5 letters and 7 postcards, he experiences the reward of having discovered the solution after only 2 wrong guesses — much better than he is likely to have done with random guessing.

Here are 3 more problems that you could use to teach children the guess-and-check strategy:

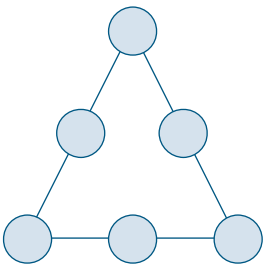
(13) A restaurant advertises twin-flavour milkshakes — that's 2 flavours of milkshake poured side-by-side into a glass. They brag that they offer 28 different twin-flavour combinations. How many different single flavours of milkshake must be available?



(14) Michelle hit the dartboard with 4 darts. Each dart hit a different number. Her total score was 25. Which numbers might she have hit to make that score?



(15) Use the numbers 1 through 6 to fill the 6 circles. Use each number only once. Each side of the triangle must add up to 9.



Work backward

Children must work backward to solve a problem if the problem states a result or an endpoint and the children have to figure out the initial conditions or the beginning (many mazes are solved by working backward from the end to the beginning). Here are 2 problems that require children to work backward from given results:

(16) Complete the following addition table:

+			3			
	12		11	15		
6		6			7	
2			5	9		
						13
5						14

(17) Erin baked some biscuits. She put half of them away for the next day. Then she divided the remaining biscuits evenly among her 2 sisters and herself, so each got 4 biscuits. How many biscuits did she bake?

Solve a similar but simpler problem

Children who know how to solve a given problem can usually solve a second problem that is somewhat similar, even if the second problem is also somewhat more difficult. The insight and understanding they gain from solving easier problems, where relationships are more apparent, carry through and let them solve harder problems. When given a problem that seems too hard, children can apply this strategy by setting the problem aside for a moment and solving a similar but simpler problem. Then they can try using the same method to solve the original problem.

Some problems are difficult just because they involve large numbers or complicated patterns, which make it hard for children to see how to solve them. Solving a similar but simpler problem first may help children figure out how to approach the original problem. In the classroom 6.2 shows how to use this similar-but-simpler strategy to solve the following problem, which is difficult because of the large number, 64:

(18) Sixty-four students play in pairs in a checkers tournament. Losers are out of the tournament. Winners play until only 1 winner is left. How many games must be played before there is 1 winner left? (See In the classroom 6.2, where this problem is used to introduce an entire lesson on solving a simpler problem.)

In the next problem, the difficulty arises from the 'complicated' number, 7.7. Students who are unsure how to solve this problem could try substituting simpler numbers, such as 10 litres per 100 kilometres and a 60-litre tank; doing this might help them recognise that it makes sense to multiply the 2 numbers.

IN THE CLASSROOM 6.2

Solve a similar but simpler problem

Objective: Solve difficult problems by first identifying and solving a related, simpler problem.

Year level: 4–5

Directions to teacher

Work with the entire class to solve the first problem together, as an example of using the 'solve a similar but simpler' problem-solving strategy. Then have children work on the later problems alone, in pairs or in small groups.

Problem

Sixty-four people enter a checkers tournament. They play individually against each other. Losers are out of the tournament. Winners play another game against a different player until just 1 winner is left. That person wins the tournament. How many games must be played until there is a winner?

Understand

How many games will be played in the first round of the tournament? (There will be 32 games and 64 people in pairs.) What do you need to figure out? (You need to know the total number of games before there is a winner.)

Plan

Is there a simpler problem you could solve first? (How many games would be played if there were only 2 players? 3 players? 4 players?) Can you find a pattern in the simpler problems that can be extended to the original problem?

Solve

Number of players	Games to play	Total number of games
2 (A and B)	A vs B	1
3 (A, B and C)	A vs B, C vs winner of AB	2
4 (A, B, C and D)	A vs B, C vs D, winners of AB and CD play each other	3
5 (A, B, C, D and E)	A vs B, C vs D, E plays winner of AB; the winner of that game plays winner of CD.	4
Any number		The number of games to play is always 1 less than the number of players.

Note to teacher: Students may find it helpful to draw stick pictures of players, labelling them A, B (or even with names) and connecting with lines to show who wins and who plays whom next. Making a table is not necessary but it helps in displaying the information discovered.

Look back

How did solving a simpler problem help with solving the original problem? (The smaller numbers were easier to work with and made it possible to see the pattern.)

Try these!

1. Jason and his 5 friends each gave 1 sticker to the 5 others. How many stickers were given out altogether?
2. Boxes of coloured markers are labelled with their colour on the top and on every side of the box except the bottom. Ten boxes are lined up against the wall on the shelf. How many labels can you see without moving any of the boxes?
3. After school at Clown Club, there were 8 kids. Every afternoon, each kid made a new funny face for each of the other clowns. They took pictures of all the faces for their clown scrapbook. How many pictures did the clowns add to their scrapbook each afternoon?

Source: Reys, R. E., Lindquist, M. M., Lambdin, D. V. & Smith, N. L. (2009). Adapted from Fennell et al. (1999, pp. 62–3).

(19) We get 7.7 litres per 100 kilometres in our car. If the tank holds 65 litres and we fill it up, about how far can we go without filling up again?

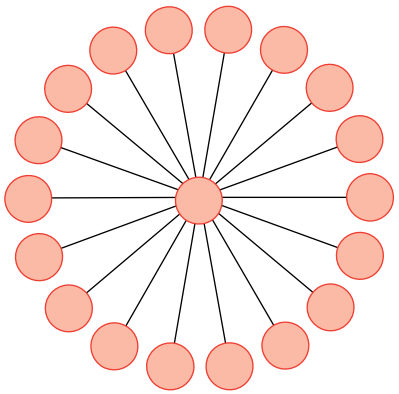
When solving a problem requires a series of steps, children may need help in recognising that they have to answer an intermediate, 'hidden' question before they can answer the final question. Answering a hidden question is another form of solving a simpler problem first, as shown in the following problem:

(20) Lara bought 6 of the fish-flavoured cat nibbles on sale. How much did she pay for all 6 nibbles? (Hidden question: What is the sale price of a fish-flavoured cat nibble?)

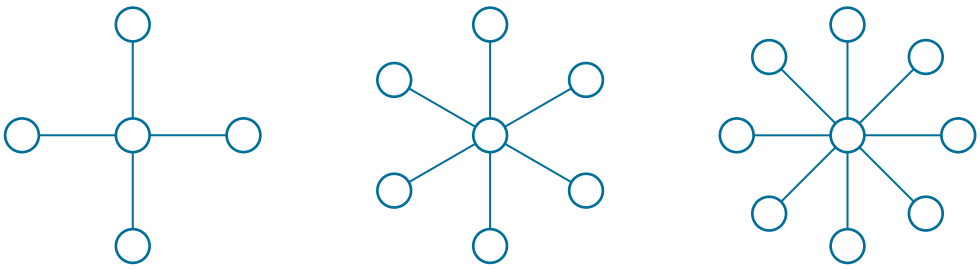


Students might find the following problem difficult because it involves so many numbers:

(21) Place the numbers 1 to 19 into the 19 circles so that the centre number plus any 2 numbers on opposite sides of the centre always make the same sum.



Students might tackle this problem by first trying simpler problems such as placing the numbers 1 to 5 or 1 to 7 or 1 to 9 in a similar pattern of circles. For example, Jeon and Bishop (2008) describe how a class of Year 5 students explored the related problem of arranging sets of '9 jumping numbers' in a 9-number Ferris wheel. Each student chose his or her own 'starting number' (for example, 10) and 'jumping number' (for example, 4) to create a set of 9 jumping numbers (in this case, 10, 14, 18, 22, 26, 30, 34, 38, 42). Then they were challenged to arrange those numbers in a 9-number Ferris wheel. The children found that exploring with a simpler set of 9 jumping numbers first (for example, 1, 2, 3, . . . 9 — with starting number 1 and jumping number 1) helped them think about how their own jumping numbers might need to be placed in the Ferris wheel.



When children do not understand a problem, you can try asking them to restate the problem in their own words. Hearing how they restate the problem can sometimes help you identify what it is that the children do not understand. At other times, restating the problem helps the children themselves figure out what the problem is asking. As a teacher, you yourself might try restating a problem as a way of eliminating unimportant words or using words that are more easily understood. Try rewording each of the following problems so that children will understand them:

(22) Find 3 different integers such that the sum of their reciprocals is an integer.
(23) I bought some items at the store. All were the same price. I bought as many items as the number of cents in the cost of each item. My receipt was $2.25. How many items did I buy?

6.4 The importance of looking back

LEARNING OUTCOME 6.4 Explaining why looking back is such an important phase in the problem-solving process. Defining the questions students should learn to ask themselves when they are solving problems and reflecting on their solutions.

Some of the most important learning that results from problem solving occurs after the problem has been solved, when students look back at the problem, at the solution and at how they found the solution. Even more important may be the learning that results from looking back at their own thought processes and at the strategies they used as they worked on the problem (Lesh & Zawojewski 2007). In fact, research indicates that helping children discuss their own thinking with you and with each other may be one of the best ways to help them become better problem solvers. In an earlier chapter, you read about the importance of metacognition — thinking about one's own thinking. You should regularly schedule looking-back time after classroom problem solving to help children develop their metacognitive abilities and thereby improve their mathematical competencies.

Looking back at the problem

It is important to help students generalise after solving a problem. Generalising involves relating the problem to other problems. Every problem solver should get in the habit of asking himself or herself this important question: How is this problem similar to and different from other problems I've seen or solved before? Problems that are very different in context or detail can be very similar in their structure. Focusing on how a problem is structurally similar to other problems often results in more significant insights than does focusing on the details of the problem or on the answer.

Looking back at the answer

When teachers tell their students 'be sure to check your work, they usually mean that the students should go through the problem-solving process again to be sure they did not make any careless errors. However, a more important type of checking is to look back at the answer to make sure it is reasonable. Problem solvers must learn to step back and ask themselves whether their answers make sense. Estimating the answer before solving the problem can help, as can thinking about whether the problem could have more than 1 answer. Healthy skepticism about answers is important for good problem solving.

Looking back at the solution process

Too often, children are given problems with only 1 right way to solve them. Many textbook exercises are like that. However, in real-life situations, 2 or more approaches are often quite possible, and the same is true of most interesting mathematics problems. Moreover, approaching a problem in different ways helps students understand the problem better. For example, think about the problems we have presented in this chapter. Maybe you do not agree with how we classified a particular problem under a particular strategy — you might have used a different strategy to solve that problem. However, even for

the problems where you felt the classification was satisfactory, there are probably other strategies that could be used to solve them.

Also, beyond thinking about different ways of approaching problems, it is also important for students to look back and put the solution process into perspective. They should consider what they did at each stage of the process — what facts they uncovered, what strategies they used and what was productive and unproductive. Sometimes, you can help children look back in this way by giving them a similar problem without numbers. This lets them focus on the relationships in the problem rather than on the specific details. As they think about how they would find the answer, no matter what numbers were involved, they must focus more clearly on the solution process. You can ask different students to talk about how they proceeded, so everyone can see that there are different ways of reaching the same answer.

Other techniques you can use to help children increase their understanding of the problem-solving process include having them write about how they solved a problem and encouraging them to evaluate their solution process to see what was successful and what was not.

Looking back at one's own thinking

People who are good problem solvers are generally good at thinking about their own thinking — that is, they are good at metacognition. They monitor the skills they have and the things they already know, they think about how they can use their skills and knowledge to solve new problems, and they make judgements about what they are doing. Older children are better at these things than younger children, but by regularly teaching this type of looking back and thereby giving children experience with it, you can help all your students develop their metacognitive abilities.

An effective teaching technique is to model the sorts of questions students should ask themselves when thinking about their own thinking, both during problem solving and while looking back afterwards. You could even post a set of questions so students can refer to them — for example:

> Was this problem like any I had ever seen before? If so, how were they similar and how were they different?
> Was this problem easier (or harder) than I expected it to be? Why?
> Did it take me more time (or less time) to solve this problem than I expected? If so, why?
> What stumbling blocks did I come up against? How did I handle them? Are there other ways I could have done it?
> How confident am I about my solution? Why?

6.5 Helping all students with problem solving

LEARNING OUTCOME 6.5 Identifying considerations for structuring problem-solving sessions to benefit a diverse range of primary school students.

As we have seen, giving students problems that are just within their reach, that challenge them to reach solutions, helps them make sense of mathematics. You must think carefully about how you organise and manage classroom teaching to ensure that all students have worthwhile problem-solving experiences. This includes paying careful attention to helping students with special needs — those with learning difficulties as well as those who are gifted in mathematics. Figuring out how to structure problem-solving sessions for a full class of diverse students will always be a challenge. Important considerations are how to manage time, classroom routines and the needs of individual students.

Managing time

Teaching mathematics effectively through problem solving requires time. Students need time to think about the problems, to mull over the relationships and to explore methods of finding solutions. They need to be encouraged to keep working and not to give up prematurely. It takes more time to tackle a problem that you do not know how to solve than to complete an exercise where you know how to

proceed. Consider tasks (A) and (B) in this chapter. Didn't (B) take you a lot more time than (A)? This is because (B) was probably an actual problem for you, whereas (A) was merely an exercise. You also need time for helping students to share and compare their solutions and to learn from one another. You can gain time for problem solving by reorganising instructional activities so that some of the time previously allotted for practising computational and other skills is redirected toward problem solving. This makes sense because students will be using and practising such skills as they solve problems.

Managing classroom routines

When you teach problem solving, you will sometimes find it useful to teach the whole class, sometimes to divide the class into small groups and sometimes to have children work in pairs or individually.

Large-group teaching (teaching the whole class) is effective for presenting and developing a new problem-solving strategy and for examining how different strategies can be used for solving the same problem. You can focus children's attention on a problem's structure, pose questions to help them use 1 strategy or find 1 solution, lead them to use other strategies or find other solutions, and encourage them to generalise from 1 problem to other problems. However, if you try to have the whole class actually solve a problem together, some students will come up with an answer before others have finished considering the problem carefully; in that case, the children slower to respond will not benefit much. Moreover, what may be an appropriately challenging problem to some students may be a trivial exercise or an impossibly difficult task for others. Discussions about problem solving are feasible with large groups, but the actual process of solving problems should be practised in small groups as well as individually and in pairs.

Small-group teaching makes it possible to group students by problem-solving ability and interests. Students in small groups have the opportunity to work cooperatively and at an appropriate level of difficulty for everyone in the group. Children feel less anxiety when everyone in a group is working together, discussing problems, sharing ideas, debating alternatives and verifying solutions. Also, small groups of students can generally solve a wider range of problems than students working alone, although the groups may take longer on each problem. In addition, research indicates that when groups discuss what a problem means and how it might be solved, they achieve better results than when they are simply told how to solve it. Groups are clearly a means of promoting communication about mathematics.

Having children work in pairs is useful because they can teach each other; this can happen no matter if you pair children of comparable abilities or of slightly different abilities. Peer teaching — with each child learning from the other — can occur in both situations.

Having children solve problems individually is also necessary, so children can progress at their own pace and use the strategies they find most comfortable. You may want to have problems available in the classroom that individual children can work on in their free time; a bulletin board, a problem corner or a file of problems can be useful for this.

Managing student needs

In any given classroom, students are likely to vary widely in their ability to learn mathematics through problem solving and in the amount and kind of help they need to increase that ability. There are a variety of ways in which you can allow for different ability levels by challenging different students differently during problem-solving sessions (Diezmann, Thornton & Watters 2003).

Compensatory strategies are specific strategies that individual students can use to deal with their own specific learning needs. These strategies can be as different as the students' needs although certain strategies can benefit almost all students. These strategies, which are particularly important when students are expected to tackle challenging problems, include the following.

- Using an approach that fits their way of learning. For example, students who tend to learn visually can try representing a problem and a solution method by making a drawing; other students can try talking through problems or using manipulatives.

- Using their strengths to communicate their thinking and to keep a record of how they solved a problem — for example, using actions, symbols, words or manipulatives. Sometimes a learning buddy can help bridge the gap between what a student is able to do or say while solving a problem and what needs to be in a written record of the solution.
- Rephrasing a problem in their own words in order to understand the problem better. (As we discussed earlier, this is a variant of the strategy of solving a similar but simpler problem.)

For students with specific needs, it may be important to do the following.

- Help them get started by reading the problem aloud or having another student explain the problem. Also consider posing a simpler problem to get a student started or having hint cards prepared ahead of time for students to consult if they wish.
- Help them create a journal or card file of 'types of mathematics problem I can solve'.
- Allow extra time for students with abstract reasoning or reading difficulties to break down problems by making notes summarising 'what I know' and 'what I need to find out'. Alternatively, you or another student could help them do this. Students with these types of difficulty are more likely to be successful at understanding and solving a problem if they can break it down in this way. (It is important to recognise that helping students understand a problem more clearly should not involve telling them how to do it; rather, you should help them figure out what is expected and what approaches they can use.)

There are various ways you can modify your teaching to help all children with problem solving. However, these modifications are especially important for students with special learning difficulties.

- Assign problems that different students can approach in different ways.
- Routinely call on individual students to rephrase problems; this will help all the students understand the problem better.
- Ask pairs or small groups of students to share their solutions with the rest of the class; then ask a student from the group (sometimes a student with special needs) to talk about what was discussed in the group as they worked on the problem.
- Allow students (especially those with vision or spatial–motor difficulties) to write on the whiteboard rather than on paper.
- Allow students (especially those with attention deficits) to work on problems in quiet places.

Many students with learning disabilities are accustomed to teaching that focuses on learning through imitation, rather than learning through solving problems, so they may need time to adjust to this new approach. Your goal is to supply just the right amount of support and help, while still ensuring that each child is challenged by problems at the appropriate level of difficulty. For gifted students, you should prepare problem extensions ahead of time so they can move on to thinking about a problem extension if they successfully complete the problem before the rest of the class. The extension might be a more general problem, a problem with larger numbers or a problem with different constraints; for example:

(24) Andrew has 20 biscuits. How many will he have left over if he shares them equally (without breaking any) with himself and 1 friend? Himself and 2 friends? Himself and 3 friends? Himself and 4 friends?

(25) Andrew has some biscuits, but we're not sure how many. He can share his biscuits equally without any left over (and without breaking any) with 1 or 2 or 3 or 4 or 5 friends. What is the smallest number of biscuits that Andrew could have?

Problem (24) might be appropriate for a Year 3 class just being introduced to division; problem (25) could be an extension for students who need more challenge.

MAKING CONNECTIONS

In many countries, the pendulum of curricular change in mathematics education has swung back and forth over the years between more attention to basic skills and more attention to problem solving. Early in the twenty-first century, researchers noted that problem solving seemed to be gaining more attention

and interest worldwide, in response to an anticipated demand for more workers with higher-order skills (Lesh & Zawojewski 2007; Maclean 2001).

Problem solving is incorporated into school mathematics instruction differently in different countries around the world. In the Trends in International Mathematics and Science Study (TIMSS), Japanese students attained some of the highest overall test scores, while Australian students' scores were quite a bit lower. Interestingly, the TIMSS also gathered videotapes of actual classroom instruction in different countries, and these give us a glimpse of how Japanese and American or Australian lessons tend to be structured differently (Shimizu 2003; Stigler & Hiebert 1999).

American and Australian lessons videotaped for the TIMSS generally had 2 phases and involved students doing many 'problems' (but the problems were often really just exercises, rather than true problems):

1. the teacher demonstrated or explained how to do sample problems
2. the students worked on numerous problems (really exercises) on their own while the teacher helped individual students who were having trouble.

By contrast, a typical Japanese mathematics lesson — which lasts 45 minutes in primary school — consisted of multiple (sometimes 4 or 5) phases but typically focused on just 1 (or at most 2) genuine problems where:

1. the teacher posed a complex problem (something challenging that the students had not seen before)
2. the students worked on the problem on their own or in pairs
3. the whole class discussed different students' approaches to the problem, orchestrated by the teacher; the teacher then summed up.

Optionally, depending on the time available and on students' facility at solving the original problem, students worked on extensions of the original problem. A key difference between Australian and Japanese teaching is the role of the teacher while children are working on the initial problem (Shimizu 2003). The Japanese teacher moves around the classroom not only to give guidance and assistance but also to take note of the different ways in which students are approaching the problem. The Japanese teacher is planning which students to call on during the upcoming class discussion. In contrast, the Australian teacher is likely to call on whichever students raise their hands, on the students who seem likely to give a correct response or on students whose minds are wandering. The Australian teacher is less likely to think about what approach the student used to solve the problem and which approaches to discuss in which order during the class discussion.

Before Japanese teachers even assign a problem, they think about what approaches students are likely to take to the problem (and what errors are likely), and they decide on the order for talking about these approaches during the class discussion. As they circulate around the room, they look for particular approaches, and they plan to call on particular students to present their thinking. For example, a teacher might first call on a student who has solved the problem using a correct but very labourious method. Then the teacher might call on students with various incorrect solutions so that the class can debate whether the answers are correct and discuss and understand common errors. Later, the teacher might call on students with elegant but non-obvious solutions.

As one researcher (Shimizu 2003, p. 207) notes:

> For Japanese teachers, a lesson is regarded as a drama, which leads up to at least one climax, or *yamaba*. In fact, a central characteristic of Japanese teachers' lesson planning is their deliberate structuring of their lessons around the yamaba. Thus, when a whole-class discussion begins, students listen carefully to the solutions proposed by their classmates and present their own ideas, because during this discussion, the lesson highlights, or *yamaba*, appear.

This same researcher (Shimizu 2003) offers 5 practical suggestions from Japanese teachers.

- *Suggestion 1: Label students' methods* during the whole-class discussion to ensure student ownership of the presented method and to make the discussion more interesting for the class.
- *Suggestion 2: Use the chalkboard effectively.* Whenever possible, record everything discussed during the lesson on the board in an organised fashion, and do not erase. This helps everyone compare different solution methods, and — at the end — offers a written bird's-eye view of the entire lesson.

- *Suggestion 3: Use the whole-class discussion to polish the students' ideas*. The Japanese word *neriage* — which translates as 'polishing up' — describes what should happen during the collaborative class discussion, as everyone involved participates in pulling all the various students' ideas together into 1 coherent mathematical idea.
- *Suggestion 4: Choose the numbers in, and the context of, the problem carefully*. The teacher must think about the variety of responses that he/she hopes the students will present and choose the problem accordingly.
- *Suggestion 5: Consider how to encourage a variety of solution methods*. If the teacher anticipates 4 different approaches, but her class comes up with only 2 of those 4, she must be ready with pre-planned questions that can be asked to lead students to thinking about the problem in other ways, if these alternative approaches are important to the goal of the lesson.

Japanese teachers understand the importance of thinking deeply about the relationship between the mathematical content to be taught and the problems they assign. They realise that anticipating student responses is a crucial aspect of lesson planning when teaching through problem solving.

A GLANCE AT WHERE WE'VE BEEN

Problem solving is one of the most important skills in mathematics; it should pervade the mathematics curriculum. Children need many experiences with genuine problems, not just with exercises. Australian students tend to have difficulty with nonroutine problems but have shown improving performance on routine problems. Problem solving helps students make sense of mathematics.

Signposts for teaching mathematics through problem solving include (1) allowing mathematics to be problematic for students, (2) focusing on the methods used to solve problems and (3) telling students the right things at the right time. Factors that impact on children's problem-solving skills include knowledge (making connections between what they know and new problems), beliefs and attitude (self-confidence and its influence on the affective domain), control (the ability to monitor and direct their own thinking) and socio-cultural factors (using nonschool experience to solve problems; the classroom atmosphere). Effective teaching through problems means choosing appropriate types of problem (problems that test the right skills develop deep mathematical knowledge and that are at the right difficulty level), finding good problems, having students pose problems and using calculators and computers appropriately.

Children should be encouraged to use a variety of problem-solving strategies. Important strategies that can help children develop their problem-solving skills include acting out the problem, making a drawing or diagram, looking for a pattern, constructing a table, guessing and checking, working backward and solving a similar but simpler problem. It is extremely important that children learn to look back after engaging in problem solving. They should look back at the problem to see how it is the same as and different from other problems, look back at the answer to make sure it is reasonable, look back at the solution process to assess whether they used the right strategy and, most importantly, look back at their own thinking, at how they thought about the problem and why.

Teachers can ensure that they help all children with problem solving, including children with special needs, by managing their time appropriately, managing the classroom routines (using large-group instruction and small-group instruction as appropriate, as well as having students work in pairs and individually) and managing student needs by using compensatory strategies to adjust instruction to the needs of individual students.

THINGS TO DO: FROM WHAT YOU'VE READ

1 Solving problems is considered a key route to learning with understanding. What is the difference between a genuine problem and a mere exercise?

2 What does teaching mathematics through problem solving mean? What signposts for teaching guide this approach?

3 Give an example of a problem for each of the following: (a) children must represent mathematical ideas, (b) children must investigate, (c) children must estimate, (d) children must reason logically and (e) children must use multiple steps or multiple strategies.
4 Give an example of a traditional mathematical story problem and show how you might change the problem to make it open ended.
5 Solve the problems presented in this chapter and then indicate what age/stage you think each is appropriate for and why.
6 Identify problems in this chapter for which calculators or spreadsheets would be useful. Are there problems for which using a calculator or a spreadsheet would take away all of the challenge? (These are important considerations when you are choosing problems for your class and deciding when students may and may not use calculators or spreadsheets.)
7 Choose a content topic for a particular year level (e.g. number, geometry, data). Make up at least 1 interesting problem for that topic that can be solved by each of these strategies: looking for a pattern, making a drawing or diagram and constructing a table.
8 Why isn't finding the answer the final step in solving a problem? Explain what good problem solvers do after they have obtained an answer.

THINGS TO DO: GOING BEYOND THIS TEXT

IN THE FIELD

1 *Problem-solving strategies.* We want students to be fluent in the use of many problem-solving strategies and be able to identify which strategy would be most useful to a given situation. We also know that students will tend to stick with the method they feel most comfortable with. Select a problem from the chapter and consider the various strategies children might use to solve this problem. Give the problem to a small group and compare the ways in which they approach the task. Was there a strategy you had thought of that the students didn't use? How could you build on the strategies students confidently use to teach new problem solving methods? Plan a problem you could use to teach this new strategy.
2 *Meta language*. Observe a primary school classroom while children are engaged in a problem-solving lesson or investigation. Listen to what the students are saying. Can they confidently describe their thinking to others or are they stumbling over the words? How is the teacher supporting the development of mathematical meta-language? Think about how you will encourage the use of mathematical language in your own classroom.

IN YOUR JOURNAL

3 Solve the following problem (for 10 people) in at least 2 different ways and write up your solutions. Then build on your work to write a solution for the second and the third versions of the problem (for 15 people and for p people).

> There were 10 people at a party. Each person shook hands with each other person exactly once. How many handshakes took place? What if there were 15 people at the party? Write a formula for the number of handshakes if there were any number of people (p) at the party?

A more challenging extension to this problem is as follows.

> A woman and her husband hosted a party for 4 other couples. The hostess asked everyone to shake hands with and introduce himself or herself to everyone else. Of course, no one shook hands with their spouse. At some point, the hostess stopped them and asked each person how many hands they had shaken. Each person gave a different response. What was the response of her husband?

WITH ADDITIONAL RESOURCES

4 Start a file with problems from this chapter and then add other problems, including nonroutine and open-ended problems. Many mathematics education websites offer long lists of interesting problems.

Solve your problems and identify the strategies that you found useful. Categorise the problems in the way you find most useful.

5 Plan an interactive bulletin board focused on problem solving.

6 Pick a textbook and make a list of the problem-solving strategies it teaches. Compare that list to the list of strategies in this text. Note other ways that problem-related instruction is evident in this textbook. Write an evaluation of the book based on your strategy comparison, your additional notes and the Problem Solving proficiency strand statement in the Australian Curriculum.

7 Read the classic children's book *Counting on Frank* (Clement 1990). Write a lesson plan in which this book could be read aloud to Years 4–6 students as a motivator for a problem-solving lesson.

8 Read O'Brien and Moss (1999). Choose your own context and problems similar to those used by these authors with Year 5 students (problems with consistent data and with inconsistent data). If possible, try these problems with students of a similar age and analyse how they approach them.

9 Read Dwyer (2003), where a teacher reports about teaching problem solving to her class of bilingual Year 3 students. Solve the problem the children solved and write about the features of this problem that allow for multiple entry points and a variety of approaches for that diverse group of students.

10 Read Westegaard (2008), where a teacher suggests many ways that quilt blocks may be used as a context for mathematical problem solving. Identify a mathematics concept that can be approached through problems involving quilt blocks and develop a problem-based mathematics lesson to engage children in investigating that concept.

11 Obtain catalogues from software companies and select software that you think would help children develop their problem-solving skills.

12 Make up problems using newspaper or magazine articles or information from reference books (e.g. *Guinness Book of World Records*) or websites. Add the problems to your file.

WITH TECHNOLOGY

13 *Number Pyramid* (Alex Varju 2011, available at the Apple app store) builds on number skills for students in K–4. Consider the strategies needed to work through the game. What other apps/programs also focus on number sense? Can you find others for older grade levels? Keep a record of these for your work with students.

14 *1010!* and *MakeThemFit!* Available at the Apple app store) build spatial and geometric sense through problem solving. How might games like this be used in the classroom? Can you find some apps/games, which develop measurement sense for students? Keep a record of these for your future teaching.

15 *Calculator*. The use of calculators to assist in problem solving can be beneficial for students. Think about the problem solving strategies outlined in this chapter and which of these would benefit from the use of a calculator. Choose some problems from this chapter for which you would recommend that students be encouraged to use a calculator. Explain your reason for choosing these particular problems and how you would use a calculator for each.

16 Begin a collection of ideas for problem solving tasks to use with students. Consider how computer software, educational apps and calculators might be integrated into these tasks. Find problems for different stages or year levels.

17 Visit the website www.mathsrepublic.com.au. This site provides teachers with problem solving questions to develop learning paths for their students. Students can work at their own pace and the program offers hints throughout the solving process. The tasks are linked to the Australian Curriculum and cover all 3 content strands. Consider how you would use this website in your own lesson planning.

18 Visit the website www.splash.abc.net.au. This site allows teachers to search for resources, which align to the content being covered in the classroom. Access is provided free and there is a mixture of interactive activities for students and support notes for teachers. Consider how you would utilise this website in your lesson planning.

19 Visit the website www.scootle.edu.au. This site offers teachers access to online resources and activities covering all strands of the curriculum. Teachers can select and create learning paths for students which cover content from the classroom. Explore the digital resources that are available on the site.

CHILDREN'S LITERATURE CONNECTIONS

Allen, P. (1998). *Mr Archimedes' Bath*. Sydney: HarperCollins.
The stories introduce young students to the concepts of Archimedes' principle. Young students can model and roleplay the stories, whilst older students can explore the more formal mathematical reasons.

Base, G. (1998). *The Eleventh Hour: A Curious Mystery*. Ringwood, Vic.: Viking Kestrel.
An elephant's birthday party comes to an abrupt end when it is discovered that all the food has been eaten. Clues to the culprit are to be found in detailed drawings on every page. Investigation of the drawings and the story itself has the potential to develop students' logical thinking, pattern recognition, problem solving and reasoning.

Hutchins, P. (1987). *Changes Changes*. New York: Aladdin Picture Books.
A story told in pictures of a toy man and woman and their adventures. Using blocks, they build the story scenes. Students can follow the story by copying the construction and supplying their own words for the pictures.

Hutchins, P. (1989). *The Doorbell Rang.* New York: Greenwillow Books.
A plate of freshly baked biscuits is to be shared between an ever-increasing number of visitors. The story introduces younger students to the concepts of division and fractions.

Hutchins, P. (1994). *Clocks and More Clocks*. New York: Atheneum Books for Young Readers.
The various clocks in Mr Higgins' house each record a different time. His problem is to determine which time is correct — until the clockmaker tells him that they are actually all correct.

McKenzie, H. (2006). *Australian Twelve Days of Christmas*. Newtown, NSW: Black Dog Books.
An Australian version of the well-known Christmas song. The song lends itself to the exploration of a variety of mathematical ideas, including pattern identification in growing sequences.

Murphy, S. J. (1998). *The Penny Pot*. New York: HarperCollins.
Jesse decides she wants her face painted at the school fair but realises that she doesn't have enough money. At the face-painting booth, there is a penny pot. People place their extra pennies in the pot after having their face painted. Soon there are enough pennies in the pot for Jessie to have her face painted. Different combinations of money are used to let the reader see how much money Jessie needs to have her face painted. There are also activities and games in the back of the book for teachers and parents.

Zaccaro, E. (2000). *Challenge Math for the Elementary and Middle School Student*. Bellevue, IA: Hickory Grove Press.
Challenge children in Years 4–8 using some of more than 1000 problems (presented with 3 levels of difficulty) in areas such as algebra, astronomy, trigonometry and probability.

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CHAPTER 7

Counting and number sense in early childhood and primary years

LEARNING OUTCOMES

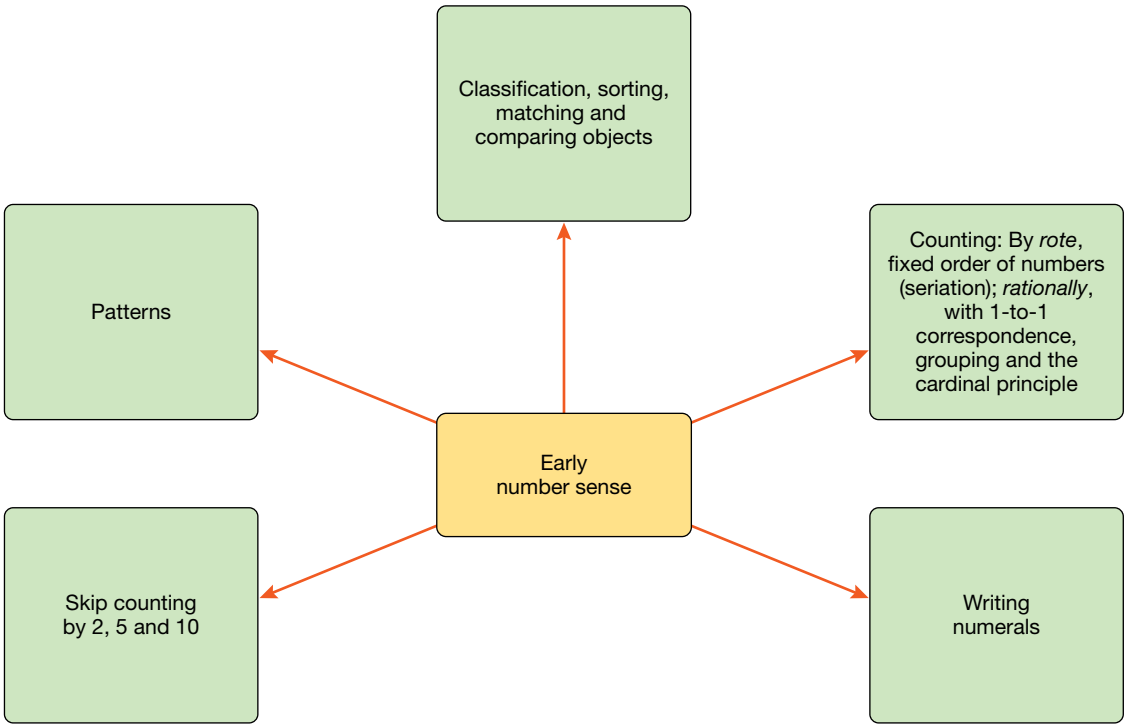
7.1 Understanding that number sense takes time to develop and stages of learning include prenumber, early number and counting.
7.2 Articulating the 4 important principles of counting.
7.3 Describing key strategies for more advanced counting.
7.4 Explaining the difference between cardinal, ordinal and nominal numbers.
7.5 Making connections between reading, writing and speaking number words and symbols.



'The study of mathematics, like the Nile, begins in minuteness but ends in magnificence.'

Charles Caleb Colton (English writer, 1780–1832)

Chapter 7 concept map



Introduction

This chapter begins the second part of *Helping Children Learn Mathematics* in that it discusses teaching strategies, techniques and learning activities related to specific mathematical topics — in this case, early number concepts. Number sense, like common sense, is difficult to define, but here are some characteristics.

Number sense includes:

- an understanding of number concepts and operations on these numbers
- the development of useful strategies for handling numbers and operations
- the facility to compute accurately and efficiently, either mentally, with pen and paper, or with technology as needed
- the flexibility to use estimation before during or after computation
- being able to detect errors and to recognise results as reasonable
- the ability and inclination to use existing knowledge to make mathematical judgements
- an expectation that numbers are useful and that work with numbers is meaningful and makes sense.

People with the above characteristics of number sense are able to understand numbers, their relationships to each other and use numbers effectively in everyday living. Primary school mathematics programs play a critical role in developing children's number sense. Students need to develop concepts meaningfully so that they can use numbers effectively both in and out of school. Helping students develop number sense requires appropriate modelling, posing process questions, encouraging thinking about talking numbers and, in general, creating a classroom environment that nurtures number sense.

7.1 Developing number sense

LEARNING OUTCOME 7.1 Understanding that number sense takes time to develop and stages of learning include prenumber, early number and counting.

Number sense is not a finite entity that a student either has or does not have. The development of number sense is a lifelong process.

In early childhood settings and primary school number sense development involves several stages:

1. prenumber and informal number
   - sorting and classification
   - patterning
2. early number development
   - conservation
   - comparison
   - one-to-one correspondence
3. number development and counting
   - connecting objects and events with number names, including oral & written, cardinal & ordinal numbers
   - group recognition
   - counting forward and backward
   - skip counting
   - establishing benchmarks of quantities, such as 5 or 10
   - place value.

These stages form the basis of whole-number development and provide the foundations for developing basic facts as well as mental and written computation involving addition, subtraction, multiplication and division of whole numbers. Computations that arise from meaningful problem-solving situations will promote children's development of number sense and encourage them to reflect on the reasonableness of their results to make sure their answers make sense. Facility with whole numbers provides the foundation for work with fractions, decimals, estimation, measurement, and data. All of these topics reflect the continuing development of number sense.

The National Council of Teachers of Mathematics (NCTM 2006) in their Curriculum Focal Points make similar connections between early number concepts, skills and process as does the Australian Curriculum: Mathematics (ACARA v8). Included in table 7.1 are the early number expectations for mathematical content and processes. The main difference between the NCTM (2006) document and the Australian documentation is that the US organisation includes prekindergarten in their focus (Appendix B). For children aged 4 and 5 therefore, the NCTM expectations are best compared alongside the Early Years Learning Framework (EYLF) for Australia (Australian Government Department of Education, Employment and Workplace Relations 2009). In particular, 'Outcome 5: children are effective communicators' (p. 43) explains that children begin to use symbols in their play, and that symbols help them communicate their ideas, thoughts and concepts. Furthermore, children 'begin to sort, categorise, order and compare collections and events and attributes of objects in their social and natural worlds' (DEEWR 2009, p. 43).

In figure 7.1, several examples of number sense in action are illustrated. What does the number 5 mean to young children? It can mean many different things. It might be their current age or their age next year. It might be how old they were when they started kindergarten. Other children might provide different examples, and the same children might give different responses on a different day, because their concepts of numbers continuously grow and change.

TABLE 7.1 **Early number proficiencies and content descriptions for Foundation and Year 1 in Australia**

Proficiencies	Content description
Foundation Year **Understanding** includes connecting names, numerals and quantities. **Fluency** includes readily counting numbers in sequences and continuing patterns. **Problem-solving** includes using materials to model authentic problems, sorting objects, using familiar counting sequences to solve unfamiliar problems and discussing the reasonableness of the answer. **Reasoning** includes explaining comparisons of quantities and creating patterns.	• Establish understanding of the language and processes of counting by naming numbers in sequences, initially to and from 20, moving from any starting point • Connect number names, numerals and quantities, including zero, initially up to 10 and then beyond • Subitise small collections of objects • Compare, order and make correspondences between collections, initially to 20, and explain reasoning
Year 1 **Understanding** includes connecting names, numerals and quantities, and partitioning numbers in various ways. **Fluency** includes readily counting number in sequences forwards and backwards, and locating numbers on a line. **Problem-solving** includes using materials to model authentic problems, using familiar counting sequences to solve unfamiliar problems and discussing the reasonableness of the answer.	• Develop confidence with number sequences to and from 100 by ones from any starting point. Skip count by 2s, 5s and 10s starting from 0 • Recognise, model, read, write and order numbers to at least 100. Locate these numbers on a number line

Source: © Australian Curriculum, Assessment and Reporting Authority 2015.

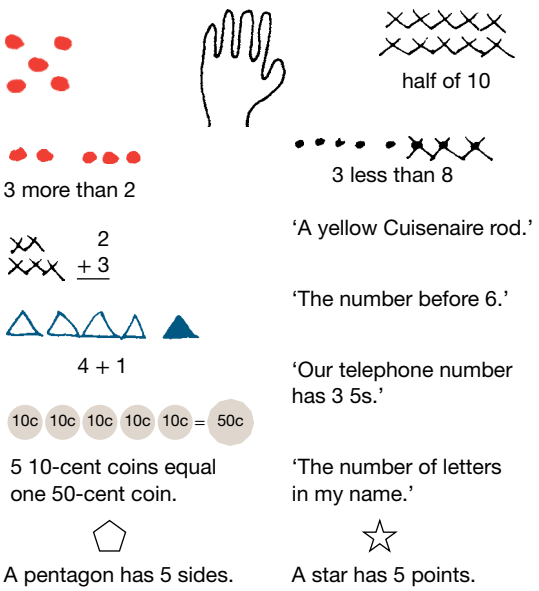




FIGURE 7.1 The meanings of the number 5 suggested by young children. The flower has 5 petals.

Children begin to develop an early sense about numbers long before they begin to count. For example, young children can answer questions about number related to their everyday experiences: How old are you? [*2*]; What channel has your favourite program on television? [*7*]; On what level of the building is your doctor's office? [*4*]; or How many sisters do you have? [*1*]. Additionally, many early experiences

introduce the number names as well as their symbols — 7 on the channel indicator or 4 on the elevator. These names and symbols are memorised through sound and sight recognition and provide an important beginning, but a child's knowledge of these concepts alone does not indicate the child's grasp of number.

For one thing, these experiences underscore a very important characteristic of number. It is an abstraction. It can't be adequately illustrated in just one situation. The multiple meanings of 5 illustrated in figure 7.1 demonstrate how quickly the concept of *5* becomes associated with different situations. Research into how children develop number sense makes it clear that the more varied and different their experiences, the more likely it is that they will develop and abstract number concepts from those experiences (Back 2014; Fuson 2003; Kilpatrick, Swafford & Findell 2001; Payne & Huinker 1993). Teaching strategies that further develop children's early understanding of number and promote number sense are a high priority. The goal of this chapter is to stimulate your thinking about number sense and its development during the early years.

Prenumber concepts

Numbers are everywhere, and young children experience a vast amount of early number situations, as shown in figure 7.1. Many of these experiences do not rely on numbers per se but provide the basis for building early number concepts and the foundation for later skills. Such experiences are called *pre-number experiences*. Early childhood practitioners need to help children take advantage of their intuitive mathematical learning. Different steps are involved in developing prenumber concepts that lead eventually to meaningful counting skills and number sense. Prenumber work is not all done before children do anything with numbers in school; rather, it typically occurs simultaneously with activities involving number. Although the learning paths that children take are bound to differ greatly, they all begin with classifying whatever is to be counted.

Classification

Classification is fundamental to learning about the real world, and it can be done with or without numbers. For example, children can be separated into 2 groups according to colour of clothing such as wearing blue or not wearing blue (which is classification) without considering number. Classification skills are prerequisite to any meaningful number work. Children need to sort and classify first. Before children can count, they must know what to count, and classification helps identify what is to be counted.

Classification is naturally integrated in other subjects, such as reading, science and society and environment (Gallenstein 2004). For example, recognising consonant and vowel letters from the alphabet requires classification, as does separating flowers from weeds or distinguishing between a leaf and a stem in plants. Asking which states of Australia meet the Pacific Ocean is also classification. Young children learn to distinguish between dogs and cats, reptiles and mammals, and between toys they enjoy and those they never use. These are all examples of classification in action. Classification not only helps children make more sense of things around them but this skill helps them become flexible thinkers. Classifying objects in different ways fosters the development of higher order thinking skills.

As children sort and classify materials, they must decide whether each object has the given characteristic. If children disagree on how an object should be classified, it forces them to defend their answers and clarify how the classification process was done. At this point, there may be no formal counting as materials are sorted but words such as *more, less, few, many, most, least, the same amount* and *none* will likely be used in describing the resulting collections.

Classification allows people to reach general agreement on what is to be counted. For example, consider a pile of buttons and the following question: How many plastic buttons have 2 holes? The answer is a number that tells *how many*. When a number is used in this way, it is called a *cardinal number*. Before finding the specific cardinal number, however, you must first decide which buttons are plastic and how many of them have exactly two holes. Once this classification is done, the members to be counted are well defined and ready to be counted.

Stories provide opportunities for classification. Books such as *An Australian 1, 2, 3 of Animals* (Bancroft 2007) provide practice in visual discrimination as well as in classification.

Attribute blocks, sometimes called *logic blocks*, provide an excellent model for classification activities and help develop logical thinking. These blocks can be made from cardboard (see Appendix C for an attribute block master), but are usually available in wood or plastic. The blocks differ in several attributes, including colour, shape and size. Consider, for example, the 24 pieces shown in Appendix C and In the classroom 7.1. These pieces illustrate 3 attributes:

Size: large, small (L, S)
Colour: blue, red, green (B, R, G)
Shape: square, triangle, pentagon, circle (S, T, P, C).

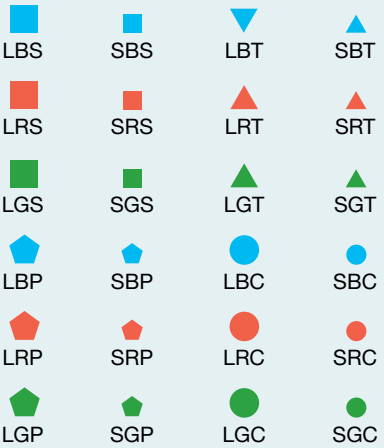
Commercially purchased attribute blocks (often found in school resource centres) often have 48 or 60 blocks and include a fourth attribute, of thickness, for use in a variety of classification activities suitable for Foundation Years students through to Year 7 and beyond.

IN THE CLASSROOM 7.1

Who am I?

Objective: Use concrete materials, such as attribute blocks, to develop classification and reasoning.

Year level: 2–4

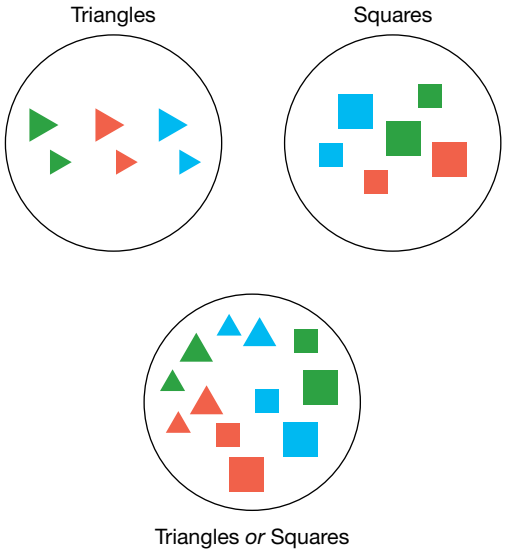


- Match me with the attribute blocks shown here:
  - A. I have 3 sides.
    I am blue.
    I am large.
    Who am I?_____
  - B. I am blue or red.
    I have 3 sides.
    I am not large.
    Who am I?_____
  - C. I am not blue.
    I am not red.
    I have 5 sides.
    I am small.
    Who am I?_____
  - D. I am not large.
    I have more than 4 sides.
    I am green.
    Who am I?_____
- Which clues describe more than 1 piece?_____
- Which clues describe only 1 piece?_____
- Your turn:
  Play 'Who am I?' with a partner.

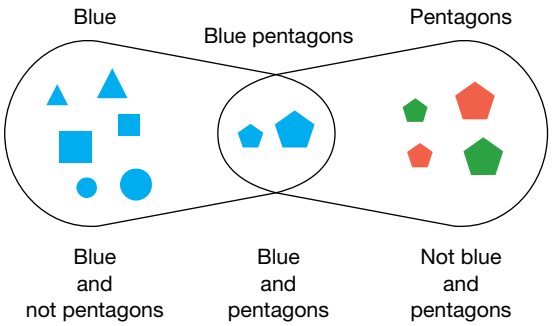
Initially, each block is described in words, for example the first block is a 'large blue square'. Symbols, such as LBS, are not used for the attribute pieces in the early stages, rather a clear verbal description

of the pieces by young children is the goal. As children manipulate the blocks and describe them, they begin to make natural connections between the concrete model and different ways of representation. Many of the 24 pieces are alike in some attributes, but no 2 pieces are alike in all attributes, which provides opportunities for 'Who am I?' games, as presented in In the classroom 7.1. A 'mystery bag/box' activity, which is more interactive, would have 1 child hide a block in a bag or box and the other children (in a small group) ask questions about the attributes of the block, for example, 'Is it blue?' until a good guess can be made. To support children with the verbal demands of the task, a similar shape can be collected from the pile of shapes and the children ask, 'Does it have something the same as this shape?'. The question is repeated a number of times until a logical guess is made. Such activities encourage children to think logically and develop communication skills. In the process, children informally explore fundamental notions, including sorting, matching, comparison, shape and sets theory.

Communication and language can be further developed, when children explore groups or sets of objects. These activities are best done with the concrete materials and using labels and circles. The formal mathematical ideas are the set operations of union and intersection but informally students are working with the language of 'and' / 'or' and a variety of characteristics (attributes). By combining *disjoint sets* (sets where the chosen attribute is not common) a natural model for addition is found. The logical connection *or* can be used to develop the union of 2 or more sets. For example, focusing on the characteristic of *shape,* the union of triangles and squares produces a set that contains all attribute blocks that are either triangles *or* squares, as shown.



The intersection of sets, where there are 2 or more common characteristics, can help children explore the logical connection *and*. For example, by using the attribute blocks the child can determine the shapes that are pentagons *and* blue. Children might place these in loops, as shown.



When playing games such as 'What is my rule?' for 2 characteristics (in this case shape and colour), children identify the main overlap, which is 'it is pentagon and it is blue'. Other subsets, such as pieces that are blue and not pentagons, or pieces that pentagons but not blue also exist. Using *not* to describe a relationship is an important step in development. Repeating the game several times with other chosen characteristics, for example size and colour, maybe large and red, would involve more and varied pieces and further strengthen the experience of classification for children.

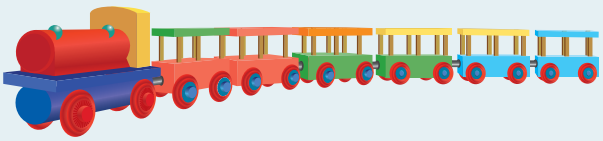
The logical connectives *and*, *or* and *not* can be used to help children classify pieces according to their attributes. For example, the 'alike-and-difference trains' shown in In the classroom 7.2 provide opportunities for students to use attribute blocks to classify, search for patterns and use logical thinking. Children at all year levels can benefit from structured activities with these materials.

IN THE CLASSROOM 7.2

Alike-and-difference trains

Objective: Use attribute pieces to recognise patterns and identify relationships.

Year level: 2–4

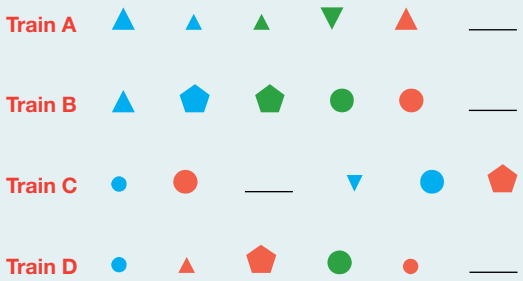


Each carriage in a train is like the carriage it follows in 1 or 2 ways, or it is different from the carriage it follows in 1 or 2 ways.

The picture shows all carriages with the same wheels. The first carriage has a green top and red body but the second has a red top and a red body. The top has changed but the body is the same. The third carriage top stays the same but the body changes.

Now consider the attribute 'trains' shown.

- Find the alike-and-difference pattern in each train, and describe the missing carriage:



Which of these carriages are 1-difference trains?——

Which of these carriages are 2-difference trains?——

- For the student, have a turn at the following open-ended tasks:

Begin with ■. Make a train (with at least 6 carriages) in which each carriage has exactly one attribute different from the carriage it follows. Compare your train with someone's train. Are they the same?

Begin with ■. Make a train (with at least 6 carriages) in which each carriage has exactly 2 attributes the same as the carriage it follows. Compare your train with a peer's train. Discuss how they are alike or different.

A final challenge could include putting approximately 20 pieces in a 'loop', that is a train that joins up at the end and beginning.

Attribute blocks virtually guarantee student involvement, but they also require teachers to assume an active role. When children are engaged in activities with attribute blocks, observation and questioning can provide information about their thinking processes. Observing children's actions reveals much about their maturity. For example, when asked to choose a piece that is blue and a triangle, 1 child might choose a blue piece but not a triangle. Another might select a triangle that is not blue. Whilst these responses might reflect poor listening skills, additional questioning may show that the children didn't understand what the word *and* means or are unable to keep 2 different attributes in mind simultaneously. Careful observation and questioning of children as they are using materials will help the teacher better understand their thinking, which will help the design of appropriate learning activities.

Many different experiences are needed to sharpen children's observation skills and provide them with the basis on which to build the notion of numbers. Consider another example, in which 2 correct answers are possible. The child is asked to count money.

Q: How much money is this?
A1: 3, if coins are counted.
A2: 4, if dollars are counted.



This example provides a reminder that a number name alone, such as 3 or 7, is rarely reported. In this case, '3 coins' describes both *cardinality* (i.e. how many) and what was actually counted. This example also provides another reminder that what is to be counted must be well defined or clearly understood. If there is any confusion about what is being counted, then counting discrepancies are certain to happen.

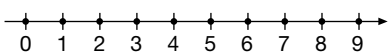
Such discrepancies occur in many different forms but are particularly troublesome with a number line. The diagram below represents 2 children standing on a number line made from a roll of tape and fastened to the floor.



Q: How far is Scott away from Barb?
Answer: Is it 4? or 3? or 5?
The solution depends on what is to be counted:
Should the intervals between the dots be counted? Should the dots be counted? Which dots exactly?

The distance between Barb and Scott can be determined by length. In this case the number of steps or the intervals are counted.

Research confirms that confusion between dots and intervals often contributes to later misunderstanding with number lines (Kloosterman et al. 2004). Confusion over what should be counted can be both an abstraction and classification problem that needs resolution to make counting meaningful. When children are ready to count with a number line, stepping the distance out and landing at the number can be emphasised, rather than the gap between numbers. This misconception needs to be resolved well before children begin working with rulers to measure distance. Thus, classification is an important step in developing number sense and early counting skills.



Patterns

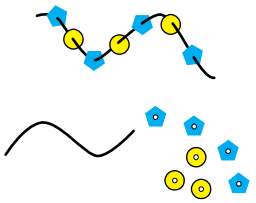
Mathematics is the study of patterns. Creating, constructing and describing patterns require problem-solving skills and constitute an important part of mathematics learning. Patterns can be based on geometric attributes (shape, symmetry), relational attributes (sequence, function), physical attributes (colour, size, texture, number) or affective attributes (like, happiness). Sometimes patterns combine several attributes. For example, a child's list of favourite colours provides a pattern involving physical attributes (colour) and affective attributes (like).

Natural and manufactured materials such as beads, buttons, pegs, leaves, shells, shapes, cubes, attribute blocks, pattern blocks and other concrete materials provide opportunities for children to stack, arrange and order objects in various ways. Number sense and mathematical exploration grow from such patterning. In the early years, patterns help children develop number sense, ordering, counting and sequencing (Coburn et al. 1992). Later, patterns are helpful in developing thinking strategies for basic facts, discussed in chapter 9, and in developing algebraic thinking. In addition, as children grow older, their experiences with patterns accelerate as they explore graphical representation, number theory and geometry. Patterns, as with puzzles, are usually intellectually inviting and stimulating for people of all ages.

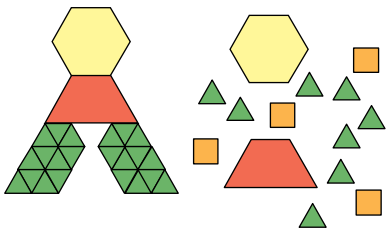
Exploring patterns requires active mental involvement and frequently, physical involvement. The opportunities to do patterning are limitless. Here are 4 different ways that patterns might be used in developing mathematical ideas.

Copying a pattern

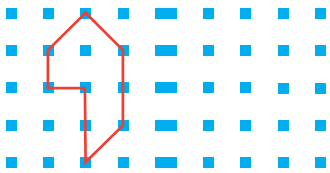
Children are shown a pattern and then asked to make one 'just like it'. The original pattern might take many different forms. For example, children might be given a string with beads and asked to make the same pattern:



Alternatively, pattern blocks (see Appendix C for a pattern block master) could be laid out for children to copy.

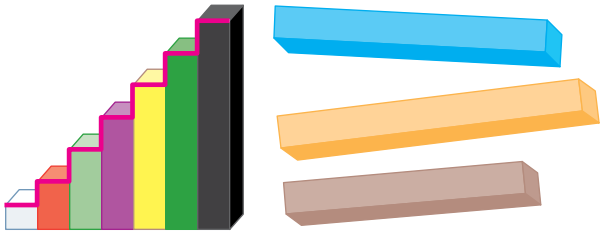


This experience requires students to choose the same pieces and arrange them in the same order. One could also model a figure on a geoboard (see Appendix C for a geoboard master) and ask children to copy the figure on an empty geoboard.



Finding the next one

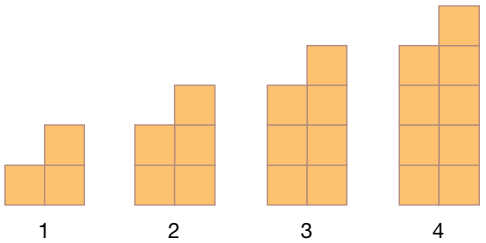
The trains in In the classroom 7.2 illustrate problems where children 'find the next one'. In that case, 'the next one' is the next carriage in the train.



Considering a somewhat easier pattern created by 'stairs' of Cuisenaire rods, children might be asked to find the next rod for the staircase. This find-the-next activity naturally leads to continuing or extending the pattern.

Extending a pattern

Children are shown a pattern and asked to continue it. For example, an initial pattern can be made with blocks or sticky notes, and children can be asked to continue the pattern:



Notice how this visual pattern might serve as the foundation for exploring several important mathematical ideas. Each picture represents an odd number. In this case 3, 5, 7, 9 blocks are used progressively. The pattern could be used to observe something common about the representations — that they are all a rectangle plus one. This latter observation might lead to the algebraic generalisation $2N + 1$ to describe odd numbers.

Making their own patterns

Children need opportunities to create their own patterns and are eager to do so. Sometimes the patterns they make are highly creative and reveal insight into their mathematical thinking.

Language and communication are important elements of patterning activities. You should encourage children to 'think out loud' as they search for patterns. Ask them to explain why they selected a certain piece or why they did what they did. Sometimes children 'see' different patterns than you anticipate. As teachers, you need to learn and understand children's patterns and encourage them to share their thinking.

Early number development

Making comparisons and recognising more and less of quantities may be done without counting, however these experiences may also provide opportunities for counting. Visual comparisons help children develop the recognition of small quantities, yet research by Piaget and others shows that for young

children this number recognition may be unstable (Kilpatrick, Swafford & Findell 2001). Several important stages of number development are required for stability.

Conservation

The phenomenon of *conservation of number* — that a given number does not vary — reflects how children think. The teacher needs to be aware of signs that illustrate when conservation of number is not present in children and the implications for early number development and counting. This idea occurs in different forms, but a typical version of Piaget's task involving counting and numbers.

There are 2 rows of blocks or counters arranged side by side, and a teacher and a 5-year-old child look at them together. The teacher asks the child to make a comparison to decide whether there are more orange blocks, more purple blocks or the same number.



Teacher: How many purple blocks?

Student: [*counting them*] 9.

Teacher: How many orange blocks?

Student: [*counting again*] 9.

Teacher: Are there more purple blocks, more orange blocks or the same number?

Student: They are the same.

Once the teacher has established that the child recognises the same number of blocks (or counters) in each row, they displace one row — altering the length or arrangement but keeping the number the same. The original question is repeated.



Teacher: How many purple blocks?

Student: 9.

Teacher: How many orange blocks?

Student: 9.

Teacher: Are there more purple blocks, more orange blocks or the same number?

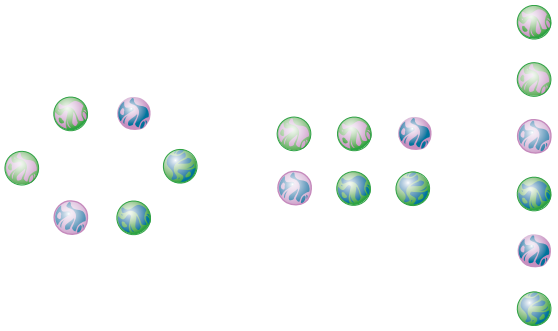
Student: More orange blocks.

Teacher: I thought you said there were 9 purple and 9 orange.

Student: I did, but this 9 [*pointing to row of orange blocks*] is bigger.

This example illustrates how a young child typically 5 years of age or younger thinks a quantity varies and depends on arrangement or configuration. Here the child believes that changing the length of the row of blocks changes the number of blocks. They might also believe that because the adult has made a change then the quantity will be effected. At this stage the child is still developing the idea that the number of objects in a group remains the same, is 'conserved', no matter how they are arranged.

There are 3 variations shown to the arrangement of marbles. Some children count 6 marbles in each of these groups but report that 1 of the groups has more. For adults it seems inconceivable that 'this 6' could be more than 'that 6', but children's logic and adults' logic can be very different.



Conservation of number as described by Jean Piaget has been the subject of much research and some controversy. The design of the conservation task and language used can incorrectly influence the child's response. Firstly, when the adult deliberately rearranges the blocks or counters, the child is likely to think something important has changed as compared to a task where the change occurs by accident (Hughes 1986). Secondly, when the adult asks a child the same question twice (Are there more . . .?), the child might assume that their first response was incorrect and change it anyway (Siegal 1997). The main issue is that pre-school children are capable of much more than what was once thought, that they bring to school a depth and variety of intuitive mathematical learning (Clarke, Clarke & Cheeseman 2006; Perry & Dockett 2005; Rogers 2000). Hence, early number activities should make sense to children and help to increase their awareness of the invariance of number.

Group recognition

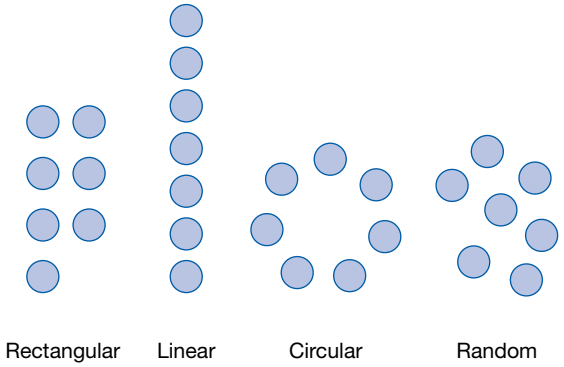
The patterns encountered in classifying and making comparisons provide many number-sense experiences. In fact, before actually counting, children are aware of small numbers of things: 1 nose, 2 hands, 3 wheels on a tricycle. Research shows that all children entering school can identify quantities of 3 things or less by inspection alone without the use of counting techniques and most are able to see groups up to 4 objects given dice, ten-frames or cards (Warren, deVries & Cole 2009).

The skill to 'instantly see how many' in a group is called *subitising*, from a Latin word meaning 'suddenly'. It is an important skill to develop. In fact, 1 instructional goal for Year 1 students is to develop immediate recognition of small groups. Sight recognition of quantities up to 5 or 6 is important for several reasons.

1. *It saves time*. Recognising the number in a small group is much faster than counting each individual member of that group.
2. *It is the forerunner of some powerful number ideas*. Children who can name small groups give evidence of knowing early order relations, such as 3 is more than 2, and 3 is 1 less than 4. Some children might also realise that 3 objects contains a group of 2 and a group of 1.
3. *It helps develop more sophisticated counting skills*. Children who recognise the number in a small group will more quickly begin counting from that point.
4. *It accelerates the development of addition and subtraction*. Early work with these operations involves manipulation with objects. Being able to recognise the quantity in a small group frees children of the burden of counting small quantities to be joined or removed and allows them to concentrate on the action of the operation.

Sight recognition is evidenced by children's skills in reading the number of dots on the face of a die or on a domino. In fact, both of these materials provide natural as well as interesting models for developing and practising this skill. As children grow older, their ability to recognise quantities continues to improve, but it is still limited.

Certain arrangements are more easily recognised or subitised. For example, there are arrangements in an array, a line, a circle and without any pattern shown.



Children usually find rectangular arrangements easiest, followed by linear and then circular, whereas random arrangements are usually the most difficult. If the arrangement does not lend itself to some grouping, people of any age will have difficulty with larger quantities. Few adults can recognise by inspection groups of more than 6 or 8, and even these groups must be in common patterns such as those found on playing cards or dominoes.

For example, look at these pictures of birds:



How many birds do you see?

Each picture shows 12 birds, but you probably used different processes to count them. The picture on the left provides no clear groups, so you could either count every bird or perhaps identify the numbers of birds in 1 part and then count the rest. In the other 2 pictures, some natural groupings are suggested: 4 groups of 3 and 2 groups of 6. It is even more difficult to recognise large groups without counting or looking for smaller groups in 1 part. Nevertheless, small-group recognition or subitising is a powerful ally in counting larger groups. Research supports the notion that subitising is a prerequisite for counting (Clements & Sarama 2007).

Comparisons and 1-to-1 correspondence

Comparison of quantities is another important part of learning to count and is also essential in developing number awareness. Comparisons are plentiful in classrooms as children use materials. Teacher-led activities frequently provide opportunities for comparisons, with questions such as: 'Does everyone have a piece of paper?' or 'Are there more pencils or desks?'.

These questions either directly or indirectly involve comparisons that may lead to the important and powerful mathematical notion of one-to-one correspondence. Look at figure 7.2(a) and consider this question: Are there more hearts or gingerbread biscuits? Counting provides a solution, particularly with the biscuits scattered on a plate; however, if the biscuits are arranged in an orderly fashion — figure 7.2(b) — you can make direct comparisons and answer the question without counting. Sometimes placing connectors (using string, drawing lines or arrows) provides a visual reminder of the one-to-one correspondence that underlies many comparisons, as in figure 7.2(c). In this case, the one-to-one correspondence confirms that there is one more heart than the number of biscuits. The process of recording tallies when organising data (see the final chapter) rests on the notion of 1-to-1 correspondence.

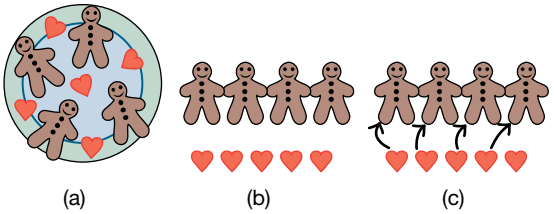


FIGURE 7.2 Models for making comparisons by (a) counting, (b) physically comparing without counting and (c) 1-to-1 correspondence

When making comparisons, students must be able to discriminate between important and irrelevant attributes. In figure 7.3(a), it is hard to determine who has more leaves — Madhu or Bevin. The leaves are very different; their sizes, shapes and colours vary. Still, the procedure for setting up a correspondence is the same.

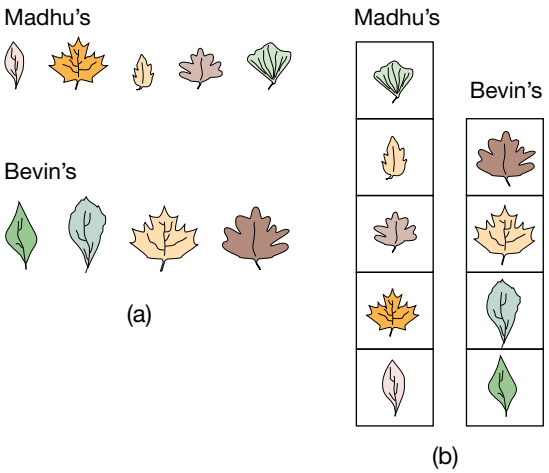


FIGURE 7.3 A framework for comparison that facilitates a 1-to-1 correspondence

To ensure that members of two sets are arranged in an orderly fashion for comparison, a method that is sometimes helpful involves using pieces of square paper of the same size. In this case, placing each leaf on a card and then stacking the cards on a common base, as in figure 7.3(b), provides a helpful framework. This method provides a graphical representation of the information, using one to one correspondence allowing quick and accurate visual comparisons. Several different, equally valid, verbal descriptions may be used for the example given in figure 7.3. 'Madhu has more leaves than Bevin' and 'Bevin has fewer leaves than Madhu' are both correct. Children need to become familiar with descriptions of relationships such as *more than, less (fewer) than,* and *as many as*. Over time, more specific comparison will include number. For the example above, a child might say 'Madhu has 1 more leaf than Bevin' or 'Bevin has 1 less leaf than Madhu', giving an exact difference in quantity.

In these cases, the notion of order and sequence are being developed. Children need to realise that 4 is the number between 5 and 3, as well as 1 more than 3, and 1 less than 5. Understanding of order relationships will be likely to evolve naturally as comparisons are made and discussed. These comparisons can be made with more than 2 groups. For example, children can write their first names on large grid paper and cut out the strips.



Then they can physically compare their names with the names of other students. The following questions are suitable. Who has the longest name? Who has the shortest name? Can you find someone with a name the same length as yours? Can you find someone whose name has 1 more/less letter than your name?

Comparing and ordering can be more effective when the information is organised and represented in a simple graph.. The graph in figure 7.4 was constructed by classifying children's names according to length. It summarises the information and presents it in an organised form. Children can answer the previous questions and create additional questions of their own.

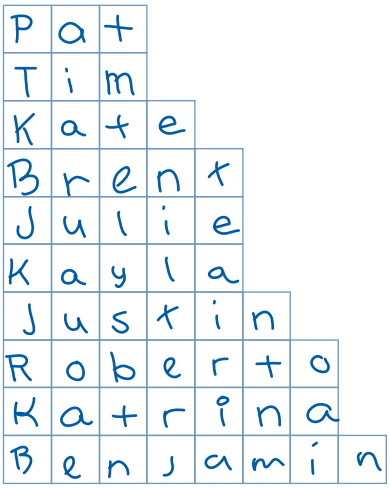


FIGURE 7.4 Classification of children's names on a graph for comparison

Please note that young children will need support ordering things once their data goes beyond just 2 categories. The ordering process for more items can become quite complex, and is best left for children in Year 2 or 3. That is why organisational techniques, such as graphing (see the final chapter), are particularly helpful and will contribute to the early development of numbers.

Counting

Patterns facilitate the counting process; however, the sound pattern for the first 9 numbers is not repeated correctly until after 20. The tens or teen decade introduces 10, 11 and 12 — which are unique. The following 7 numbers that follow are familiar but in a different pattern to the later decades. That is, if 13 was to be 'three teen' or tenty-three, 14 to be tenty-four and 15 to be 'five teen' or tenty-five, then the pattern would be more obvious and logical for young counters. Most children learn these number names by listening to counting programs or books and by imitating adults and older children. As young children practise counting, they often say nonconventional sequences of number names. It is not unusual to hear a young child count '1, 2, 5, 8, 15, 20, 6, 100'. This counting may sound strange, but it is perfectly natural. It reflects the child's struggle to remember both the number names and their order, both of which are necessary in order to count.

Eventually children may count apples, blocks, cards, stones, twigs, steps or petals on a flower. Once bitten by the counting 'bug' a child will happily count at any opportunity. For example, counting the petals on the flowers shown in figure 7.5 provides a very interesting setting for practising counting and a reminder that numbers are everywhere in nature.





FIGURE 7.5 Models from nature for counting practice

Items such as blocks or petals are *discrete objects* (i.e. materials that lend themselves well to handling and counting). *Continuous quantities*, such as the amount of water in a glass or the mass of a person, are measured with units or tools, rather than counted.

What is counting? It is a surprisingly intricate process by which children call number values by name. A close look at the counting process shows that finding how many objects are present involves 2 distinct actions. A child must say the number-name series, beginning with 1, and point to or think about a different object as each number name is spoken. Children exhibit several different but distinct stages of counting, which now follow.

7.2 Counting principles

LEARNING OUTCOME 7.2 Articulating the 4 important principles of counting.

How do children count? Given a particular counting situation, such as finding out the number of shells. A child who is a 'rational counter' says each number name as the shells are counted, as indicated in figure 7.6. A rational counter also realises that the last number named, '7', reports the total, or the cardinality, of the group of shells being counted.

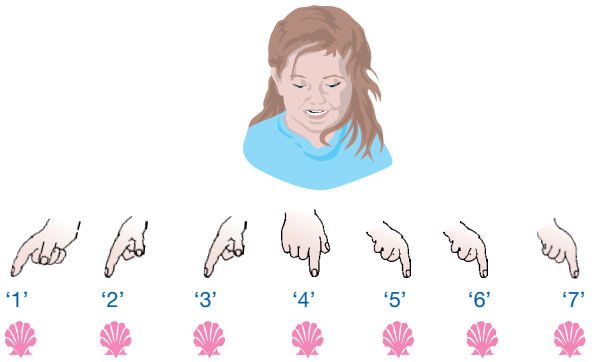


FIGURE 7.6 Rational counting: correct sequence, with correct correspondence

As an adult, it is unlikely that you remember learning to count. By observing young children it is possible to be reminded that counting strategies vary and are developed over a period of time. There are 4 essential principles required for the counting process.

1. *Each object to be counted must be assigned 1 and only 1 number name*. As shown in figure 7.6, a *1-to-1* correspondence between each shell and the number name is established.
2. *The number-name list must be used in a fixed order every time a group of objects is counted*. The child in the figure starts with '1' and counts '2, 3 . . . 7', in a specific order. This is known as the *stable-order rule* or *seriation*.
3. *The order in which the objects are counted doesn't matter*. This is known as the *order-irrelevance rule*. Thus, the child can start with any object and count them in any order. This is particularly pertinent when objects are randomly organised.
4. *The last number name used gives the number of objects*. This principle is a statement of the *cardinality rule*, which connects counting with *how many*. Regardless of which object is counted first or the order in which they are counted, the last object named always tells the cardinal number of the objects being counted.

Children reflecting these counting principles may show confusion or uncertainty when counting, yet research shows that with encouragement and ample opportunity to count, young children will develop efficient counting strategies without direct instruction (Clements & Sarama 2007). Knowing these principles will help the teacher (or parent) recognise the levels of children's counting skills. Careful observation of children, together with a good understanding of these principles, can pinpoint counting errors. Once the trouble is diagnosed, teaching and learning can focus on the specific problem.

SPOTLIGHT ON EARLY CHILDHOOD

Number concepts: Kate counts people in her house

Background

Young children working with blocks in a preschool setting, demonstrated a sound understanding of *numeration* concepts such as sorting, classifying and comparing. They were able to order objects according to size and quantity and they counted well, many to an advanced level. Estimation of quantity appeared in the children's play often. *Patterns*, both regular and irregular, were a feature of the children's constructions and their embellishment of buildings and surroundings.

Principles of counting

In the following scenario, Kate (4 years, 2 months) and Melanie (3 years, 6 months) are both in the block corner. Initially Kate counts to 21 without error, but the presence of Melanie seems to encourage her to rehearse this skill as she repeats the number sequence.

Kate counts the people inside the house and she is quite excited about the task.

Kate: Look how many people we're getting in the house. 1, 2, 3, 4, 5, 6, 7, 8, 9 . . .

A little later, Kate places a large plastic person inside the building.

Kate: Now that's the last one.

Melanie looks inside. Kate gives her a nonverbal 'stop' signal.

Kate: I'll count them!

She points to each person as she goes.

Kate: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21.

Melanie: Wow!

Kate (to Anna): You know what? I counted 21 people in my house.

Kate counts again, pointing to each person.

Kate: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19.

Melanie (interrupting): Did you count that one?

Kate counts again.

Kate: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. . .

When Kate can no longer reach inside the building to point and count the people, she stands and goes around to the other side of the building where she resumes, still pointing at each plastic figure.

Kate: . . . 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21.

Melanie: Did you count that one?

Kate: I did! It was 2!

Kate: There's 21 in this house.

Anna: 21 people, wow!

Melanie: There are 21 people in my family.

Kate: There are 21 people in my house.

Mathematical analysis

During this scenario, Kate demonstrated a sound understanding of the principles of counting (Reys et al. 2009, p. 144). She used *1-to-1 correspondence*, by pointing at each object as she gave the number name, even moving to a more comfortable position to do so. She recited the number names in the same fixed order each time she counted, using the principle of *seriation*. Finally, she used the *cardinal principle* when putting in 'the last 1' and then used the last number name to declare the total of the group. Kate is a rational counter.

Melanie's question about whether Kate counted a particular person and Kate's response illustrates the complexity of rational counting and Kate's understanding of the order in which objects are counted. Kate is sure she counted the person because she pointed to each in a fixed order. However, in her response Kate used the cardinal number (2) instead of the ordinal number (second) in describing the position of that person in her number sequence.

Source: (Rogers 1997).

Counting stages

There are several identifiable counting stages, and each reflects 1 or more of the counting principles. For example, some children may count the objects correctly and still not know how many objects have been counted. In response to the question 'How many shells are on the table?' a child might correctly count '1, 2, 3, 4, 5, 6, 7', as shown in figure 7.6, and yet be unable to answer the question. This child does not realise that the last number name indicated how many.

The shells in figure 7.7 are shown in a linear arrangement because this configuration is easier to count than objects in a rectangular or scattered formulation. Sometimes it is helpful to encourage children to organise their objects so that a beginning and end are clear. This can decrease the likelihood of counting errors based on the third principle that result in double counting or omitting objects.

Rote counting

A child counting by rote will have 1 or more principles not yet fully learned. Perhaps the child knows some number names but not necessarily the proper sequence, as shown in figure 7.7(a). In this case, the child provides number names, but these names are not given in the correct counting sequence. These children may 'count' the same objects several times and use a different counting sequence each time. Children who exhibit this error need to spend more time on the *stable-order rule*, as described in the second counting principle.

Rote counters may know the proper counting sequence, but they may not be able to maintain a correct correspondence between the objects being counted and the number names. Figure 7.7(b) shows

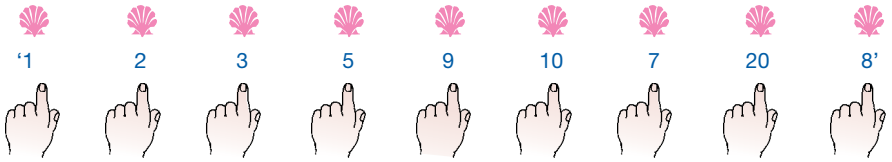
an example in which the rote counter is saying the number names faster than pointing, so that number names are not coordinated with the shells being counted. Rote counters may say the number names until they perceive all the objects are counted even when some have been omitted. It is also possible that the rote counter points faster than saying the words, as illustrated in figure 7.7(c). This rote counter is pointing to the objects but is not providing a name for each of them. Asking children to slow down their counting and use 1-to-1 correspondence will help children avoid this kind of rote counting error.

It is important that children demonstrate all 4 counting principles to be effective at counting. Otherwise, children may not have their number names in the proper sequence, or they may not consistently provide a number name for each object being counted. A 1-to-1 correspondence may not be shown, which is a critical distinction between rote and rational counting. Using 1-to-1 correspondence in counting represents significant progress and establishes 1 of the prerequisites for rational counting.

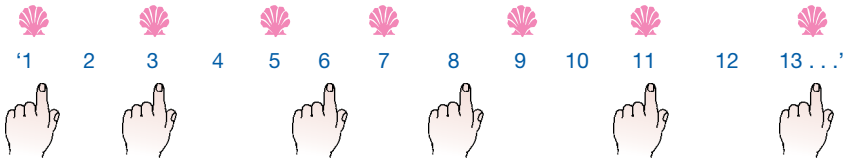
Rational counting

In rational counting, the child gives a correct number name as objects are counted in succession; however, in rational counting the child not only uses 1-to-1 correspondence but also is able to answer the question about the number of objects being counted. Rational counters exhibit all 4 counting principles.

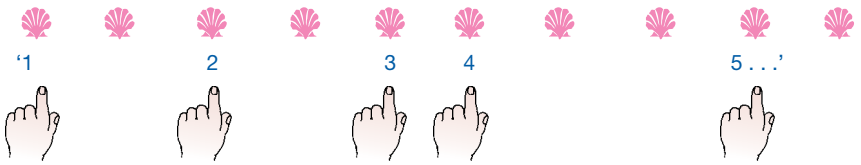
Rational counting is an important skill for every primary-level child. Children will notice their own progress in developing this skill and become proud of their accomplishments. Early in Year 1 some children will count to 10, others to 20, some to 50 and a few to over 100 (Fuson 2003). No upper limit should be imposed. Learning experiences should provide regular practice and encourage each child to count as far as he or she can.



(a) Incorrect sequence, correct correspondence



(b) Correct sequence, incorrect correspondence (counts too fast)



(c) Correct sequence, incorrect correspondence (points too fast)

FIGURE 7.7 Rote counting errors

7.3 Counting strategies

LEARNING OUTCOME 7.3 Describing key strategies for more advanced counting.

Once mastery of rational counting to 10 or 20 has been reached, more efficient and sophisticated counting strategies should be encouraged.

Counting on

In counting on, the child gives correct number names as counting proceeds and can start at any number and begin counting. For example, a child may count a pile of 8 counters and when 3 more counters are added to the pile, count on '9, 10, 11'; or beginning with 28 counters they count '29, 30, 31. If bridging decades they might be skilled at counting from 98 counters and say '99, 100, 101'. Initially however, children will count all objects before counting on from a given amount. For example, when a child adds 3 counters to 8 counters (already counted), he or she counts all of them again by beginning at 1 and ending with 11. It is a natural progression for a child to start from the biggest number and count on but this strategy takes time to develop. Counting on practice leads children to discover valuable patterns. Counting on is also an essential strategy for developing addition.

Being able to count on or back requires children to recognise the starting number and the previous numbers. Thus, if we start counting on from 8, a child needs to know the numbers that come immediately before and after 8 as well as the sequence of numbers preceding or 'nested' within the number 8. Research suggests that even children reflecting all the counting principles show confusion or uncertainty about the nested inclusion of previous numbers (Fuson 2003).

Counting back

When children count back, they give correct number names as they count backward from a particular point. For example, to count back to solve the problem, 'Brian had 22 rabbits and 3 were lost', a child might count '21, 20, 19' and conclude there were 19 left. At an early stage, counting back can be related to rockets blasting off (counting down — 5, 4, 3, 2, 1, blast off); later, it becomes helpful in developing subtraction.

Many children find it difficult to count backward, just as many adults find it difficult to recite the alphabet backward. Learning experiences in counting should include practice with counting backward as well as forward. Counting backward — '5, 4, 3, 2, 1' — helps children establish sequences and relate each number to another in a different way. In the classroom 7.3 provides an activity that uses a calendar to practise counting forward and backward.

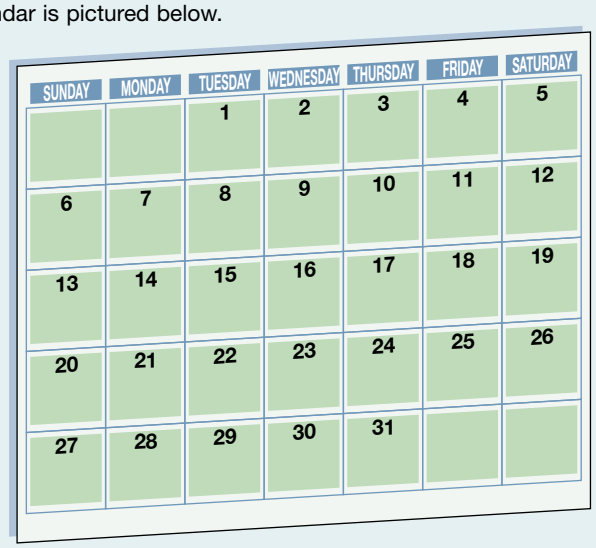
IN THE CLASSROOM 7.3

Numbers on a calendar

Objective: Use a calendar to count forward and backward.

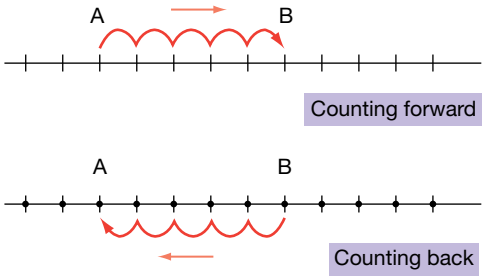
Year level: 1

- A page from a calendar is pictured below.

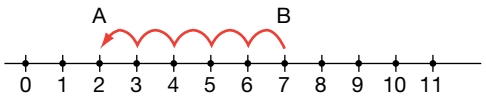


1. A calendar has many patterns. What patterns can you see?
2. What day and date is 7 days after the 7th?
3. What day and date is 1 week after the 5th?
4. If you start Monday and count on 7 days, what is the day?
5. If you start Sunday and count back 4 days, what is the day?
6. What day and date is 5 days before the 30th?
7. What day and date is 5 days after the 21st?
8. Count on 7 days after the 11th. What is the date? What is the day?
9. Count back 7 days from the 27th. What is the date? What is the day?
10. Make up some counting problems on your own that use the calendar. Give them to a friend to solve.

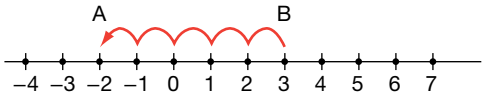
Another useful model is the number line shown earlier in the chapter, but children need to understand what is being counted on it. Once understood, the number line can be used for counting on from any starting point by moving to the right, and counting back from any starting point by moving to the left.



Just as counting on models addition, counting back models subtraction. For example, using the number line and counting back 5 steps from B models the action of $7 - 5 = 2$.



For a more complex example with older students, the number line can be used to illustrate positive and negative numbers. Counting back 5 steps from B, where B is 3, would model the problem $3 - 5 = -2$.



The model of a number line, which never ends; going in each direction infinitely, can naturally be used for addition and subtraction of integers.

These examples demonstrate the importance of labelling a number line accurately and reminding children that since we can't show all the numbers, we are only looking at a piece of a number line. The piece of the number line being used is chosen to focus on certain numbers. Figure 7.8 shows how number lines might be shown differently with number strings and equally spaced marks. Children can use the patterns shown to fill in the extra numbers.

The calculator provides a valuable instructional tool to help children improve their ability to count backward. Many children are surprised to learn that it is as easy to count backward on a calculator, as it is to count forward. In the classroom 7.4 uses a calculator to count forward and In the classroom 7.5 shows how counting backward with a calculator introduces negative integers.

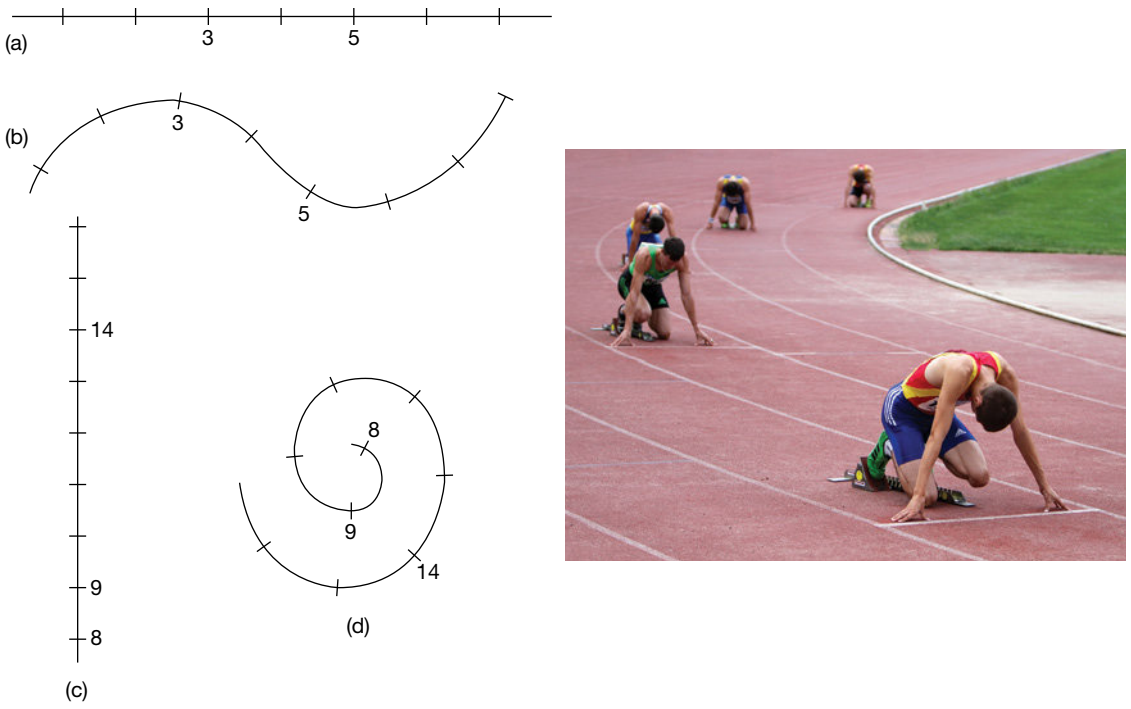


FIGURE 7.8 Examples of different number strings and a real life example of a staggered starting line for a 200-metre race

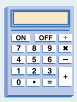
You might ask: Why introduce zero and negative numbers when counting? Since both counting on and back are important skills, children will naturally encounter zero and negative numbers, so it makes sense to introduce them. Furthermore, most young children already have an intuitive knowledge of non-positive numbers as a result of real world experiences. Counting back on a calculator provides 1 tool for representing them. Other models include thermometers, losses and gains and elevators. Research suggests that students are capable of understanding negative numbers far earlier than was once thought (Kilpatrick Swafford & Findell 2001).

IN THE CLASSROOM 7.4

Counting on . . . and on

Objective: Use calculators to develop confidence with number sequences.

Year level: 1

- Use your  to count.

  Enter 1 + 1.
  Now press = = = = . . . and count as long as you want.
- Time yourself:
  How long did it take to count from 1 to 100? _____
  Guess how long it will take to count from 100 to 200 by 1s?_____
  How long did it take to count from 100 to 200 by 1s?_____
  Guess how long it will take to count from 1 to 1000 by 1s?_____

IN THE CLASSROOM 7.5

Counting on . . . and back

Objective: Use calculators to develop confidence with number sequences.

Year level: 1

- Enter 1 + 1 in your  . Then press = = = = . . .

  Stop when you get to 8. Predict the next number.

  Count to 15. Cover your display. Press the equal button 4 times. What number do you think is hidden? Check your display to see if you are right.
- Enter a starting number, for example 15, on your calculator. Then press –1 and = = = = . . .

  What happens? How far do you need to go to arrive at zero on the display?

  Now try these on your calculator:

  4 + 1:_____,_____,_____,_____,_____,_____

  4 – 1:_____,_____,_____,_____,_____,_____

  10 – 1:_____,_____,_____,_____,_____,_____

  10 – 2:_____,_____,_____,_____,_____,_____

Skip counting

In skip counting, the child gives correct names in order, but instead of counting by 1s, they count by 2s, 5s and 10s or other values. The starting point and direction are optional although for younger children starting from the key number is the best practice. Skip counting occurs frequently in everyday life. For example, a shop has chocolates advertised as 2 for $1. A child counts '2, 4, 6, 8, 10', indicating how many they can buy for $5. Looking at the time on an analogue clock, a child might count '5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60' to show they understand minutes in an hour. While travelling in a car a child might see a road sign that shows 60 and ask 'What does that sign mean?'. Their parent or caregiver might count out '10, 20, 30, 40, 50, 60 — that is 60 km per hour'.

In addition to providing many patterns, skip counting on the hundred chart, In the classroom 7.6 and 7.7 provide counting practice, early recognition of odd and even numbers and readiness for multiplication and division.

IN THE CLASSROOM 7.6

Hunting for numbers

Objective: Use a hundred chart to investigate number sequences.

Year level: 1–3

- Study the chart and answer the questions.
  - What number is hidden by the ■?
  - What number is before/after the ■?
  - What number is 2 spaces after the ■?
  - What number is 2 spaces before the ■?
- Put a ● on any number.
  - Begin at the ● and count forward 5.
  - Begin at the ● again and count backward 5.
- Put a ▲ on a different number.
  - Begin at ▲ and count forward 5.
  - Begin at ▲ again and count backward 5.
- Discuss any patterns you see.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42		44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

IN THE CLASSROOM 7.7

Skip counting

Objective: Use a hundred chart to investigate number sequences.

Year level: 1–3

- Use your calculator and a hundred chart to count.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

  - Start at 3 and count by 3s.
  - Circle every number you counted.
  - Describe a pattern you see.
- Which of these numbers would be counted? 13, 51, 61, 62, 63, 100, 113
- Skip count again starting at 4 and counting by 4s. Use a different colour or a new chart.
  - Describe the pattern. It might help to think of chess.

Different sporting scores provide opportunities for skip counting, such as by 2s and 3s in basketball, and by 6s in football. Skip counting, coupled with counting on and counting back, provides excellent preparation for counting change with money. Thus, given these coins children would be encouraged to choose the largest-valued coin and then begin counting on — '10, 15, 20, 25'.



10c 5c 5c 5c

Counting change is a very important skill whose usefulness children recognise. It holds great appeal for them. It should be introduced and extended as far as possible in the primary years. Interestingly enough, most of the difficulties associated with counting change can be traced to weaknesses in counting, particularly skip counting. Throughout the early years of primary schooling and beyond, teachers can and should take advantage of every opportunity to encourage accurate and rapid skip counting.

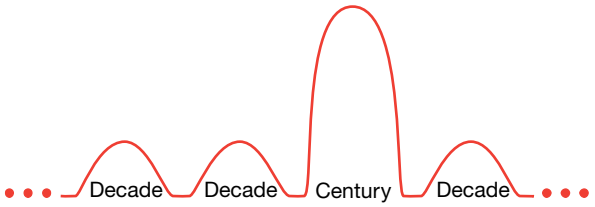
Counting practice

Counting practice should include counting on and counting back. There are a large number of counting books available to children and teachers. For example, *Anno's Counting Book* by Mitsumasa Anno; *One Moose, Twenty Mice* by Clare Beaton; *The Very Hungry Caterpillar* by Eric Carle; and *I Can Count the Petals of a Flower* by John and Stacy Wahl provide a variety of rich and stimulating contexts for counting. These books engage children in counting and provide insight into how children count. Furthermore, books, such as *How Many Snails*? *A Counting Book* by Paul Giganti and *Count on Your Fingers*

African Style by Claudia Zaslavsky, focus on numerical relationships. These books are all useful for counting-based discussions between children and adults. For example, *Anno's Counting Book* provides a sequence of pictures representing numbers 0–12, months of the year and times of the day.

Starter questions can be asked to encourage active participation and maintain a high level of interest. Further activities can include asking children to study and discuss each page of the book in groups; searching for mathematical language; and helping children create their own picture book (Griffiths & Clyne 1988).

Research has shown some predictable trouble spots for children when counting. For example, children often slow down, hesitate or stop when they reach certain numbers, such as 29; however, as soon as they establish the next number as 30, their counting pace quickens, until they are ready to enter the next decade (set of 10 numbers). Bridging the next century (set of 100 numbers) poses a similar challenge. As children count '. . . 98, 99', they may pause and be uncertain how to name the next number. Bridging to the next 10 or 100 is among the common transitional points of counting difficulty as children try to establish the pattern of numbers.



Bumps in the road for successful counters

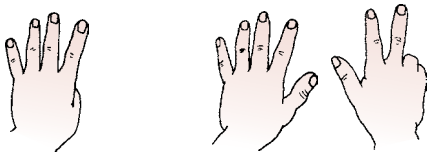
Calculators are beneficial tools to illustrate how numbers move in a repetitive pattern and how transitions are made across the decades, 39–40, 69–70, and beyond 99–100, 109–110 and 199–200. Not only is the calculator a valuable instructional tool that helps improve children's ability to count, but it is also a powerful counting tool that they love to explore. Early counting with the calculator should emphasise the physical link between pressing the keys and watching the display. Because the display changes constantly, the students begin to recognise patterns. Calculator counting involves a physical activity, pressing a key each time a number is counted, through which students can relate the size of a number to the amount of time needed to count it.

Children are usually surprised to learn that it is easy to count by any number on the calculator. A calculator can start at zero and count by ones (In the classroom 7.4) or can begin at any starting point and skip count forward or backward by any number (In the classroom 7.5). It just depends on the + or – sign and the number chosen for skip counting.

Calculator counting opens exciting mathematical explorations and promotes both critical thinking and problem solving. According to Huinker (2002, p. 316), 'using calculators as learning tools can empower young children with the capacity to investigate number ideas in ways that were previously inaccessible to them'.

Developing number benchmarks

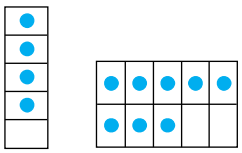
Number benchmarks are perceptual anchors that become internalised from many concrete experiences, often accumulated over many years. For example, the numbers 5 and 10 (the number of fingers on 1 and 2 hands) provide 2 early number benchmarks. Children recognise 4 fingers as being 1 less than 5 and 8 as being 3 more than 5 or 2 less than 10.



'1 less than 5'

'3 more than 5'
'2 less than 10'

The five-frame (5 × 1 array) and the ten-frame (5 × 2 array) (masters are available in Appendix C) can illustrate the pictorial representation of these early benchmarks. 4 is shown as one less than 5 and 8 as 2 less than 10 or 3 more than 5. Ten-frames can also help to illustrate the pairing of numbers and odd and even arrangements.



In Japan, the early benchmarks of 5 and 10 are later used with the ten-frame and the Japanese *soroban* (which is similar to an abacus) to promote counting, quick recognition of quantities and mental computation (Shigematsu, Iwasaki & Koyama 1994).

Understanding 1–5

Development of the numbers 1 through to 5 is principally done by sight recognition of patterns, coupled with immediate association with the oral name and then the written symbol. Real world examples enrich the learning of number as shown in the example, figure 7.9, which illustrates a picture of a real-world object, a tricycle, and the question, 'How many wheels?'

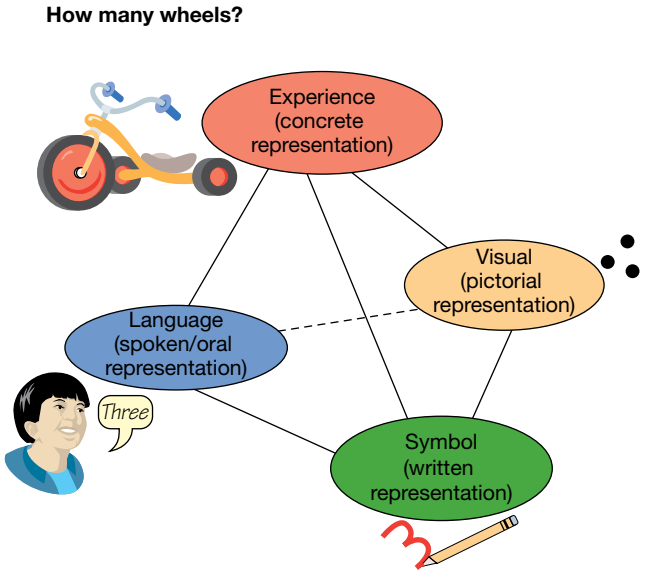


FIGURE 7.9 The ELViS (**E**xperience, **L**anguage, **V**isual and **S**ymbol) model for concrete, pictorial, oral and written representations

It is important that the number of wheels be linked to both the number name and the written symbol. It is also important to provide different arrangements of the quantity with dots, blocks and other objects, as well as different forms of the numerals including digital technologies, to broaden children's experiences.

Many valuable relationships are established as the numbers 1 through 5 are developed, but none are more useful than the notions of *1 more* and *1 less*. These connections are fundamental in early counting and also in learning place value with larger numbers. The notion *1 more* and *1 less* evolves from many different real-world experiences, such as the following: 'David has 1 less cupcake than Jean-Paul';

'Mia has 1 more apple than Beth'; 'Their school has 1 less player on the team'; and 'The music group has 1 more child than the art group'.

The concepts of *1 more* and *1 less* can be modelled in different ways. Figure 7.10 shows 3 such models. Having many experiences with such models and patterns helps children abstract numbers and establish useful connections between them.

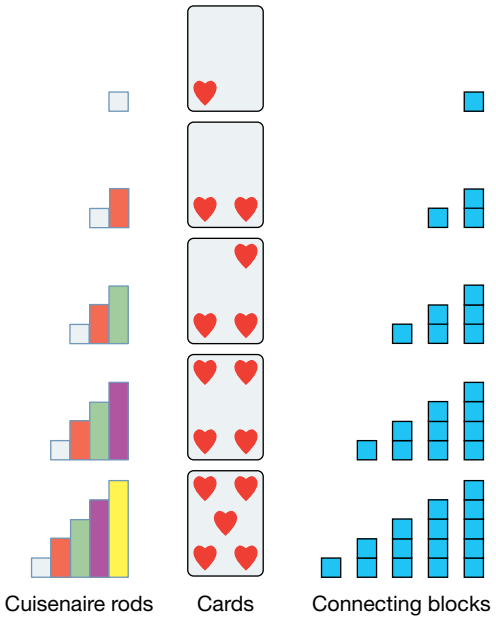


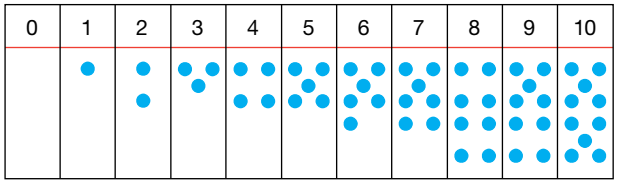
FIGURE 7.10 3 models for 1 more or 1 less

These arrangements provide a basis for developing the concept of 5 in concrete and pictorial form as well as discussing *1 more* or *1 less*. Using these models for discussion can help children establish important connections that link numbers such as 4 and 5. For example, children might say that the yellow rod is 1 step longer than the purple rod or that the 5 card is just like the 4 card except it has an extra heart in the middle.

The hearts on the cards and the towers of blocks provide clear reminders of the numbers represented and the notion of *1 more* or *1 less*. Both the staircase of rods and the towers of blocks vividly illustrate not only these concepts but also an important, yet subtle, difference between the models. The staircase of rods illustrates the concept of *more*, but it is not absolutely clear how much more until the length of the rods has been made clear. If you used any colour rod without identifying the size of the unit rod, it would not be possible to associate the rod with a unique number. Thus the rods are a different model for developing numbers than the cards or blocks.

Some models illustrate zero more clearly than others. For example, a rod of no length is more difficult to grasp than a card with zero hearts. Care should be taken to introduce zero as soon as it becomes natural to do so, using models appropriate for the purpose, including the number line, illustrated earlier. Help children distinguish between zero and nothing by encouraging the use of zero to report the absence of something. For example, when reporting the score of a game, it is better to say 'Sydney Swans 3 goals, Adelaide Crows 0' than to say '3 to nothing'.

As the numbers through to 10 are developed, it is important that various patterns among them be discovered, recognised, used and discussed. A variety of patterns suggesting different relationships are shown in the following number chart.



For example, the number 7 is shown with 5 dots and 2 more dots. The number 10 is composed of 2 groups of 5. Whilst recognising patterns in premade cards is important, helping children create their own dot cards is a useful activity as it allows them to explore their own patterns.

On the number chart above, 6 is shown as 5 dots and 1 more dot, but other representations are possible, as shown in figure 7.11, and they should be explored and discussed with students. Figure 7.11(b), for example, shows that 6 can be represented as 1 group of 3, 1 group of 2 and 1 group of 1 in a triangular arrangement. It can also be shown as 2 groups of 3, as in parts (a) and (d), or 3 groups of 2, as in part (c). No mention is made of addition or multiplication in this context, but such observations provide helpful connections when these operations are developed.

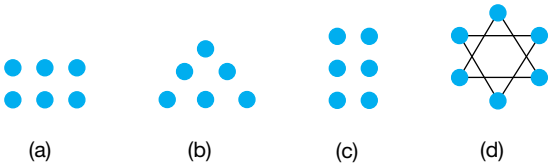
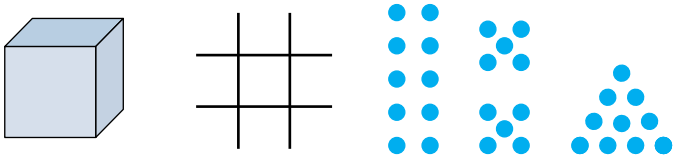


FIGURE 7.11 Some representations of 6

Similar illustrations and applications for the numbers 7 through to 10 should be presented. For example, 7 days in a week may suggest a natural grouping of 5 and 2 (weekdays/weekend), 8 vertices of a cube might be visualised as 2 groups of 4, and the number of squares in a noughts and crosses grid suggest 3 groups of 3. All of these different representations of the same number provide opportunities to check on students' conservation of number and real world application.

Mon Tue Wed Thu Fri Sat Sun



Most children realise very early that 10 is a special number. At the early stage of number development, the most unusual thing about 10 is that it is the first number represented by 2 digits, 1 and 0. In addition to having 10 fingers and toes, children encounter the number 10 in many situations, such as in playing games and changing money. These experiences can be extended to include discussion about different representations of 10, such as: 'Can you find 2 groups of 5'; 'Can you find 5 groups of 2'; 'Does the group of 4, 3, 2 and 1 remind you of bowling'; or 'Do you see the triangle arrangement'. The number 10 provides the cornerstone for our number system, and the significance of 10 is developed further in chapter 8.

A ten-frame is certainly one of the most effective models for facilitating patterns, developing group recognition of numbers, and building an understanding of place value. The ten-frame can be made from an egg carton shortened to contain 10 pouches, or it may simply be outlined on paper or cardboard and laminated (refer Appendix C). This frame is a powerful organiser and helps provide the basis for future thinking strategies for addition, multiplication by 2 and 5 and for mental computation. Initially, children might use counters to make different representations of the same number in the ten-frame, as illustrated

in figure 7.12. Earlier, 8 counters on a ten-frame were illustrated with the thinking strategies for 5 and 10, but they are also shown below with the additional strategy of doubles (4 and 4). Encountering a variety of groupings on the ten-frame should stimulate discussion about different patterns.

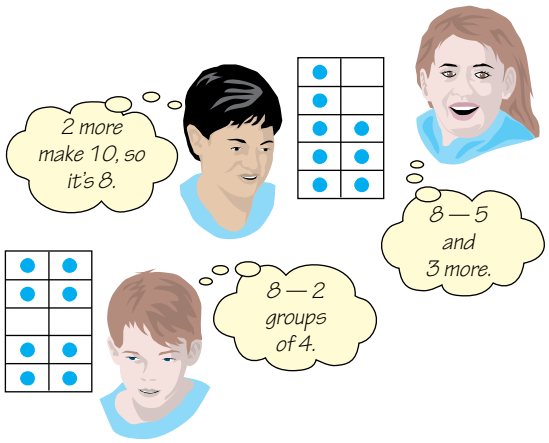


FIGURE 7.12 Representations of 8 on a ten-frame

Figure 7.13 shows some of the possible arrangements with connections that might be constructed as children examine different representations on the ten-frame. These relationships encourage children to think flexibly about numbers, thereby promoting greater number sense. Experiences with the ten-frame also facilitate the development of addition, subtraction, multiplication and division, as well as place value.

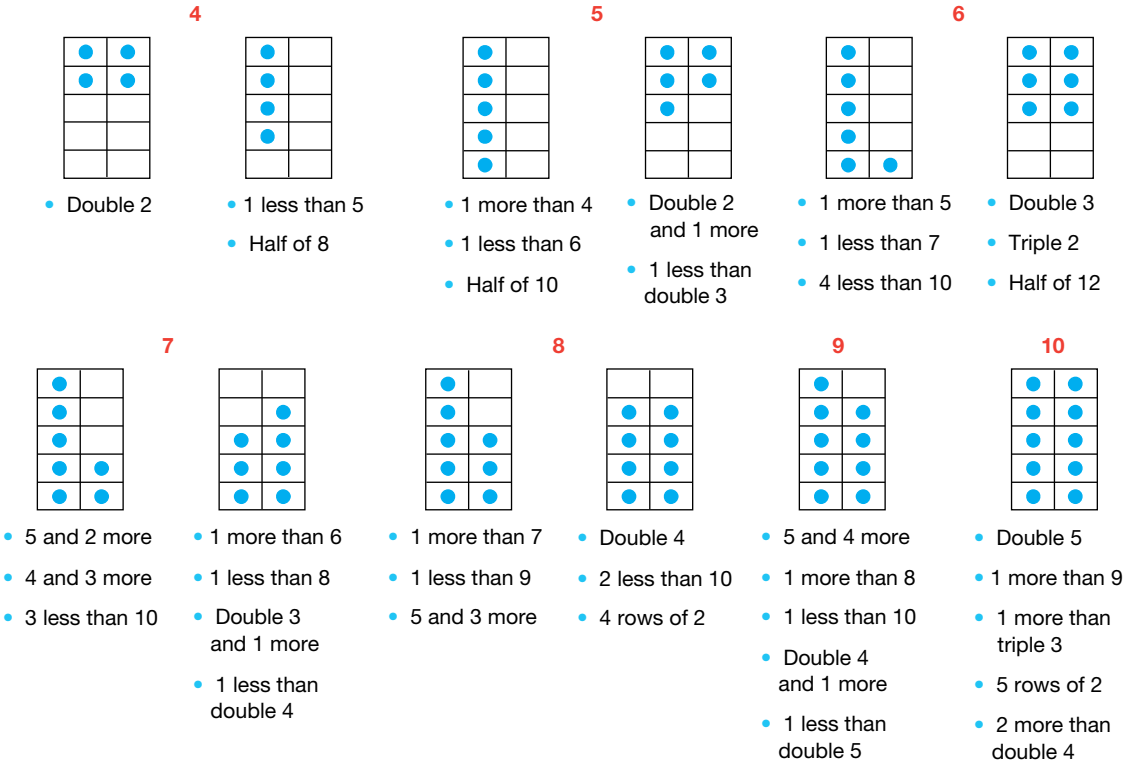


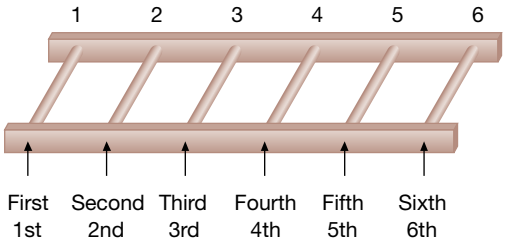
FIGURE 7.13 Connections form representations on the ten-frame

7.4 Cardinal, ordinal and nominal numbers

LEARNING OUTCOME 7.4 Explaining the difference between cardinal, ordinal and nominal numbers.

Thus far we have discussed some important considerations in number development. A major goal has been counting and then finding a correct number name for a given group. This aspect of number provides a *cardinal number*, which answers the question 'How many?' Another important aspect of number emphasises arranging objects or events in an order and is known as *ordinal number*; it answers the question 'Which one?' or 'Which position?'

An emphasis on ordering or arranging things in a given sequence leads to ordinal numbers. The order may be based on any criterion, such as size, time of day, age or position in a race. Once an order is established, however, the counting process not only produces a set of number names but also names each object according to its position. Thus, in counting the rungs on this ladder, the number 1 is first, 2 is second, 3 is third, and so on.



Children often know ordinal numbers such as first, second and third before they begin school. Encounters with statements in their everyday life provide early and valuable experience with ordinal numbers. Examples similar to the following would be commonly understood: The first letter of the alphabet is A, Bella is second in line, Charlie was third in the race, Daniel is the youngest or last-born child in my family but I am the first-born.

It is important that the development of early number concepts provide children with opportunities to learn both ordinal and cardinal numbers. The early childhood educator should not be concerned about which concept to teach first, both need attention. It is worth noting however that knowledge of ordinal relationships, along with logical thinking, leads to more challenging experiences, such as those suggested in figure 7.14. The complexity of language used in the first example is suitable for older children, as these questions are likely to generate much discussion. Number cards would be a suitable manipulative to help children solve the problem and further clarify notions of ordinal numbers. The second image below is more open-ended and could be used by younger children to act out with concrete materials and create their own story.

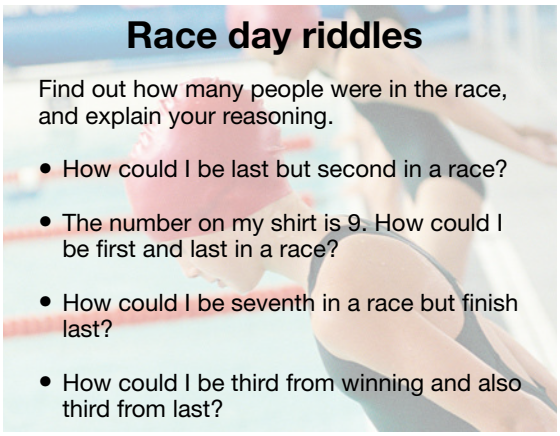




FIGURE 7.14 Thinking about ordinal relationships

Another aspect of number provides a label or classification and is known as *nominal*. Examples are the number on a player's uniform, the licence plate of a car, a postcode identifying your address and a telephone number. Nominal numbers provide essential information for identification but do not necessarily use the ordinal or cardinal aspects of the number.

When using cardinal, ordinal and nominal numbers, children do not need to distinguish between the terms. Distinctions can be made informally by asking questions within problem situations: 'How many pieces are on the chessboard'; 'Which runner is third in the race'; or 'What is your phone number?' These questions not only help children think about numbers but also illustrate that numbers have different uses.

7.5 Writing numerals

LEARNING OUTCOME 7.5 Making connections between reading, writing and speaking number words and symbols.

In the early years, children will need to focus on number development and relationships among numbers. The process of counting by naming numbers in sequence, identifying quantities of objects, instantly recognising small groups (subitising) and using 1-to-1 correspondence come well before the recording of symbols for numbers. Young children typically have difficulty writing numerals as well as letters, and they should not be pushed to symbolise too soon. Spending time initially recognising numerals and connecting them with other number meanings rather than writing numerals is developmentally appropriate. The lack of development of the small muscles needed to write offers 1 problem, and the limited eye–hand coordination of many young children constitutes another difficulty. Both of these issues make it difficult or impossible for young children to write numerals. If children are pressured into premature symbolisation, it can create unnecessary frustration and anxiety. Similarly learning the right shape and flow of the numerals takes time and can not be rushed.

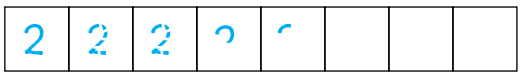
Young children usually recognise a number symbol and say it correctly long before they write it. Many young children initiate early writing of numerals on their own, and they get a feeling of great accomplishment from it; however, children's writing skills develop much more quickly in Year 1 or 2 than in kindergarten.

Children can begin by tracing the digits. Here is a recommended stroke sequence:



Note that 2, 3, 4 and 7 start at the top left; 5, 6, 8, 9 start at the top right.

Textbooks provide guidance to children in different forms. Usually a starting point is indicated as well as the direction:



Encouraging children to draw the appropriate number of objects beside the numeral being written will help them connect the number concept with its symbolic representation. When children are learning to write numerals, encouraging them to practise this writing skill, just like learning to write letters is recommended. Dedicated teaching and learning time needs to be given, separate from learning mathematical concepts, skills and processes, so that children can master this skill and use it freely before applying it in written computational situations. If time is not allocated wisely, then writing in mathematics consumes children's concentration, and they might forget about the mathematics being learned at the time. This is one of the inherent dangers of overemphasis on worksheets. Children can spend too much time focused on the recording or decoration of the mathematics and not on the exploring and thinking about aspects of their learning.

Many children develop the skill for writing numbers on their own or through guided intervention at home. At school they will need monitoring and some occasional help to form the symbols correctly. Other children will need systematic step-by-step procedures to help them. Although there is no one best way to form a numeral, some patterns may help. Guiding a child's hand until the child takes the initiative in writing helps him or her get started. Later, outlines of numerals for children to trace are helpful. Here are some additional suggestions that will support students.

- *Cut out shapes of numerals*. Use an interactive whiteboard or overhead projector to create a large numeral and have a child trace the numeral's silhouette.
- *As 1 child is tracing the projected numeral, have class members trace it in the air*. As the tracing is being done, describe it verbally such as 'go to the right and then down'. This activity can be extended to using only dots to form the outline or pattern for the children to follow.
- *Have a child who can make numerals stand behind someone who cannot*. Ask the skilled child to use a finger and gently 'write' a numeral on the other child's back. The child in front should identify the numeral and write it on the whiteboard, trace it on a poster or write it in the air. This approach calls on the tactile sense and helps some children better develop their writing skills.
- *Use numerals that have been cut from sandpaper and pasted to cards, or take some cord and glue it in the shape of numerals*. Place a mark on each numeral to show the child where to begin tracing it with his or her finger. This approach is particularly helpful with children who persist in reversing numerals.
- *Cover numerals to be traced with a transparency*. Then give the child a water-soluble pen and have him or her practise tracing the numerals.
- *For children having difficulty writing numerals, digital technology is helpful*. The calculator display provides a visual reminder of the number symbol and can reduce the emphasis on writing complicated numerals, if used wisely.

The fact that numerals take different forms should be mentioned but not overstated, as children will come across them naturally.



Some familiarity with these forms will help avoid confusion when a 4 appears as a printed number or as a digital display on a clock or calculator. The wide use of digital numbers in everyday life does create the need for children to be familiar with them.

MAKING CONNECTIONS

Children enjoy learning words and symbols from other countries. This expands their worldview and may cast a new light on their own number system. Quality teachers show an interest in the cultures of their students, and build on the cultural diversity of the students in their classrooms. Intercultural understanding is 1 of 7 general capabilities included in the Australian Curriculum (ACARA v8.1).

While there have been many different interpretations of Aboriginal counting systems, it is clear that the context or purpose for counting is of major importance (Perso 2003). An example comes from the traditional Pitjantjatjara community of Pukatja (formerly Ernabella), in the remote north-west region of South Australia. Counting occurs as '1, 2, 3, many' but with the special use of the hand for 'mara', meaning '5'.

Counting and early number development exists in every culture. While in some western countries the symbolism is similar to that used in Australia (the United States and Germany), in some Asian countries (Japan and China), it is different. Figure 7.15 shows how the numbers 0–10 are represented and named in several different countries. A look at this figure shows some connections, such as *quattro* in Italian meaning '4' in English and connected with a quadrilateral, meaning a polygon with 4 sides. Likewise, *null* in German is used in English to reflect the cardinal number of an empty set.

Children in all of these countries learn to count, applying the same counting principles that were discussed earlier. In Asian countries, such as China, Japan and Taiwan, students learn their own system for counting and communicating their results. In addition, the international availability of technology, including calculators, means that students in these countries need to become familiar with Hindu-Arabic notation. Thus Asian students must learn to recognise and use both Hindu-Arabic numerals and those used in their own country as well as the proper vocabulary for each of them.

The Japanese counting system is complex to us. Figure 7.15 shows the number symbols and 2 different number names used in Japan for the numbers 4 and 7. These numbers have alternate number names (e.g. 4 may be *shi* or *yon*) and although either is fine, there are preferred uses. For example, July or the seventh month is *Shichigatsu*, whereas 7 yen is *nana* yen. Figure 7.15 also shows 2 different columns with number names up to 10, which means Japan has 2 counting systems for the first 10 numbers. Figure 7.15 has a blank for the name of zero in Japan for the counting numbers. Counting begins with 1, so there is no Japanese name for 0 when counting. After 10 only 1 system is used, and we will explore it a bit more in the next chapter.

Ordinal numbers in English require students to memorise new words, such as *first*, *third* and *sixth*. In China, making ordinal numbers is easier. It is accomplished by simply adding *di* to the front of a number. So to make '1' become 'the first', *yi* becomes *di yi*, and to make '6' become 'the sixth', *liu* becomes *di liu*. These examples of how culture influences counting provide a vivid reminder of the challenges for helping children from different backgrounds learn to count.

Number	English	Italian	German	Japanese symbol	Japanese	Japanese	Chinese symbol	Chinese
0	Zero	Zero	Null	〇	Rei		零	Ling
1	One	Uno (a)	Eins	一	Ichi	Hitotsu	一	Yi
2	Two	Due	Zwei	二	Ni	Futatsu	二	Er
3	Three	Tre	Drei	三	San	Mittsu	三	San
4	Four	Quattro	Vier	四	Yon, Shi	Yottsu	四	Si
5	Five	Cinque	Fünf	五	Go	Itsutsu	五	Wu
6	Six	Sei	Sechs	六	Roku	Muttsu	六	Liu
7	Seven	Sette	Sieben	七	Nana, Shichi	Nanatsu	七	Qi
8	Eight	Otto	Acht	八	Hachi	Yattsu	八	Ba
9	Nine	Nove	Neun	九	Kyuu	Kokonotsu	九	Jiu
10	Ten	Dieci	Zehn	十	Jun or Ju	To	十	Shi

FIGURE 7.15 Symbols and number names for 0 to 10 in several different countries

A GLANCE AT WHERE WE'VE BEEN

Good number sense is a prerequisite for all later computational development. Young children need to recognise small groups of objects (up to 5 or 6) by sight (*subitising*) and name them properly. *Subitising* saves time, helps develop more sophisticated counting skills and accelerates the later development of addition and subtraction. Activities involving sight recognition of the numbers of objects in small groups provide many opportunities to introduce and use key terms such as *more*, *less*, *after*, and *before*, *1 more* and *1 less*. To foster better number sense, learning the numbers through to 5 should focus on patterns

and developing recognition skills. Models such as the five- and ten-frame, number lines, the hundred chart and calculators provide a powerful tool for helping children explore and construct relationships for numbers up to 10 and beyond.

Counting skills usually start before children begin school but need to be developed using careful and systematic learning activities before written work is appropriate. Rational counting is a goal for all young children and is characterised by several principles, including using a fixed number-name list, assigning 1 and only 1 name to each object counted and realising that the last object counted represents the number in the group.

Counting processes reflect various levels of sophistication, beginning with rote counting and eventually leading to rapid skip counting forward and backward. Counting skills are extended in the primary years and often are further refined throughout our lives. Oral counting leads to ways of writing and representing cardinal, ordinal and nominal numbers.

Competence with and understanding of the numbers 0 through to 10 are essential for meaningful later development of larger numbers. The relationship between grouping of objects, the number names, the written symbols and the order between numbers must be well understood. This knowledge is the basis for the successful study of primary and early childhood mathematics, and it prepares children for the necessary understanding of large numbers and place value.

THINGS TO DO: FROM WHAT YOU'VE READ

1 What are some common early number expectations identified by the National Association for the Education of Young Children and the NCTM *Focal Points*? Are they similar to the Australian Curriculum and the Early Years Learning Framework? If so, how?

2 Suppose you send a note home to parents encouraging them to help their children improve sight-recognition skills. A parent responds, 'Why should my child learn to recognise a group of 5? After all, you can just count them.' How would you respond?

3 We identified 4 fundamental principles of counting in this chapter.
- (a) Describe in your own words what each of the principles means.
- (b) Explain why that principle is essential to becoming a rational counter.
- (c) Choose 1 of the principles and describe an activity that would help a child progress toward developing that principle.

4 Distinguish between rote and rational counting.

5 What is meant by conservation of number? Why is its development an important part of number sense?

6 Describe how counting could be used to answer these questions.
- (a) How many floors are between the seventh and fifteenth floors?
- (b) If Alfie has read to the bottom of page 16, how many pages must he read to reach the top of page 21?

7 Examine the number strings in figure 7.8.
- (a) Describe some patterns that might result from these number strings.
- (b) How would you use these number strings to promote counting forward? Backward?
- (c) Describe how you would ramp the number strings up or down to adapt to children with different levels of counting skills.

THINGS TO DO: GOING BEYOND THIS TEXT

IN THE FIELD

1 Gather together a group of your peers (4–6) and discuss counting in other languages. Do any of them speak a second language? What similarities and differences did you find? How are the numbers 10 to 20 treated in these languages? Would these numbers be an advantage or disadvantage to their mathematical learning?

2 Go to the 'Count Me in Too' framework at www.curriculumsupport.education.nsw.gov.au/countmein. Undertake some activities or assessments with young children.

3 Using a copy of the *Early Numeracy Interview Booklet* from www.education.vic.gov.au/studentlearning/teachingresources/maths/interview/moi.htm, undertake section A, 'Counting with a 5-year-old child'. Questions 1–7 on pages 22 and 23 are recommended. However, a detour for younger children on pages 24 and 25 is available if the child is struggling.

You will need permission to undertake the assessment in a school or with the parent/guardian of a child under supervision. Using the *Early Numeracy Interview Booklet* and the Points of Growth and Observation Notes as a guide, conduct a 30–40 minute interview with the child. Read pages 13 and 14 of 'Conducting the interview' carefully and make sure you have the time, space and equipment well organised. Depending on the level of the child, work through as much of section A as possible. Record and analyse the results, carefully deciding what the interview has informed you about the child and how this assessment would help you in future planning if you were the teacher of this child.

4 *Using attribute pieces*. In the classroom 7.1 and 7.2 provide several patterning and logical reasoning activities using attribute pieces with children. Explore 1-thing different trains and loops (beginning joins to end) with several children. Repeat the process with 2-thing different trains and loops. Did your children find 1-difference trains to be as difficult as 2-difference trains? Did students feel comfortable knowing that more than one correct answer exists?

IN YOUR JOURNAL

5 Observe children counting in the playground. How do children demonstrate their counting abilities? In your journal, describe their counting activities and the strategies they appear to use.

6 There are controversies related to teaching counting in the classroom. In your journal, respond to this statement: 'Counting is an immature habit that teachers need to help students break.'

WITH ADDITIONAL RESOURCES

7 Examine Burton et al. (1993), Cavanagh et al. (2004) or McIntosh et al. (1997), Back (2014) or wwwmathsolutions.com. How do these articles characterise number sense? Select and report on teaching strategies designed to foster number sense.

8 Read the article about zero by Anthony and Walsh (2004). Describe some of the misconceptions children have and share some of the suggestions made to help children gain a better understanding of zero.

9 Compare 2 current textbook series.

(a) Find how far each series expects children to be able to count at the beginning and by the end of Year 1.

(b) Find examples of activities designed to develop number sense. Describe the activities and identify the year level. Do you think these activities would be effective? Explain why.

(c) Find an example of a visual pattern that connects numbers to geometry.

10 Review *The Wonderful World of Mathematics: A Critically Annotated List of Children's Books in Mathematics* (Thiessen, Matthias & Smith 1998). Find 3 multicultural books related to counting that are highly recommended.

11 In his ground-breaking research, Martin Hughes (1986) interviewed children to determine how they recorded symbols for quantities of objects hidden from view. Read about Hughes' box task and discuss his ideas that children know much more about numbers before they start school than we give them credit for.

WITH TECHNOLOGY

12 *Counting with calculators*. In the classroom 7.4 and 7.5 provide structured calculator counting activities. Try 1 or more of them with some children in a classroom that has calculators available. Were the children experienced at using the calculators or were they viewed as a novelty? Were the children comfortable making estimates for the counts? Describe some patterns that emerged.

13 *Virtual calculators*. Investigate the NCTM website at http://www.nctm.org/Classroom-Resources/Interactives/ and try the Calculator and Hundreds Boards activity. Explore this resource by entering the number 5 and then repeatedly enter the symbol on the calculator. How does the hundred board change? Describe how this virtual tool can be used as an instructional tool. Identify advantages of using this tool over the handheld calculator and the paper hundred chart.

14 Explore the following 2 websites that provide opportunities to count in other languages: 1–10 in over 5000 languages at www.zompist.com/numbers.shtml and 1–100 in 18 languages at www.marijn.org/everything-is-4/counting-0-to-100/french. Look for similarities and differences in 3 languages that are of interest to you.

CHILDREN'S LITERATURE CONNECTIONS

Anno, M. (1977). *Anno's Counting Book*. New York: Thomas Y. Crowell.
How would you describe a picture that refers to zero? This book begins with a barren winter landscape — a hazy, blue sky above a hazy, white hill. On the next page the scene brightens: 1 tree, 1 bird, 1 house. Turn the page again and the snow has started to melt and you find 2 buildings, 2 trucks, 2 trees, 2 children, 2 dogs and 2 adults. Suddenly there is almost more than you can count on each page! And the numbers continue to increase.

Bancroft, B. (2007). *An Australian 1, 2, 3 of Animals*. Surry Hills, NSW: Little Hare Books.
The author and illustrator Bronwyn Bancroft is a descendant of the Bundjalung people of New South Wales and Queensland. The artwork produced for this children's counting book is both colourful and full of wonderful patterns.

Beaton, C. (1999). *One Moose, Twenty Mice*. Sydney, NSW: Peribo Pty Ltd, 2000 also Barefoot Books.
This counting book contains beautifully illustrated pages filled with craftwork and patterns.

Carle, E. (1985). *The Very Hungry Caterpillar*. Vic.: Penguin.
A counting book where the caterpillar eats fruit, food and a leaf then builds a cocoon, finally becoming a butterfly.

McGrath, B. B., Alderman, P. & Estrada, P. (2003). *Soccer Counts*, Watertown, MA: Charlesbridge.
This is a beautifully written and illustrated children's book with rhymes throughout to capture interest. The story follows a game of soccer from beginning to end, with both genders represented. Each page has extra information about the history and rules of this popular sport. To make it metric the following conversions will help: a regulation ball has a circumference of 56 centimetres (p. 1); the goal is eight feet high, which is 2.44 metres (p. 8); the free kick of 10 yards is equal to 9.14 metres (p. 10); and the penalty kick is 11 metres (p. 12).

Murphy, S. (2000). *Dave's Down-to-Earth Rock Shop*. New York: HarperCollins.
Josh and Amy learn that they can sort their rock collection by many different ways such as colour, size, type and hardness. They also learn that the same objects can be organised in many different ways. This is a great book for students to learn about attributes. There are activities and games in the back of this book for parents and teachers to use.

Murphy. S. (1997). *Just Enough Carrots*. New York: HarperCollins.
This is the story of a bunny and his mother and how they shop in a grocery store for the lunch guests that are coming to their house. As they get the items they need, the bunny learns about counting and comparing different amounts from their cart of items to the other shoppers and their carts. This is a great introduction to comparing different amounts. There are activities and games in the back of this book for teachers and parents to use.

Schlein, M. (1996). *More Than One*. New York: Greenwillow Books.
This book will let the reader see how the number 1 can be more than 1 depending on how it is attached to another number. For example: 1 basketball team, 1 month, 1 week, 1 year etc. This is a great book to introduce number sense.

Wadsworth, O. & Vojtech, A. (2002). *Over in the Meadow*. New York: NorthSouth.
Each number from 1–10 is represented by a rhyme of a mother and her offspring in the meadow. Each illustration has many other things to count, which adds interest to every page.

Zaslavsky, C. (1980). *Count on Your Fingers African Style*. New York: Crowell.
The focus is on numerical relationships in the African marketplace where people buy and trade using many different languages, including finger counting. This book explores the practicality of maths within the context of African culture and helps children see that maths can be fun and creative.

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ACKNOWLEDGEMENTS

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CHAPTER 8

Extending number sense: place value

LEARNING OUTCOMES

8.1 Articulating the 4 key characteristics of our number system and the importance of these for future learning.

8.2 Recognising that mathematical materials can help children develop a deeper understanding of place value.

8.3 Identifying effective methods of introducing the concept of place value to children.

8.4 Describing what happens when numbers are composed and decomposed, initially to 100, then 1000 and beyond.

8.5 Understanding the use of models, language and pattern for recording and comparing large numbers in base ten.

8.6 Explaining how an understanding of place value helps develop students' ability to read and write numbers.

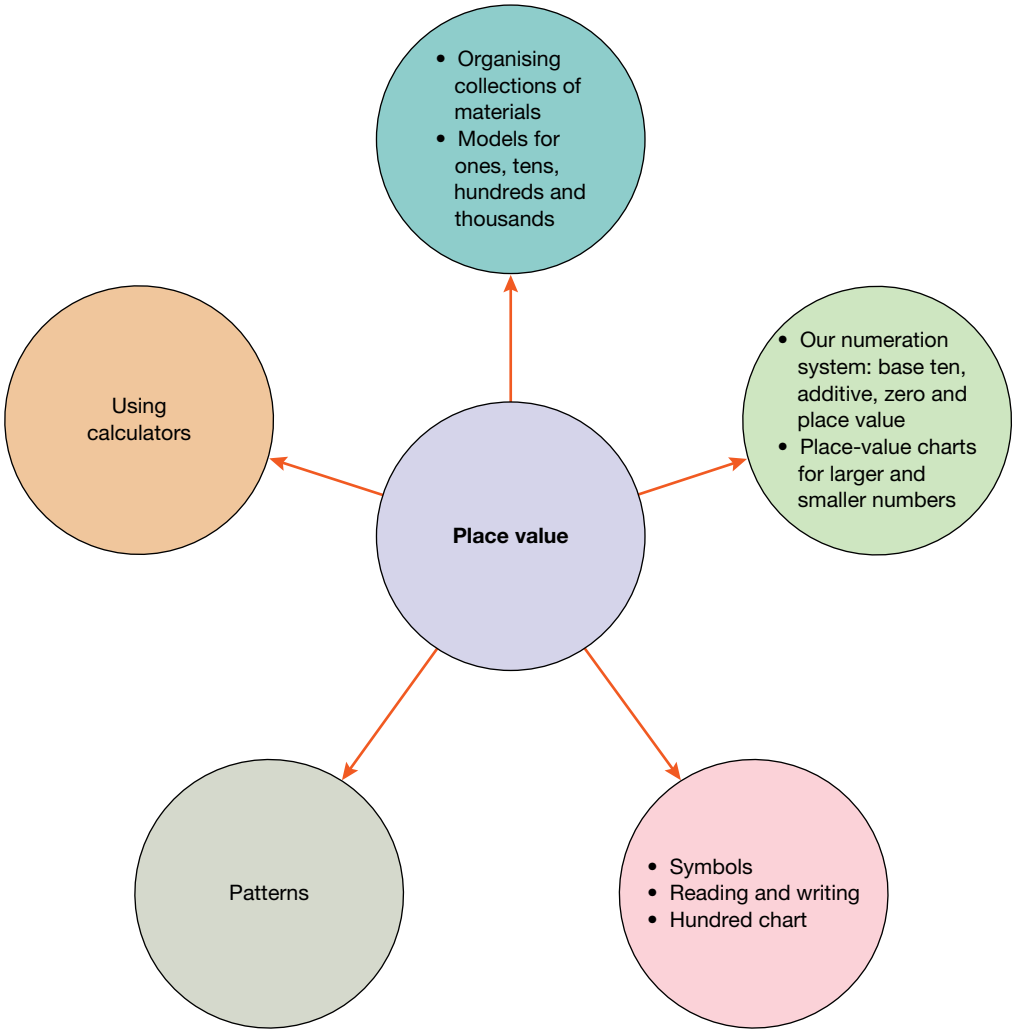
8.7 Identifying how rounding relates to place value and naming numbers.



'The creator of the universe works in mysterious ways. But uses a base ten counting system and likes round numbers.'

Scott Adams (American cartoonist)

Chapter 8 concept map



Introduction

To be numerate individuals, children need to make sense of numbers and the ways in which numbers are used in and out of school. This sense-making of numbers is a hallmark of the National Council of Teachers of Mathematics, Standards and Expectations (NCTM 2000) in the same way as it is for the Australian Curriculum: Mathematics (Australian Curriculum Assessment and Reporting Authority v8). Both curriculum documents include key concepts and skills related to understanding and representing numbers and number systems. Place value is critical to this understanding or sense making and is one of the cornerstones of our number system. The famous mathematician Gauss underscored the importance of place value when he said, 'Without place value, we would get no place with numbers'. The crucial nature of place value is illustrated in table 8.1 which includes the content and processes for Years 1 to 6 in the sub-strands of Number and place value, Money and financial mathematics, decimals and measurement in Australia.

TABLE 8.1 **Place value proficiencies and content descriptions for Year 1 to 7 in Australia**

Proficiencies	Content description
Year 1 **Understanding** includes partitioning numbers in various ways. **Problem-solving** includes using familiar counting sequences to solve unfamiliar problems and discussing the reasonableness of the answer.	**Number and place value** • Develop confidence with number sequences to and from 100 by ones from any starting point. Skip count by tens starting from zero. • Recognise, model, read, write and order numbers to at least 100 and locate these numbers on a number line. • Count collections to 100 by partitioning numbers using place value. **Money and financial mathematics** • Recognise, describe and order Australian coins according to their value.
Year 2 **Understanding** includes connecting number calculations with counting sequences, and partitioning and combining numbers flexibly. **Fluency** includes readily counting numbers in sequences. **Problem-solving** includes formulating problems from authentic situations, making models and using number sentences that represent problem situations.	**Number and place value** • Investigate number sequences, initially those increasing and decreasing by . . . tens from any starting point, then moving to other sequences. • Recognise, model, represent and order numbers to at least 1000. • Group, partition and rearrange collections up to 1000 in hundreds, tens and ones to facilitate more efficient counting. **Money and financial mathematics** • Count and order small collections of Australian coins and notes according to their value.
Year 3 **Understanding** includes connecting number representations with number sequences, partitioning and combining numbers flexibly.	**Number and place value** • Recognise, model, represent and order numbers to at least 10 000. • Apply place value to partition, rearrange and regroup numbers to at least 10 000 to assist calculations and solve problems. **Money and financial mathematics** • Represent money values in multiple ways.
Year 4 **Understanding** includes making connections between representations of numbers, partitioning and combining numbers flexibly, extending place value to decimals. **Problem-solving** includes comparing large numbers with each other.	**Number and place value** • Recognise, represent and order numbers to at least tens of thousands. • Apply place value to partition, rearrange and regroup numbers to at least tens of thousands to assist calculations and solve problems. **Fractions and decimals** • Recognise that the place value system can be extended to tenths and hundredths. Make connections between fractions and decimal notation.
Year 5 **Understanding** includes making connections between representations of numbers, comparing and ordering . . . decimals and representing them in various ways. **Fluency** includes using estimation to check the reasonableness of answers to calculations. **Problem-solving** includes formulating and solving authentic problems using whole numbers and measurements.	**Number and place value** • Use estimation and rounding to check the reasonableness of answers to calculations. **Fractions and decimals** • Recognise that the place value system can be extended beyond hundredths.

Year 6 **Understanding** includes representing decimals in various ways and describing connections between them, and making reasonable estimations. **Problem-solving** includes formulating and solving authentic problems using decimals, percentages and measurements. **Reasoning** includes describing results for continuing number sequences.	**Fractions and decimals** • Add and subtract decimals, with and without digital technologies, and use estimation and rounding to check the reasonableness of answers. • Multiply decimals by whole numbers and perform divisions by non-zero whole numbers where the results are terminating decimals, with and without digital technologies. • Multiply and divide decimals by powers of 10. **Using units of measurement** • Connect decimal representations to the metric system. • Convert between common metric units of length, mass and capacity.
Year 7	**Real numbers** • Round decimals to a specified number of decimal places.

Source: © Australian Curriculum, Assessment and Reporting Authority 2015.

As discussed in the previous chapter, early number sense and counting naturally leads to place value as an organisational structure for number. The same principles are involved as larger numbers are counted. Representations of larger numbers are based on learning the pattern or sequence in a meaningful way. So when counting from 9 to 10 or 99 to 100 or 999 to 1000, place value is an important transition between symbols and language. In a similar manner, in adding 1 to 9 to get 10 or 1 to 99 to get 100 or 1 to 999 to get 1000, changes in place value are experienced. These transitions from 1-digit to 2-digit numbers, 2-digit to 3-digit numbers, and 3-digits to 4-digits are fundamental to place value. Place value is important throughout all of our work with whole numbers, the 4 operations, the metric system and decimals.

8.1 Our numeration system

LEARNING OUTCOME 8.1 Articulating the 4 key characteristics of our number system and the importance of these for future learning.

Although we say 'our' numeration system, it is actually a multi-cultural mix and 'ours' only to the extent that it is a part of our cultural heritage. History tells us that 'our' numeration system is really the result of continuous development and refinement over many centuries (Zaslavsky 2003). The number system we use, called the Hindu-Arabic system, was primarily invented in India by the Hindus and transmitted to Europe by the Arabs, but many different countries and cultures contributed to its development.

The Hindu-Arabic numeration system has 4 important characteristics.

1. *Place value*. The position of a digit represents its value; for example, the 2 in $23 names 'two tens' or 'twenty' and has a different mathematical meaning from the 2 in $32, which names 'two ones'.
2. *Base of ten*. The term *base* simply means a collection. In our system, ten is the value that determines a new collection and is represented by the numeral 10. The place value system has 10 digits, 0 through 9. Thinking of 10 digits on our hands is helpful for some children.
3. *Use of zero*. A symbol for zero exists and allows us to represent symbolically the absence of something. For example, 309 shows the absence of tens in a number containing hundreds and ones; however, the number 309 if regrouped contains 30 tens and 9 ones.
4. *Additive and multiplicative property*. Numbers can be written in expanded notation and summed with respect to place value. For example, 123 names the number that is the sum of $100 + 20 + 3$. The number 123 is also $1 \times 100 + 2 \times 10 + 3 \times 1$. Each digit is multiplied by a power of 10, that is ones is 10^0, tens is 10^1, and hundreds is 10^2.

These properties make the system efficient and contribute to the development of number sense but make it very abstract. However, once children understand these four characteristics, *place value, base ten, zero* and *additive and multiplicative*, the formation and interpretation of numbers — either large or small — is a natural development.

One other well-known historical number system, Roman numerals are still in use today in our society, but not for computation. Roman numerals are different from the Hindu-Arabic system in that they use groupings of both 5 and 10, and letters as symbols. The system lacks place value and a symbol for 0. The Roman system of numeration was originally additive because lower-value symbols followed the higher-value symbols — for example, CXVI has a value of one hundred C, plus 10 X, plus 5 V, plus 1 or 116. However, in later developments, a lower-value symbol placed in front of a higher one signified a subtraction. Therefore CXIV represents 100 and 10 plus one-less-than-five, or 114. This is not a place value system. As a result, computation with Roman numerals was a difficult and cumbersome process, which relied on the abacus for efficiency rather than symbolic algorithms.

The abacus has been used extensively throughout the world as a counting, place value and calculating tool in both modern and ancient times. Each bead represents a number according to base 10, 16, 20 or 60 depending on the system of numeration or measurement in the countries where it is used. Artists have depicted the abacus in use against written algorithms, more than 500 years ago, however there is evidence of the abacus dating back further, to ancient Babylonian and Native American cultures, more than 2500 years ago. It is well documented that the abacus (or soroban) is a powerful tool. There was a competition held in Tokyo on November 12, 1946 between the Japanese soroban and an electric calculator, in which the Japanese master using the abacus won four to one against the American operator of the calculator on an extensive range of complex problems for the four operations and combinations of these.

8.2 Nature of place value

LEARNING OUTCOME 8.2 Recognising that mathematical materials can help children develop a deeper understanding of place value.

A thorough understanding of place value is necessary if computational algorithms for addition, subtraction, multiplication and division are to be learned and used in a meaningful way. Place value understanding develops from a variety of experiences, such as counting and mental computation. For example money is base ten, additive and multiplicative, and uses zero: $1 plus $0.25 (or 25c) gives you $1.25 coins. Development of place value promotes number sense and rests on 2 key ideas.

1. *Explicit grouping or trading rules are defined and consistently followed*. These ideas are implicit in the bulletin board display shown in figure 8.1. A display can provide a constant reminder of the importance of grouping by tens for place value. Our base ten number system is characterised by trading up 10 ones for 1 ten, 10 tens for 1 hundred, 13 hundreds for 1 thousand and so on. It is also possible to trade down 1 ten for 10 ones, 1 hundred for ten tens and so on. The 2-way direction of these trades (e.g. 10 tens up for 1 hundred or 1 hundred down for 10 tens) should be stressed because there are times when each type of trade either up or down is needed. These equivalent trades illustrate the composing and decomposing of numbers known as regrouping. It should also be noted that similar trades are followed with numbers less than 1 in decimal form. Thus, 1 whole can be traded down for 10 tenths ($1 = 10 × 10c) and vice versa 10 tenths equals 1 whole. Similarly, 10 hundredths can be traded up for 1 tenth (10 × 1c = 10c) or 1 tenth traded for 10 hundredths, and so on.
2. *The position of a digit determines the number being represented*. For example, the 2 in 3042 and the 2 in 2403 represent completely different quantities: 2 ones in 3042 and two thousands in 2403. The zero plays a similar yet different role in each of these numbers. It has positional value in each case, but it reports the lack of a quantity for that place. There are no hundreds in 3042 and no tens in 2403. Although the notion of zero will continue to be expanded and developed throughout primary school mathematics, children should experience the role of zero in place value early and often.

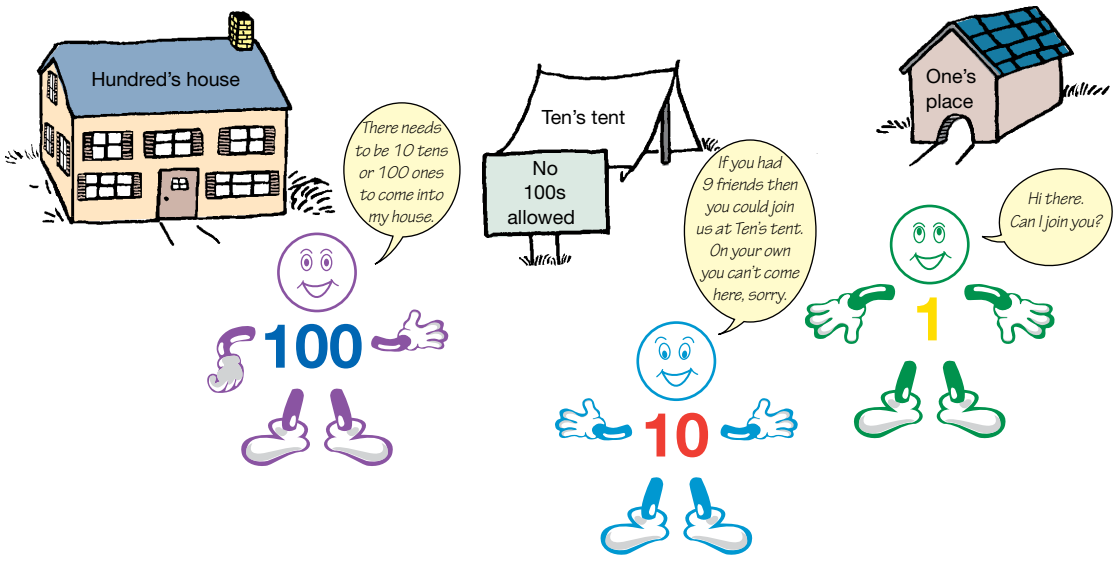


FIGURE 8.1 A bulletin board display for place value

In the Hindu-Arabic number system, place value means that any number no matter how small or large, can be represented using only 10 digits (0–9). Think about the problem of representing numbers without place value! Each number would require a separate and unique symbol. Your memory storage would quickly be exceeded, and you would probably have to use only the few numbers with symbols you could remember. However, place value is difficult for some children to grasp. Oral counting or rote recitation of numbers by young children is often interpreted as understanding place value but can be far from the case. Many children who can count correctly beyond ten have little understanding of the concept of place value. In most cases, the confusion or misunderstanding can be traced to a lack of counting and trading experiences with appropriate materials and the subsequent recording of these results. Early and frequent hands-on counting activities, trading activities and experience with recording numbers, are essential prerequisites for meaningful understanding of place value.

Children often encounter multi-digit numbers before starting school. For example, children might distinguish between the 1- and 2-digit numbers on a digital clock, a timer for a microwave oven, a calendar or they might recognise these types of numbers from their street address or apartment. From experience, children might know that these numbers can mean different things and apartment numbers 201 and 102 are not the same.

Modelling ungrouped and pre-grouped materials

Two types of material can be used to help young children develop place value understanding: (1) ungrouped materials and (2) pre-grouped materials. Ungrouped materials include beans, cubes or straws that children can form into groups, as shown in figure 8.2. If there are enough beans to make a group of 10, then those beans can be placed in a cup or bag or glued on a stick. Once these beans are glued on a stick, then they can be considered to be pre-grouped materials. The stick, bundle or tower does not need counting each time, as the child knows there are 10 objects present in that group.

Pre-grouped materials are formed into groups during or before the child uses them. Figure 8.2 shows examples of pre-grouped materials where the teacher may have prepared groups of 10 in advance. Experience with concrete materials is essential in establishing and developing the concept of place value. Research suggests that learning experiences should focus on concrete models that are simultaneously connected to oral descriptions and symbolic representations of the models (Rogers 2009; Thompson 2000; Wearne & Hiebert 1994).

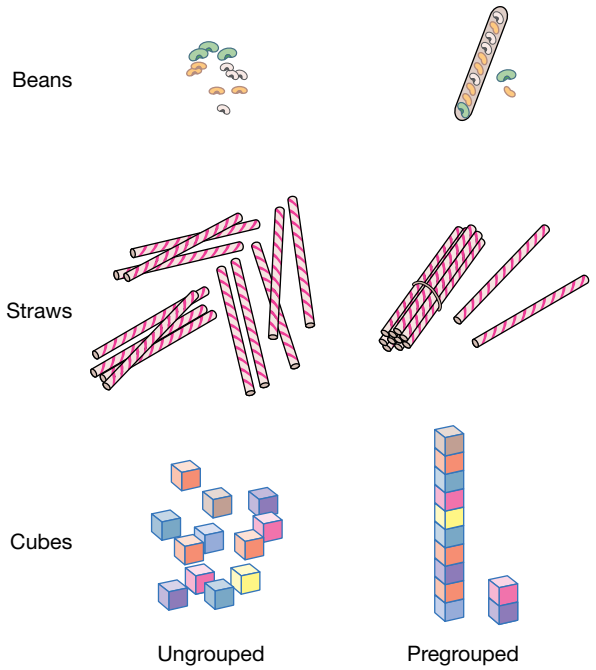


FIGURE 8.2 Ungrouped and pre-grouped models for developing place value

Modelling proportional and non-proportional materials

Place value models may be either proportional or non-proportional. Base-ten blocks, beans glued in groups of 10 on a stick and pop-sticks bundled together are examples of proportional models (see figure 8.3). When using proportional models the material for a group of 10 is 10 times the size of the material for 1, similarly 100 objects are 10 times the size of 10, and so on. A connection to place value through measurement of height provides another proportional model, meaningful to children. Materials such as interlocking cubes can be used by children to measure their height and then formed into towers of 10 cubes for easy counting. Metre sticks, decimetre rods and centimetre cubes can all be used to model a 3-digit number.

On the other hand, when using non-proportional models, size relationships are not maintained. Money is a real world example of a non-proportional place value model because 10 $1 coins are different in size and character to 1 $10 note even though 10 $1 coins is a fair trade for a $10 note in our monetary system. Similarly, 10 $10 notes are the same value as 1 hundred dollar note but they are not proportional in size or similar in features.

Figure 8.3 illustrates some proportional and non-proportional models that are effective in helping students understand not only place value but larger numbers as well. All of the models shown in figure 8.3 represent the same 3-digit number, 123. The value of using different embodiments is that a child is less likely to associate place value with a particular model. In fact, a key learning objective is to develop concepts to a level that does not depend on any 1 physical model. Once children can see the commonality between the various models, they are well on their way to a deeper understanding and abstraction of the main idea.

Since non-proportional materials are not related by size, the child needs to learn and understand the exchange relationship among the pieces, just like when using money for purchases. For example, 1 blue counter (or block) might represent the same amount as 10 yellow counters; 1 green counter might represent the same as ten blue counters, and so on. When playing place value games in ten-land the basic rule of trading would be, '10 yellows equals 1 blue'. Use of a trading mat can help keep the counters in order. In a similar

way beads on an abacus are arranged in a fixed place value order. The abacus, a cultural-historical counting and calculating tool, illustrated at the beginning of this chapter and coloured counters are similar non-proportional materials. In each model, the beads or counters placed in place value columns, provide the basis for representing numbers and exchanging values through addition and subtraction. The distinction of colour for the beads or counters is important for the early establishment of proper trades; however, it should be dropped when appropriate to move attention shifts from the colour of the beads or counters to their position on the place value chart. Only the position of the bead or counter has long-range significance.

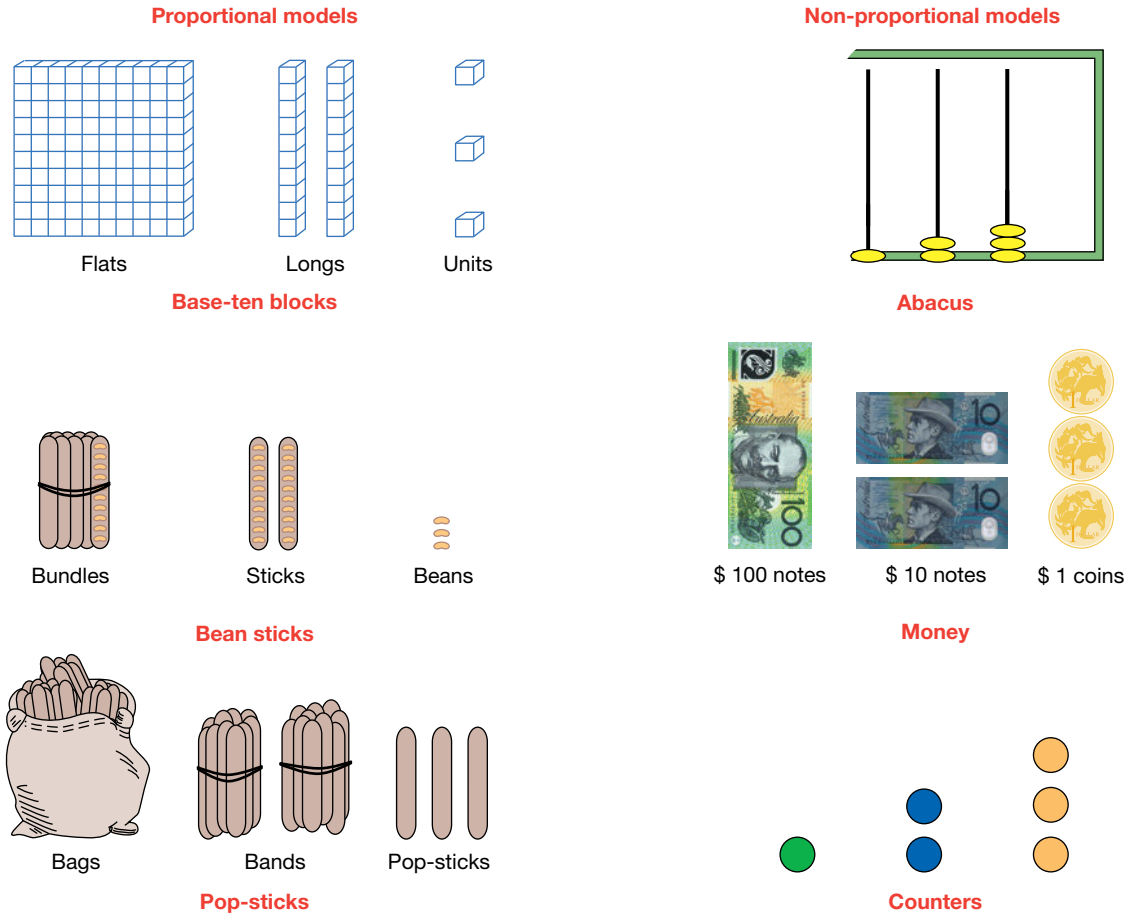


FIGURE 8.3 Place value models: Proportional models and non-proportional models

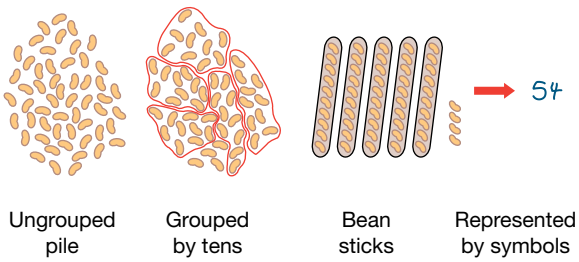
When learning about place value relationships, a smaller base can be used for exchanging or trading. Hence, experience in different exchanges — such as in the games three-land, four-land (or lizard land) and five-land — can be introduced to young children (Rogers 2009; Swan & White 2006; White 2002; Anderson 1988). In each case, an amount of counters (or blocks) of one colour is exchanged for one counter of a higher value. This is explained in more detail in the Spotlight on early childhood. Games with concrete materials like these can be most beneficial in developing abstract ideas about trading and place value. Although non-proportional models do not demonstrate size relationships, children 3–5 years of age can work with 4, 5 or 6 objects easily without relying heavily on counting skills. In contrast, working with 10 objects in a group before bundling or exchanging can be difficult for young children. Furthermore, the use of base-ten blocks does not necessarily result in the development of the number system (Price 1998) because the blocks are premade and the learners need to understand the place value relationships for themselves.

Both proportional and non-proportional models (as shown in figure 8.3) are important and should be used, as each has their own advantages and disadvantages. It is recommended that a range of activities and materials be used (AISSA 2004) and that base-ten relationships be developed alongside trading activities. In everyday life, children often focus on size proportionality, which is why they might believe a 50-cent coin is worth more than a $1 or $2 coin. Exchanging experiences with money such as using a collection of coins in a purse or pouch and making the statement 'my coins are too heavy, let's trade some' will lead young students to a deeper understanding of the abstract nature of our numeration system.

All types of modelling are useful. Ungrouped and grouped, proportional and non-proportional materials all have their place. It is important to remember that students learn in different ways according to prior experiences. Beginning with ungrouped proportional material, such as pop-sticks or beans, is a good starting point for most students. Using small quantities and bundling or trading with one-to-one representations can then progress to non-proportional materials, such as coloured counters or interlocking bricks/cubes, which help develop the relationships of place value through trading experiences.

Grouping and trading

Place value cannot be easily understood in abstract terms, children need experience in counting piles of objects; trading for tens, hundreds and thousands; and talking about the results. The bean sticks and coloured counters with place value mats provide 2 early models for counting and grouping. As children work with these models, they use the materials and others (for example, beans, buttons, pop sticks, cubes or bricks) to practise our numeration system. These ungrouped materials provide valuable experience and prepare children for using pre-grouped materials. Notice in the diagram that there is a logical transition from an ungrouped pile of beans, to grouping by tens, to representing that same quantity with pre-grouped bean sticks, to recording symbols.



The transition from 10 beans to 1 stick with 10 beans illustrates how the number 10 is composed of 10 ones.

While ungrouped materials provide practice in counting and grouping, it is tedious and time consuming to count beans and to build models as quantities increase. For example, a large pile of beans is impressive and useful for demonstration but often not practical for use by individual students. Pre-grouped materials make it easy to build larger numbers but the use of these shouldn't be rushed.

Trading and grouping by tens gives rise to problem-solving situations that contribute to number sense and provide opportunities for developing mental computation. Asking children to group by tens as they count the larger piles serves several valuable purposes. First, if a child loses count, correction is easy when the smaller groups have been formed. It is also easier to check for errors by inspecting groups of 10 than to recount the entire pile of beans. Hence, the most important purpose of this practice is having children explore how an unknown quantity of objects can be determined by organising the collection from scratch. These experiences should not be rushed as the process of grouping by tens and exchanging is the foundation of place value.

SPOTLIGHT ON EARLY CHILDHOOD

Trading games

Giving children in Years 1 and 2 the opportunity to participate in place value games can help inform their teachers of what the children are thinking and how much they understand of our number system. To play these games, children work best together in small groups. Each child has a trading mat (Appendix C), and each group has a collection of coloured counters and either specially labelled or traditional dice. For younger students, a banker can be nominated to perform the exchanges.

The basic processes for the games are as follows.

1. Each child rolls the die (or dice).
2. They count (or say) the total on the dice.
3. They collect yellow counters from their pile or from the banker; and place them on their mat.
4. An exchange for a counter of larger value is made, if possible.

The basic conversions for trading in base ten are 10 yellows equals 1 blue, 10 blues equals 1 green (100 yellows), and 10 greens equals 1 red (1000 yellows). For the game 'ten-land' a 10-sided die (numbered 0–9), a dodecahedron die (numbered 1–12) or 2 traditional 6-sided dice can be used. The aim of the game is to become the first player to collect 100; the winner is then given the option of playing as the next game's banker. Play should allow all children in the group to win a green in ten-land.

To begin trading games from the very beginning with young children, smaller-sized collections are better and children should be given the opportunity to work in different 'lands'. Three-land, four-land (or Lizard Land) and five-land are great starting places. The basic rules vary, depending on the size of the land: 3 yellows equals 1 blue in three-land, 4 geckos equal 1 lizard in four-land, and 5 yellows equals 1 blue in five-land. The die should have numbers smaller than the land initially before children progress to the extension ideas. That is, in three-land the die should have just 1 and 2 numbered on it, in lizard land 1, 2 and 3 on it, and in five-land 1, 2, 3 and 4 on it. Later, students can proceed to a traditional die numbered 1 to 6. The games are relatively fast in smaller lands because children can subitise the collection easily and they do not get bogged down in counting. Working in small groups of 4 or less allows the game to move quickly.

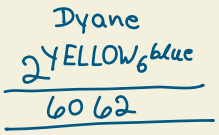
Finally, children should be warned about collecting counters and not exchanging; these are known as illegal collections. By the end of their turn, the place value mat should always have groups of counters less in number than the land the children are playing in. For example, a collection of 1/2/6 is fine in ten-land and represents the number 126 but it is an illegal collection in three-land. After exchanging the 3 yellows for 1 blue (twice) and 3 blues for 1 green, the 1 green counter, 2 blue counters and 6 yellows, would become 2/1/0 in three-land.

When it is time to conclude the activities, children can be encouraged to record their collection on a small place value sheet. After participating in three-land, four-land, five-land, six-land and ten-land activities regularly, children in Years 1 and 2 can be asked to discuss their numbers as a whole class. Seeing what students wrote on their record sheets in pictures or symbols — as well as asking them to compare the size of their numbers, add on or subtract 10 from their number and round up or down — can lead to valuable insights.

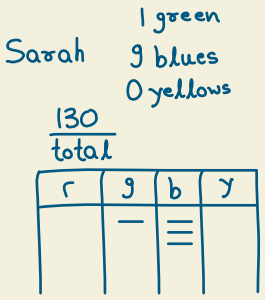
The following recordings by Dyane, Sarah, Jessica and Bill, all Year 1 students working in ten-land, were collected after a lesson late in the school year. These students had participated regularly and eagerly in various trading games with their classmates (Bruschi 1990).

In these recordings, yellow represents ones; blue represents tens; green represents hundreds; and red signifies thousands.

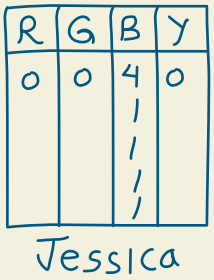
Dyane had written with words 2 yellow 6 blue and the symbols 60 and 62.



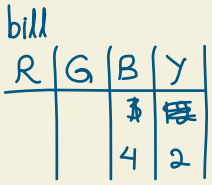
Sarah had shown a pictorial representation of her collection with rectangle counters on a place value mat and written 1 green 9 blues and 0 yellows in words (she has made an error in her writing but 3 blues actually is the same as 9 yellows). The symbol 130 is correct.



Jessica had symbolised 40 with an excessive use of zero as a placeholder.

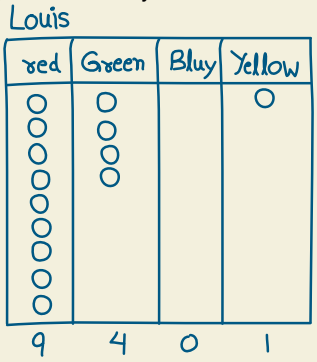


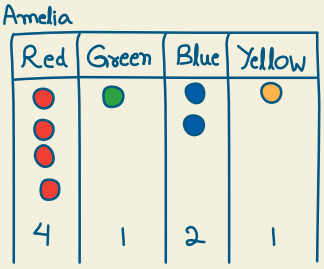
Bill wrote down his result as 3 tens and 12 ones and then traded in his mind, recording 4 tens and 2 ones beneath, with 42 as his result.



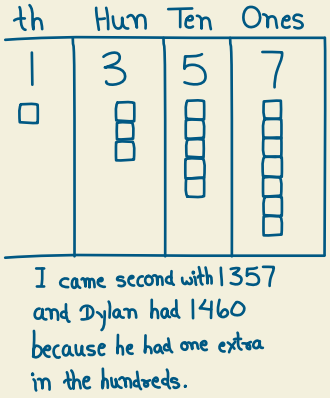
Further research with students in schools and work with pre-service and experienced teachers has confirmed the benefits of these trading activities for the development of mathematical language, mathematical thinking and reasoning, recording of multi-digit numbers and social interaction (Rogers 2009). The following recordings of Louis, Amelia and Nicholas are examples of more advanced activities involving place value. The children were in Year 3 but benefitted greatly from review of concepts that had not yet made sense to them. A variety of levels of play over many weeks, including prior knowledge activities and extension games into subtraction and multiplication, are recommended with these place value materials.

Louis and Amelia, were both Year 2 students, who recorded the larger numbers 9401 and 4121 respectively, using pictures and symbols.





Nicholas, a Year 3 student, interpreted his result of 1357 as smaller than the 1460 scored by Dylan.



Children's attitudes towards mathematical learning as a result of their early place value experiences were extremely positive. Anecdotal evidence collected from students confirmed for the researcher (Bruschi 1990), the many benefits of place value activities. One junior primary student, Anthea, wrote the following:

> Thank you for teaching us trading. I like it very much. It was very easy when we were in three land. But it got harder than three land. When we got up to ten land it was very hard. I like ten land the most.

8.3 Beginning place value

LEARNING OUTCOME 8.3 Identifying effective methods of introducing the concept of place value to children.

A place to start

Misconceptions such as those described in figure 8.4 are commonplace when children are learning about numbers beyond 10. These examples illustrate confusion involving different aspects of place value. Later errors in computation can often be traced back to a lack of fully understanding place value (Whitenack et al. 2004). The importance of place value makes getting a good start essential.

Counting	Becoming confused when counting the teen numbers because of the lack of pattern in the numbers from 11 to 19
Bridging the decade or hundred	Making the transition when counting, for example, counting aloud 'thirty-eight, thirty-nine, thirty-ten' or writing '38, 39, 3010'
Reversing digits when writing numbers	For example, writing 52 and 25 and not recognising any difference in these numbers
Writing numbers read out loud	For example, writing 'one hundred and sixty-four' as 100604

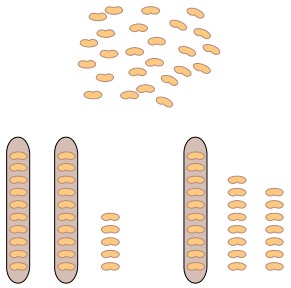
FIGURE 8.4 Some common errors related to place value

For students having difficulty with the teen numbers, it is better to skip to the twenty and thirty decades and start with the larger numbers. Figure 8.4 illustrates that the names for the numbers 11–19 in English are not consistent with the names for other numbers, even though the symbolisation or visual

pattern is wholly consistent. This lack of consistency in counting the teen numbers is confusing; children need to explore the vocabulary of larger numbers before the place value counting patterns are evident. The numbers 11–19 do not exhibit the place value characteristics in their names that other numbers do. To do so, they would have to be renamed onety-one (1 ten and 1 one), onety-two, onety-three . . . onety-nine, which would make them consistent with larger numbers, such as thirty-one, thirty-two, thirty-three up to thirty-nine. In some other countries, such as Japan, the naming pattern for the numbers 11–19 is consistent with the naming of larger numbers (Yoshikawa 1994), which gives children a great start with place value understanding. Vocabulary for counting, naming and representing numbers is an issue among many countries.

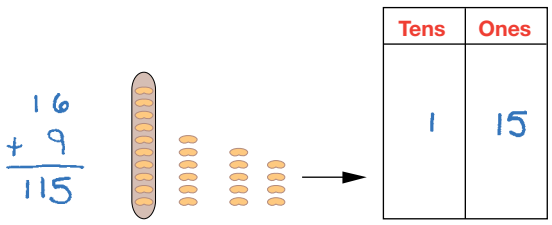
What does a number such as 25 mean? The number 25 can be thought of as 25 ones, but in written form it is composed of 2 tens (20) and five 5 (5). It can be decomposed into 1 ten and 15 ones as well as 5 groups of 5 ones. It is important that children have experiences thinking of numbers in various ways. The ability to compose and decompose numbers in different ways reflects good number sense. For example, when thinking of money, several combinations of coins — such as 1 20-cent coin and 1 5-cent coin; 2 10-cent coins and 1 5-cent coin; 1 10-cent coin and 3 5-cent coins; or 5 5-cent coins — may be imagined. With certain items, such as eggs, the collection might vary. Twenty-five eggs is 1 more than 2 dozen and 4 six-packs plus 1 extra. These are different but equivalent representations of the quantity 25 and illustrate 2 different ways of composing or decomposing 25 meaningfully. With metric measurements, 25 centimetres might be thought of as 2 decimetres and 5 centimetres; however, it could also be thought of as 0.25 metres or one quarter of a metre. Children who have developed good number sense know when 1 particular form of a number is useful.

Of course, if place value is to be called on, the tens and ones model is needed. The number 25 in monetary form could be 2 10-dollar notes and 1 5-dollar note. When using the model of bean sticks, 25 beans could be represented several different ways:



Which way is better? It depends on what is to be done with the 25 beans because there are times when each form may be useful. The grouping at the top, for example, would be easier to divide among five people equally. Either of the groupings on the bottom would be easier to count and to add or subtract with.

The notion of representing a quantity with the least number of pieces for a particular model is critical in place value. Establishing its importance at an early stage can eliminate some later errors such as:



Because 10 or more ones exist on the place value mat, a trade must be made. Making the 10-ones for one-10 trade results in the least number of pieces and thus 25 becomes the most meaningful representation.

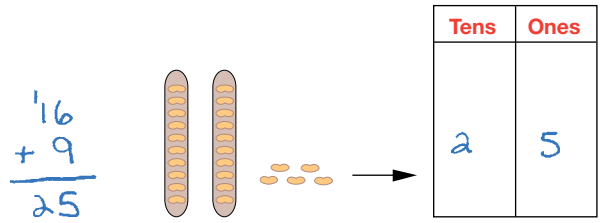


Figure 8.5 highlights the advancement from a concrete model to a symbolic representation via a pictorial or visual representation. The bridges from the physical models to the symbolic representation must be crossed back and forth many times if meaningful learning and deep understanding is to occur. Careful attention must be given to making a link between the model and the language. As children become fluent in talking about their models, it will become natural for them to describe 25 in different ways.

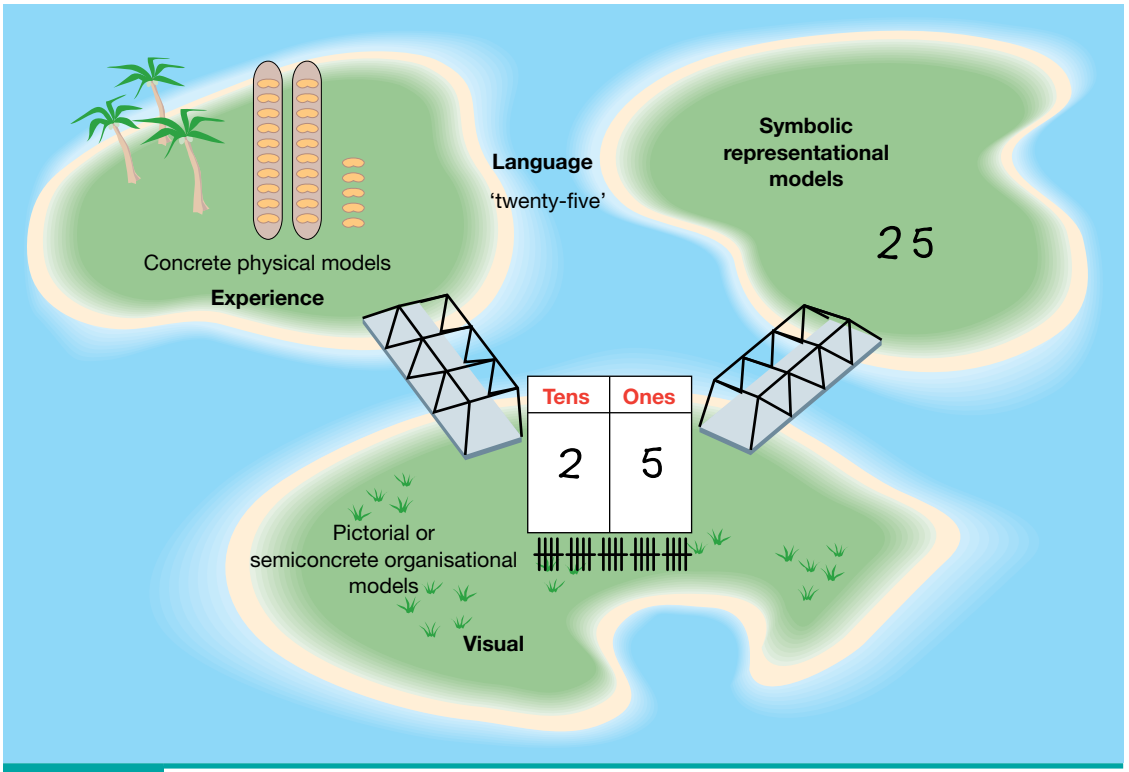
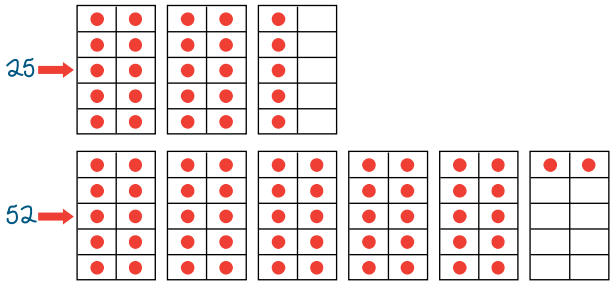


FIGURE 8.5 The idea of place value can help children cross from concrete models of numbers to their pictorial and symbolic representations. The language of the number encompasses all.

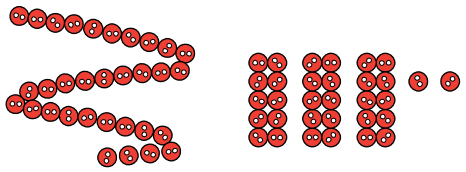
It can occur that children reverse the digits of numbers when they are still learning about place value. On the other hand, this error might be symptomatic of a disability known as *dysnumeria*. In either case, it is important that children understand the consequences of such reversals. Consider, for example, the

numbers 25 and 52. The ten-frame (Appendix C) provides a convenient model for representing these 2-digit numbers.



Seeing the numbers visually as a quantity will remind children of the physical differences between 25 and 52. The two numbers are formed with the same digits but are completely different because of place value. Children should compare the modelled numbers and talk about them in an effort to better appreciate the magnitude of the differences. Once again trading games in ten-land will help children overcome this kind of problem as it takes longer to collect and exchange 52 counters than it does 25 counters.

When quantities of objects are organised into groups of 10, it not only illustrates place value, but it facilitates counting. For example, these 2 piles show the same number of buttons. One has all of the buttons in a string. The other has composed the buttons into groups of tens and ones. Decide how many buttons are in each pile.



If you want to easily know how many buttons there are in each pile, which arrangement would you use? Explaining which pile and why that arrangement was chosen leads to a discussion of how grouping by tens facilitates counting and organising larger quantities.

8.4 Consolidating place value

LEARNING OUTCOME 8.4 Describing what happens when numbers are composed and decomposed, initially to 100, then 1000 and beyond.

Students require a great deal of active experience with making quantities; composing and decomposing numbers; recording multi-digit numbers and connecting models, pictures and symbols in their primary school years. This learning cannot be rushed as it might lead to a dependence on rules not fully understood. Later confusion with the four operations, decimals and measurement can be often linked to inadequate place value understanding and insufficient practice with a variety of number embodiments. Furthermore, skills such as estimation when solving problems is enhanced by deep place value understanding. Constructing knowledge about our numeration system is a complex process for learners. In the classroom 8.1 features a model of a dartboard that shows an excellent way of bridging understanding for visual learners. This model helps to develop place value understanding and provides practice in important mental computational skills via a pictorial model. Scores are found by counting the darts in each circle, which provides practice for counting by ones, tens and hundreds and then combining the totals.

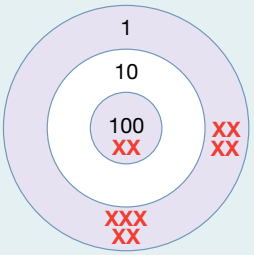
IN THE CLASSROOM 8.1

What's in a score?

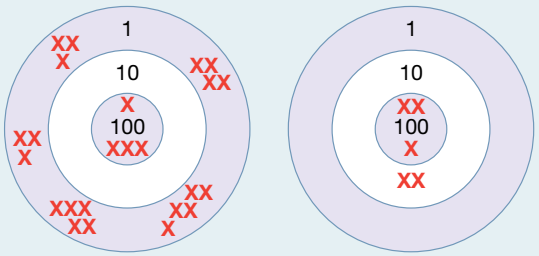
Objective: Count by multiples of ones, tens and hundreds to develop place value understanding.

Year level: 2–3

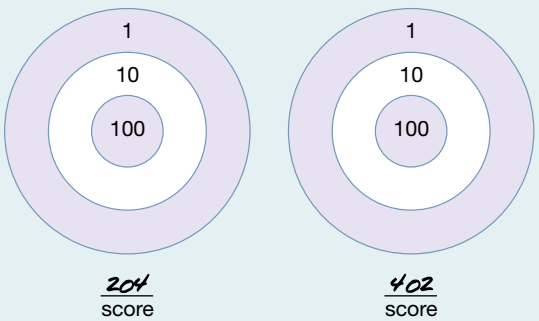
Here is how darts landed for this score.



- Determine the score for the following boards.



- Place darts on these boards to show how the following scores might have been made.



Create some problems for the dartboard to share with a friend and challenge him/her.

Another model, the base-ten blocks, together with number expanders and a place value mat, as shown in figure 8.6, can be used to make connections between concrete, pictorial and abstract representations of numbers. The additive property of place value is illustrated for the number 123. A variety of strips, such as those in figure 8.6, which connect the base-ten blocks to number symbols in expanded form, can be used effectively to review key ideas.

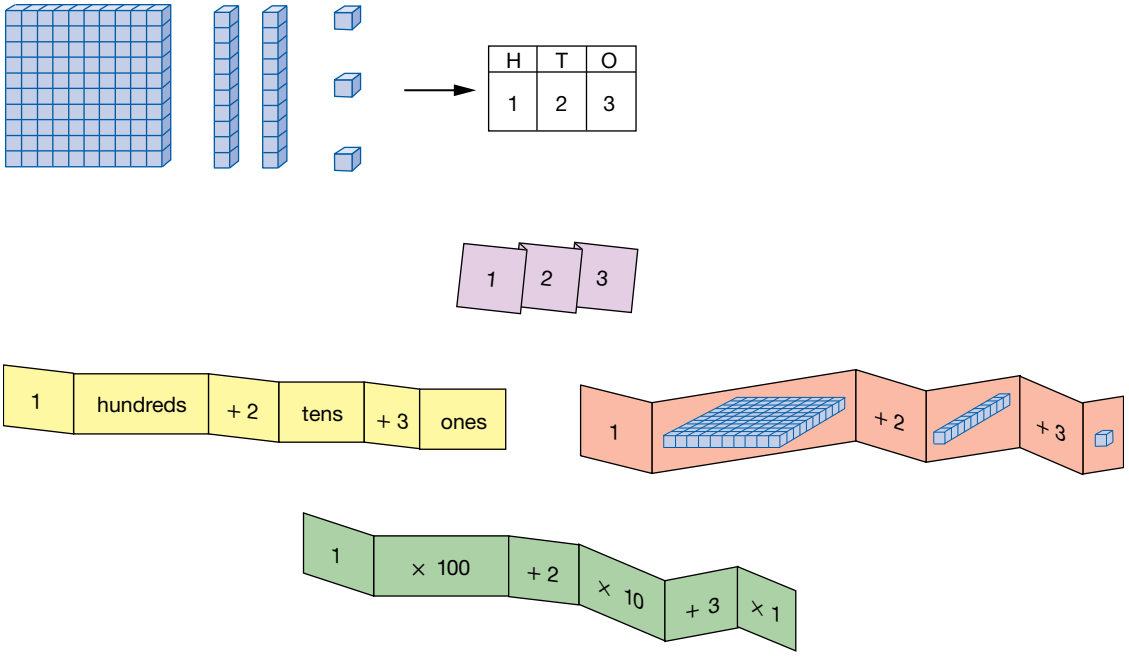


FIGURE 8.6 Connecting models and symbols that reinforce place value

While the number 123 is composed of 1 hundred, two 2 and 3 ones, it can be useful to decompose the number differently on occasion. For example, if the school had 123 students to take on an excursion with 3 buses available (that is, 123 is divided by 3), it would be easier to think of 123 as 12 tens and 3 ones. The 12 tens shared equally put 40 students in each bus and the 3 ones shared gives 1 per bus. The result of this operation is 41 children per school bus. Encouraging children to name numbers in different ways promotes number sense and real life purpose.

Regrouping and renaming

Recent research (Rogers 2014) has highlighted the difference between regrouping and renaming with *rename* being 'possibly the most critical aspect of place value' (Rogers 2014, p. 295). Whilst the skill of regrouping develops primarily with models and materials as illustrated in figures 8.7 and 8.8, the ability to rename is more abstract. It relies on other levels of place value understanding and demands deeper thinking, in particular multiplicative thinking.

One of the most important ideas for young children to gain initially is that of regrouping. By observing what happens when 1 is added to an amount of 9 (or 19, 29, or 149 for example), regrouping or trading needs to occur. Adding 1 more will create a group of 10, which needs to be traded for 1 ten. The pattern involved in bridging from one decade to another should be recognised and clearly understood by children.

Regrouping and place value are intertwined in later development of estimation, mental and written computation. Regrouping happens whenever bridging occurs, as from 1 ten to another (such as 29 to 30) or from 1 hundred to another (such as 799 to 800). Regrouping or decomposing also happens when, for example, 6 tens and 7 ones (67) are considered as 5 tens and 17 ones, or 245 is thought of as 24 tens and 5 ones or 40 ten-cent coins are traded for 4 dollars. Money is more complex to understand but it

is a real world material that children might have greater confidence and experience with exchanging or regrouping. The importance of clearly understanding the regrouping (composing and decomposing) process cannot be overemphasised. Understanding is most likely to develop when children experience this process with physical models and practise trading up and trading down (composing and decomposing).

Whenever trading occurs, there are accompanying changes in how we record the number. Understanding this changed notation requires many experiences with problems involving trading and the related recording process. A variety of different models will support students in understanding place value and regrouping and renaming.

Figure 8.7 shows how regrouping affects digits and place value using the example of money. Figure 8.8 further illustrates the regrouping process with 2 different models — proportional and non-proportional. Extending to thousands can be done with proportional models to illustrate the dramatic size increase that occurs as new places are used. However, children soon recognise that it becomes cumbersome to model large numbers with proportional models (see the powers of 10 place value chart in Appendix C).

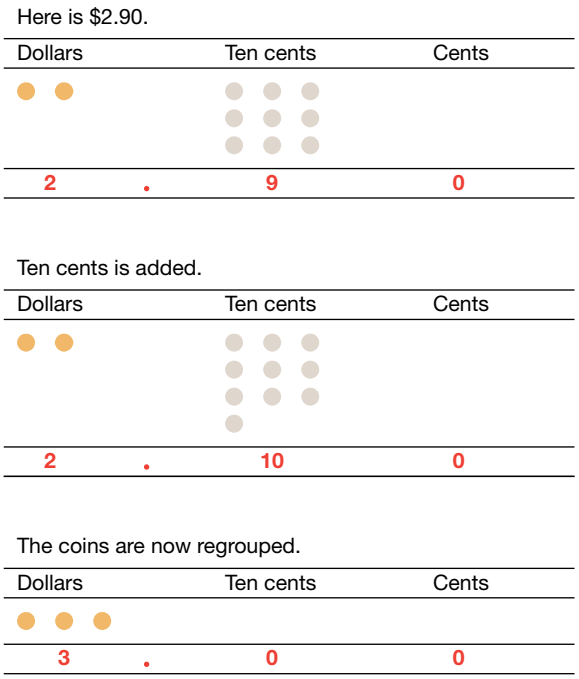


FIGURE 8.7 Non-proportional model illustrating the relationship between regrouping and place value

Although it is an abstract tool, the calculator can be successfully used to illustrate regrouping with very large numbers. For example, estimating how long a calculator takes to count to 1 thousand or 1 million can help students develop a better grasp of large numbers. As the magnitude of numbers increases, physical models of them become more difficult to represent.

A calculator provides many opportunities to practise and develop important place value concepts. Wipe out is a place value game that involves either addition or subtraction using a calculator. The goal is to change (wipe out) a predetermined digit by subtracting or adding a number.

This activity can be made into a competitive game for 2 people. The players take turns entering a number and naming a specific digit the other player must change to 0.

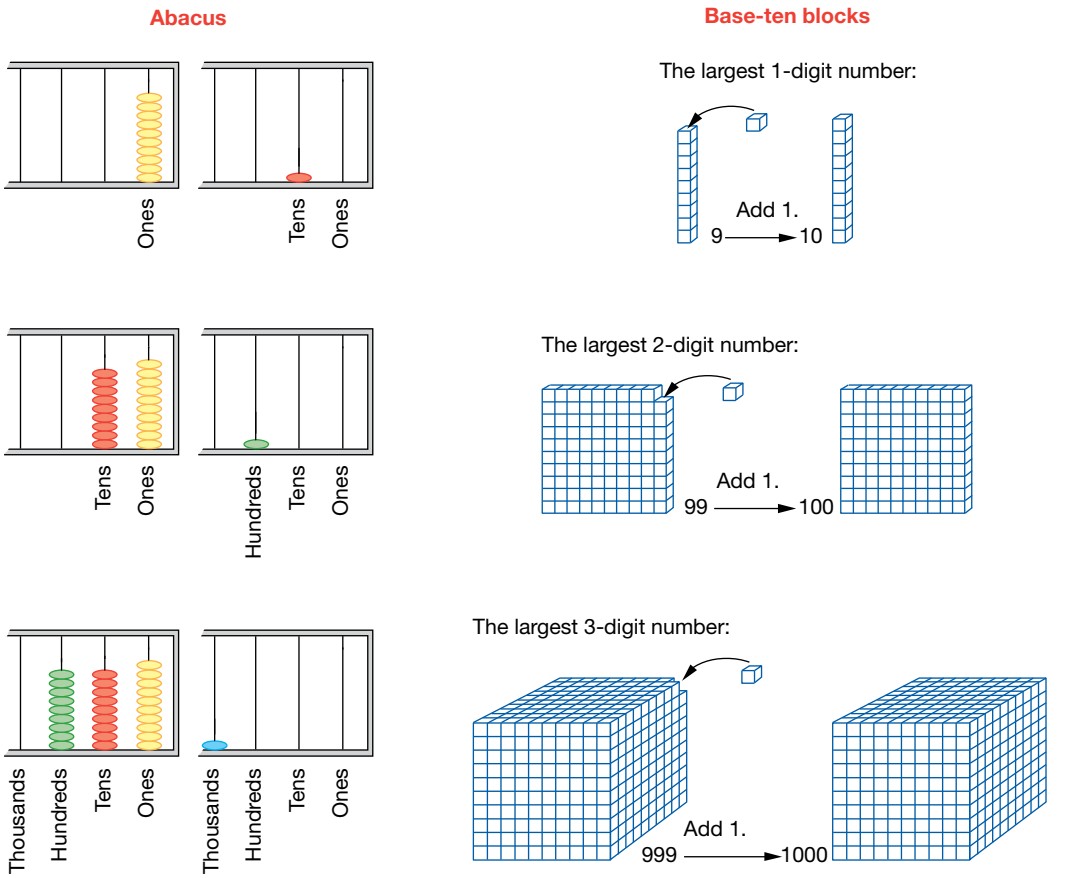
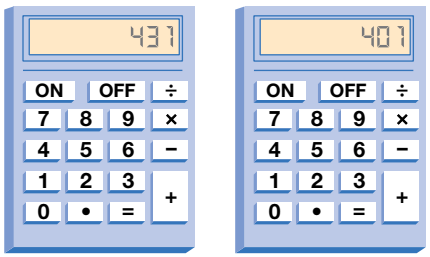


FIGURE 8.8 Non-proportional and proportional models illustrating further relationships between regrouping and place value

For example, Kelly enters 431, naming the 3 to be wiped out:



Tanya wipes out the 3 by subtracting 30, which also leaves the other digits unchanged.

A player scores a point for changing the digit to 0 on the first try. A record of the game in table form reinforces the identification of the correct place value:

Wipe out record Name: Tanya

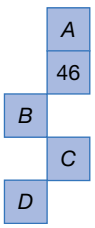
Entered	Wiped out	Keys pressed	Display	Score
431	3	-30	401	1
24	4	+6	30	1
849	8	-800	49	1
206	2	-200	6	1

Games such as Wipe out can be played without a calculator, it is much more exciting when one is used. Children can use mental computation and estimation without getting bogged down with written computation. The focus remains on place value. Furthermore, children are often surprised by what happens when they make a place value error, or discover a misconception they have previously held, which increases their place value understanding.

In the classroom 8.2 shows 2 slightly different hundred charts that can be used to explore concepts and further consolidate understanding of tens and ones. One chart goes from 1 to 100; the other from 0 to 99 (master copies are in Appendix C). One advantage of the latter chart is that the tens digit in each row is constant, whereas in the first chart the tens digit always changes in the last column. Using the 1-to-100 chart eliminates the problem of students starting on 0 and being confused with their counting. Either of these charts can be used for exploring place value, looking for patterns, applying mental computation and developing algebraic thinking.

The hundred chart provides many opportunities for counting on with tens and doing mental computation. For example, 43 + 30 can be determined by counting mentally 43, 53, 63, and 73, which is a natural method for counting by tens on the hundred chart. Similarly, 43 + 29 can be found by counting by tens 43, 53, 63, 73 and then dropping back one number, to 72. Counting by tens and then dropping back or bumping up illustrates how to adjust numbers and be flexible when thinking about numbers.

Consider the following diagram and how patterns and generalisations lead to algebraic thinking:



Thinking mathematically with place value and patterns, these solutions can be found directly from the hundred chart:

$$A = 46 - 10$$
$$B = 46 + 10 - 1$$
$$C = 46 + 20$$
$$D = 46 + 30 - 1$$

If the diagram is shifted to an unknown position on the hundred chart, then 46 changes to a number N. Although the starting number N has changed, the solutions are similar to before:

$$A = N - 10$$
$$B = N + 10 - 1$$
$$C = N + 20$$
$$D = N + 30 - 1$$

Such activities illustrate how a variable and the power of patterns and place value can promote algebraic thinking.

IN THE CLASSROOM 8.2

Objective: Use a hundred chart to explore place value and pattern.

Year level: 3

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Fifth column

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

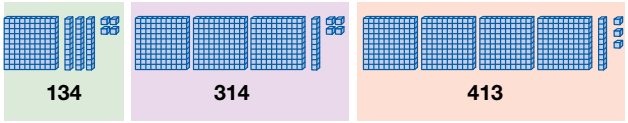
Fifth column

- Use the hundred charts to answer these questions:
  - What is the same for all the numbers in the fifth column in the first chart? In the second chart?
  - How are the numbers in the fourth and sixth columns the same? How are they different?
  - Explain where to stop when starting from any square in the first 3 rows and counting forward 10 more squares.
  - Starting on a different square in the first 3 rows and counting forward 10 more squares, where is the stopping space?
  - After doing this several times, explain the pattern that has been found.
- Cut out an L shaped piece like this  and lay it over 3 squares in the charts.
  - What numbers are covered? Describe them and the pattern found.
  - Move this piece to a different place and explain what numbers are covered.
  - Repeat this process again with different shapes like 2 in a row 
  - And 4 in a square  .

- Here is a part of the hundred chart with one known number and 4 unknowns:
  - Use what is known about a hundred chart to:
    - Find *A* _____ *B* _____ *C* _____ *D* _____
    - Explain 2 different ways to find *C*.
  - Supposing that 46 is replaced by *N*, find *A* _____ *B* _____ *C* _____ *D* _____.

	A		
	46		
B			
		C	D

Using concrete materials to model three-digit numbers provides opportunities to show how the rearrangement of digits impacts upon a number. Notice the physical difference between 134, 314 and 413.



Looking at the different arrangements shows that 413 is almost 100 more than 314 and that 134 is more than 100 less than 314. This ability to focus on the lead, or 'front-end', digits is an important part of number sense and estimation.

Furthermore, developing the right language to recognise and write a number is an essential part of place value understanding. It becomes increasingly more difficult for each place value. For example: Thirty-four or 34 can be named as 3 tens and 4 ones or 34 ones which is possible for children in the early years to read, write and compare. But when progressing to one hundred and thirty four or 134, the name can be 1 hundred 3 tens and 4 ones or 1 hundred and 34 ones or 134 ones, which are all different relationships. When progressing to thousands, the idea is far more complex and needs time to develop. A new 'family' of numbers is created. Three-digit numbers for the 'ones' family consists of HTO (hundreds, tens and ones) but the next 'thousands' family consists of another three digits with an additional HTO with up to 6 digits as shown in figure 8.10. Naming and renaming these larger numbers demands much higher order thinking and can take years to develop.

A large-scale, long-term assessment exercise asked students to: 'Write a number given 3 digits (say, 7, 2 and 5) and the place value of two of the digits (say 7 tens and 5 hundreds)'. Figure 8.9 shows that between one-half and two-thirds of the students were able to provide a correct answer (572) and that Year 8 students did not score much differently than Year 4 students (Kouba & Wearne 2000; Warfield & Meier 2007).

Year level	1990	1992	1996	2000	2003
Year 4	50	54	58	67	65
Year 8	55	60	60	62	63

FIGURE 8.9 Percentage of Year 4 and Year 8 students who correctly answered a place value item involving 3 digits

8.5 Extending place value

LEARNING OUTCOME 8.5 Understanding the use of models, language and pattern for recording and comparing large numbers in base ten.

Research reports that many children lack an understanding of the relative sizes of numbers greater than 100 (Rogers 2014, Fuson 2003; Payne & Huinker 1993). Misunderstandings can result from many factors — one of which may be that students are forced to symbolise too soon. That is, there

is insufficient opportunity to model large numbers, which helps children develop a visual awareness of the relative sizes of numbers. For example, a number such as 123 has several different equivalent representations:

One hundred and twenty-three
12 tens and 3 ones
$100 + 20 + 3$
$1 \times (100) + 2 \times (10) + 3$
$1 \times (10^2) + 2 \times (10)^1 + 3$

For large numbers, children can use variations of the models shown in figure 8.3. Namely base-ten blocks, beans and bean sticks, pop sticks, counters and the abacus to gain more practice in determining the size of 3-digit numbers. For larger numbers into the thousands another place value mat can be placed to the left of the original mat showing hundreds, tens and ones. This thousands mat will show hundred thousands, ten thousands and thousands (see figure 8.10). A third mat would take the representations into the millions. Base-ten blocks, pop sticks and counters can all be used to model the large numbers with most of these models being non-proportional due to the size involved. In addition to these physical models, children's literature like books suggested at the end of the chapter, will help to provide real world connections to larger numbers.

Figure 8.10 provides an illustration of how 2 numbers can have the same digits but be different. The question might be put, 'How are 2130 and 1032 the same and different?' As children engage in discussion to answer this question, their knowledge of place value and their sense of numbers will grow. Using the same digits to represent different numbers helps children appreciate the importance of representing the place values accurately. Although the numbers 2130 and 1032 use the same digits, the models that represent these numbers are dramatically different. Furthermore the symbolic representation of them is different.

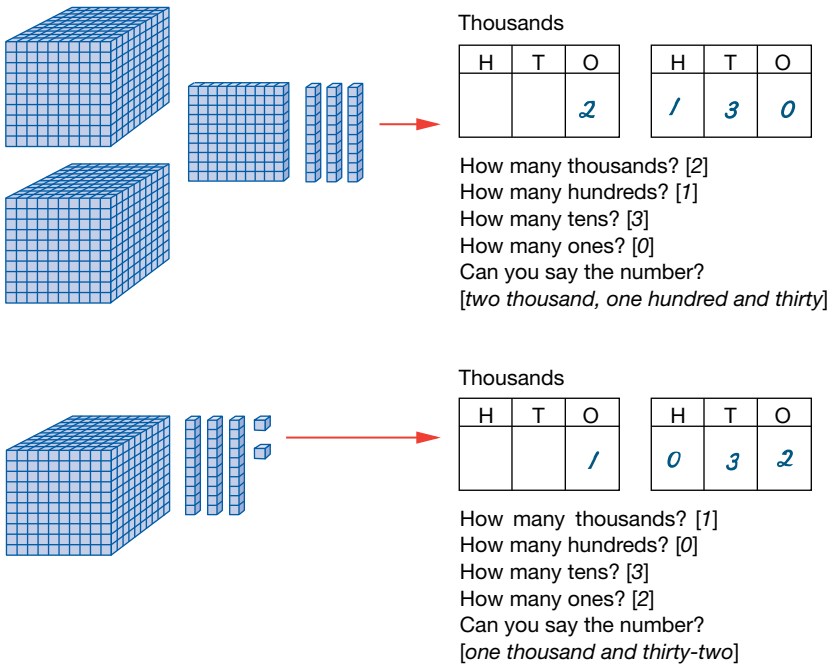


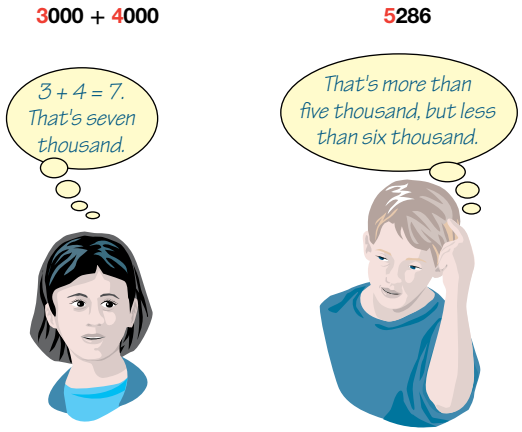
FIGURE 8.10 Using a thousands mat to demonstrate how 2 numbers can have the same digits but be different

For example:

- two thousand, one hundred and thirty is 2130 or in expanded form is $2 \times (1000) + 1 \times (100) + 3 \times (10)$ or in scientific notation it is $2(10^3) + (10^2) + 3(10^1)$ — there is no ones value in this number
- one thousand and thirty-two is 1032 or in expanded form is $1 \times (1000) + 3 \times (10) + 2$ or in scientific notation it is $(10^3) + 3\ (10^1) + 2(10^0)$. There is no hundreds value in this number.

Notice how the language used to describe these numbers in scientific notation uses powers of ten. The names *ten cubed* and *ten squared,* can be seen directly in the base-ten model shown in figure 8.10. This special form of expanded notation is introduced to students in upper primary, Year 7 according to the Australian Curriculum. Expanded notation can help develop number sense and alerts children to the importance of the front-end, or lead digits, which in this case is thousands.

When considering larger numbers, children should be encouraged to focus on the front-end digits as these are helpful for comparing numbers, placing them in order of size and for doing mental computation or estimation.



The front-end approach can be naturally extended and applied to the whole number. For example, students could be asked to decide which number is larger when the front-end digits are the same. In the example below the thousands and hundreds are the same so the student will look at the tens place next to compare the 2 numbers.

5	4	1	2
5	4	8	9

The same question 'Which is larger' for the example below should be easily answered when the last 2 digits are the same.

2	4	5	6
1	4	5	6

The following real-world problem helps students compare and order large numbers:

> Norway has an area of 385 252 square kilometres.
> The state of Victoria has an area of 237 629 square kilometres.
> Greece has an area of 131 940 square kilometres.
> Is it possible to find out which country/state has the greatest area by comparing the first front-end digit in this case? Why? Which one has the greatest area?



This problem reminds students of the importance of place value when comparing numbers.

Note that, the front-end approach is used only when the numbers have the same amount of digits. If you were to compare Victoria with Tasmania, which have areas of 237 629 and 68 332 square kilometres respectively, the number of digits would be different. Similarly, with Western Australia, with an area of 2 527 621 square kilometres, compared to Victoria. It would not be appropriate to use the front-end approach to compare those values.

Counting and patterns

Practice in skip counting, particularly by 2s, 5s, and 10s helps children to build on prior number knowledge and decreases bumps in the road when learning about place value. As described earlier, careful and considered use of hundreds charts can also be a valuable tool in smoothing out the bumps.

Calculators are useful in place value and pattern recognition. Seeing each value displayed on the calculator helps students develop important insight into what digits are changing and when. When children are counting by 1s, they observe that the digit on the right (ones place) changes every time they 'count', while the next digit (tens place) changes less frequently, and it takes much longer counting to change the third digit (hundreds place). Figure 8.11 demonstrates an understanding of place value by displaying results of counting by 1s, 10s, 100s and 1000s. If given the opportunity to explore and skip count by 1s, 10s and 100s with the calculator, as illustrated in the opening snapshot of a lesson, children will discover a great deal about place value. At an appropriate time, children can be encouraged to record their calculator counting using number rolls or long strips of card.

Calculator counting provides many opportunities to discuss patterns related to place value. Such counting can also contribute to a better grasp of large numbers, thereby helping to develop students' number sense. For example, having students count by 1s to 100 and 1000 with the calculator helps them better understand the magnitude of these numbers. By expanding this activity, students will come to realise that it takes about the same amount of time to count to 1000 by 1s on a calculator as to count to one million by 1000 or to 100 by one tenth or 0.1 on a calculator.

For example, one Year 4 student after doing calculator counting said, 'That means there are as many thousands in one million as ones in one thousand'. This is a profound observation — of the type that leads to a better understanding of both place value and large numbers. It also reflects a growing sense of numbers.

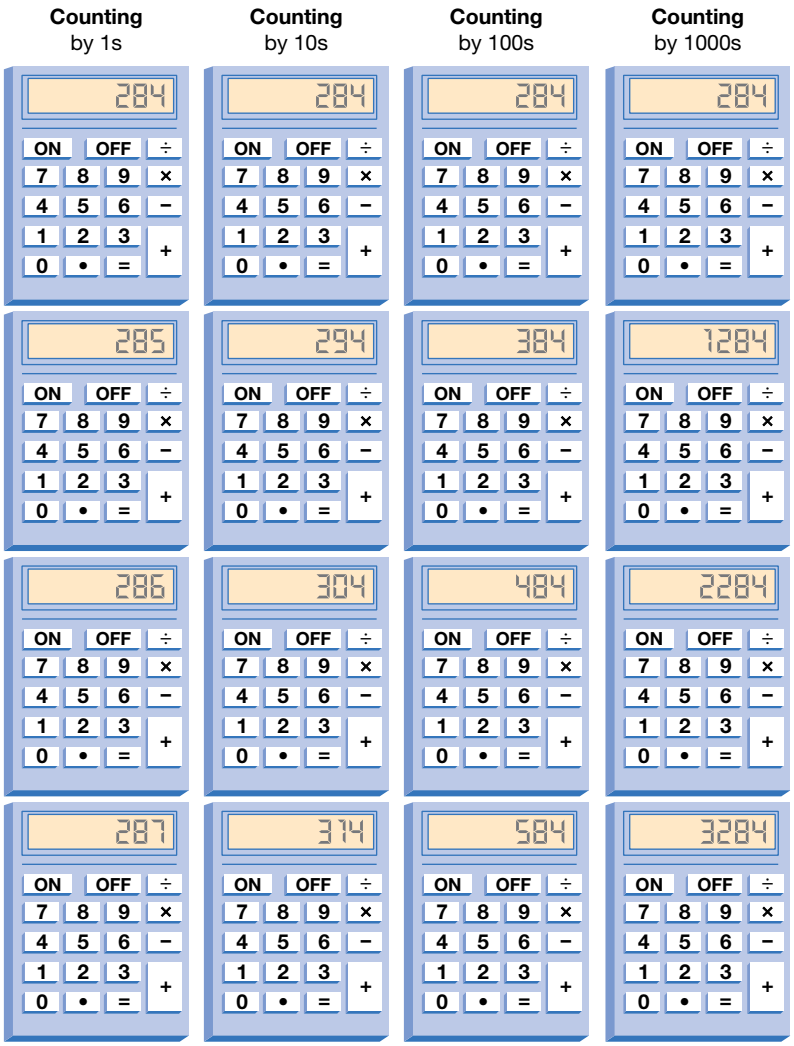


FIGURE 8.11 Calculator counting to illustrate place value patterns

IN THE CLASSROOM 8.3

Objective: Count by hundreds to develop place value and pattern recognition.

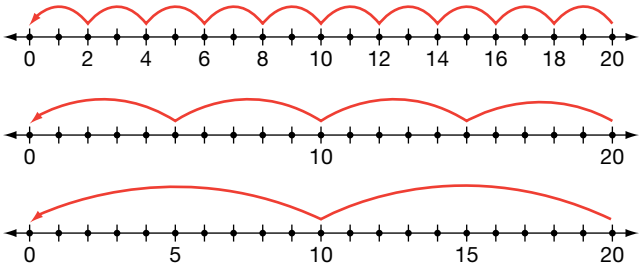
Year level: 2–3

- Use your calculator or do these mentally.
- Begin at the start number and add 100.
- Decide which target you will hit.

Start	Rule	Which target?	Guess	Check
20	+100	200 220	_____	_____
35	+100	335 500	_____	_____
41	+100	410 441	_____	_____
86	+100	586 580	_____	_____
97	+100	897 970	_____	_____
169	+100	469 696	_____	_____
123	+100	321 323	_____	_____

As figure 8.11 shows, counting on by tens, hundreds or thousands never changes the ones place; however, when counting by tens, the tens place changes on each count and the next digit (hundreds) changes every 10 counts. Observing these patterns in counting larger and larger numbers helps students recognise place value properties. Further practice such as In the classroom 8.3 encourages the recognition of patterns to help students add multiples of 100. Similar investigations could be given with other powers of 10.

For older students, counting back provides similar place value patterns. The calculator is an excellent tool for this counting down. The diagram below also illustrates how the number line model might be used to count backwards from 20 by 2s, 5s and 10s.



The number line would need to be extended to count back by more 10s. Yet the calculator facilitates counting back by 2, 3, 5, 10 or any number starting at any place as shown in figure 8.12. Place value patterns resulting from counting back are easily apparent. Observing these patterns in counting larger and larger (or smaller and smaller) numbers helps students recognise that place value properties are reflected in both positive and negative integers.

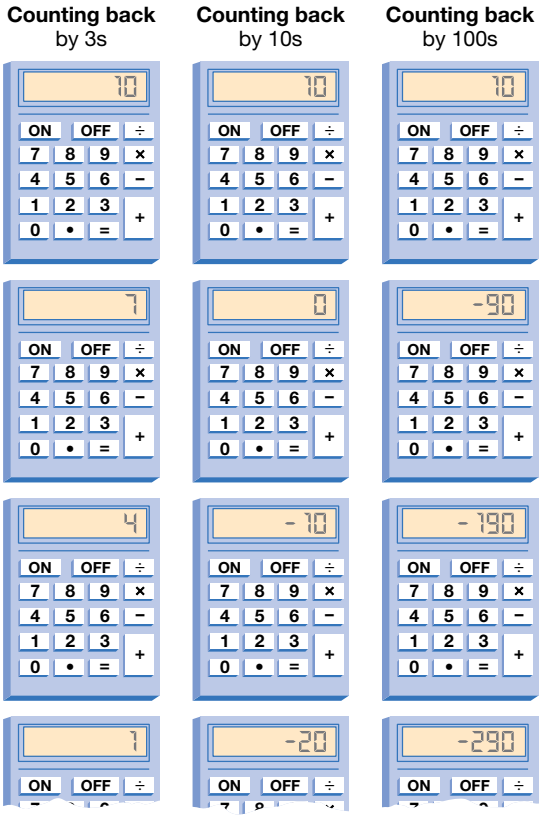
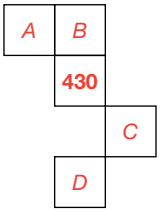


FIGURE 8.12 Counting back to illustrate place value with negative integers

Earlier in this chapter, it was seen that the hundred chart can provide a useful model for counting and pattern recognition related to place value. Activities involving counting with multiples of 10 were illustrated in In the classroom 8.2. In a similar way, In the classroom 8.4 shows an extension of these ideas to larger numbers. Consider the section of grid shown:



Thinking mathematically with place value and patterns, these solutions can be found directly from the chart:

$$A = 430 - 100 - 10$$
$$B = 430 - 100$$
$$C = 430 + 100 + 10$$
$$D = 430 + 200$$

If the diagram is shifted to an unknown position on the hundred chart, then 430 changes to a number N. Although the starting number N has changed, the solutions are similar to before:

$$A = N - 100 - 10$$
$$B = N - 100$$
$$C = N + 100 + 10$$
$$D = N + 200$$

Activities such as these can further develops students' number sense and provide practice in important mental computation skills. The hundred or thousand chart is an important symbolic representation of our numeration system; however, it needs to be used with caution for large numbers.

IN THE CLASSROOM 8.4

The power of 10 on the thousand chart

Objective: Use a thousand chart to identify patterns and skip count by tens and hundreds.

Year level: 2–3

10	20	30	40	50	60	70	80	90	100
110	120	130	140	150	160	170	180	190	200
210	220	230	240	250	260	270	280	290	300
310	320	330	340	350	360	370	380	390	400
410	420	430	440	450	460	470	480	490	500
510	520	530	540	550	560	570	580	590	600
610	620	630	640	650	660	670	680	690	700
710	720	730	740	750	760	770	780	790	800
810	820	830	840	850	860	870	880	890	900
910	920	930	940	950	960	970	980	990	1000

- Counting by 10:
  - Start on any square in the first 3 rows.
  - Count forward 10 squares, and note the finishing place.
  - Start at a different square and count forward 10 squares.
  - After doing this several times, describe the pattern that was found.
  - Describe a quick way to count 'a hundred more' on this thousand chart.
- Counting by 100:
  - Start at 240 and use the thousand chart to add 300. How was it done?
  - Start at 240 again and describe how to add 290 with the thousand chart.
- Create a new adding tens or adding hundreds problem for a buddy to solve.
- Explain what you like/dislike about using a hundred or a thousand chart.

8.6 Reading and writing numbers

LEARNING OUTCOME 8.6 Explaining how an understanding of place value helps develop students' ability to read and write numbers.

Reading and writing numbers are an abstract or symbolic activity and should follow a great deal of experience with modelling with concrete materials, recording with pictures and talking about numbers. Children who can write multi-digit numbers do not necessarily understand their true meaning or the relationship between the numbers in each of the place value columns. This recommendation is based on research with young children and alerts us to the danger of a premature focus on symbols (Payne & Huinker 1993). A sustained development of number sense should accompany reading and writing numbers. This approach ensures that the symbols the students are writing and reading are meaningful to them.

Consider some ways in which understanding place value helps develop reading and writing numbers. Take the example of the number 123. The places (hundreds, tens, ones) as well as the value of each (1, 2, 3) are easy to identify. The symbol 1 means 1 hundred or a group of 10 tens or a group of 100 ones. The 23 is both 2 tens 3 ones and 23 ones, and 123 is 1 hundred 2 tens 3 ones; 12 tens 3 ones; and 123 ones. These representations are equivalent and may be shown on the place value mat:

H	T	O
1	2	3

1 hundred 2 tens 3 ones

H	T	O
	12	3

12 tens 3 ones

H	T	O
		123

123 ones

In the classroom 8.5 provides practice to further develop place value.

IN THE CLASSROOM 8.5

Place value and the power of moving digits

Objective: Use number cards to form different multi-digit numbers.

Year level: 2–3

Materials: A copy of the 0–9 cards from Appendix C or three 0 to 9 dice.

Use cards with 0, 1, 2,...9 and give the student 2–5 cards.

Ask the student to use any 3 of their cards to create a number that is:

(a) a 3-digit number _____ _____ _____
(b) the largest 3-digit number possible with their cards _____ _____ _____
(c) the smallest 3-digit number possible with their cards
(d) the largest 3-digit even number possible with their cards _____ _____ _____
(e) a number with an even number in the tens place _____ _____ _____
(f) a number more than 200 _____ _____ _____
(g) a number less than 600 _____ _____ _____
(h) a number between 100 and 400 _____ _____ _____

The skill of reading numbers in different ways or renaming them (and understanding these representations) can be useful in many operations with whole numbers. For example, renaming provides a stepping-stone to mental computation; more specifically, it leads to multiplying a number by 10, as shown below.

H	T	O
3	8	0

Reads: 3 hundreds 8 tens 0 ones.
[*Realises there are no ones in the ones place.*]
Renames: 38 tens, which is 380.

This idea can extend naturally to larger numbers but is much more difficult for students to grasp because it is a multiplicative base ten concept. For example, twenty-three thousand is 23×1000 or 23 000. Likewise fifty million is $50 \times 1000\,000$ or 50 000 000.

A common misconception is to write number words as they sound. For example, sixty-one would be incorrectly written as 601 and one hundred and twenty-three as 100203. If a child made this mistake, the teacher could use the place value mat and counters as a model to demonstrate the difference.

Forty-one represents →

Tens	Ones
●● ●●	●
4	1

not this →

Hundreds	Tens	Ones
●● ●●		●
4	0	1

Modelling several numbers on the mat can help clarify this notion.

Similar problems exist in naming and representing larger numbers. Battista (2012) states that once students can successfully read three-digit numbers, they can be supported to read and write large numbers by focusing on the place value of hundreds, tens and ones and the family names (which are ones, thousands, millions, billions).

Open-ended questions such as those shown in figure 8.13 encourage children to estimate and think about numbers. As they share their answers and talk about different ways of understanding millions and billions, their number sense grows. Reading books listed in the Children's literature connections such as *Can You Count to a Googol*? by Robert Wells provide real-world examples of larger numbers and provide examples to representing place value symbolically. *The Wonderful World of Mathematics: A Critically Annotated List of Children's Books in Mathematics* (Thiessen, Matthias & Smith 1998), *Math and Literature: Grades K–1* and *Math and Literature: Grades 2–3* (Burns & Sheffield 2004a, 2004b) identify more books to give children additional insight and appreciation of larger numbers.

How big is BIG?

A million . . .

- dollars is _____ $100 notes.
- days is about _____ years.
- kilometres is about _____ times around the equator of the Earth.

A billion . . .

- dollars would buy about _____ hectares of land.
- seconds is about _____ years.
- people live in _____ .

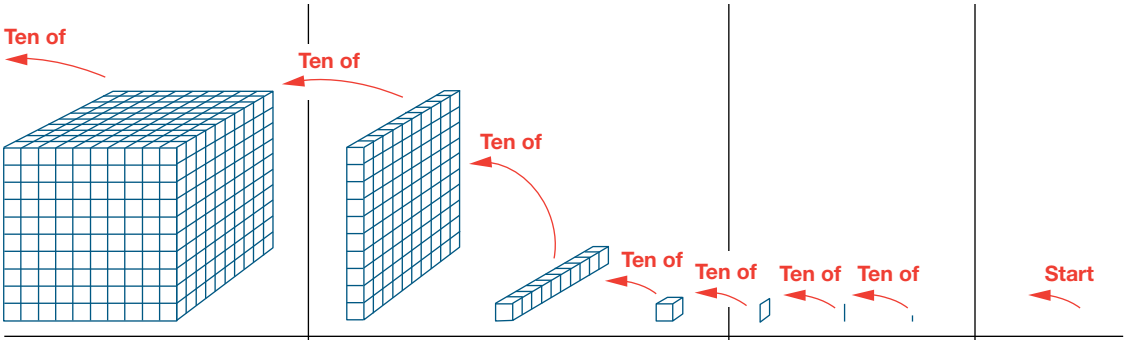


FIGURE 8.13 Questions to promote thinking about the size of numbers

It is helpful for students to link different models to larger numbers. For example, students might begin with a 1-cubic-centimetre cube. This would follow with 10 of these equal to a 10 (long), 10 of those equal to a 100 (flat), and 10 of those equal to a 1000 (block). If they then make larger models up to 1000 blocks (or a cubic-metre model, using metre length pipes/dowels and joiners) then that will hold a million centimetre cubes (1000 000 cubes). Caution needs to be expressed with these models, particularly the thousand blocks, as some students might be confused as to their real value. According to Rogers (2014) who confirmed the work of Fuson et al. (1992) the 1000 cube might be seen as 600 not 1000 because the child is seeing the surface area (6 100 flats) rather than a solid object made from 10 layers of 100.

For students of Year 7 level and beyond, the base-ten blocks can be used to help students make the connection between the concrete model and the symbolic representation, as in figure 8.14. The large cube 1 metre in size that has the value of 1 000 000 is the biggest block shown on the left. The smallest cube 1 centimetre in size has the value of 1. It is shown as 10^0 and is the smallest block shown on the far right. This model helps students mentally 'see' that 10 000 is a long piece made up of 10 large cubes, where each cube is 1000. Although this model can be constructed with physical materials, children quickly appreciate the power of constructing mental images to represent larger numbers. When students begin to develop an intuitive grasp of larger numbers and begin to use millions and billions intelligently, they are ready to write and read these larger numbers. Place value mats can be naturally expanded to represent larger numbers:

Billions			Millions			Thousands			Ones		
H	T	O	H	T	O	H	T	O	H	T	O



One	Hundred	Ten	One	Hundred	Ten	One	Oral number name
Million		Thousand					
1 000 000	100 000	10 000	1000	100	10	1	Place-value notation
10^6	10^5	10^4	10^3	10^2	10^1	10^0	Powers of ten notation

FIGURE 8.14 Connecting the symbolic representation of 1 000 000 with a concrete model

This provides students an opportunity to generalise that naming of numbers — as hundreds, tens, ones — applies not only to 3-digit numbers but to larger numbers as well. The only difference is that each block of 3 digits introduces groupings of thousands, millions, billions and much larger numbers. Zeros continue to be read as place keepers but also need care not to cause confusion.

To develop facility in reading large numbers, children need learning activities and practise in actually naming them aloud in groups. For example, the number 12 345 678 is read using the period names (families) as 'twelve million, three hundred and forty-five thousand, six hundred and seventy-eight'. This example is a clear application of an organising strategy: the digits within each period are read as hundreds, tens and ones, as with 'three hundred and forty-five thousand'. For this reason, children need to think of larger numbers (those of more than 3 digits) in blocks of 3 digits.

Recognition and understanding of the hundreds, tens and ones pattern provide a powerful organisational strategy that can be called on in naming numbers. Only the key terms — *ones*, *tens* and *hundreds* — along with recognition of the periods for thousands, millions and billions are needed to name very large numbers.

In many countries, it is agreed that commas are not used to separate blocks of three digits. Instead, for example, the number 2346457 is written with spaces between as 2 346 457, two million, three hundred and forty-six thousand, four hundred and fifty-seven. The blocks of digits remain visible but are separated by spaces rather than commas. As teachers, we can encourage children to be sensitive to the importance of writing numbers clearly and distinctly. We can justify the lack of commas by explaining that in some countries of the world commas are used to separate whole numbers and decimals rather than a decimal point.

For practice in reading and writing, newspapers provide a rich context to explore numbers of all sizes. Examine a newspaper and highlight all of the numbers reported in headlines and related stories. You may be surprised at the high frequency with which numbers occur. Naming numbers is clearly an important skill. Yet with the widespread use of calculators, a more efficient way to read multi-digit numbers is becoming common. For example, 32 764 is read as 'three two seven six four' and 4.3125 as 'four point three one two five'. Each of these readings is correct and much easier to say than the respective naming of periods. However there is the danger that children will say the digits without any realisation of the place value of the numbers involved. Whilst important, such interpretations are not necessary at

every stage of the problem-solving process. If a number is being copied from a calculator display, then a direct translation of digits is without a doubt the best way to read the number. Rather than requiring children to read numbers in a specific way, it is far better to recognise the value of each technique and encourage children to choose wisely — namely, to select the technique that is most appropriate for a given situation.

8.7 Rounding

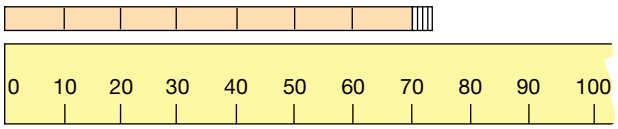
LEARNING OUTCOME 8.7 Identifying how rounding relates to place value and naming numbers.

An important aspect of developing number sense is recognising that some numbers are approximate (for example, the population of a town or city) and some are exact (such as the number of births per year in Australia). Approximate values are associated with estimation, frequently involve rounding and are encountered regularly in our daily lives.

Rounding integrates understanding of approximate values with place value and naming numbers. Numbers are usually rounded to make them easier to use or because exact values are unknown. How numbers are rounded depends on how they are used. For example, the actual attendance at an international cricket match recently was 54 321. Although the attendance could be rounded to the nearest ten (54 320) or the nearest hundred (54 300), it is more likely to be reported as 'about 54 000' or 'over 50 000' or 'less than 60000' because these values are convenient and easier to communicate.

As children develop rounding skills, they should come to realise that rounding rules may vary and are not universal. Often half way is seen as the point where the rounding of a number goes up. For example, the idea that less than 5 goes down and equal to or more than 5 goes up, is a common rule. The number 75 normally would round up to 80 and 85 would round up to 90.

The precision of the rounded numbers needs to reflect the problem context. For example, in the diagram below, a metre stick could be used as a number line to compare against the length of rods. The question is to round off the length of 74 rods made with 7 groups of 10-rods and 4 centimetre rods as illustrated. The answer will vary depending on the accuracy required and the place value in particular.

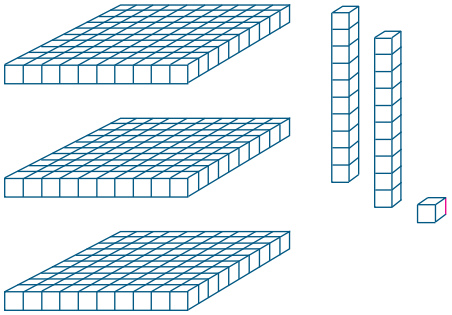


Is the length of rods closer to 70 or 80 centimetres? The answer is 70 centimetres.

Is it closer to 0 or 1 metre? The answer is 1 metre.

If you round to the nearest 10 then the length is 70 centimetres. If you round to the nearest 100 (that is, 1 metre), the length is 1 metre. How numbers are rounded depends on how the values will be used. For example, if a strip of cloth was cut to cover the length of rods, it would not make sense to round down to the nearest 10 centimetres and cut a length of 70 centimetres because it would be insufficient cloth. If however, a metre of cloth were cut, it would provide plenty of material to cover the length of rods. Children need to think about numbers in the real world context when practising the skill of rounding rather than indiscriminately applying rounding rules without a context.

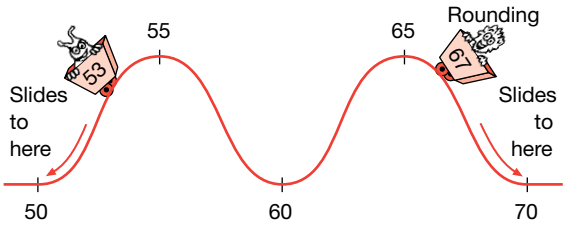
Base-ten blocks are a useful concrete model for developing and scaffolding children's ability to round numbers to the nearest 10 or 100. The question below is to round off the number 321 to the nearest 100. The number in this case represents a real world context of children in a primary school and the problem is whether there is enough seating in the school hall for a particular event. The questions below focus attention on the quantity and the idea of *closer to*, which is essential in rounding numbers.



Is this amount more than 300? [*Yes*]
Is this amount more than 400? [*No*]
Is this closer to 300 or 400? [*Closer to 300*]

Answer in context: If our school hall can fit up to 350 people comfortably, do you think we can have the planned event including everyone in the hall? *Justify* your answer, considering all factors. Note that, this model was extended to help children become more aware that 350 is halfway between 300 and 400.

A visual representation such as the roller-coaster model could be used to develop rounding skills. If children understand the concept of a number line, the roller coaster can provide an effective tool for rounding.



Some children might have experienced a real world ride on a roller coaster; therefore they might know what happens when the roller coaster stops at certain points. The model also suggests that something special happens at the top, that is, the coaster could roll either way. The force of forward motion however makes the observer think that the cart will go forward rather than backwards. This observation provides an opportunity to discuss a common rule of rounding, such as 'if the number ends in 5, you go over the hill to the next valley'. In rounding, attention is given to the back-end digit or digits. Children should view rounding as something that not only makes numbers easier to handle but, more importantly, makes sense.

Real world situations such as money will help students to explore different ways to round numbers but there needs to be clarity about the 2 aspects of dollars and cents. The example below considers both possibilities.



How would you round these prices? [*To the nearest ten cents*]
How would you round these prices? [*To the nearest dollar*]
If you only had money in notes, what would be needed to make each purchase? [*Round up to $25 and receive change*.]

These symbolic but relevant examples will encourage students to think about both the advantages and consequences of rounding numbers. Meaningful rounding is about knowing how precise an answer is needed for a particular situation and what place value is required to round to. These skills will improve through practice in many different problem contexts.

MAKING CONNECTIONS

Young children have varying cultural, linguistic and home experiences on which to build mathematics development of numbers. For example, patterns in naming numbers appear throughout Asian languages such as Japanese, yet in English and Italian the number names don't always form consistent patterns. Figure 8.15 shows that in English and Italian the number names from 11 to 20 do not explicitly name the number of tens as do the naming of numbers from 20 and larger. On the other hand, figure 8.15 shows that in Japan and China the numbers from 10 to 20 clearly name the ten and a related number. In China and Japan, place value patterns abound. For example, in Japan multiples of ten from 20 to 90 are (digit) *ju*; hundreds from 200 to 900 are (digit) *hyaku*; thousands from 2000 to 9000 are (digit) *sen*. These patterns consistently reinforce place value and facilitate naming numbers.

Counting to ten

English	One	Two	Three	Four	Five	Six	Seven	Eight	Nine	Ten
Italian	*Uno*	*Due*	*Tre*	*Quattro*	*Cinque*	*Sei*	*Sette*	*Otto*	*Nove*	*Dieci*
Chinese	*Yi*	*Er*	*San*	*Si*	*Wu*	*Liu*	*Qi*	*Ba*	*Jiu*	*Shi*
Japanese	*Ichi*	*Ni*	*San*	*Yon, Shi*	*Go*	*Roku*	*Nana, Shichi*	*Hachi*	*Kyuu*	*Juu, Ju*

Eleven to twenty

English	Eleven	Twelve	Thirteen	Fourteen	Fifteen	Sixteen	Seventeen	Eighteen	Nineteen	Twenty
Italian	*Undici*	*Dodici*	*Tredici*	*Quattordici*	*Quindici*	*Sedici*	*Diciasette*	*Diciotto*	*Diciannove*	*Venti*
Chinese	*Shi yi*	*Shi er*	*Shi san*	*Shi si*	*Shi wu*	*Shi liu*	*Shi qi*	*Shi ba*	*Shi jiu*	*Er shi*
Japanese	*Ju ichi*	*Ju ni*	*Ju san*	*Ju yon*	*Ju go*	*Ju roku*	*Ju nana*	*Ju hachi*	*Ju kyuu*	*Ni-ju*

Some larger numbers

English	Twenty-one	Twenty-two	Twenty-three	Twenty-four	Twenty-five	Fifty	Hundred	Thousand	Ten thousand	Hundred thousand
Italian	*Ventuno*	*Ventidue*	*Ventitre*	*Ventiquattro*	*Venticinque*	*Cinquanta*	*Cento*	*Mille*	*Diecimila*	*Centomila*
Chinese	*Er shi yi*	*Er shi er*	*Er shi san*	*Er shi si*	*Er shi wu*	*Wu shi*	*Pai*	*Chien*	*Wan*	*Shi wan*
Japanese	*Ni-ju-ichi*	*Ni-ju-ni*	*Ni-ju-san*	*Ni-ju-yon*	*Ni-ju-go*	*Go-ju*	*Hyaku*	*Sen*	*Man*	*Ju-man*

FIGURE 8.15 Counting to 20 in different languages

Figure 8.16 shows characters used to represent numbers in China. Notice how the larger numbers shown reflect place value and are both additive and multiplicative. For example, $751 = 7 \times (100) + 5 \times (10) + 1$. But 751 written with Chinese characters is 七五一, which literally means 7 hundreds, 5 tens and 1. So if the Chinese characters are known, it is easy to express any Hindu-Arabic number and vice versa (Uy 2003). It is also interesting to note that in both Chinese and Japanese, four groups of digits are used instead of three.

In our number system, the unit changes every multiple of 1000. We think ones, tens, hundreds and then move to ones, tens, hundreds in the thousands place; then ones, tens, hundreds in the millions place, and so on. In China and Japan, their number systems are based on 10 000 rather than 1000. Thus, in Japan 10 000 is called a *man*. In Australia, 3000 is read as 'three thousand' whereas in Japan 30 000 is read *san-man*.

Australia is located in a geographic region near to Asia and many Australians have travelled to and are familiar with Asian languages. The hypothesis that Asian counting and the language of number words is a great advantage to students when learning mathematics has been recognised by many scholars.



FIGURE 8.16 Chinese numeration system

While mathematical concepts are universal across cultures, the language associated with these cultures is not. For example, in English we name the numbers from left to right (but we write them with place value from right to left). That is tens are spoken before the ones for 2-digit numbers larger than 20; thus, 35 is spoken as 'thirty-five'. In Arabic, numbers are named differently. For example, a 2-digit number, say 45, is spoken 'five forty' which is consistent with place value. The number 345 is named 'three hundred five forty'; 8345 is named 'eight thousand three hundred five forty'. As larger numbers are named, those digits are spoken first but the digit in the ones place is named before the digit in the tens place.

Patterns for naming large numbers in many countries can also be confusing. For example, in the United States a million is a thousand thousands (1 000 000). However, in the United States the billion is named by the short scale, that is a thousand million or 1 000 000 000 or 10^9. Traditionally in the UK, a billion is 10^{12}, represented by 1 000 000 000 000. As of 1999, in Australia for finance, government, education, media and literature, one billion is equal to 1 000 000 000 or 10^9 (following the short-scale approach).

Culturally, mathematicians have often underestimated the depth of mathematical knowledge intuitive to Aboriginal children, as different ideas are valued in some cultures. The concept that Aboriginal and Torres Strait Islander people count just 1, 2, 3, 4 and 'many' is a false generalisation according to the work of John Harris (1987). When the situation demands, Aboriginal and Torres Strait Islander people use numbers precisely and can represent large amounts.

Teachers working in rural and isolated communities and in traditional Aboriginal lands would have a wealth of knowledge about this topic which researchers have not fully documented. These cultural differences in naming numbers underscores the need to know your students well and strive for similar understandings when numbers are named.

A GLANCE AT WHERE WE'VE BEEN

Children need to have a clear understanding of our number system if they are to have a strong foundation for future mathematical learning and the ability to be numerate. They need to be able to distinguish the 4 characteristics of our number system: (1) the role of zero, (2) the additive and multiplicative properties

of numbers, (3) a base of ten and (4) place value. Many concrete experiences involving counting, modelling, comparing, grouping and trading, and representing numbers (particularly in groups of ten) are necessary.

As children develop their skills with the aid of various models (such as bean sticks, base-ten blocks, counters and place value mats and the abacus), they need to learn how to organise the results in a systematic fashion and record them. Representing numbers pictorially on place value mats will serve as a visual reminder of the quantities involved and provide a bridge toward the symbolic representation of larger numbers. Establishing these bridges from the concrete to the abstract is particularly critical in developing place value. Likewise is the development of language for place value. Reading, writing, speaking and renaming numbers in a variety of ways is crucial. The importance of place value is second to none in all later development of number concepts.

Place value concepts are developed over many years. Trading activities plant the seeds early, but recognition of the power and importance of place value is developed, refined, extended, expanded and applied throughout the study of mathematics. Place value concepts help children compose and decompose numbers and begin to recognise equivalent representations. In this chapter there have been some examples of how place value is reflected in the language and representation of different cultures. There are many further historically and culturally rich examples, which could be studied. While language variations exist, the utilisation of place value concepts and processes in these cultures is universal.

Understanding place value is essential to counting and facilitates operating with very large and very small numbers. Place value is not taught and mastered over a few days, weeks or years. This means that place value is not completely developed before the four operations and estimation are introduced. Experiences with adding, subtracting, multiplying and dividing whole numbers develops additional competence and understanding of place value. Furthermore, working with decimals in later years will extend place value concepts and skills. For example, systematic work with smaller numbers, such as tenths, hundredths and thousandths, is based on the same notion of trading as whole numbers, and similar patterns in naming and renaming numbers result. Place value is essential for working with concepts of money and units of metric measurement. Thus place-value concepts initiated in the early years, will grow throughout the primary years will be integrated and extended throughout school mathematics.

THINGS TO DO: FROM WHAT YOU'VE READ

1. Identify 4 characteristics of our numeration system. Select one of these characteristics and expand on it. Possibly make a concrete, visual and abstract representation of it.
2. Show how 201 and 120 could be represented with 4 different place-value models or materials. Label them as structured or unstructured, proportional or non-proportional.
3. Discuss money as a non-proportional example of place value. Why could models of money lead to confusion? Explain.
4. Justify the importance of trading experiences and discuss how opportunities for trading up and trading down (composing and decomposing numbers) can lead to a deeper understanding of place value.
5. Study table 8.1 and compare the Australian Curriculum with expectations shown by the NCTM Standards and the Focal Points (Appendices A and B).
6. It has been suggested that the calendar concepts of decades, centuries and millenniums can be used to demonstrate notions of place value. Describe how this could be done. Why would a monthly calendar help young children recognise 2-digit numbers?
7. Using your calculator, enter the largest number possible in the display and name this number. How many digits does it have? Add 1 to this number and describe what your calculator does.

THINGS TO DO: GOING BEYOND THIS TEXT

IN THE FIELD

1. *Mathematics Online interview*. Using the *Mathematics Online Interview* (Victoria State Government, Education and Training if you have access) or the online interview booklet at www.eduweb.vic.gov.au/edulibrary/public/teachlearn/student/mathscontinuum/onlineinterviewbklet.pdf, undertake questions 8–17 of Section B: Place Value (pp. 27–30) with a young child. Reflect on what their responses inform you about the child's ability to recognise, model, represent and order numbers to 1000 and beyond.
2. *Early number sense*. Ask a Year 2 child some questions, such as: What number comes before 17? 30? 200? What number comes after 39? 209? How many tens are in 40? 47? Try to capture the words students use to describe how they determined their answers. Explain how their answers help you better understand their thinking and their level of number sense.
3. *Checking place-value concepts*. Give Year 2 or 3 students some of the following tasks: Write three hundred and forty-eight. Write a large number and a smaller number. Name each number. Rename one number in different ways. How many tens are in 250? How many hundreds are in 1600? Explain how the answers to these questions helped you better understand the child's development of place value.
4. *Patterns on a thousand chart*. In the classroom 8.4 provides an extension of the hundred chart. (See Appendix C for other variations.) Show the chart to at least 3 children (Years 3–5) and say 'please tell me about some patterns that you see'. After each description, say 'describe other patterns you see'. Be sure to allow them time to think and reflect; pattern recognition takes time. Make a list of the patterns for each child and compare their observations.
5. *Models for large numbers*. Different models exist for helping children develop meaningful concepts of large numbers. Several articles (Ellett 2005; Joslyn 2002; Losq 2005; Thompson 2000) show and describe useful physical models. Examine one of these articles and describe the approach taken to help children better understand large numbers.
6. In Australia, the curriculum resource maths300, www.curriculum.edu.au/maths300, provides excellent support for teachers planning learning activities for students of many age levels. Investigate, using the keyword search, lessons involving place value. Choose a lesson that appeals to you and modify the plan for a group of students that you know well. Implement the lesson(s) with your students and evaluate the success of the activities and the learning that appears to have occurred.

WITH TECHNOLOGY

7. Go to the Math Forum at www.mathforum.org/mathtools and then go to Math Topics. You will need login details. There you will find a number of virtual resources related to place value — including multi-base arithmetic blocks (MAB), which allow you to explore units, longs, flats and blocks in an interactive mode. Click the Math 2 link, then place value to find the place value tool that provides a very fast counter where children can practise naming digits of very large numbers. Discuss how children can practise naming digits of very large numbers.
8. Go to the National Library of Virtual Manipulatives at www.nlvm.usu.edu. There you will find MAB, Abacus and Chip Abacus tools in the K–2 resources. You can use the MAB to provide exploration in several different bases. The Sieve of Eratosthenes provides a hundred chart that can be extended to 1000 and allows for exploring patterns involving multiples (including multiples of 10). In what ways could you use these resources in a primary classroom?
9. *Ancient cultures*. Investigating ancient cultures' number systems helps us better understand and appreciate the qualities of our number system. Here are some links that provide information on different systems of numeration, such as Babylonian, Mayan and Roman.
   - www.math.wichita.edu/history/topics/num-sys. html#egypt
   - www-history.mcs.st-and.ac.uk/HistTopics/Babylonian_numerals.html
   - www.mathforum.org/alejandre/numerals.html

10 *Web surfing*. Use Google or some other search engine to search for '100 charts' or '1000 charts'. Identify several useful sites you found. Briefly describe three different resources available at these sites.

11 *Counting with a calculator*. In the classroom 8.3 provides a structured counting activity that can be facilitated with a calculator. Use this activity as a guide and ask primary school students to count by hundreds and predict which results will occur. Ask them to count with the calculator and compare results. Describe any difficulties or discoveries that you observed.

12 Observe a Foundation or Year 1 class working with calculators, counting in ones, tens or hundreds and discuss similarities and differences.

CHILDREN'S LITERATURE CONNECTIONS

Berry, D. (1994). *The Rajah's Rice: A Mathematical Folk Tale from India.* New York: W. H. Freeman.
When Chandra, an Indian village girl who bathes the rajah's elephants, cures the beasts after they fall ill, the rajah offers her jewels as a reward. She refuses, accepting only a measure of rice for the hungry villagers: 2 grains on the first square of a chessboard, 4 on the second and so on, doubling the amount for each subsequent square. Although the amount seems insignificant at first, it grows at an alarming rate, since doubling has little effect on small numbers but an increasingly enormous effect as the numbers grow larger. The rajah's storehouse is soon empty and he must then admit that he cannot fill her seemingly modest request.

Clements, A. (2006). *A Million Dots.* New York: Simon & Schuster.
The author shows the reader what a million looks like by using dots. Given examples of different things like shoeboxes, the author lets the reader think about how much a million looks like through the illustrations presented in the book.

Coerr, E. (1977). *Sadako and the Thousand Paper Cranes.* New York: Putnam.
This book is about a Japanese girl and her determination to overcome leukaemia by folding 1000 paper cranes, as legend holds that a person is granted a wish once they fold this many. However, Sadako died before she completed this task. Her classmates completed her quest and for every thousand cranes folded, a wish is made for world peace.

Cuyler, M. (2005). *100th Day Worries*. New York: Simon & Schuster Children's Publishing.
Jessica is worried about collecting 100 objects for school but her whole family helps by providing groups of 10 objects. This story has a heart-warming ending.

LoPresti, A. S. (2003). *A Place for Zero: A Maths Adventure.* Watertown, MA: Charlesbridge Publishing.
This fairytale story describes how Zero finds a way to make numbers bigger than 9 and 99 and discovers real meaning for himself.

Milbourne, A. (2007). *How Big is a Million?* London: Usbourne Books.
Pipkin, the young penguin, is searching for a million because it is such a large quantity to imagine. Fish, penguins and snowflakes help along the way.

Murphy, S. J. (2004). *Earth Day — Hooray!* New York: HarperCollins.
The Maple Street School Save the Planet Club is having a can drive to recycle aluminium cans. As the club members collect the cans and sort them into different amounts, the reader will learn about place value. There are activities and games in the back of this book for parents and teachers to use.

Princzes, E. (2002). *One Hundred Hungry Ants.* Boston US: Houghton Mifflin.
One hundred hungry ants hurry to sample the delights of a picnic, but marching in single file seems too slow for 100 empty tummies. The smallest ant of all suggests they travel in a variety of different ways to meet their destination sooner.

Pulley Sayre, A. & Sayre, J. (2003). *One Is a Snail, Ten Is a Crab: A Counting by Feet Book.* Cambridge US: Lothrop, Candlewick Press.
This is a beautifully illustrated and entertaining book that follows a cast of different animals and people on a seaside adventure. Place value, additive and multiplicative thinking are all encountered in this book.

Schwartz, D. M. (1985). *How Much is a Million?* New York: Lothrop, Lee & Shepard Books.
The reader will see how much a million, a billion and a trillion can be. The pictures in the text help the reader better understand the size of these large amounts.

Schwartz, D. M. (1989). *If You Made a Million.* New York: Lothrop, Lee & Shepard Books.
Marvelosissimo the Mathematical Magician and his team of cheerful kids (and their multitude of animal friends) take on some jobs. For each job, they'll be paid an appropriate amount of money. However, soon questions arise — what does that much money look like, and how can it be spent, saved or used to pay off a loan?

Wells, R. E. (2000). *Can You Count to a Googol?* Morton Grove, IL: Albert Whitman.
This resource book will let children see how our decimal numeration system is developed. Starting with the number 1, the reader will see how our number system can develop many numbers using zeros that will contain the following numbers: billion, trillion, quadrillion, octillion and finally a googol. Quantities and distances are related to the numbers.

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CHAPTER 9

Operations: meanings and basic facts

LEARNING OUTCOMES

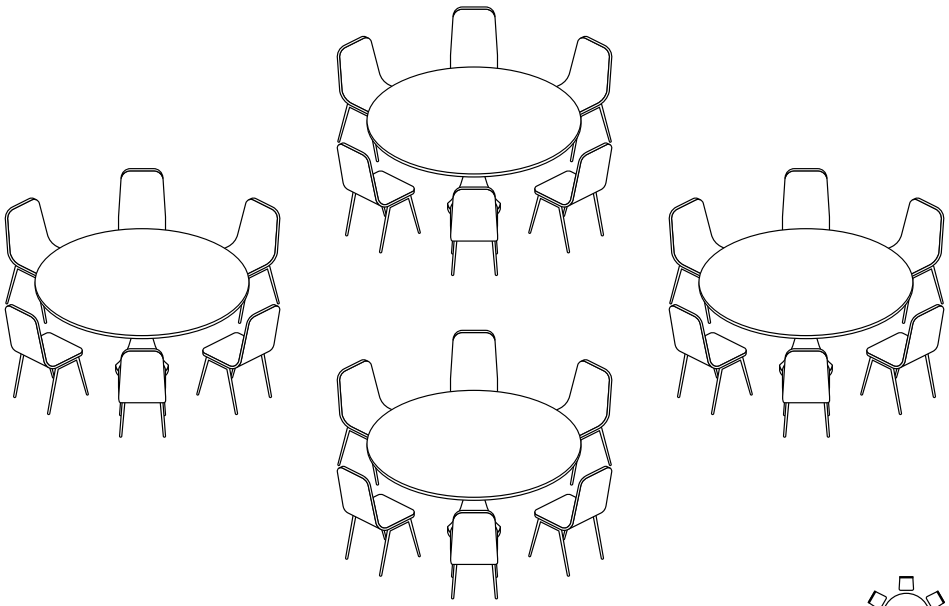
9.1 Articulating the prerequisite knowledge that children need before engaging in work on the 4 operations.

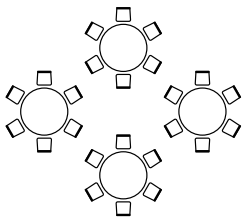
9.2 Understanding the meaning of the 4 operations and why they are the foundation for later mathematics.

9.3 Explaining the mathematical properties with a focus on commutativity.

9.4 Understanding the 3-phase process for helping children learn basic facts.

9.5 Identifying a variety of key thinking strategies for the basic facts to help children learn and practise addition, subtraction, multiplication and division.

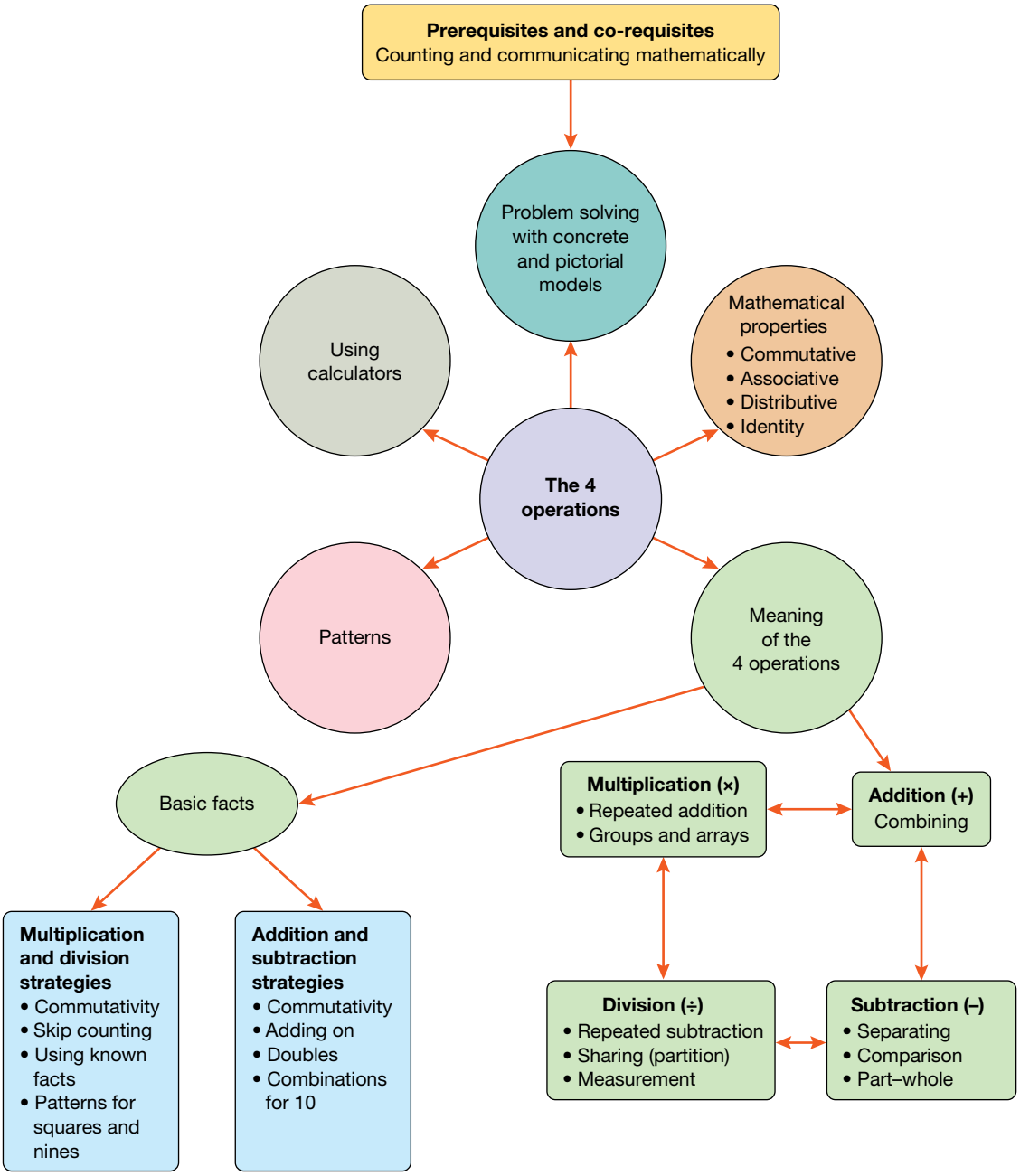




‘Can you do divisions?
Divide a loaf by a knife – what’s the answer to that?’

Lewis Carroll, *Through the Looking Glass*

Chapter 9 concept map



Introduction

Classroom activities that are well planned, actively involve students in a problem situation, promote reasoning and discussion, and allow students of various abilities to effectively engage in learning are important experiences for helping children learn mathematics in the primary years. Furthermore, open-ended problems where different strategies can be applied will help students develop a deeper understanding of the relevance and meaning of the 4 operations.

An understanding of addition, subtraction, multiplication and division — and knowledge of the basic number facts for each of these operations — provides an essential foundation for all later work with number computations and measurement calculations. To be effective in this later work, children must develop broad concepts for these operations. This development is more likely to happen if each operation is presented through a variety of concrete and visual representations. Such experiences help children recognise that an operation can be used in different types of situations. Furthermore, children need to understand the properties that apply to each operation and the relationships between all 4 operations.

Learning the basic number facts is 1 of the first steps children take as they refine their ideas about addition, subtraction, multiplication and division. By using these facts, together with an understanding of place value and mathematical properties, a child can perform any operation with whole numbers. A famous mathematician Poincaré (1854–1912) using an analogy of a house built with bricks, once stated that basic facts are an essential building block for mathematics, although mathematics itself consists of much more than just facts. Both understanding the operations and having immediate recall of basic facts are essential for efficient and accurate pencil and paper computations (algorithms) with all sorts of numbers (whole numbers, fractions, decimals and measurement units). However, operation sense and basic fact knowledge are just as essential when a calculator is readily available because it is important to monitor the reasonableness of answers obtained by pushing buttons. Operation sense and basic facts also form the building blocks for estimating answers or doing exact computations mentally in many everyday situations where it would be inefficient to use either paper and pencil or a calculator. So, no matter what type of computation a child is doing — paper and pencil, calculator, mental exact or mental estimation — both operation sense and quick recall of the basic number facts are important.

Table 9.1 lists the content descriptions related to operation meanings (for whole numbers) and basic facts that are identified in the Australian Curriculum. Throughout this chapter the knowledge, concepts, skills and processes will be further developed in accordance with Australian Curriculum expectations.

TABLE 9.1 **The 4 operations and basic facts: Proficiencies and content descriptions for Foundation to Year 4 in Australia**

Year	Content description
Foundation **Problem solving** includes using materials to model authentic problems, using familiar counting sequences to solve unfamiliar problems and discussing the reasonableness of the answer.	**Number and place value** • Represent practical situations to model addition and sharing.
Year 1 **Understanding** includes partitioning numbers in various ways. **Problem solving** includes using materials to model authentic problems.	**Number and place value** • Represent and solve simple addition and subtraction problems using a range of strategies including counting on, partitioning and rearranging parts.
Year 2 **Understanding** includes partitioning and combining numbers flexibly and identifying and describing the relationship between addition and subtraction and between multiplication and division. **Problem solving** includes formulating problems from authentic situations, making models and using number sentences that represent problem situations. **Reasoning** includes using known facts to derive strategies for unfamiliar calculations, and comparing and contrasting related models of operations.	**Number and place value** • Explore the connection between addition and subtraction. • Solve simple addition and subtraction problems using a range of efficient mental and written strategies. • Recognise and represent multiplication as repeated addition, groups and arrays. • Recognise and represent division as grouping into equal sets and solve simple problems using representations. **Patterns and algebra** • Solve problems using number sentences for addition and subtraction.

(continued)

TABLE 9.1 *(continued)*

Year	Content description
Year 3 **Understanding** includes partitioning and combining numbers flexibly. **Fluency** includes recalling multiplication facts. **Reasoning** includes generalising from number properties and results of calculations.	**Number and place value** • Recognise and explain the connection between addition and subtraction. • Recall addition facts for single-digit numbers and related subtraction facts to develop increasingly efficient mental strategies for computation. • Recall multiplication facts of two, three, five and ten and related division facts. **Money and financial mathematics** • Represent money values in multiple ways and count the change required for simple transactions to the nearest five cents.
Year 4 **Understanding** includes making connections between representations of numbers, partitioning and combining numbers flexibly. **Fluency** includes recalling multiplication tables. **Problem solving** includes formulating, modelling and recording authentic situations involving operations. **Reasoning** includes generalising from number properties and results of calculations, and deriving strategies for unfamiliar multiplication and division tasks.	**Number and place value** • Investigate number sequences involving multiples of 3, 4, 5, 6, 7, 8 and 9. • Recall multiplication facts up to 10 × 10 and related division facts. **Patterns and algebra** • Solve word problems by using number sentences involving multiplication and division where there is no remainder.

Source: © Australian Curriculum, Assessment and Reporting Authority 2015.

9.1 Helping children develop number sense and computational fluency

LEARNING OUTCOME 9.1 Articulating the prerequisite knowledge that children need before engaging in work on the 4 operations.

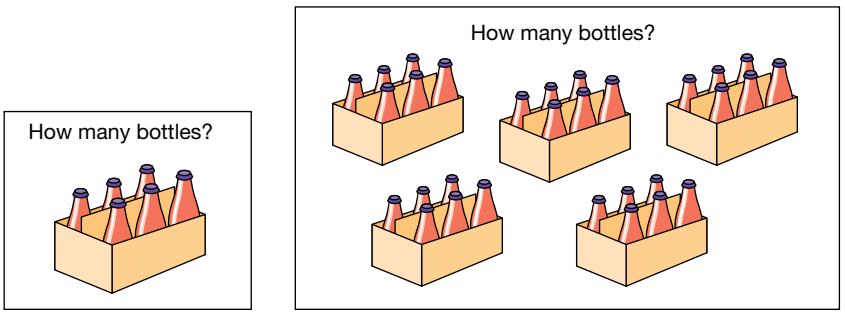
Ultimately, the learning goal for the 4 operations is that children will know how to add, subtract, multiply and divide in both educational and real world situations they encounter. More importantly, children need to know *how* to apply each operation *when* problem solving. Additionally, students need to quickly recall the basic facts for addition and multiplication and their related facts, (for subtraction and division) as needed because this skill frees the brain to focus on higher order mathematical tasks (O'Connell & SanGiovanni 2011).

How can teachers help children attain these skills and understandings? The best place to begin is by finding out what each child already knows and can confidently do with number. Then, according to King & Bay-Williams (2015), they need to develop through the 3 phases suggested by Baroody (2006) leading towards mastery of the basic facts. These phases will be explained throughout the chapter, the first being modelling and/or counting. Most children entering school have different levels of readiness for formal work on the operations. There are 4 prerequisites for such work that seem particularly important: (1) facility with counting, (2) experience with a variety of concrete situations, (3) familiarity with many problem-solving contexts and (4) experience using language to communicate mathematical ideas.

Facility with counting

Children use counting to solve problems involving addition, subtraction, multiplication and division long before they come to school, as research has indicated (Baroody, Lai & Mix 2006; Clements & Sarama 2007). Any problem with whole numbers can be solved by counting, provided there is sufficient time. However, it is not always efficient to solve problems by counting, therefore, children need to be able to use other procedures to cope with more difficult computations. Figure 9.1 illustrates this idea by comparing the counting method with additive and multiplicative thinking, which is more efficient.

Counting nevertheless remains an integral aspect of children's beginning work with operations. They need to know how to count forward and backward as well as how to skip count by 2s, 5s and 10s as well as other groups (see an earlier chapter). They need to count as they compare and analyse sets and arrays. However, they need more than counting to become proficient in the 4 operations.



We can count — 1, 2, 3, 4, 5, 6.

We could count, but it's more efficient to use a combination of counting (6 bottles in a carton, 5 cartons) and multiplication ($5 \times 6 = 30$).

FIGURE 9.1 An example showing the efficiency of using operations

Experience with a variety of concrete situations

Children need to have many experiences in problem situations and in working with physical objects to develop understanding about mathematical operations. Understanding improves if they can relate mathematical facts and symbols to an experience they can visualise. Concrete materials serve as a referent for later work with the operations as well as for constructing the basic facts. Materials also provide a link to connect each operation to real world problem-solving situations. Whenever a child wants to be sure that an answer is correct, materials can be used for confirmation.

Familiarity with many problem contexts

Problem situations are used for mathematics learning to develop conceptual understanding, encourage higher-level thinking and problem solving skills, and apply a variety of mathematical ideas. The benefits of beginning with problem situations is well established. Nonroutine problems require reading, comprehension, finding a solution strategy and solving the problem. Additionally, children need to become familiar with checking that their answer(s) fit in the original problem.

As with other mathematical content, a variety of problem contexts or situations should be used to familiarise students with the 4 basic operations, building knowledge until computational mastery is achieved. Children need to think of mathematics as problem solving as a means by which they can resolve problems through applying what they know, constructing possible methods to reach solutions and then verifying that

the solutions make sense. Not all students realise that mathematics is a tool that has real life applications. Most children already know that the 4 operations are used when doing computation in everyday life but school-based learning in mathematics needs to be connected to those experiences. Students also need to realise that the 4 operations can be extended to algebraic thinking, that symbolic representations such as $5 + 4 = ?$, $6 \times 8 = n$ or $a/b = 3$ are problems they can solve. Encouraging students to have the attitude, 'If I don't know an answer, I can work it out' is worth striving towards.

Experience in talking and writing about mathematical ideas

Children need to talk and write about mathematics; putting experiences into words helps with making meaning. Both concrete materials and problems can be vehicles for communicating about mathematics. All early phases of learning about the operations and basic facts should reflect the important role that language plays in their acquisition. According to the National Council of Teachers of Mathematics (2000) in their *Principles and Standards for School Mathematics*, communication is a way of sharing ideas and clarifying understanding. Through thinking, writing, talking and listening students are able to further develop mathematical language.

The move to symbols is often made too quickly and the use of materials dropped too soon. Instead, the use of materials should precede and then parallel the use of pictorial and abstract representations. Early on, when children are manipulating materials they can be encouraged to record answers in their own way. When children talk with peers and adults and record what is happening in a given situation, it helps them to see more clearly the relationship of the ideas and symbols to the concrete materials and the problem itself.

The oral and written language that children learn as they communicate about what they are doing with materials helps them to make a pictorial connection between concrete and symbolic understanding of the mathematical operations. By modelling, talking, drawing and writing, the referent for each symbol is strengthened. Children should begin their work with operations after talking among themselves and with their teacher about a variety of experiences. They need to be encouraged to continue talking about the mathematical ideas they meet as they work with the operations. As soon as feasible, they can put their ideas on paper — at first by drawings alone; but as soon as they are able to write symbols and words, children should also be encouraged to write number sentences and narrative explanations of their thinking.

SPOTLIGHT ON EARLY CHILDHOOD

Meanings for the 4 operations

In their daily activities, children construct meaning from operations with numbers. They explore ways of deconstructing and combining numbers.

Interview with a Year 2 student (7 years, 4 months)

Teacher: So what does adding mean to you?

Student: Like 12 plus 3, start with the big numbers and add on, 12 + 1 + 1 + 1 = 15.

Teacher: Great. What about subtract?

Student: Don't know.

Teacher: Well at school you might call it 'take away'.

Student: Oh yeah. Like 3 take 2 is 1. I have these 3 mandarins and I give 1 to my brother and my Dad and there is 1 left.

Teacher: What about comparing? Max has 6 and you have 2. How many more does Max have?

Student: Well he has 4 more; I understand that.

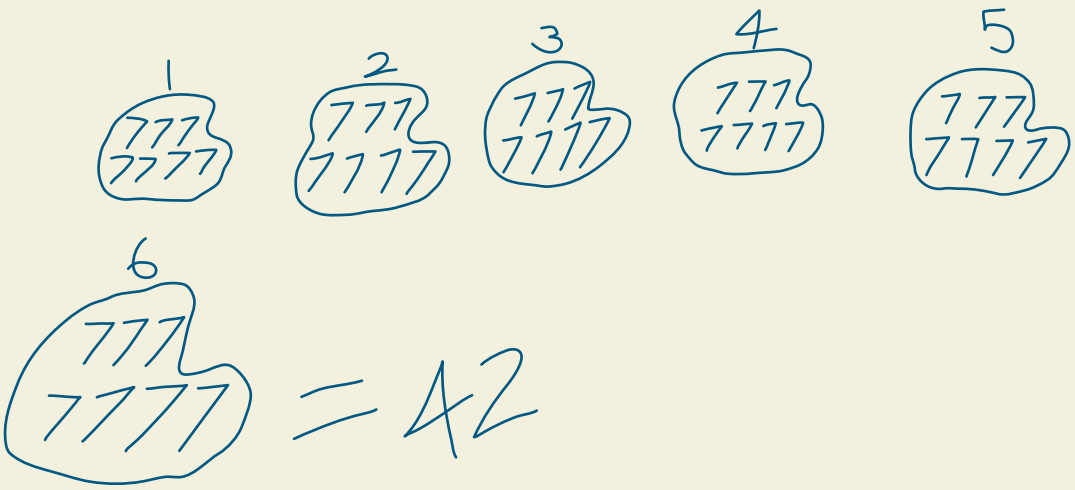
Teacher: Great. Now, what does multiplying mean to you?

Student: Like 6 groups of 7.

Teacher: How would you work it out if you didn't know the answer?

Student: I would draw the groups like this...

[The student draws little symbols of the number 7 with 7 in each group, circling each complete group and writing the numbers of the groups until there are 6. Then the result, which equals 42, is written in the symbols '= 42' on the page.]



Teacher: Fantastic! I really like the way you worked out 6 times 7 is 42. Well done. Do you mind if I share this [*pointing to the paper*], with my university students?

Student: That's fine.

Teacher: So what about divide, what does that mean to you?

Student: Don't know.

Teacher: Well, sharing is the same idea.

Student: Oh yeah. Like 5 mandarins between 2 people can't work out. [*The student eats 1 mandarin*.] Now, 4 shared between 2 people is 2 each.

Teacher: Fantastic, I really appreciated you sharing these ideas with me.

9.2 Developing meanings for the operations

LEARNING OUTCOME 9.2 Understanding the meaning of the 4 operations and why they are the foundation for later mathematics.

Children encounter the 4 operations in natural ways when they work with many diverse problem situations. By representing these problem situations (for example, acting them out, using physical models or drawing pictures), they develop meanings for addition, subtraction, multiplication and division. Mastery of basic facts and later computational work with multi-digit numbers should be based on a clear understanding of the operations.

Conceptual understanding of the 4 operations, the ability to strategise basic facts leading to mastery and computational proficiency are 3 main outcomes of teaching and learning mathematics in the

number strand. The following general sequence of activities is appropriate for helping children develop meaning for the 4 basic operations.

1. *Concrete — modelling with materials*. Use a variety of verbal problem situations and manipulative materials to act out and model each operation.
2. *Pictorial — representing with pictures*. Provide representations of objects in pictures, diagrams and drawings to move a step away from the concrete toward symbolic representation.
3. *Abstract — representing with symbols*. Use symbols (especially numeric expressions and number sentences) to illustrate each operation.

The 4 operations are clearly different, but there are important relationships among them that children will come to understand through modelling, pictorial and symbolic experiences. They will understand that:

- addition and subtraction are inverse operations; that is, 1 operation undoes the other:

$$5 + 8 = 13 \rightarrow 13 - 5 = 8$$

- multiplication and division are inverse operations:

$$4 \times 6 = 24 \rightarrow 24 \div 4 = 6$$

- multiplication can be viewed as repeated addition:

$$4 \times 6 \rightarrow 6 + 6 + 6 + 6$$

- division can be viewed as repeated subtraction:

$$24 \div 6 = 4 \rightarrow 24 - 6 - 6 - 6 - 6 = 0$$

These relationships can be developed through exploration with a variety of different experiences, followed by discussion and practice.

Addition and subtraction

The concepts of addition, subtraction, multiplication and division all need to be developed through concrete, pictorial and abstract representations. The language of the operations encompasses all of these representations, which children move through, linking each to the others. Figure 9.2 illustrates these representations with examples for addition and subtraction.

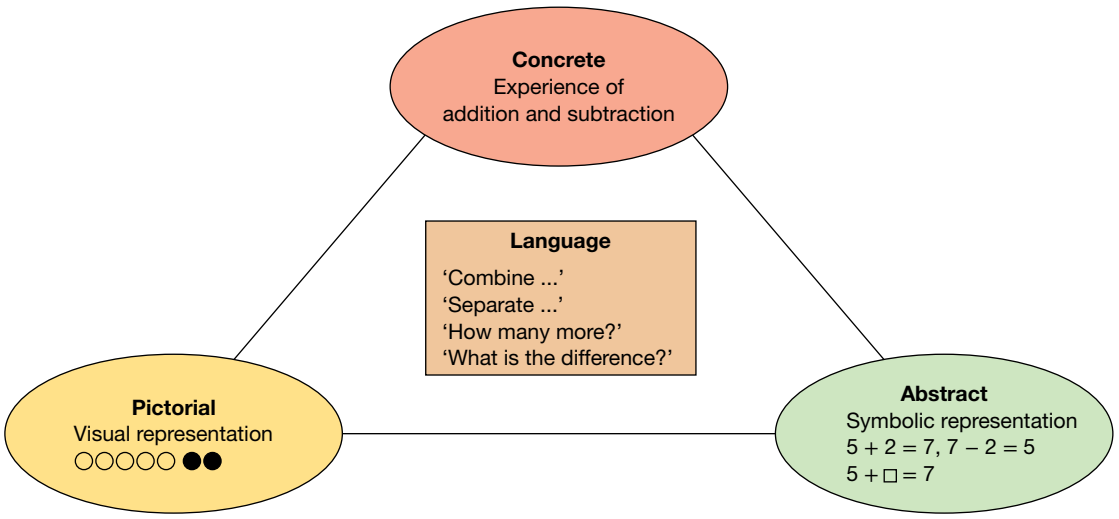


FIGURE 9.2 Developing the concepts of addition and subtraction

Let us consider the following real-world problem:

> Alice has 5 teddy bears and 2 puppy dogs. How many soft toys does Alice have altogether?

This problem can be written as $5 + 2 = 7$ (or $2 + 5 = 7$) where the first addend represents 1 type of toy and the second addend represents another type of toy. The third number, known as the sum, gives the total number of toys.

Figure 9.3 shows a variety of models (including counters, multilink cubes, a balance scale and number line) that can be used to represent addition. Each model depicts the idea that addition means combining amounts or 'finding how many in all'. The models for addition can also be used for subtraction. Each model can be applied to the 3 different situations that lead to subtraction.

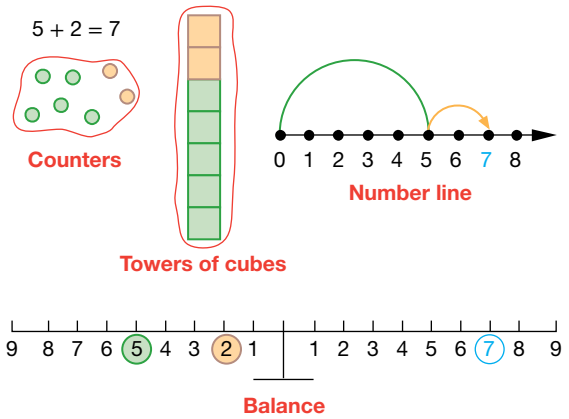
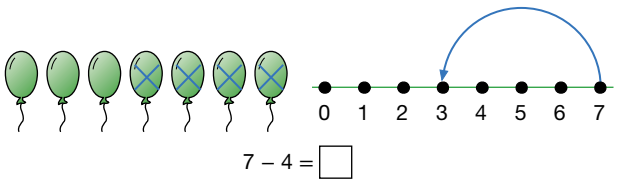


FIGURE 9.3 Some models for addition

Separation problems

Separation, or take away, involves having 1 quantity, removing a specified amount from it, and noticing what remains. Research indicates that this subtraction situation is the easiest for children to learn; however, persistent use of the words *take away* can lead children to believe that there is only 1 subtraction situation. Confusion and misunderstanding can result when students meet the other 2 situations. This is why it is important to read a subtraction sentence with the right language such as $8 - 3 = 5$ as '8 minus 3 equals 5' rather than '8 take away 3 equals 5'. Take away is just one of the 3 types of subtraction situations and should not be overemphasised. The 2 representations below, pictures and the number line, help to demonstrate the separation idea for subtraction.

> Wan had 7 balloons. She gave 4 to other children. How many did she have left?

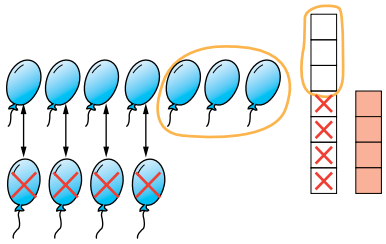


Comparison problems

Comparison, or finding the difference, involves having 2 quantities, matching them 1 to 1 and noticing what is the difference between them. The total number of objects in the group and 1 part is known. There is an unknown part, which needs to be found, by adding on. Problems of this type represent subtraction,

even though nothing is being taken away. It is a problem that involves addition or an answer to the question, 'how many more do we need?' The 2 representations below, pictures and cubes, help to demonstrate the comparison idea for subtraction.

Wan had 7 balloons. Richard had 4 balloons. How many more balloons did Wan have than Richard?

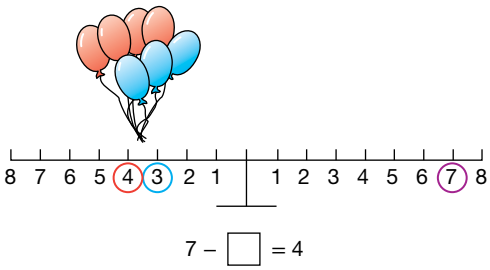


$7 - 4 = \square$

Part–whole problems

The final type of subtraction situation is known as *part–whole*. In this type of problem, a quantity of objects can logically be separated into 2 parts. The number of objects in the entire set is 1 of the parts. The unknown quantity is the number in the remaining part. Nothing is being added or taken away — there is simply a static situation involving parts and a whole. The 2 representations below, pictures and a number balance, help to demonstrate the part–whole idea for subtraction.

Wan had 7 balloons. There were 4 red balloons the rest were blue. How many were blue?



$7 - \square = 4$

It is useful to help students recognise that many subtraction problems can be thought of as 'missing-addend' problems because the strategy of 'think addition' can be used to find the answer. For example, in the preceding problem, the question, '4 plus what equals 7?' makes sense.

The importance of providing many and varied experiences in which children use physical objects to model or act out examples of each operation cannot be overemphasised. The formal language of *addend* and *sum* can be gradually introduced so that children are aware that each quantity has a name but there is no need to overemphasize this aspect. In a number sentence like $5 + 6 = 11$, the parts of the equation (5 and 6) are known as the addends and the amount in total is known as the sum.

Consider a task that involves children in thinking about combining 2 piles of coloured tiles together to reach a given total. Some children might focus on the concrete by actually placing tiles together and counting. Other children might be comfortable drawing pictures or writing numerals as they work on the problem. All students will be working towards finding answers in a missing addends problem. The open-ended nature of the problem helps them relate number sentences (such as $6 + 5 = 11$) with an amount, where more than 1 solution might be possible.

For children in the early stages of understanding, the situation need not be represented by written symbols. Moving tiles, counting and thinking about additive relationships are the initial components of

the above activity. The teacher might choose to introduce symbols, to students who are ready, as a complement to the physical manipulation of objects. Symbols provide a way of recording what's happening with the materials that goes beyond pictorial representation. As work with numbers and operations progresses and student understanding increases, the amount of abstraction that the teacher encourages and expects will naturally increase.

Multiplication and division

The same sequence of experiences used to develop understandings of addition and subtraction can also be followed for multiplication and division. The concepts of multiplication and division need to be developed through concrete, pictorial and symbolic representations. The language of the operation encompasses all these representations. Figure 9.4 illustrates these representations with examples for multiplication and division.

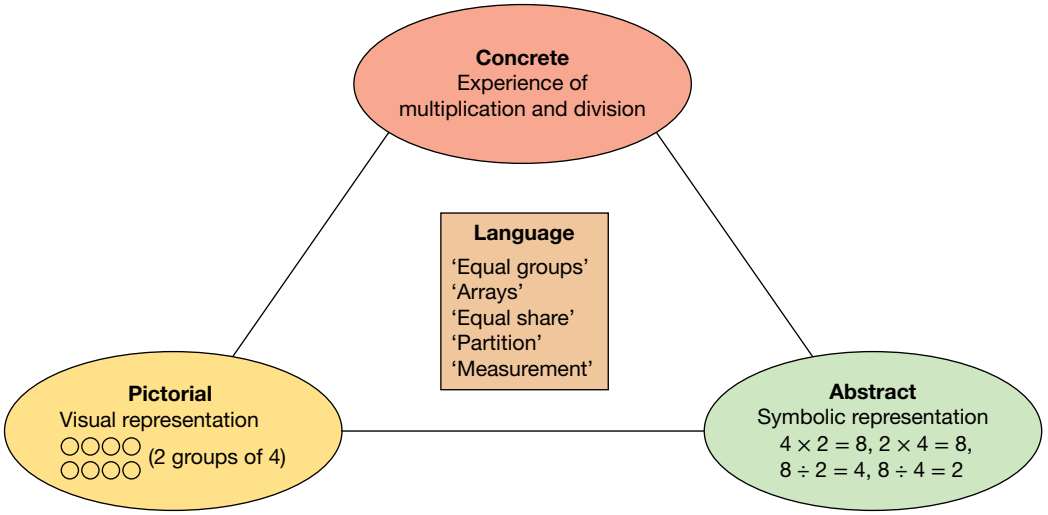


FIGURE 9.4 Developing the concepts of multiplication and division

Let us consider the following real-world problem:

> Andrew has 2 packets of stickers. Each packet contains 24 stickers. How many stickers does he have altogether?

It is important to notice that the numbers in multiplication and division problems represent different sorts of things. This problem can be written as $2 \times 24 = 48$, where the *first factor* (2) represents how many groups or sets of equal size (here, how many packets) are being considered, while the *second factor* (24) represents the size of each group (here, how many stickers per group). The third number (48), known as the *product*, indicates the total of all the parts (here, the total number of stickers). The old saying 'you can't add apples and oranges' points out that in addition and subtraction problems, a common label must be attached to *all* the numbers involved. In the example used previously Alice had teddy bears and puppy dogs, which were added together as soft toys, which became a common label. By contrast, in the sticker problem, for multiplication, the labels for the numbers are different. In this example, Andrew had packets (2 of them), there were stickers per pack (24) and total stickers (48). Hence, there is no common label for all numbers involved.

Figure 9.5 illustrates some of the most commonly used models for illustrating multiplication situations. These are, sets of objects, arrays and the number line. Research indicates that children do best when they can use various representations for multiplication and division situations and can explain the relationships among those representations (Kouba & Franklin 1993, 1995; Verschaffel, Greer & De Corte 2007).

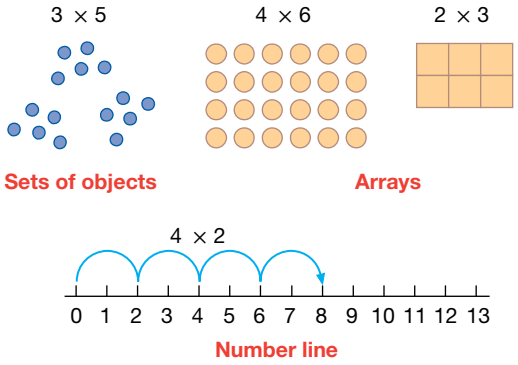


FIGURE 9.5 Commonly used models for multiplication

There are 4 distinct multiplicative situations: (1) equal groups, (2) multiplicative comparisons, (3) combinations and (4) areas or arrays (Greer 1992). Problems involving the first 2 of these situations are most commonly found in primary school, although students should eventually become familiar with all 4 types. These 4 multiplicative situations are described below to help the teacher, understand and recognise their variety. Children do not need to know these labels, but it is important that they encounter a broad range of problem situations involving multiplication and division.

Equal-groups problems

Equal groups involves the most common type of multiplicative situation, where the problem deals with a certain number of groups, all the same size. When both the number and size of the groups are known (factors), but the overall amount is not known (product), then the problem can be solved by multiplication. For the inverse situation: when the product in an equal-groups problem is known, but either of the factors (number of groups or the size of the group) is unknown, then the problem can be solved by division. There are 2 distinct types of division situations that regularly occur in real life, depending on which part is unknown. These 2 types of division are known as measurement and partition division.

Comparison problems

Comparison involves another type of multiplicative situation. With comparison problems for subtraction discussed above, there were 2 different groups of objects that needed to be matched 1-to-1 to decide how much larger one is than the other. In similar fashion, comparison problems with multiplicative structures involve 2 different groups, but the relationship is not 1-to-1. Rather, in multiplicative comparison situations, 1 set involves multiple copies of the other. An example of this type of problem might be:

> Hanna spent $35 on Christmas gifts for her family. Geoff spent 3 times as much. How much did Geoff spend?

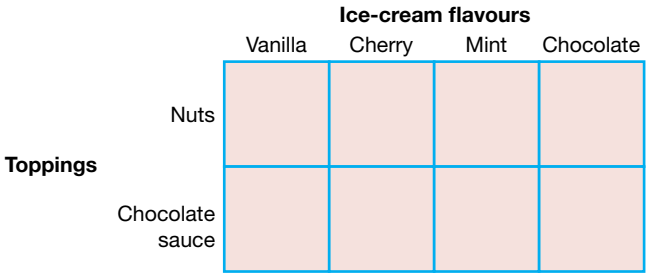
In this case, Hanna's expenditures are being compared with Geoff's, and the problem is solved by multiplication. The question does not involve 'how much more?', as it did with additive/subtractive comparison. Instead, the structure of the problem involves 'how many times as much?' If the problem is changed slightly to include information about how much Geoff spent but Hanna's expenditure is unknown, or the comparison factor unknown, then the problem could be solved by division.

Examples of these problem situations could be as follows.

1. Hanna spent a certain amount on Christmas gifts for her family, and Geoff spent 3 times as much. If Geoff spent $105, how much did Hanna spend?
2. Hanna spent $35 on Christmas gifts for her family and Geoff spent $105. How many times as much did Geoff spend compared to Hanna?

Combination problems

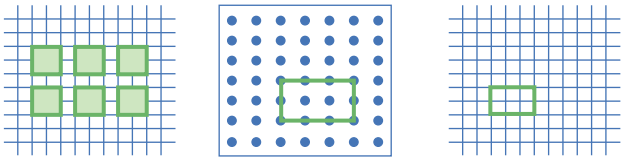
Combinations involve another practical sort of multiplicative situation. Here, the 2 factors represent the sizes of 2 different groups and the product indicates how many different pairs of these groups can be formed, with 1 member of each pair taken from each of the 2 different groups. There are many real world examples of combination problems and they are often solved with the use of a table or an array. Common real world problems of this type include matching clothes to wear (for example, t-shirts and pants), sandwich fillings to bread, and menu choices for a fixed price. The example below illustrates the number of different ice-cream cone combinations, which are possible with 4 different ice-cream flavours and 2 toppings if each serving can have exactly 1 ice-cream flavour and one topping. An example of 1 solution would be mint ice-cream with chocolate sauce.



Area and array problems

Finally, *areas* are a typical example of multiplicative situations. The area of any rectangle (in square units) can be found either by covering the rectangle with unit squares and counting them all individually or by multiplying the width of the rectangle (number of rows of unit squares) by the length (number of unit squares in each row). Once again an array model is a logical method for solving this type of problem. In a rectangular array where there is an arrangement of discrete, countable objects (such as chairs in an auditorium), the total number of objects can be found by multiplying the number of rows by the number of objects in each row. This is still an area problem as all the chairs cover a similar amount of floor space. The array model for multiplication can be especially effective in helping children visualise multiplication facts as areas. The connection between number and 2D-shape (rectangles), by making and naming rectangles using tiles, geoboards or graph paper provides a natural and enduring visual image for the concept of multiplication.

These illustrations show a 2-by-3 or 3-by-2 rectangle.



Thus, each rectangle contains 6 small squares. Asking children to build and name a variety of rectangles with different dimensions is a quality experience for the concept of multiplication. According to King & Bay-Williams 2015) children can create and label array cards using centimetre grid paper, which when used in meaningful problem situations and games, can help children see and learn the multiplication facts. In the classroom 9.1, using tiles as a concrete and visual model, illustrates several learning experiences designed for the purpose of visualising multiplication situations.

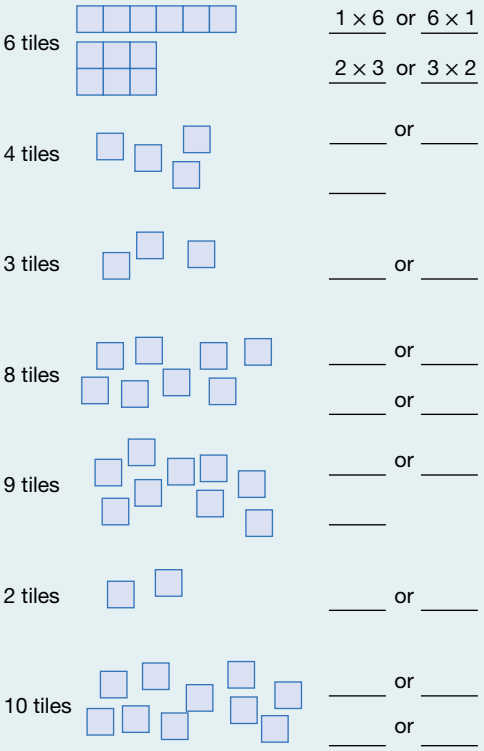
IN THE CLASSROOM 9.1

Rectangles and more rectangles

Objective: Build rectangles with tiles to develop visual representations for multiplication as repeated addition, groups and arrays.

Year level: 2

- Use the tiles to make different rectangles with all the tiles? Record with a picture and symbols, if possible.
- The first example shows the solution for 6 tiles.
- What do you notice in each case? Special cases? Explain to someone.



Extension: Find and record all the rectangles possible using up to 20 tiles. What did you discover?

Just as the 3 models of equal groups of objects, the number line and arrays are useful in representing multiplication, they can also be useful in representing division, with the relationship to repeated subtraction frequently revealed. For division, there are 2 unique situations, which need to be considered: *measurement* and *partition*.

Measurement (repeated-subtraction) problems

In measurement (or repeated-subtraction) division situations, how many objects in each group is known but the number of groups needs to be determined.

> Jenny had 12 grapes. She gave 3 to each friend. How many people received grapes?

Here, you can imagine Jenny beginning with 12 grapes and making piles of 3 repeatedly for each person until all the grapes are gone. She is measuring how many groups of 3 she can make from the original pile of 12. The symbolic representation is $12 \div 3 = 4$.



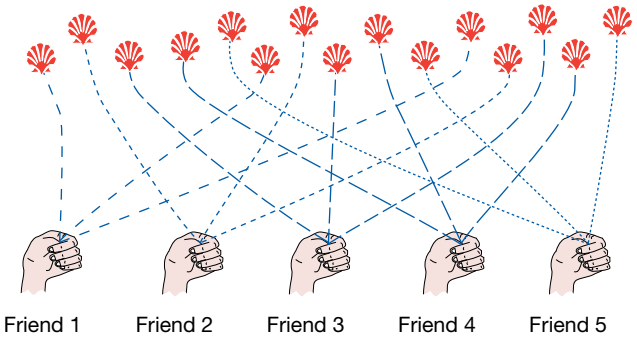
Another measurement subtraction example is measuring how many metre-length hair ribbons can be made from a 10-metre roll of ribbon. Imagine repeatedly stretching out a metre and cutting it off, thus measuring how many hair ribbons you can make. For this example, $10 \div 1 = 10$.

Partition (sharing) problems

In *partition* (or sharing) division situations, a collection of objects is distributed equally into a given number of groups and sharing determines the number in each group. By contrast with measurement situations, how many groups is known, but the number of objects in each group is not known.

> Garry had 15 shells. If he wanted to share them equally among 5 friends, how many should he give to each?

Imagine Garry passing out the shells to his 5 friends. Firstly he gives 1 each, 1 for you, 1 for you, 1 for you, 1 for you, and 1 for you. Then a second shell is given to each person, and so on, until they are all distributed. Finally, he might check to see how many each person received. The symbolic representation is $15 \div 5 = 3$.



Partitioning (or sharing) is difficult to show in a diagram, but it is natural and easy for children to act out. Equal amounts of food for each child in a family, dealing cards for a game, and transporting school children in cars with a number of parent drivers, are all examples of a partition situation.

When helping children learn about multiplication and division it is not desirable to focus on the formal language of each situation or name problems as measurement or partition division. However, it is important for the teacher to know about the different meanings of multiplication and the 2 types of division situations so that students have opportunities to work with examples of each and do not become overly preoccupied with just 1 situation. It is vital that students are able to identify when a problem requires multiplication or division, and that the result reflects the original context of the situation.

9.3 Mathematical properties

LEARNING OUTCOME 9.3 Explaining the mathematical properties with a focus on commutativity.

An understanding of the mathematical properties commutative, associative, distributive and identity and how they affect each operation (table 9.2) is essential when building children's understanding of the four operations. While understanding these properties is not a prerequisite to work with the 4 operations, it needs to be developed as part of a deeper understanding of the operations.

TABLE 9.2 Mathematical properties for primary-school children

Property	Mathematical language	Child's language	How it helps
Commutative	For all numbers a and b: $a + b = b + a$ and $a \times b = b \times a$	If $4 + 7 = 11$, then $7 + 4$ must equal 11, too. If I know 4×7, I also know 7×4.	The number of addition or multiplication facts to know from memory is greatly reduced. In each case not 100 facts but 55.
Associative	For all numbers a, b and c: $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$	When I'm adding (or multiplying) 3 or more numbers, it doesn't matter where I start.	When more than 2 numbers are being added (or multiplied), combinations that make the task easier can be chosen. For example, $37 \times 5 \times 2$ can be solved as $37 \times (5 \times 2)$ or 37×10 rather than $(37 \times 5) \times 2$.
Distributive	For all numbers a, b and c: $a(b + c) = ab + ac$	8×12 is equal to $8 \times (10 + 2)$ and $8 \times 10 + 8 \times 2$. Numbers can be split apart to multiply pieces, before adding. $8 \times (10 + 2) = 8 \times 10 + 8 \times 2$.	The more difficult basic facts e.g. 6×, 7×, 8× can be split into smaller, easier-to-remember parts. For example, 8×7 equals $8 \times (5 + 2)$ and $(8 \times 5) + (8 \times 2) = 40 + 16 = 56$.
Identity	For any whole number a: zero is the identity for addition $a + 0 = a$ and 1 is the identity for multiply $a \times 1 = a$	Zero added to any number is easy; it's just that number. One times any number is just that number.	The 19 addition facts involving 0 and the 19 multiplication facts involving 1 can be easily remembered once this property is understood and established.

In primary school, children are not expected to know and apply these properties until Year 7 in Australia. Rather, the objective is to help children understand the commutative, associative, distributive and identity properties in order to use them when it is efficient. Additionally these properties can lead students to better strategies for number fact recall and application.

Table 9.2 gives the meaning of each property, states what children should understand, and provides examples to illustrate how the property can make learning and using the basic facts easier.

Understanding these properties implies knowing when they apply. For example, both addition and multiplication are commutative, but neither subtraction nor division are.

Addition and subtraction: $7 + 3 = 10$ and $3 + 7 = 10$ but $7 - 3$ is *not* equal to $3 - 7$
Multiplication and division: $7 \times 4 = 28$ and $4 \times 7 = 28$ but $28 \div 7$ is *not* equal to $7 \div 28$

Commutativity can be a common cause for misconception with some children. The idea of 'subtract the smaller number from the larger' or 'divide the larger number by the smaller' regardless of their order can be quite common if the property is not understood. Care needs to be taken to ensure that children understand why the order is important for subtraction and division.

9.4 Overview of learning the basic facts

LEARNING OUTCOME 9.4 Understanding the 3-phase process for helping children learn basic facts.

As children develop an understanding of the 4 operations, learning begins to focus on certain number combinations. These are generally referred to as the *basic facts*. The tables of facts, which follow, illustrate all the combinations in abstract form. However they should not be used to the detriment of the learning stages, which build on prior knowledge of counting, encourage modelling with materials, use strategies that meaningfully build on known facts, and lead to mastery. If used incorrectly tables of facts for the 4 operations can become a crutch, which impede the process of learning the number facts meaningfully.

- *Basic addition facts* each involve 2 1-digit addends and their sum. There are 100 basic addition facts (from 0 + 0 up to 9 + 9; see figure 9.6). Please note that the facts table does not go up to 12 as we use a decimal system for money and measures. There are 2 different ways to read a basic fact from the table below. Both methods are the same because of the commutative law but mathematically the horizontal then vertical approach is best (just like the x and y co-ordinates). To read, 4 + 9 = 13, find the first addend (4) along the top row and the second addend (9) down the left column. By reading vertically down the 4-column and horizontally across the 9-row then the sum (13) can be found at the joining place.

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

FIGURE 9.6 The 100 basic facts of addition

- *Basic subtraction facts* are derived from the inverse relationship of addition and subtraction. The 100 basic subtraction facts result from the difference between 1 addend and the sum for all 1-digit addends. Thus the 100 subtraction facts are also pictured in figure 9.6, the same table that pictures the 100 basic addition facts. To read the basic subtraction fact 13 − 4 = 9, find the cell in the 4-row that contains the sum 13, then read up that column to find the difference (9) at the top of the table. Note that a number sentence containing any addend greater than nine is not included in the table as it is not a basic fact.
- *Basic multiplication facts* each involve 2 1-digit factors and their product. There are 100 basic multiplication facts (from 0 × 0 up to 9 × 9). It is worth noting that many tables of facts for multiplication do not include zero as it is a null factor for all other factors, hence 1 × 1 up to 9 × 9 yields 81 basic facts. In the same manner as before there are 2 different ways to read basic facts from figure 9.7. To read, 6 × 5 = 30, find the first factor (6) along the top row and the second factor (5) down the left column. By reading vertically down the 6-column and horizontally across the 5-row then the product (30) can be found at the joining place. This is the same as 5 × 6 = 30.

×	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

FIGURE 9.7 The 81 basic facts for multiplication

- *Basic division facts* are derived from the inverse relationship of multiplication and division. There are only 81 basic division facts because division by 0 is not possible, therefore no facts with 0 exist.

Development of the addition and subtraction facts typically begins at age 6 or Year 1 with a view to mastery during Year 3 according to the Australian Curriculum (v8.1). Teachers would normally begin modelling, representing and solving addition and subtraction problems with children 1 year or more prior to when fluency is expected. The expectation for basic multiplication facts is split into a number of year levels: the meaning of multiplication is developed in Year 2; multiplication facts for 2, 3, 5, and 10 are expected by the end of Year 3; and sequences for 3, 4, 6, 7, 8 and 9 are developed in Year 4 leading towards mastery of all basic facts. The Numeracy Benchmarks (Curriculum Corporation 2000) were nationally agreed minimum acceptable standards for Years 3, 5 and 7 in Australia. By Year 3, students were expected to remember, or work out, basic addition facts to $10 + 10$, matching subtraction facts (e.g. $9 + 4 = 13$, $13 - 9 = 4$ and extensions of these facts (e.g. $23 - 9 = 14$). By Year 5, students were expected to know, or work out, multiplication facts to 10×10 and use them to calculate extensions of these facts. For example, $6 \times 8 = 48$ can be extended to $60 \times 8 = 480$.

Why does it take some children years to master their basic facts? The truth is it is a very difficult process that needs time. According to O'Connell and SanGiovanni (2011), asking students to memorise numerous pairs of number facts without meaning can be discouraging and confusing. It is far better to approach the learning of basic facts through a strong conceptual understanding, developing strategies to further understand number facts, and by engaging in appropriate practice using problem based situations. This approach is far more effective in the long term and far less stressful than drill and practice.

Some children however might need additional support if their difficulty is due to a learning disability, which makes it problematic for them to memorise the facts. Use of a calculator will benefit this child so that they can proceed with learning mathematics. Additionally, as research by Clark and Kamii (1996) indicates for multiplication, a child may have trouble with multiplication facts because he or she has not yet developed the ability to think multiplicatively. If the child is not a multiplicative thinker then according to Siemon and Breed (2005), they will need support to move from additive thinking (that is predominantly using repeat addition or equal groups), towards multiplicative thinking (which can be supported by the understanding and use of arrays).

In the majority of cases, children's difficulties in mastering basic facts could stem from 1 (or both) of the following 2 causes, and in these cases teachers can definitely provide help. Firstly, the underlying numerical understandings might not have been developed. Thus, the process of remembering the facts quickly and accurately can be stressful to students, and has become no more than rote memorisation or meaningless manipulation of symbols. As a result, the child has trouble remembering the facts. Secondly, the skill of strategising facts until they are well known might not have been taught by teachers

or understood by children, which results in inefficient strategies. Teachers can do something about both of these problems by using a 3-phase process for helping children learn basic facts.

1. Start where the children are.
2. Build understanding of the basic facts.
3. Focus on how to remember facts.

Start where the children are

Many children come to school knowing some basic facts. For instance, the chances are great that they can say in context '1 and 1 are 2', '2 and 2 are 4', and maybe even '5 and 5 are 10'. They might know that 2 and 1 more is 3, and that 6 and 'nothing more' is still 6. However, they probably don't know that $6 + 7 = 13$, nor do they have a clear concept of the meanings of symbols such as + and =.

Similarly, they may know that if there are 3 objects and 2 are removed, then only 1 remains. However, they might not know the meaning of the symbolism $3 - 2 = 1$ or of the sentence '3 minus 2 equals 1'. In a shopping context they might act out buying 3 pieces of liquorice at 5c each and give the cost as 15c but not know the equation $3 \times 5 = 15$. They might be able to show that 8 apricots shared fairly among 4 children means each child receives 2 apricots, but they might not know that $8 \div 4 = 2$.

In other words, young children can probably solve many simple number problems involving facts, but they are not likely to be able to recognise or write the facts in symbols yet. Nor do many children understand that a number sentence in horizontal form with an unknown such as $4 + 2 = n$ asks the same question in vertical form, as shown:

$$\begin{array}{r} 4 \\ \underline{+2} \end{array}$$

The teacher's role is to help children organise what they know, construct more learning to fill in the gaps and, in the process, develop meaning for the basic facts and for the abstract symbolism we can use to represent them.

In the beginning it is important to determine what each child knows (their prior learning) by using responses from group discussions, observations of how each child works with materials, how they record their ideas with paper-and-pencil, and their responses during individual interviews or performance-based tasks. Many teachers use an inventory at the beginning of the year, the term or the unit of work, which they administer individually for younger children or a group questionnaire in later years. The purpose of such an inventory would be diagnostic in nature, to discover the following.

- *Whether the children have the concept of an operation.* Given a story problem involving simple addition, subtraction, multiplication or division, can they model the situation? Solve the problem? Explain why their solution makes sense? Write a corresponding number sentence? Identify the operation they used?
- *What basic facts they understand.* Given a number fact, can they draw a picture to illustrate it? Or given a picture, can they identify a related number fact?
- *What strategies they use to find the solution to problems.* Can they answer the question 'How did you know $7 + 9 = 16$?' and beyond just saying 'I know it'?
- *If there is fluency with basic facts.* Which facts do they remember, without stopping to figure them out?

Teachers can then use the information gained from discussion, observation, samples of work and interviews to plan teaching and learning experiences for their wide range of children. Do some children need more work with manipulative materials to understand what multiplication means? Do some children need help in seeing the relationship between addition and subtraction, e.g. equivalence $17 - 8 = 9$ and $8 + 9 = 17$? Do the children recognise that counting on from a number is quicker than counting all? Which children need practice in order to move beyond counting and grouping to mastering the basic facts? The teacher can organise children in ways that are differentiated and help to

meet individual needs (as suggested in chapter 3). The teacher can provide well-structured activities, opportunities for discussion and direct instruction to fill in the missing links and strengthen understanding and competency.

Build understanding of the basic facts

The teacher's emphasis when helping children learn their basic facts should be on facilitating children to organise their thinking and to see relationships between the facts. Children need to develop strategies for remembering the facts before they engage in drill and practice experiences to develop fluency. Researchers point out the following:

> The view on . . . how these number facts should be learnt and taught has drastically changed in the past decades. Whereas learning single-digit arithmetic was for a long time based mainly on memorizing those facts through drill-and-practice to the point of auto-matization . . . current instructional approaches put great emphasis on the gradual development of these number facts from children's invented and informal strategies (Verschaffel, Greer & De Corte 2007, p. 560).

Generally, the facts with both addends (addition), or both factors (multiplication), greater than 5 are more difficult for children, but what is demanding for an individual child is really the important point. Although many textbooks, websites and computer programs emphasise practice on the generally difficult facts, they often encourage the child to keep a record of the basic facts that they know confidently and those that are more difficult for them individually, suggesting extra practice on those. The teacher should suggest or reinforce this idea. Figure 9.8 (for addition) and figure 9.12 (for multiplication) indicate a bold line for facts smaller/greater than $5 + 5$ and 5×5 to encourage those children having difficulties to focus on the smaller facts first.

How can the basic facts for an operation be organised meaningfully? Many references present facts in small groups or 'families' (e.g. facts with sums to 6: $0 + 6 = 6$, $1 + 5 = 6$, $2 + 4 = 6$ and $3 + 3 = 6$ plus their commutative equivalents). Other references organise the facts in 'relationships' (e.g. facts in the '2–3–5 family' are $3 + 2 = 5$, $2 + 3 = 5$, $5 - 3 = 2$ and $5 - 2 = 3$). Further references categorise the facts by 'thinking strategies' (e.g. all facts where 1 is added or facts involving 'doubles' such as $7 + 7$). All of these approaches are valid and effective and no particular order for teaching the basic facts has been shown to be superior to any other. Thus, the teacher can use professional judgement regarding what each group of children needs and can choose whether to use or modify the sequence given in teacher resources and references.

A variety of thinking strategies can be used to recall the answer for any given basic fact. Thinking strategies are efficient methods for deriving answers and helping to retain them (Baroody 2006). The more efficient the strategy, the more quickly the student will be able to construct the correct answer for the sum, difference, product or quotient of 2 numbers and, eventually, to develop fluency with the facts so he or she can quickly recall them.

Research has shown that certain thinking strategies help children learn and master the basic facts (Fuson 2003, King & Bay-Williams 2015). Understanding of the facts develops in a series of stages characterised by the children's development and abilities. Some of these thinking strategies involve using concrete materials or counting. Others are more mature because a known fact is used to figure out an unknown fact. Teachers can lead children to develop these more mature, efficient strategies to help them recall facts. The next section on thinking strategies for basic facts provides additional detail on how these skills can be developed.

Many children rely heavily on counting — in particular, finger counting — and fail to develop more efficient ways of recalling basic facts. For example, a child might count 4 fingers and then 5 more to solve $4 + 5$. This strategy is perfectly acceptable at first; however, children need to 'trust the count' (Willis, 2002) that is, the counting process should not have to be repeated every time $4 + 5$ is encountered. Teachers want the child to move beyond counting on from 4 (which is relatively slow and inefficient) to

more strategic thinking like '4 + 4 = 8, so 4 + 5 is 9' or some other efficient strategy. Eventually, the child must be able to recall '4 + 5 = 9' immediately and effortlessly. Some children discover effective strategies for the basic facts on their own, while others might need scaffolding. When the teacher is satisfied that the children are familiar with a particular strategy (able to model it with materials or pictures and beginning to use it mentally), it is time to practise the strategy. In other words, before students start working on committing facts to memory, they should be able to:

- state or write related facts, given 1 basic fact
- explain how they got an answer, or prove that it is correct
- solve a fact in 2 or more ways.

Focus on how to remember facts

Consider this scene, in a Year 3 classroom:

> Pairs of children are keying numbers on a calculator and passing it back and forth. Other pairs are seated at a table, some playing a card game and others playing board games. Several are busily typing numbers on computer keyboards. Still others are working individually with flashcards or number triangles. What are they all doing? They could be practising basic facts.

If children are to become skilful with methods for calculating addition, subtraction, multiplication and division of larger numbers (algorithms) and proficient at estimation and mental computation, they must know the basic facts with immediate, automatic recall. The ultimate goal for number proficiency is computational fluency and efficiency.

The following key principles for practising the basic facts might be helpful.

- Children should attempt to memorise facts only after understanding is attained.
- Children should participate in practice with the intent to develop fluency. Knowing from memory (remembering) not memorisation should be emphasised.
- Practice activities should be short (5–10 minutes) and available almost every day.
- Children should work on just a few facts in any given session and constantly review previously learned facts.
- Children need to develop confidence in their ability to remember facts fluently and should be praised for good efforts. Teacher observation and records of their progress are recommended.
- Practice activities should be varied, interesting, challenging and presented with enthusiasm and relevance.
- Children should never feel pressured in front of their peers or the teacher, nor timed in a way that lets down a team effort.

Computer software provides a natural complement to more traditional materials and activities, such as flashcards, games and audiotaped practice, for establishing the quick recall of basic facts however they need to be used wisely. Most software and applications for practice of basic facts keep track of the number of exercises attempted and the number answered correctly. Some programs display the time taken to give correct answers, thus encouraging students to compete against their own records for speed as well as mastery. This approach does not suit every kind of learner and can lead to anxiety in some children so caution is needed. Requiring short response time (within 3–4 seconds) can be useful because it promotes more efficient thinking strategies and encourages children to develop fluent recall. Many children enjoy software and applications that display a cumulative record of their individual progress. This feature allows children to diagnose the basic facts they know and don't know and it can provide a source of motivation. If each student is competing against him or herself, then the goal of mastery (or a 100 per cent score) is likely to be an incentive.

When using flashcards for practice, the child should start by going through the entire set and separating the cards into piles of those they know confidently and those they are unsure about. The known pack could be set aside for a quick review at the end of the session. The remainder of the learning time

could be used to focus on strategies for those cards that are not familiar to them. The next time the child works with the cards, he or she would review the cards, moving any newly learned facts to the confident pile and/or any uncertain or forgotten facts to the unsure pile. Then practice begins again, with the unsure pile of cards. This approach can help children see their progress and focuses their attention on those facts that need further practice. If another child nearby is struggling with similar basic facts they could discuss their strategies and/or call over a more experienced peer or the teacher for support.

Teachers need to take note of whether certain basic addition and multiplication facts occur more (or less) frequently during practice sessions. For example, basic facts involving numbers larger than 5 might occur less frequently than facts involving numbers from 2 to 5; and problems involving 0 and 1 may be quite rare. However children will need a variety of random facts to strategise, practise and commit to memory. Therefore, it is important to ensure that equitable practice is provided, especially the more difficult facts involving the numbers 6, 7, 8 and 9.

Several types of games are shown in In the classroom 9.2. Children of all age groups find games an enjoyable way to practise what they know. These activities can be used as a supplement to other practice activities that can be found in teacher references, journals, computer software, applications and other sources.

IN THE CLASSROOM 9.2

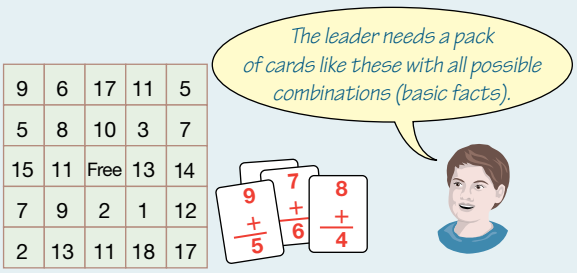
Games for practising basic facts

1. Addition bingo

Objective: Use a Bingo game to practise basic addition facts.

Year level: 1–2

Each player needs a different Bingo card and some buttons or counters for markers.



It's easy to play:

- The leader draws a card and reads the addends on it.
- Each player covers the sum on his or her Bingo card.

Some sums are given more than once on a Bingo card; however, a player may cover only 1 answer for each pair of addends.

The winner is the first person with 5 markers in a row!

2. Multig

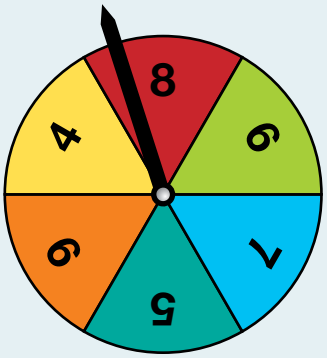
Objective: Use a spinner game to practise basic multiplication facts.

Year level: 3–4

Use the playing board (enlarged) or make 1 on heavy construction paper. Each player needs some buttons or counters for markers.

- Take turns and spin twice. Multiply the 2 numbers. Find the answer once on the board and put a marker over it.

- Each turn is worth 1 point when the answer is found.



Don't forget the spinner. You can't play this game without it.

56	25	40				
36	49	20	81			
30	64	35	48	32		
56	42	63	28	54	45	
32	48	54	72	24	16	35
16	24	28	36	40	30	25	20
63	45	81	56	49	42	64	
20	25	72	45	24	36		
40	49	32	28	54			
30	16	72	48				
42	35	63					

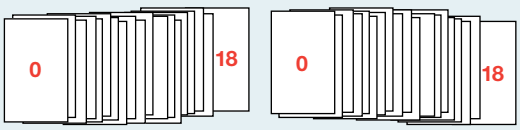
- Score 1 bonus point for each covered ♦ that touches a side or corner of the ♦ you covered.
- If you can't find an uncovered ♦ to cover, you lose your turn.
- Opponents may challenge the answer any time before the next player spins.
- The winner is the player with the most points at the end of 10 rounds.

3. Zero wins

Objective: Use a card game to develop number sense and practise addition and subtraction facts.

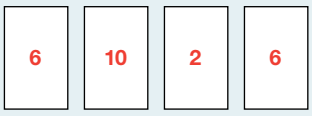
Year level: 2–3

Make 2 identical sets of 19 cards with a number from 0 to 18 on each.



Follow these rules:

- After shuffling, the leader deals 4 cards to each player and places the remaining cards face down in the centre of the table.
- Players need to add or subtract the numbers on their 4 cards so they equal 0. For example, suppose you had these cards:



- With these cards you could solve the problem in many ways, for example:
  10 – 6 + 2 – 6 = 0
  or
  6 + 6 – 10 – 2 = 0
  or various other number sentences.
- On each round of play, the players may exchange 1 card if they wish, and each player takes a turn being first to exchange a card on a round. To make an exchange, the first player draws a card and discards a card, face up. Other players can draw from either the face-down pile or the face-up discard pile.
  The first player to get 0 on a round wins the round!

4. Test your facts

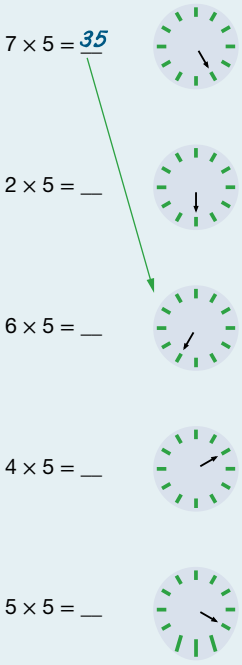
Objective: Use a variety of activities and puzzles to practise multiplication and division facts.

Year level: 3–4

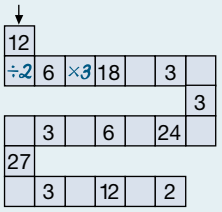
Fill in the empty boxes:

5	7	*35*
4	*9*	36
5	8	
9		81
	6	48

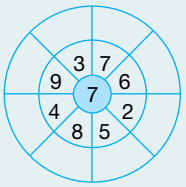
Complete these 5x facts and match them with the clock minutes.



Fill in each empty box to make the next number correct:



Multiply each number in the middle ring by the number in the centre. Write the answer on the outside.



9.5 Thinking strategies for basic facts

LEARNING OUTCOME 9.5 Identifying a variety of key thinking strategies for the basic facts to help children learn and practise addition, subtraction, multiplication and division.

In the following 4 sections, the thinking strategies for basic facts for addition, subtraction, multiplication and division are discussed, along with teaching suggestions.

Thinking strategies for addition facts

The 100 basic facts for addition were shown previously in figure 9.6. These facts should not be presented to children to memorise from the completed table. Rather, children should gradually and systematically learn the facts building upon what they know. If they choose, they could record the facts they know confidently on the table as they progress. It is worth noting that children can be given the addition table of facts, to look for patterns as an exploratory activity once the rows and columns have been explained. The following questions could be useful.

- What patterns do you see in the table? (They will probably notice that each row or column shows numbers counting up by 1s, but they start at different starting points)
- What is the largest sum? What are its addends? Can you write the addition sentence for this sum?
- Circle all the sums of 5. Why does 5 appear more than once in the table as a sum? Can you write all the addition sentences for this sum? For another sum of your choice?
- What sum appears most often in the table? Can you write all the addition sentences for this sum?

Asking questions like this can help children recognise the orderliness of the basic addition facts. This overview can also help children see their goal as they begin to work on fluency with facts. Thinking strategies for teaching basic addition facts include commutativity; adding 1, 2, doubles and near-doubles; combinations to 10; and adding to 10 and beyond. For many facts, more than one strategy is appropriate.

Commutativity

The task of learning the basic addition facts is simplified because of the commutative property. Changing the order of the addends does not affect the sum. Children encounter this idea in concrete form, for example, when they notice that putting 2 blue objects and 3 yellow objects together gives the same quantity as putting 3 blue objects and 2 yellow objects together.



When working with the basic addition facts, children might record them using symbols but not yet realise they are the same facts.

$$\begin{array}{r} 2 \\ +5 \end{array} \quad \text{and} \quad \begin{array}{r} 5 \\ +2 \end{array}$$

or (where the square is an unknown number)

$$2 + 5 = \square \quad \text{and} \quad 5 + 2 = \square$$

It is important to understand that the same 2 numbers have the same sum, no matter which number comes first. Children need to be able to put this idea into their own words, although 'turn around facts' (NSW Department of Education and Families 2011) is often used in recent times. Students do not need to know the term *commutative property*, but they need to use the idea as they work with basic facts. A calculator can also help children verify that the order of the addends is irrelevant. Have them key into the calculator combinations such as '5 + 8 =' and '8 + 5 ='. Using a variety of different combinations will give evidence to the idea that the order does not affect the outcome. A visual representation of the commutative law using the addition facts table can help to create a moment of realisation, especially if the child can fold the table along the diagonal to see each side is exactly half. Figure 9.8 illustrates that the same 45 numbers appear above and to the right of the diagonal as the 45 numbers below and to the left. This mirror image helps demonstrate that the commutative property reduces the number of facts to be learned by almost half. Each yellow square below the diagonal can be matched with a white square above the diagonal with the same addends and sum. There are now 55 distinct addition facts to learn not 100 facts.

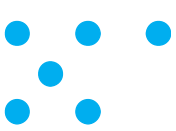
+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

FIGURE 9.8 Addition facts derived by the commutative strategy

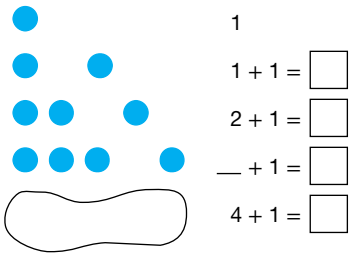
Adding 1, 2 and 0

Adding 1 to a number is easy for most children as it is just counting 1 extra. In fact, most children learn this idea before they come to school, and have an initial understanding of it. To reinforce their initial concept, experiences with objects come first, followed by pictorial representation and paper-and-pencil recording of the number fact with the symbols for plus and equals (in Year 1 and 2).

$5 + 1 = \square$



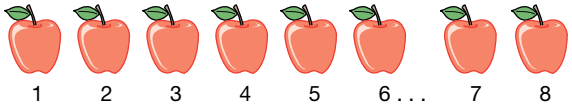
Recognition of the pattern is then encouraged:



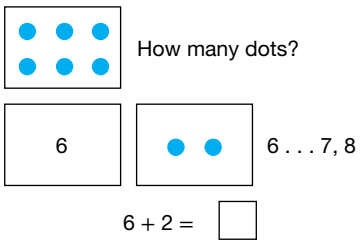
The strategy for adding 2 to a number logically extends from adding 1; children count on 2 more objects or numbers in sequence. These facts should be learned using concrete examples in which children see that any time they add 2 objects the number grows more.



Adding 1 or 2 can be applied effectively to addends greater than 5. As illustrated below, the problem 6 + 2 = is solved by starting from the larger addend, 6, and counting on, 7, 8.



Some children might not start from the larger addend, which makes this strategy hard. It is more difficult to start at 2 and count on 6 i.e. 2, 3, 4, 5, 6, 7, 8 than to just count on a few numbers. Thus, the strategy might need to be learned by children using activities such as this one, with dot cards:



Adding on more than 1 or 2 numbers is both slow and prone to error. Unfortunately, some children develop the habit of using a counting on strategy with numbers more than 2, and sometimes with the aid of concrete tools easily available (such as the fingers on their hands). Failure to progress beyond this point, or having little confidence to do so, means the student is unlikely to develop more efficient addition strategies. According to Hopkins (2009), 17.5% of Year 7 students displayed simple addition fact difficulty due mostly to slow counting and inaccuracy. It is important to help students move beyond counting to become fluent, efficient and accurate. These skills are extremely important for more advanced work in number. When students are slowed down by inefficient strategies for recall of basic facts, they are often impeded in doing more advanced work in mathematics.

Finally, adding 0 to any number doesn't change it, because 0 is the identity element. When nothing is added the quantity is unchanged. For some children this concept is difficult to understand. Concrete modelling of the situation is tricky because it is difficult to picture adding nothing. Therefore, explicit

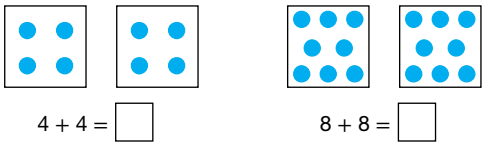
work on the 0 facts can be postponed until children have mastered some of the other facts. Overall, if the 27 facts described by the addition strategies (for 0, 1 and 2) are now remembered by the child, only 28 addition facts remain.

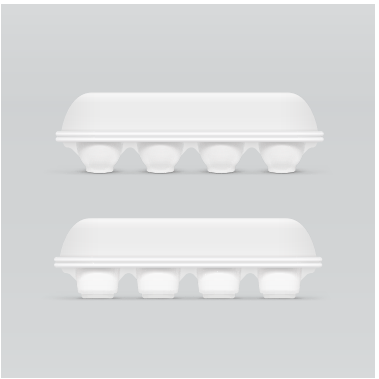
Adding doubles and near-doubles

Doubles are basic facts in which both addends are the same number, such as 4 + 4 or 9 + 9. Most children learn these facts quickly, often repeating them before they come to school. The concept of doubles builds on the ideas of 1-to-1 correspondence and groups of 2 (pairs), usually gained during kindergarten or by the Foundation year of school.

Connecting doubles facts to familiar real world situations often helps students remember them (e.g. 2 hands shows 5 + 5 = 10, an egg carton shows 6 + 6 = 12, 2 weeks on a calendar shows 7 + 7 = 14).

Students can profit from work with objects followed by drawings:





Another strategy, *near-doubles*, can be used for the facts that are 1 more or 1 less than the doubles.

7 + 8 = □	Think
So	7 + 7 = 14
	7 + 8 is 1 more
	7 + 8 = 15

7 + 6 = □	Think
So	7 + 7 = 14
	7 + 6 is 1 less
	7 + 6 = 13

The 3 thinking strategies, commutativity, adding 1, 2 or 0, and doubles and near-doubles, are illustrated in figure 9.9.

(a) Addition facts derived by adding 0, adding 1 and adding 2 strategies

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5							
4	4	5	6							
5	5	6	7							
6	6	7	8							
7	7	8	9							
8	8	9	10							
9	9	10	11							

(b) Addition facts derived by adding doubles and near-doubles strategies

+	0	1	2	3	4	5	6	7	8	9
0	0	1								
1	1	2	3							
2		3	4	5						
3			5	6	7					
4				7	8	9				
5					9	10	11			
6						11	12	13		
7							13	14	15	
8								15	16	17
9									17	18

FIGURE 9.9 Addition facts derived by 2 key strategies

Combinations to 10

The combinations to 10 facts are those 9 pairs of numbers that together make 10: 1 + 9, 2 + 8, 3 + 7, 4 + 6, 5 + 5, 6 + 4, 7 + 3, 8 + 2 and 9 + 1. Note that recognition of the commutative property can reduce this list to just 5 facts (since each of the 9 facts except 5 + 5 has a commutative equivalent (for example, 8 + 2 and 2 + 8 are the same). Of the 9 combinations to 10, 5 have already been mentioned in the previous strategies (either *adding on* or *doubles*). The 'new' facts here are 3 + 7 (and 7 + 3) and 4 + 6 (and 6 + 4). Combinations to 10 (or bridging to 10), is an effective strategy that needs to be practised in different ways.

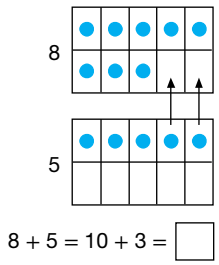
Adding to 10 and beyond

With the strategy of *adding to 10 and beyond*, 1 addend is broken apart so that part of it can be used with the other addend to make 10. Then the remaining part of the first addend is added to the 10 to go beyond into the teen numbers. This strategy is used most easily when 1 of the addends is 8 or 9 because adding 1 or 2 easily makes the sum of 10. Once children are fluent with combinations to 10, then *adding to 10 and beyond* can be useful in many situations. These are the most difficult addition facts to develop and will take time and practice to build student understanding.

	Think
8 + 5 = □	8 + 2 = 10,
So	5 is 2 and 3 more
	(8 + 2) + 3 = 10 + 3 =13

	Think
8 + 5 = □	5 + 5 = 10
So	8 is 5 and 3 more
	(5 + 5) + 3 = 10 + 3 =13

Working from the bigger number, the child recognises that 8 + 2 = 10, and then mentally breaks 5 apart into 2 + 3. The 2 is used with the 8 to 'add to 10' and the 3 is used to go 'beyond'. It sounds complex but builds on a strong foundation of combinations to 10. Alternatively the doubles strategy combined with the combinations to 10 would work well in this situation. The child recognises that 5 + 5 = 10, and then mentally breaks 8 apart into 5 + 3. The 5 is used with the 5 to 'add to 10' and the 3 is used to go 'beyond'. To help children learn this strategy a ten-frame (or 2 ten-frames) can be effective because they provide a visual image of adding to 10 and going beyond:



The research of Funkhouser (1995), indicates that working with five-frames as a base and then moving to ten-frames may be particularly helpful for children with learning disabilities.

Children need to know the combinations to 10 well in order to use the *adding to 10 and beyond* strategy efficiently. They also need to realise how easy it is to add any single digit to 10 to get a number in the teens, without having to count on. They should be able to respond immediately to questions such as 3 + 10 = ?, 10 + 7 = ? and 9 + 10 = ?

The thinking strategies *combinations to 10* and *adding to 10 and beyond* for addition are illustrated in figure 9.10.

In many cases, more than 1 strategy can be used to remember a basic fact. This will become clear when working with children. They will think in different ways when recalling a known fact, and this will strengthen their understanding of the relationships involved. Referring back to figure 9.6, introduced earlier in this chapter, all 100 of the basic facts have been covered with 1 or more of the strategies. That is, once the strategies for adding 1, adding 2, adding 0, adding doubles and

near-doubles, combinations to 10 and adding to 10 and beyond have been learned, children will have a strong foundation for solving problems involving addition and subtraction with larger numbers. During the process, children should be encouraged to look for patterns and relationships because almost all of the 100 basic addition facts can be developed from a variety of relationships with other facts.

+	0	1	2	3	4	5	6	7	8	9
0										
1										10
2									10	11
3								10	11	12
4							10		12	13
5						10			13	14
6					10				14	15
7				10					15	16
8			10	11	12	13	14	15	16	17
9		10	11	12	13	14	15	16	17	18

FIGURE 9.10 Addition facts derived by the combinations-to-10 and adding-to-10-and-beyond strategy

Please note, children might invent additional strategies of their own. They often use the basic facts that are their 'favourites' or those they are completely confident with, in preference to strategies discussed in this section. Encourage creativity!

$6 + 7 = \square$

Think
5 + 5 = 10, so 13 is it!
6 is 1 more than 5
7 is 2 more than 5

$6 + 8 = \square$

Think
I know my doubles
7 + 7 = 14
1 up from 6 and 1 down from 8

Thinking strategies for subtraction facts

For each basic addition fact, there is a related subtraction fact. In some mathematics programs, the 2 operations are taught simultaneously. The relationship between them is then readily emphasised and learning the basic facts for both operations proceeds easily. Even when they are not taught simultaneously, the idea of related facts is frequently used. Figure 9.11 shows several versions of materials for practising related facts. Students can practise with fact triangles by covering the number in 1 corner with their thumb and figuring it out by viewing the other 2 numbers. In a similar way, students can practise with folding fact cards by folding 1 door shut to conceal 1 number, and figuring it out by viewing the other 2 numbers.

Think addition is the key thinking strategy for learning and recalling the subtraction facts. Encouraging children to recognise, think about, and use the relationship between addition and subtraction facts is most efficient. Finding the answers to subtraction facts by thinking about missing addends.

$15 - 7 = \square$

Think
$7 + \square = 15$
7 + 8 = 15
So 15 – 7 = 8

Other strategies that are basically the inverse of the addition facts can all be encouraged.

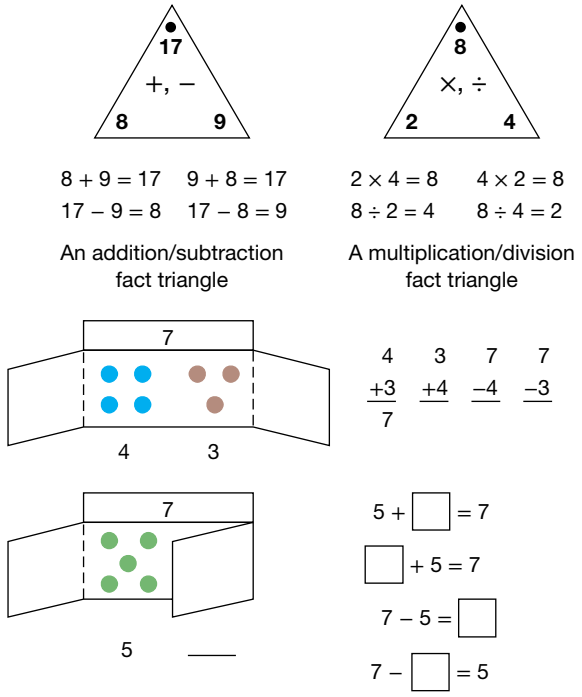


FIGURE 9.11 Examples of related facts

Subtracting 1 and 2

Once children have learned strategies for adding 1 and adding 2, learning the related facts for subtracting 1 or 2 is not a problem. Counting backwards with small numbers is easily done if children know the sequence from 1 to 20 well. Further work with materials and observing patterns similar to addition are useful if larger numbers are involved.

Doubles and halves

Children might need support to learn the strategy of halves for subtraction facts compared to doubles for addition facts. It rests on the assumption that children know the doubles for addition. Acting out the problem by splitting groups in half is a worthwhile task.

16 − 8 = □

Think
8 + □ = 16
8 + 8 = 16
So 16 − 8 = 8

Counting on

The strategy of counting on can be used easily and efficiently when there is a small difference. If the numbers are quite close together then the missing addend question: 'How much more would I need?' is a natural thinking process. The problem 9 – 5 = can be thought about as addition or counting on 5, 6, 7, 8, and 9. The answer was a further count of 4 numbers, 9 – 5 = 4. Research has shown that counting on is a very powerful subtraction strategy for many students (Fuson 2003). When observing young children, this strategy might be seen with fingers on 1 hand rather than done mentally. This is an intuitive approach but as confidence builds, children should be encouraged to think beyond concrete materials to develop fluency. Children should be encouraged to use known addition facts to reach a solution to the subtraction problem.

Thinking strategies for multiplication facts

Multiplication is frequently viewed as a special case of addition in which all the addends are of equal size. The solution to multiplication problems can be found by repeat addition (or skip counting), which is known as additive thinking. Observing number patterns in addition is a useful starting point for multiplication; however using multiplication effectively requires a deeper understanding. Multiplicative thinking takes time to develop but is flexible, efficient and applicable to more advanced mathematics and real world situations.

Learning about multiplication begins in Year 2 as children develop ideas about repeat addition, groups, and arrays (Australian Curriculum, v8.1). In Years 1 and 2, children count by 2s, 5s, 10s, and later by other numbers such as 3s. These experiences provide a foundation for understanding the patterns that occur with the basic multiplication facts.

Use of the calculator, as described in an earlier chapter, can aid teachers in developing ideas about these patterns of multiplication. Using the constant function on calculators, children realise that 2 6s equal 12, 3 6s equal 18 and so on. Children can quickly do a repeat addition pattern with any number.

The basic multiplication facts pair 2 1-digit factors with a product, that is, factor × factor = product. The 81 combinations for multiplication facts were represented previously in figure 9.7. These basic multiplication facts, presented in abstract form as a table or chart of facts, should not be presented to children to learn from memory until they have been meaningfully introduced. Rather, children should gradually and systematically learn the facts building upon their prior knowledge.

The basic facts are best developed through problem situations, experiences with manipulatives, pictorial representations and various thinking strategies. The table becomes the end result of this process of the development of multiplication facts (and related division facts). If they choose, children could record the facts they know with confidence on the table as they progress.

There has been a growing interest in research about the development of children's strategies for multiplying and dividing single-digit numbers (Verschaffel, Greer & De Corte 2007). Thinking strategies for multiplication facts provide efficient ways for a child to attain each fact. Initially, working with number facts up to 5 × 5 can be a useful strategy. Other strategies include commutativity, skip counting or repeated addition, splitting the product into known parts, and using patterns for the square numbers and the 9s.

Commutativity

Commutativity applies to multiplication just as it does to addition. It is, therefore, a key strategy for helping students learn the multiplication facts. In the Classroom 9.1 emphasises this property. The calculator is also useful in reinforcing the idea. Children can multiply 4 × 6 then 6 × 4, for example, and notice that the answer in both cases is 24. After they have tried many combinations, students should be able to verbalise that the order of the factors is irrelevant.

Figure 9.12 illustrates that the same 36 numbers appear above and to the right of the diagonal as the 36 numbers below and to the left. This mirror image helps demonstrate that the commutative property reduces the number of facts to be learned by almost half, in the same way as addition. Each yellow square below the diagonal can be matched with a white square above the diagonal with the same factors and product. There are now 45 distinct multiplication facts to be learned, not 81, if the student recognises the power of commutativity. Furthermore, of these facts fifteen are less than 5 × 5 = 25, which can be a good starting point for some children and reduce the feeling of being overwhelmed.

×	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

FIGURE 9.12 Multiplication facts derived by the commutative thinking strategy

Skip counting

The strategy of skip counting works well for the multiples children know confidently such as, 2s, 5s, and 10s but it can be applied to 3s (or other numbers) if children have learned to skip count by them. The 2 examples below are for facts of 2 and 5.

$8 \times 5 = \square$
So

Think
5, 10, 15, 20, 25, 30, 35, 40
$8 \times 5 = 40$
8:40, wow, 8 on the clock is 40 minutes past!

$8 \times 2 = \square$

Think
2, 4, 6, 8, 10, 12, 14, 16
$8 \times 2 = 16$
8 pairs of shoes is 16 shoes

Notice that the child in the first example has made a connection to skip counting around the clock face (as you do when counting 5-minute intervals after the hour), which is a good way to reinforce and recall multiples of 5. The facts that can be efficiently established with the skip counting strategy for 2s and 5s are represented in figure 9.13. Once students are confident with the skip counting strategy and these facts are learned, the number of basic facts are reduced from 45 to 21 for multiplication.

×	1	2	3	4	5	6	7	8	9
1									
2		4	6	8	10	12	14	16	18
3		6			15				
4		8			20				
5		10	15	20	25	30	35	40	45
6		12			30				
7		14			35				
8		16			40				
9		18			45				

FIGURE 9.13 Multiplication facts derived by skip counting for 2s and 5s

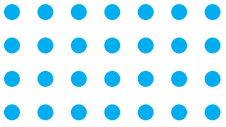
Repeated addition

The strategy of repeated addition extends the concept of skip counting to factors less than 5, that is triple and quadruple (3× and 4×) numbers. At this stage the child might still be using additive thinking as it applies to groups less than 5.

$3 \times 6 = \square$ Think
$6 + 6 + 6 = 18$
$3 \times 6 = 18$

Because this strategy is based on 1 meaning of multiplication, children need to engage in learning that takes them beyond additive thinking. Concrete objects and pictorial representations are a useful methodology. Arrays, grids, number lines and groups of objects provide additional experiences to help develop this strategy and further advance the concept of multiplication. Figure 9.14 illustrates an array of rows and columns for the number fact 4×7 (and 7×4 when the array is rotated).

Look at these dots:



Ring them in sets of 7. How many sets of 7 are there? How many dots in all?

Count them: 7, __, __, __
Add them: $7 + 7 + 7 + 7 = \square$
Multiply them: $4 \times 7 = \square$

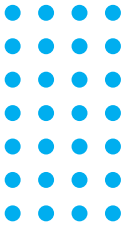
Do the following calculations using a calculator:

$3 + 3 + 3 + 3 + 3 =$
$5 + 5 + 5 + 5 + 5 =$

Do you know a simpler way? Show it here:

_____
_____

Jackson recorded these dots, which were counters he used to slove a problem:



Ring them in groups of 4. How many groups of 4 are there? How many dots in all?

Count them by 4s . . .
Add . . .
Multiply . . .

Explain which is the best way and why.

FIGURE 9.14 Visualising multiplication

The basic facts that can be efficiently established with the repeat addition strategy for 2s, 3s and 4s are represented in figure 9.15. Once students are confident with these facts and have begun to commit them to memory, the number of basic multiplication facts remaining for students to strategise and learn, are just 10 (for 6s, 7s, 8s and 9s).

×	1	2	3	4	5	6	7	8	9
1									
2		4	6	8	10	12	14	16	18
3		6	9	12	15	18	21	24	27
4		8	12	16	20	24	28	32	36
5		10	15	20					
6		12	18	24					
7		14	21	28					
8		16	24	32					
9		18	27	36					

FIGURE 9.15 Multiplication facts derived by repeat addition for groups less than 5

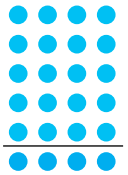
Splitting the product into known parts

As children gain confidence with some of the basic facts, they can use what they know to derive others. The strategy, known as *splitting the product* is based on the distributive property of multiplication. It can be approached in terms of '1 more group', '1 less group', 'twice as much as a known fact' or 'known facts of 5'.

The idea of *1 more group* can be used for almost all multiplication facts. If a certain multiple of a number is known, the next multiple can be determined by adding that same number once more. For example, to find 6×4 if 5 times are already known, then think 5×4 (20) and add 1 more 4 (to get 24). The computation is slightly more difficult if the addition requires regrouping. For example (shown below), where 8×4 is found by adding 1 more group of 8 to the known fact $8 \times 3 = 24$.

	Think
$8 \times 4 = \square$	$8 \times 3 = 24$
So	$8 \times 4 = 24 + 8 = 32$
	$8 \times 4 = 32$

Each basic fact can be used to help learn the next multiple of either factor. Illustrating this strategy using an array model (as shown) can be helpful.



$5 \times 4 = \square$

$1 \times 4 = \square$

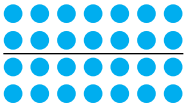
$6 \times 4 = \square$

Ask children to name each part of the array and write the multiplication fact for the whole array. Turn around the array and repeat.

There are a number of different ways to derive the same number fact and *twice as much* is an efficient strategy. Research reports that doubles (and halves) are a powerful strategy for 4s, 6s and 8s (Flowers and Rubenstein 2010/2011). In the example above where 8×4 was derived from $8 \times 3 = 24$ doubling is also an excellent method, because even numbers are involved, again with a variety of perspectives. This strategy can be applied to multiples of 4, 6 and 8 because an array with any of these numbers can be split in half. The product is found by working with twice as much as each part.

	Think
$8 \times 4 = \square$	$8 \times 2 = 16$, double is $16 + 16 = 32$, $8 \times 4 = 32$
or	$4 \times 4 = 16$, double is $16 + 16 = 32$, $8 \times 4 = 32$
or	$2 \times 4 = 8$, double is $8 + 8 = 16$, double again is 32

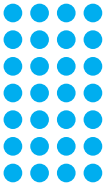
It is worth noting that some children may have trouble doing the computation mentally when regrouping (renaming) is required to do the doubling by addition. It is acceptable to jot down the 16 and 16 for example, and add them with pen and paper until they become well established as a basic fact. Alternatively, using models helps provide a visual image of this strategy. In the first case illustrated below, the array could be shown horizontally (4 rows of 7) or vertically (7 rows of 4), due to commutativity. Different children will find 1 or other of these representations easier to work with than the other. Both illustrations are given below.



2 7s are ____

2 7s are ____

$4 \times 7 = \square$

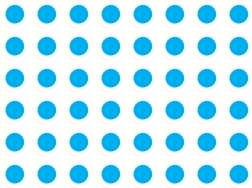


7 by 2 is ____

7 by 2 is ____

$4 \times 7 = \square$

As children progress with this strategy, they learn to split an array, write about each part, and then record the multiplication fact.



Solving the array 6×8 has a variety of options.

1 more than a 5 fact: $5 \times 8 = 40$ and $1 \times 8 = 8$, so $6 \times 8 = 40 + 8 = 48$.
Twice as much: $6 \times 4 = 24$ and 6×8 is twice as much, so $6 \times 8 = 24 + 24 = 48$.
Alternatively $3 \times 8 = 24$ and 6×8 is twice as much, so $6 \times 8 = 24 + 24 = 48$.

Further examples for this fact include using square numbers and the basic facts for 9, which are discussed in the next sections.

Working from facts of 5 is a strategy that will help children derive and learn other basic facts for multiplication. As shown in In the classroom 9.2, the facts for 5 are found in real world situations. They are illustrated on any analogue clock, and 5 facts are half of 10. The 5 facts can be helpful for solving any problem with large factors (such as 6, 7, 8 and 9) but are particularly useful for the even multiples 6 and 8 because both 5 6s and 5 8s are multiples of 10. In other words, 6×5 is the same as $(3 \times 2) \times 5 = 3 \times (2 \times 5)$ which is 3×10. Similarly, 8×5 is the same as $(4 \times 2) \times 5 = 4 \times (2 \times 5)$ which is 4×10 so it is rather easy to think tens by halving. Furthermore, a product can be split into the 5 part and remembered, then the remaining part can be added on. For example, one way to figure out 7×6, if you know $5 \times 6 = 30$, is illustrated.

	Think
$7 \times 6 = \square$	$5 \times 6 = 30$
	$2 \times 6 = 12$
So	7×6 is $30 + 12 = 42$

The facts that can be solved by splitting the product into known parts are shown in figure 9.16.

×	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4							24	28	32	36
5							30	35	40	45
6					24	30	36	42	48	54
7					28	35	42	49	56	63
8					32	40	48	56	64	72
9					36	45	54	63	72	81

FIGURE 9.16 Multiplication facts derived by splitting the product into known parts

Patterns

Although all of the basic facts for multiplication have been met using the above strategies, children frequently learn and retain the facts by looking for patterns and relationships. One of the most useful and interesting patterns concerns 9s. Looking at the products of the facts involving 9, there are many patterns. The digits of the products always sum to 9. That is $1 + 8 = 9$, $2 + 7 = 9$, $3 + 6 = 9$ and so on up to $8 + 1 = 9$. Furthermore, the 10s digit of each product is always 1 less than whatever factor was being multiplied by 9. That is $1 \times 9 = 9$ and 0 is 1 less than 1 ten, $2 \times 7 = 18$ and 1 is 1 less than 2 tens, $3 \times 9 = 27$ and 2 is 1 less than 3 tens, etc. There is a very good reason for this, just like the reason for the pattern that the product itself is less by the factor in tens, That is 9 is 1 less than 10, 18 is 2 less than 20, 27 is 3 less than 30 and so on up to 81 is 9 less than 90. If coloured on a hundred chart (see Appendix C), then the basic facts for 9 create a diagonal pattern leading back to 9 in the first row.

$1 \times 9 = 9$
$2 \times 9 = 18$
$3 \times 9 = 27$
$4 \times 9 = 36$

The 10s digit is 1 less than 4.

$0 + 9 = 9$
$1 + 8 = 9$
$2 + 7 = 9$
$3 + 6 = 9$

The sum of the digits of 36 is 9.

	Think
$5 \times 9 = \square$	The 10s digit is 1 less than 5
	The sum of the digits is 9
So	4 in the tens, $9 - 4 = 5$ in the ones
	$5 \times 9 = 45$

	Think
$7 \times 9 = \square$	$7 \times 10 = 70$
	$7 \times 1 = 7$
	$70 - 7 = 63$
	$7 \times 9 = 63$

Alternately, 7 in the 10s place less 1 10 means that the tens part is 60. Add 3 1s to make the answer 63 (sum of 9).

A quick trick for 9 × tables is:

9
18
27
36
45
54
63
72
81
90

Start writing the ones place for each fact at 9 and counting down to 0:

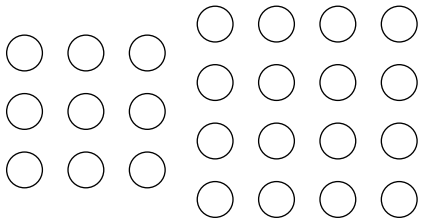
9, 8, 7, 6, 5, 4, 3, 2, 1, 0

Now, in the tens place from the end, working backwards, count down from 9 to 1:

9, **1**8, **2**7, **3**6, **4**5, **5**4, **6**3, **7**2, **8**1, **9**0

Pattern for square numbers is another useful strategy that students can use when learning the multiplication facts. Square numbers give a visual representation just as doubles and near-doubles do for addition.

Consider $3 \times 3 = 9$ and $4 \times 4 = 16$. Each of these makes a square when drawn in an array.



Hence, any number that can be arranged in a square pattern is known as a *square number*. Visually and mathematically they are the same. The facts that can be derived from square numbers are shown in figure 9.17.

x	1	2	3	4	5	6	7	8	9
1	1								9
2		4							18
3			9						27
4				16					36
5					25				45
6						36			54
7							49		63
8								64	72
9	9	18	27	36	45	54	63	72	81

FIGURE 9.17 Multiplication facts derived by the patterns for 9s and square numbers

Challenge children to find these and other interesting patterns in a table of multiplication facts such as the one represented early in the chapter, in figure 9.7. They might notice, for instance, that the columns (and rows) for 2, 4, 6 and 8 contain all even numbers, although the columns for 1, 3, 5, 7 and 9 alternate between even and odd numbers. Why? Challenge children to think creatively!

Although 'finger multiplication' (see figure 9.18), is a 'trick' (Kolpas 2002) for helping children learn basic facts for 9 (and others) it is best used for extension or interest re cultural and historical methods rather than as an aid to memorising. This seemingly magical trick, finger multiplication works because of the patterns that are inherent in the basic facts table and the base-ten number system. The method for facts 6×6 up to 10×10 is a simplified version of Chinese sticks multiplication, which is an intriguing method for 2-digit by 2-digit multiplication. Ultimately, however, it is more useful for students to focus on understanding and mastering the basic facts through reasoning and patterns than it is to be dependent on a concrete aid (fingers), which is less fluent.

Multiplying by 1 and 0

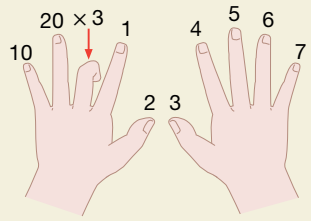
The basic facts with 1 and 0 are generally learned from experience working with multiplication. Children need to make the generalisation that 'multiplying by 1 does not change the other number' because it is the identity. One group of 7 is 7. The null factor, 0, can also be generalised. No amount of objects is 0, 'multiplying by 0 results in a product of 0'. The preceding strategies have now accounted for all the multiplication facts.

Using finger multiplication for basic facts of 9

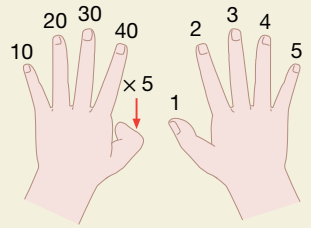
1. Hold both hands in front of you with palms facing away.
2. To do 3 × 9, bend down the third finger from the left.
3. The fingers to the left of the bent finger represent tens. The fingers to the right of the bent finger represent ones. Read off the answer: 2 tens and 7 ones, or 27.
4. To do 5 × 9, bend down the fifth finger from the left. Try this method for any 9-facts: 1 × 9 through 10 × 9.

Finger multiplication for basic facts with factors 6–10 only

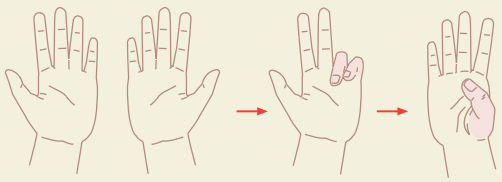
1. Hold both hands in front of you with palms facing you.
2. Think of a multiplication in which both factors are 6 or more (up to 10). For example, you can do anything from 6 × 6 up to 10 × 10.
3. For each factor, find the difference from 10.
   - Let's try the example 8 × 9.
   - For the factor 8, think 10 − 8 = 2, and bend down 2 fingers on the left hand.
   - For the factor 9, think 10 − 9 = 1, and bend down 1 finger on the right hand.
4. Look at all fingers that are still up, and count by 10 (10, 20, 30, 40, 50, 60, 70).
5. Look at the fingers that are bent down. Multiply the number of fingers bent down on the left hand by the number bent down on the right hand (2 × 1 = 2).
6. So, 8 × 9 = 70 + 2 or 72.
7. Try this method for any multiplication facts from 6 × 6 up through 10 × 10.



3 × 9 = 20 + 7 = 27



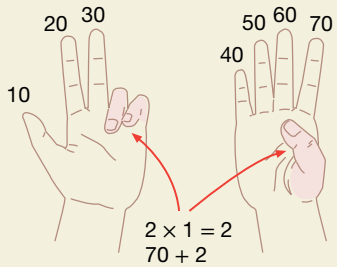
5 × 9 = 40 + 5 = 45



Multiply 8 × 9

Difference between 8 and 10 is 2 (fold down 2 fingers)

Difference between 9 and 10 is 1 (fold down 1 finger)



2 × 1 = 2
70 + 2
8 × 9 = 72

FIGURE 9.18 2 forms of finger multiplication

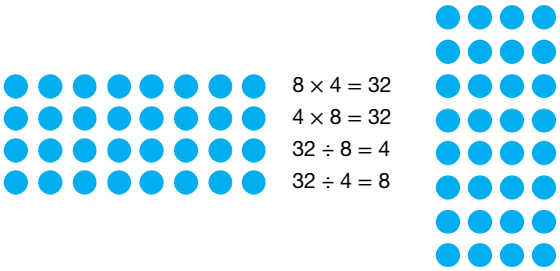
Thinking strategies for division facts

Helping children learn division traditionally took a large portion of time in the primary school curriculum. With the increased use of calculators, many educators see the value of reducing the time and attention once given to it. Nevertheless, children still need an understanding of the division process and fluency with division facts. These facts help them to respond quickly to simple division situations and develop a deeper understanding of the nature of division. When the inverse relationship of multiplication and division is known, it is clear that each basic multiplication fact has a related division fact.

Just as ‘think addition’ is an important strategy for subtraction, ‘think multiplication’ is the key thinking strategy to help children understand and recall the division facts. Finding the solution to a division problem involves finding an unknown factor when the product and another factor are known. The multiplication table (figure 9.7) illustrates all the division facts but it is read differently. For the division problem $54 \div 9 = \square$, the table can be read 2 different ways because of commutativity. One method is to look for the 9 at the top of the multiplication table and follow it down until the number 54 is found, then read across to find the other factor, 6. Students generally learn multiplication and division facts concurrently. Learning division facts such as $48 \div 6 = 8$ by remembering (and connecting with) multiplication facts such as $6 \times 8 = 48$ is efficient.

It is important to realise that most division problems that arise naturally in real world situations, or are given in computations, do not directly involve a multiplication fact that has been learned. For example, consider the computation $49 \div 6$. There is no basic fact involving 49 and 6. So what do you do? In this situation most people quickly and automatically mentally review the 6-facts that they know in order to find the multiplication that comes closest to 49. The thinking process might go like this: $6 \times 7 = 42$ — too small, $6 \times 8 = 48$ — just a little too small, $6 \times 9 = 54$ — too big. From this mental review, it can be concluded that $49 \div 6 = 8$, with 1 remainder. Children need practice in thinking this way, mentally finding the answers to problems involving 1-digit divisors. The answer will also be 1 digit but a decision needs to be made about the remainder. Is it 1 extra, 1 portion to be shared, or does it affect the answer. In division problems the context of the situation is always relevant.

Just as related facts can be developed for addition and subtraction, they can also be useful for multiplication and division. This array and the vertical version of it illustrate all 4 facts.

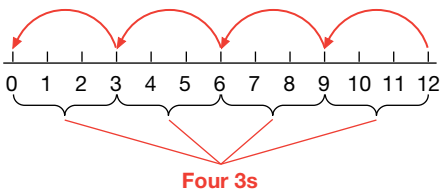


Because of the inverse relationship to multiplication, division can be stated as a fact, where the square represents an unknown factor.

$$42 \div 6 = \square \quad \rightarrow \quad 6 \times \square = 42$$

Thus, children need to search for the missing factor in a multiplication problem. In this case $6 \times 7 = 42$ is the known multiplication fact and children can use what they know to learn the more difficult division fact. Moreover, division is related to subtraction, and division problems can be solved by repeated subtraction. Whilst this concept is simpler, based on additive thinking, it can be an important stage for children to work through. Seeing how many times the number is subtracted to get down to zero makes sense to most children initially.

What is $12 \div 3$? Since $12 - 3 - 3 - 3 - 3 = 0$ (subtracting 4 3s), then $12 \div 3 = 4$



Repeated subtraction represented on a number line

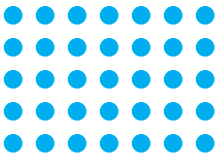
Recognising that division can be thought of as repeated subtraction, just as multiplication can be thought of as repeated addition, is helpful, particularly later when children encounter division with multi-digit numbers. One of the main division algorithms for pen and paper calculation is constructed on the understanding that division is equivalent to repeated subtraction.

However, as was mentioned earlier, counting backward is difficult for many children. Skip counting backward, in order to figure out a division problem mentally, is even more difficult than counting backward by 1, to figure out a subtraction problem mentally. The following 2 examples of skip counting backwards illustrate different levels of difficulty depending on regrouping.

	Think
$15 \div 3 = \square$	15 . . . 12, 9, 6, 3, 0
	I subtracted 5 times
So	$15 \div 3 = 5$

	Think
$28 \div 7 = \square$	$28 - 7 = 21$, $21 - 7 = 14$, $14 - 7 = 7$, $7 - 7 = 0$
So	$28 \div 7 = 4$
	Wow! Like when I think about the calendar.

A more productive approach is 'think multiplication' just as it was productive to 'think addition' in order to perform subtractions. If the multiplication facts are known, then $35 \div 7 = \square$ is recalled by thinking 'what times 7 is 35?' Some facts might be easier than others to remember. If the required fact is not immediate then splitting the product into known parts is a helpful strategy as is a visual image. Creating an array of dots with 7 in each row will be beneficial.



Pictorial representations of 35 objects into 7 rows

As with multiplication, work with arrays helps children relate the symbols to the action.

Thinking strategies for division are more difficult for children to learn than are the strategies for the other operations. There is more to remember and regrouping is often necessary. When using skip counting as a division strategy, for instance, the child needs to keep track of the number of times a number is named even as the child simultaneously struggles to count backward. However, 'think multiplication' is an efficient and accurate strategy for division, which avoids the difficulties that other division strategies involve. Being able to recall multiplication facts fluently therefore is an essential prerequisite to being able to divide efficiently.

A final word of caution is needed when 0 is involved in division. Dividing 0 by some number and dividing some number by 0 are very different situations. If you divide 0 by 6 ($0 \div 6$), the result is 0. Check this by multiplying your answer (0) by your divisor (6): $0 \times 6 = 0$. This works because 0 is a null factor for multiplication.

However, *division by 0 is impossible*. For example, to solve $6 \div 0 = \square$ requires a solution so that $6 = \square \times 0$. However, there is no value for the unknown ($\square$) that would make this sentence true. Therefore, $6 \div 0$ has no solution. A similar difficulty arises if you try to divide 0 by itself. That's why *division by 0 is undefined* in mathematics. The teacher is likely to have difficulty remembering which is possible in these 2 divisions; therefore children will struggle with the same issue.

A great activity to practise basic facts for the 4 operations with children of various ages using numbers 1–30 (from a calendar) and beyond is 'Today's Number is . . . (McIntosh, DeNardia & Swan 1994) as illustrated in In the classroom 9.3.

IN THE CLASSROOM 9.3

Today's number is . . .

Objective: Practise basic facts for the 4 operations using numbers 1–30.

Year level: Various

Today is 9 April so the number of the day is '9'. Here are some things that can be said about it mathematically.

- It is the number before 10 but after 8.
- It is the last single-digit number when counting.
- It is a square number.
- A nonagon is a shape with 9 sides.
- The sum of 9 can be made from 0 + 9, 1 + 8, 2 + 7, 3 + 6, 4 + 5 and the reverse of those facts.
- 18 – 9 = 9, 17 – 8 = 9, 16 – 7 = 9, 15 – 6 = 9, 14 – 5 = 9, 13 – 4 = 9, 12 – 3 = 9, 11 – 2 = 9, 10 – 1 = 9 and 9 – 0 = 9. These are just the basic facts for subtraction.
- When working with multiplication, there are 2 basic facts: $3 \times 3 = 9$ and $9 \times 1 = 9$.
- It is not a prime number.
- Division is interesting, there are many patterns for these basic facts: $81 \div 9 = 9$, $72 \div 8 = 9$, $63 \div 7 = 9$, $54 \div 6 = 9$, $45 \div 5 = 9$, $36 \div 4 = 9$, $27 \div 3 = 9$, $18 \div 2 = 9$ and $9 \div 1 = 9$.
- Any number is divisible by 9 if its digits add up to 9 — for example, 7236 is divisible by 9 (add digits 7 + 2 + 3 + 6 = 18 and add again 1 + 8 = 9). The year 2016, is also divisible by 9.

To extend this activity further, include brackets, indices and dice.

Encourage students to create the result of 9 in many varied and interesting ways from the numbers resulting from rolling 3 normal dice. For example, if the dice roll outcome is 2, 3 and 5, then:

- $9 = 5 + 3 + (2 \div 2)$
- $9 = (5 - 2) \times 3$
- $9 = 2^2 + 3 + (5 - 3)$.

Numbers: Facts, Figures and Fiction (Phillips 2004) is a great resource for extension ideas.

MAKING CONNECTIONS

Researchers studying how people naturally learn basic addition facts in countries all around the world have noted some fascinating worldwide commonalities, as well as some significant differences. It appears there is a natural learning progression — across all cultures — for single-digit addition and subtraction. All learners naturally build their later (more sophisticated) addition methods from their earlier methods 'by chunking, recognizing and eliminating redundancies, using parts instead of entire methods, and using their knowledge of specific numbers' (Fuson 2003, p. 73).

The first stages of this natural progression involve different forms of counting (which have already been discussed in a previous chapter). People initially add 2 piles of objects by counting each separately (1, 2, 3 and 1, 2, 3, 4) and then *counting all* (1, 2, 3, 4, 5, 6, 7). Later, they learn to *count on* — by simply stating the first addend (3) and counting up from there (4, 5, 6, 7). Soon, they learn to *count on* even without objects, keeping track of the numbers by using their fingers or auditory patterns.

In time, people begin to shortcut the counting process of addition by breaking up larger numbers into smaller, easy-to-remember number combinations. For example, they quite naturally begin to use doubles, and near-doubles and other combinations of small numbers this way. For example, they think of 3 + 4 as 3 + 3 + 1 = 6 + 1 = 7.

In many parts of the world — especially Asian countries and some European countries — people progress next (often quite naturally) to the general 'make-a-10' thinking strategy. The languages spoken in

many of these countries support this strategy. Many languages other than English have teen numbers that transparently contain 10 and some extras; their counting into the teens sounds like 'nine, ten, ten-one, ten-two, ten-three' and so forth. By contrast, our 2-digit counting tends to slow down the understanding of place value and addition facts because it is not logical. Prior to 13 (ten, eleven, twelve), the number words reveal absolutely no connection to ten! Two of the next 3 numbers (thirteen, fifteen) do suggest a connection to the correct sequence, but the variants of the unit words (*thir/three*, and *fif/five*) need to be seen as well as the new word *teen* meaning 'ten'. For the remaining numbers, fourteen, sixteen, seventeen, eighteen, and nineteen, the connection to ten is easier but the word for ten (*teen*) comes after the word for the number of ones. Logically these numbers should all be said ten (and) three, ten and four, ten and five and so on.

The facts considered 'basic' in some countries are not the same as those we call 'basic facts' in Australia. For example, basic addition facts in South Korea focus on combinations that total 10 or less. This is a reflection of their work with a ten-frame and decomposing larger numbers into combinations of 10. Thus 7 + 8 is not considered a basic fact in Korea, but children there decompose 1 of the addends to create a basic fact. So, to add 7 and 8, Korean children would think '7 + 3 is 10' or '8 + 2 is 10', and then — in both cases — they would think 'and 5 more is 15'. This approach may seem different to you, but the high levels of performance that Korean children demonstrate is testimony to its effectiveness. With help from teacher resources that stress this natural make-a-ten approach, most children in Asian countries are fluent in adding 2 single-digit numbers by the end of Year 1.

Huayu Sun's (2008) study of Chinese children aged 3–6 revealed a definite developmental sequence for addition strategies. Furthermore, mastery of the abacus by Kindergarten 3 children (the Chinese system has Kindergarten 1, 2 and 3; Kindergarten 3 is equivalent to our Foundation year) meant that by age 5, children recalled most of the basic facts for addition and subtraction and could quickly calculate facts beyond their current recall. Thus, the study found the abacus was a successful strategy used by children in China from a young age.

In many English-speaking countries, children are not naturally prompted by the language to move to the more efficient make-a-10 addition strategy. Activities such as Today's number is . . . shown in In the classroom 9.3 should be used regularly to practise basic facts in a random way. The internet has a range of suggestions for 'rainbow facts' which basically use a pictorial image for the addends that make 10. Teacher resource materials such as *Think Mathematically How to Teach Mental Maths in the Primary School* (McIntosh, DeNardia & Swan 1994) and *Mental Computation: A Strategies Approach* (McIntosh & Dole 2005) are excellent resources available in Australia, as is the *Count Me In Too* website by the NSW Department of Education and Communities, It can be found in the Kindergarten to Year 6 drop-down box in the Mathematics section.

A GLANCE AT WHERE WE'VE BEEN

Problem-solving experiences can help students develop meaningful skills in number computation with whole numbers. In this chapter, we first considered how to help children develop and model meanings for the 4 basic operations of addition, subtraction, multiplication and division. Experiences which encourage additive and multiplicative thinking, based on relevant problem contexts, and which include thinking, talking and writing about mathematics are important. These experiences should move from concrete modelling with materials, to pictorial representations, and to abstract representations with symbols. Addition means finding out 'how many in all?', but a variety of related situations lead to subtraction (separation, comparison and part-whole). Multiplication situations include equal-groups, comparison, combinations and areas/arrays. Measurement (repeated-subtraction) and partition (sharing) situations lead to division. It is beneficial for children to learn about mathematical properties at the same time that they are developing their understanding of the 4 operations and becoming proficient with basic facts.

The second major part of this chapter focused on the basic facts for each operation. A variety of thinking strategies for the basic facts for each operation were presented. It is essential for children to derive answers to basic facts they do not yet know by using logical strategies that build on what they do know. This approach will help children move from simple strategies based on counting to more mature, efficient ways of developing the basic facts. Specific suggestions for practice to help children become fluent with the basic facts will lead towards mastery. The ultimate goal is quick, confident recall of the number facts for the 4 operations (over an extended time frame as developmentally appropriate to children) without creating anxiety or meaningless repetition.

THINGS TO DO: FROM WHAT YOU'VE READ

1 Discuss each of these statements, justifying an agree or disagree stance.
   (a) When you teach multiplication, you begin preparation for learning division.
   (b) Children should not be allowed to count on their fingers when they start addition.
   (c) With the wide use of calculators, there is little need for children to attain prompt recall of the basic facts.

2 Which number properties when applied to addition and multiplication are especially helpful in learning the basic facts?

3 Describe the thinking strategies a child might use with each of the following number facts:

$8 + 3 =$ $8 \times 5 =$ $7 + 8 =$ $18 \div 3 =$ $16 - 7 =$

4 For each of the following addition strategies: adding 1, adding 2, adding doubles or near-doubles, adding to combinations of 10, adding to 10 and beyond), list 3 different facts and explain what a child might think when using that strategy.

5 When is counting back an effective strategy for subtraction? Give an example. Give an example of a fact where counting back is not such an effective strategy and explain another strategy that would be better to use.

6 Explain why division by 0 is considered 'undefined' by mathematicians.

7 There are 2 types of division that arise in real-world problems: measurement (or repeated subtraction), division and fair-sharing (or partition) division.
   (a) Write a partition (or fair-sharing) division word problem that would correspond to $24 \div 6$. Draw a picture to illustrate the solution to your problem and in simple language explain why your numerical answer makes sense.
   (b) Write a measurement (or repeated-subtraction) division word problem that would correspond to $24 \div 6$. Draw a picture to illustrate the solution to your problem and, in simple language, explain why your numerical answer makes sense.

8 For each real-world situation created in question 7, explain what the meaning of a remainder would be if the problem was $25 \div 6$.

THINGS TO DO: GOING BEYOND THIS TEXT

IN THE FIELD

1 Plan an interactive bulletin board to help children learn about 1 or more thinking strategies for subtraction.

2 Examine a recently published teacher resource for Years 2, 3 or 4. Analyse how basic facts for the 4 operations are presented in that reference. How are the facts grouped? Are thinking strategies presented? If so, how do they compare with the strategies described in this chapter? How would you use this resource for helping children learn the basic facts? If you think you would want to supplement this teacher resource, explain what sorts of supplementary materials and experiences you would use and why.

3 *Literature context for the 4 operations*. Find a popular children's book that would be useful as a motivator for a problem based lesson revolving around 1 of the 4 operations (addition, subtraction, multiplication or division). Identify the operation that the story evokes and write a problem that you might challenge students to work on after hearing the story. If possible, read the story to a group of children, ask them to solve your problem and analyse their written work for evidence of understanding the operation(s) which the story involves. A classic example is *The Doorbell Rang* by Pat Hutchins, for division and multiplication. Australian Curriculum Lessons (Maz26 2014) can be found at, www.australiancurriculumlessons.com.au/2014/01/25/the-doorbell-rang-a-lesson-on-multiplication-division-and-literature which includes a video link to the story.

4 Imagine yourself as the teacher of children aged 6 or 7 years. You have made the following statement: 'Today's number is 15.'

Represent the thinking and recording those young children might give for the result of 15 (fifteen) using each of the following concepts and/or operations:

- sorting and counting
- addition
- subtraction
- grouping (multiplication)
- place value.

Your illustrations can describe operations that result in 15 from a concrete, pictorial and/or abstract level of thinking. Please include relevant mathematical language.

5 What behaviours and emotions do you think might arise when children undertake a timed test on the basic facts? You might do some secondary research about this topic in preference to an observation in the field. How does the teacher use the results of the test? Talk to at least 5 children or adults about their memories of timed tests on basic facts (addition or times tables). What do they say about their experiences? What other approaches do you believe are more effective?

6 *Basic fact difficulty*. Talk to Year 5 or 6 teachers about what they do to support students who are not fluent in basic facts. Do these students receive help with learning strategies? Do they use calculators routinely for classwork and homework? How many in the class have problems with basic facts? Which facts cause the most difficulties? Discuss how not knowing the basic facts could be holding them back with mathematical learning in other topics.

WITH ADDITIONAL RESOURCES

7 Read 1 of the following research reports or journal articles:
'The Number line: An auxiliary means or an obstacle' (Skoumpourdi 2010)
'The empty number line in Dutch second grade' (Klein, Beishuizen & Treffers 2002)
'The empty number line: A useful tool or just another procedure?' (Bobis 2007).
These papers describe the use of a number line with no zero point shown as a computational support for children doing multidigit addition and subtraction. What instructional and psychological reasons do the authors give for using the empty number line as a central model for addition and subtraction?

8 Pick a concept or skill from this chapter and compile a brief bibliography of recent articles with research or teaching ideas on the topic.

9 Find a calculator activity, software package or game on the internet that focuses on basic facts. Share the ideas you find.

WITH TECHNOLOGY

10 Find a computer program that claims to help children learn basic facts and try it out. There are many such programs on the market. Most focus on encouraging development of quick responses, but only few encourage use of thinking strategies. Write a half-page review of the computer program (similar to those published in teacher journals). In your review you should identify the

name and publisher of the program, the cost and the type of hardware required to run it. Explain how the program works. Include answers to the following questions.

- Does the program help children learn strategies for basic facts?
- Does the program keep track of student progress?
- Does the program modify the facts presented to the student according to which ones he or she has solved correctly previously?
- Do you think the program would be effective in promoting quick recall of facts? Why?
- Do you think children would enjoy using this program? Why?
- How would you use such a program if your classroom had only 1 or 2 computers?
- Would you recommend this program to other teachers? Why?

11 maths300.com is a professional development resource with 190 lessons, well-trialled and loved by many mathematics educators. The following lessons are particularly pertinent to number facts development and practice and are well worth a look for planning and implementing with a group of children, or for building your own knowledge and experience.

- Birth Year Puzzle, Lesson 177 (for children born in 1999, 2000 and 2001 you might substitute the current year);
- Chart Strategies, Lesson 156
- Dice Footy, Lesson 161
- Domino Trails, Lesson 95 and Dominoes, Lesson 66
- Magic Squares, Lesson 98
- Row Points, Lesson 78
- Trial, Record and Improve, Lesson 84.

CHILDREN'S LITERATURE CONNECTIONS

Fromental, J. L. (2007). *365 Penguins*. New York: Harry N. Abrams Inc.
A surprise package becomes a great puzzle and a major concern for a family as the number of penguins grows.

Giganti, P. & Crews, D. (1992). *Each Orange Had Eight Slices*. New York: Greenwillow Books.
In this book, examples from everyday life show how addition leads to multiplication.

Hulme, J. N. (1995). *Counting by Kangaroos: A Multiplication Concept Book.* New York: Scientific American Books for Young Readers.
Multiplication is illustrated using Australian animals. Groups of 3 squirrel gliders, 4 koalas, 5 bandicoots…, 10 wallabys all crowd a house.

Merriam, E. (1993). *12 Ways to Get to 11.* New York: Simon & Schuster.
The book shows how to get to 11 by using different combinations of numbers. Many different pictures are used to show the different combinations of the number 11.

Murphy, S. (2006). *Mall Mania.* New York: HarperCollins.
Shopping, counting, and a birthday present all add up to a surprise ending on Mall Mania Day! A lighthearted look at addition strategies

Murphy, S. (1999). *Spunky Monkeys on Parade.* New York: HarperCollins.
At the Monkey Day Parade, the reader sees monkeys marching as cyclists, tumblers and band members. They create a spectacle as they move in 2s, 3s and 4s. This skip counting book will let the reader see many different ways of adding and multiplying.

Murphy, S. (1997). *Elevator Magic.* New York: HarperCollins.
When the elevator goes down, the subtraction starts and so does the magic. Ben sees crazy things every time the door opens. Ride along as he subtracts his way down to the lobby, and decide for yourself if it's elevator magic.

Sayre, J. & Sayre, A. (2004). *One is a Snail, Ten is a Crab*. London: Walker Books.
This is a counting-by-feet book that can be used to develop addition facts and multiplication concepts, together with early numeration.

Sierra, J. (1997). *Counting Crocodiles*. Orlando, FL: Gulliver Books, Harcourt Brace.
A clever monkey, using her counting ability, outwits the hungry crocodiles that stand between her and a banana tree on another island on a sea.

Wakefield, A. (1996). *Those Calculating Crows!* New York: Simon & Schuster Books for Young Readers.
Using different plans to prevent crows from invading his corn crop, Farmer Roy tries to scare off the crows. However, the crows are crafty. Can they count? This is based on a true story by some hunters.

The best children's books is a comprehensive website to support the use of children's books with mathematics. The following links go directly to each concept.
www.the-best-childrens-books.org/teaching-subtraction.html
www.the-best-childrens-books.org/teaching-addition.html
www.the-best-childrens-books.org/teaching-multiplication.html
www.the-best-childrens-books.org/teaching-division.html

Similarly the Christchurch city libraries have created a list with brief descriptions of children's mathematics picture books http://my.christchurchcitylibraries.com/maths-picture-books/.

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CHAPTER 10

Mental computation, calculators and estimation

LEARNING OUTCOMES

10.1 Understanding the myths and facts associated with using calculators.

10.2 Describing effective strategies for mental computation and guidelines for developing mental computation skills.

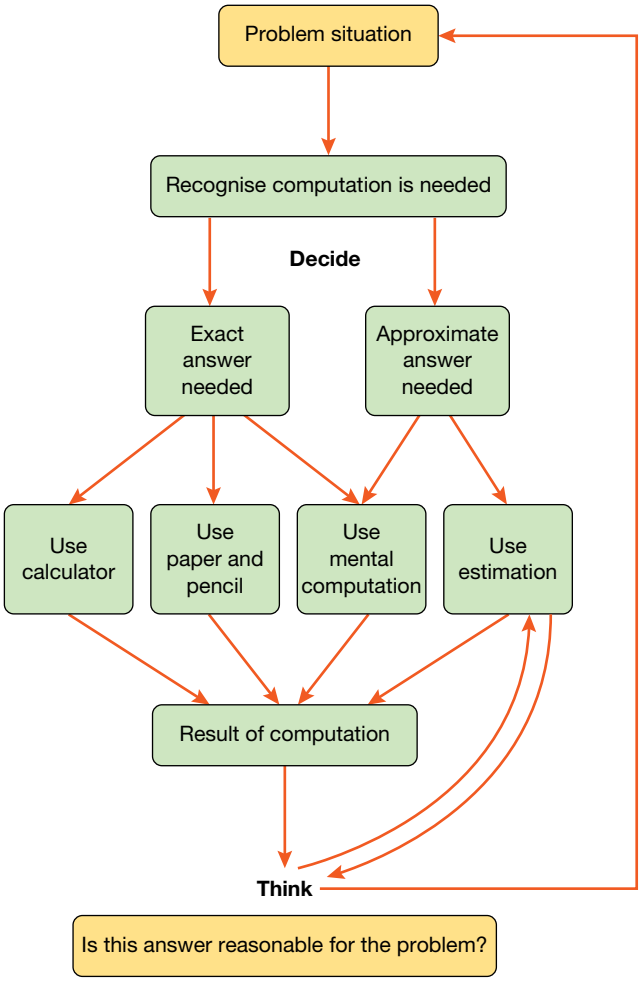
10.3 Explaining different approaches to computational estimation and how teachers can encourage estimation.



'"Give a man a fish, he'll eat for a day. Teach a man to fish, he'll eat for a lifetime." It's the same for mathematics education for twenty-first century life.'

Keith J. Devlin, *Introduction to Mathematical Thinking*

Chapter 10 concept map



Introduction

Solving mathematics problems has changed considerably over time according to people's needs in real world situations. As the situation demands, counting and solving problems mentally will make way for methods involving written strategies and tools (such as the abacus or calculator), when dealing with more complex problems. For more than 25 years, the calculator has become a widely accepted tool for mathematics learning in many countries. Whilst curriculum documents and professional organisations continue to recommend the use of calculators and computers in Australian schools (AAMT 1987, 1996), calculators tend to be underutilised as a teaching resource (Groves, Mousley & Forgasz 2006). Finding the balance between mental, written and calculator (or computer) methods is an essential aspect of the Australian Curriculum, yet many teachers appear to be reluctant to change traditional teaching methods and beliefs about the teaching and learning of mathematics to include the appropriate use of calculators (Bayetto & Yates 2007).

According to the discussion paper *School Mathematics for the 21st Century: Some Key Influences*, mental computation is an important component of children's facility with number that has clear and practical uses in the future (AAMT 2009). Mental computation can deepen children's understanding of how numbers work and increase fluency with mathematics. Research indicates that students believe being good at mental computation is an important real life skill that will be needed outside of school rather than just in school (Groves, Mousley & Forgasz 2006).

For example, using a written algorithm or calculator to solve 45 × 1000 is not appropriate because there are easy and efficient ways to produce this result. Similarly, buying 2 equivalent books, marked at 'half price' when each was originally priced at $10.50 requires no further calculation. The advice given by Joseph Ray, a textbook author of the 1800s 'If you can do the arithmetic in your head, do it' continues to be relevant 2 centuries later. Similarly, the idea given by Keith Devlin in the chapter opening, that if children learn to use mathematics effectively in the twenty-first century then there is no need to keep relearning the procedures.

Efficiency with different methods of solving problems is not merely useful; it is essential, and this is reflected in the Australian Curriculum (table 10.1). Historically, school mathematics emphasised the use of written computation far more than other methods. This, together with the fact that learners are more likely to use methods with which they are familiar, means that students often tend to use written computation even when they could use a more efficient method. Note that mental computation is best developed early and over several year levels even before written methods (Rogers 2009). Furthermore, according to the Australian Curriculum (v 8.1), proficiency with mental computation is a prerequisite for computational estimation.

TABLE 10.1 **Mental, written and calculator strategies: Proficiencies and content descriptions for Year 2–7 in Australia**

Year	Content description
Year 2 **Understanding** includes partitioning and combining numbers flexibly. **Problem solving** includes formulating problems from authentic situations, making models and using number sentences that represent problem situations. **Reasoning** includes using known facts to derive strategies for unfamiliar calculations.	**Number and place value** • Solve simple addition and subtraction problems using a range of efficient mental and written strategies.
Year 3 **Understanding** includes partitioning and combining numbers flexibly. **Reasoning** includes generalising from number properties and results of calculations.	**Number and place value** • Represent and solve problems involving multiplication using efficient mental and written strategies, and appropriate digital technologies.
Year 4 **Understanding** includes partitioning and combining numbers flexibly. **Problem solving** includes formulating, modelling and recording authentic situations involving operations. **Reasoning** includes generalising from number properties and results of calculations, deriving strategies for unfamiliar multiplication and division tasks.	**Number and place value** • Develop efficient mental and written strategies and use appropriate digital technologies for multiplication and division where there is no remainder.
Year 5 **Fluency** includes using estimation to check the reasonableness of answers to calculations. **Problem solving** includes formulating and solving authentic problems using whole numbers. **Reasoning** includes investigating strategies to perform calculations efficiently.	**Number and place value** • Use estimation and rounding to check the reasonableness of answers to calculations. • Solve problems involving multiplication of large numbers by one- or two-digit numbers using efficient mental, written and appropriate digital technologies. • Solve problems involving division by a one-digit number, including those that result in a remainder. • Use efficient mental and written strategies and apply appropriate digital technologies to solve problems.

(continued)

TABLE 10.1 *(continued)*

Year	Content description
Year 6 **Reasoning** includes explaining mental strategies for performing calculation.	**Number and place value** • Select and apply efficient mental and written strategies and appropriate digital technologies to solve problems involving all four operations with whole numbers.
Year 7 **Fluency** includes investigating best buys. **Problem solving** includes formulating and solving authentic problems using numbers. **Reasoning** includes applying the number laws to calculations.	**Number and place value** • Apply the associative, commutative and distributive laws to aid mental and written computation. **Money and financial mathematics** • Investigate and calculate 'best buys' with and without digital technologies.

Source: © Australian Curriculum, Assessment and Reporting Authority 2015.

In terms of planning and teaching, a balanced approach is required so that children learn a variety of methods for solving problems and learn when is a good time for using each method. The following framework, illustrated in the concept map might be helpful.

Beginning with a problem to solve, students need to:

1. recognise that an appropriate method is needed to efficiently find the solution
2. decide on a strategy to solve the problem
3. be aware that estimation can be useful before, during or after a mental, written or calculator strategy
4. check the reasonableness of the result, which is an important part of problem solving.

Furthermore, the concept map for this chapter shows that 2 essential decisions are needed when problem solving. First, deciding on the type of result needed and, second, deciding on the best method for gaining that result. The first decision involves the question of whether an estimate is appropriate or an exact answer is needed. The second decision involves such questions as whether a calculator would be helpful, whether the numbers in the problem allow a mental strategy to be used or whether paper-and-pencil calculation is most appropriate or convenient. Thus, the learning goals for helping students solve maths problems might be:

- developing competence with each of the computational methods
- choosing a method that is appropriate for the computation at hand
- applying the chosen method correctly
- using estimation to determine the reasonableness of the result.

In addition, by helping students choose and use computational methods appropriately and effectively, the teacher will help them build upon their number sense.

This chapter begins by discussing some of the issues involved with finding the right balance of time for teaching and learning the different strategies. The use of calculators in computation is considered foremost, together with certain myths about the disadvantages of calculator use. The teaching of mental strategies comes next, with an outline of the benefits that students derive from using a variety of these. The chapter continues with suggested methods for teaching children about estimation and when certain strategies work best. Finally, various cultural differences in the choice and use of different computational methods are considered.

Note that this chapter focuses primarily on calculators, mental strategies and estimation, while written strategies are considered extensively in the chapter which follows.

Pre-service and in-service teachers need to think about the emphasis they will place on each of the computational methods and the amount of time that might be allocated to each. A fully developed perspective will take into consideration the changing times and the real world application of mathematics problems. More than 80% of all mathematical computations in daily life involve mental computation

and estimation; whereas written computation comprises approximately 12% and the use of calculators accounts for about 8% (Northcote & McIntosh 1999, p. 20). Ironically, research shows that teachers may devote a disproportionate 70–90% of instructional time in primary school maths to focus on written computation (Reys & Reys 1998). According to McIntosh (2004) an overemphasis on formal written methods means that many students entering secondary school might have inadequate understandings of place value, number relationships and operations. Mental computation strategies, which require active thought can build a deeper understanding of number.

Many proposals have been made regarding computation, ranging from prohibiting calculator use to eliminating the teaching of written algorithms. While recognising the importance of each method of solving number problems, a better balance of teaching and learning is needed, with greater emphasis on collaborative and investigative approaches to learning. What percentage of time should be allocated to helping children learn each method? This remains an open question, as clearly the answer depends in part on the developmental levels of the students and their use of mathematics outside of school. Ideally, 10 to 15 minutes (or more) of every 60 minutes of maths per day would be allocated to mental computation and discussion of student strategies for mathematical thinking. Figure 10.1 shows the approximate percentages of instructional time spent on different computational methods in the past (1875–1975) and present (as of 2015), along with a proposal for a better balance in the future (2025).

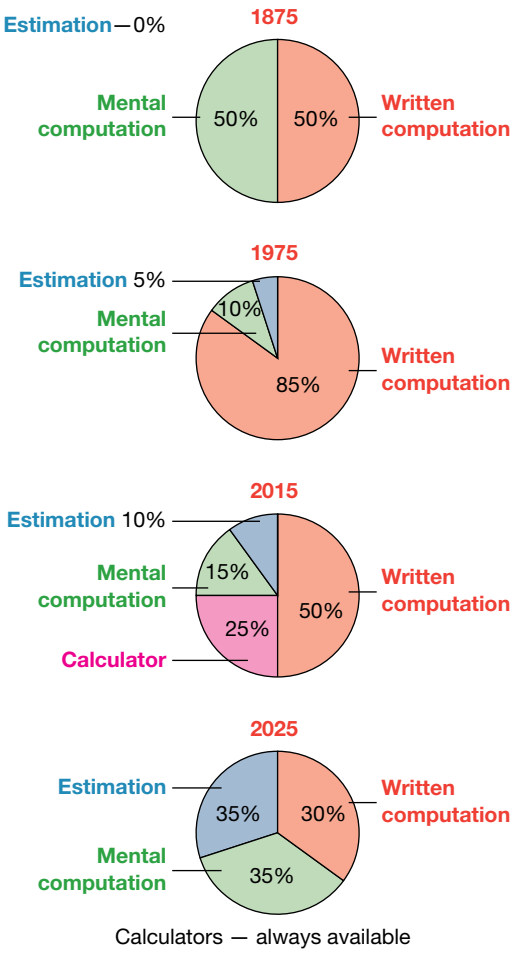


FIGURE 10.1 Time spent on teaching different methods of computation in primary schools in the past and, possibly, the future

The decline in instructional attention to written computation over the years is indisputable. Time previously devoted to developing proficiency with tedious written algorithms has become available for other purposes. Much of this time has been and can continue to be dedicated to helping students develop their number sense and their ability to select and use different methods, including calculators, mental computation and estimation. It is essential for the teacher to reflect upon their experience, think about the future needs of their students, and develop ways of finding the right balance between computational methods when helping children learn about solving problems involving number.

10.1 Calculators

LEARNING OUTCOME 10.1 Understanding the myths and facts associated with using calculators.

The Australian Association of Mathematics Teachers statement on the use of calculators and computers for mathematics in Australian schools recommends that:

> All students have ready access to appropriate technology as a means both to support and extend their mathematics learning experiences (1996, p. 1).

Implications for school and classroom practice are discussed in the statement as students' mathematical knowledge grows and changes over time. The calculator needs of young children in the early years are different to children in the primary years and students in the middle and secondary years. Specifically, the AAMT (1996, p. 5) states:

> Students of mathematics at all levels must be encouraged to become intelligent and critical users of technology, able to make informed decisions on the use of different mathematical strategies. Such decisions include *when* to use available technology, and *which technology* is most appropriate to the task. Students must develop an awareness of the *limitations* of the approach chosen, and a readiness to adapt and respond to technological changes . . . The effective use of technology encourages a move away from teacher-centred approaches and towards a more flexible and student-centred environment.

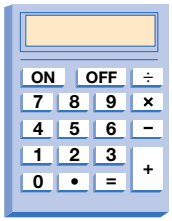
It is important for the teacher to help students understand how to use calculators appropriately, which includes showing students that not all problems can be solved with a calculator. As with any tool, students need to be given time to explore the calculator on their own. They need to be shown advantages and disadvantages of calculator use as well as how to maintain calculators.

In addition, the teacher should communicate with parents so they know that calculators are being used wisely in the classroom and that children understand their use. Often, a simple note home to parents, like the 1 shown in figure 10.2 is effective.

Calculators have been used in schools for more than 25 years; however, several myths, or incorrect beliefs, about using them persist (which is 1 reason why communicating with parents is important). In this section, we address these myths as follows.

- *Myth:* Using calculators does not require thinking.
- *Fact:* Calculators do not think for themselves. Students must still do the thinking.
- *Myth:* Using calculators lowers mathematical achievement.
- *Fact:* Calculators can raise students' achievement.
- *Myth:* Using calculators always makes computations faster.
- *Fact:* It is sometimes faster to compute mentally.
- *Myth:* Calculators are useful only for computation.
- *Fact:* Calculators are also useful as instructional tools.

Dear Parents:



We are using calculators in our Year 2 class this year in many different ways. Sometimes we use them to develop skills in counting and recognising patterns; other times to explore new topics, such as decimals; and often when solving problems that require tedious computations. We believe that a variety of early experiences with calculators will help young children make wise decisions about the use of calculators both in and out of school.

If you are curious about how we are using calculators, please ask your child to tell you about some things he or she has been doing with this tool. If you have further questions, please let me know.

FIGURE 10.2 A sample note to parents about using calculators in the primary years

Using calculators requires thinking

Teachers need to emphasise that calculators are a mathematical tool, they don't think — they only follow instructions. For example, suppose students were using a calculator to solve the following problem:

> A bus holds 32 children. How many buses are needed to hold 1000 children?

Is a calculator needed here? Would the calculator know the number operation required to solve the problem? Can the calculator check the result? Clearly, the student has to think about these questions.

This problem is difficult for primary students to solve with or without a calculator. To use the calculator effectively, students need to realise that the process required is division and that 1000 (number of children) must be divided by 32 (capacity of 1 bus). Up to this point, the learning is the same with or without the calculator. Next, the student can be encouraged to consider estimating using place-value understanding. For example $32 \times 10 = 320$ (too small), $32 \times 100 = 3200$ (too large) but about a third of this would be close. Therefore $32 \times 30 = 960$ (mental computation) will give a realistic estimate. Now, the students are ready to decide which calculator buttons to select and the order in which they need to be pressed.

[1] [0] [0] [0] [÷] [3] [2] [=]

Finally, when the result 31.25 appears on the calculator, the student needs to check and interpret the answer. Was it close to our estimation? Yes. What does the decimal part mean? Can I have 0.25 of a person on each bus? No. If 31 buses are used then 8 people would be left behind. If 32 buses are used then there is space for all the children and some extra seats for parents or adult volunteers.

This example shows that students do need to think for themselves. It also confirms that the calculator can extend the students' achievement beyond their current skills, is faster and more accurate than pen-and-paper algorithms, and affords children the time to think mathematically. The important problem solving processes of understanding, doing and checking become more significant than the calculation.

Using calculators can raise student achievement

Research has consistently indicated that the use of calculators in a balanced program can improve students' skills in mathematics, in terms of the 4 operations, problem solving and retention (Ellington 2003, Groves 1997; Smith 1997; Hembree & Dessart 1992). Moreover, 'students using calculators possess a better attitude toward mathematics and an especially better self-concept in mathematics' (Hembree & Dessart 1992). Many parents and some teachers worry that students will become dependent on calculators and hence won't learn to compute. Research shows this is not the case (Ellington 2006). The role of the teacher is to help students learn to make the most of digital technologies (Australian Curriculum Assessment and Reporting Authority 2013) as they adapt to new ways of doing things effectively and learn to choose technology appropriately.

Calculators are not always the fastest way of doing computations

As seen in relation to the bus problem above, calculators can make computations fast and easy, simplifying a problem solution. When it comes to straightforward calculations, the situation can be quite different. It is true that students can use a calculator to compute more complex problems such as long multiplication and division (2345×67 or $34\,567 \div 89$) very quickly. However, problems involving calculations that build on basic addition and multiplication facts and familiarity with numbers, such as 4×40, $99 + 99 + 99 + 99$, $\frac{1}{2} + \frac{1}{4}$ and $100 - 25$ are much quicker to perform mentally than with a calculator. The time required to press the correct keys on the calculator is far greater than a mental calculation. Students should always use automatic recall of the basic addition and multiplication facts (and related subtraction and division facts), as a first choice. Activities like the one shown in the classroom 10.1 are enlightening to students of various ages and ability levels. Helping children learn which process (MAL, PAL or CAL) to choose is important. In many cases, calculations are more quickly and effectively done mentally than with a calculator.

Calculators are useful for more than doing computations

The most obvious use of calculators is to solve number problems, but the processes involved in using the calculator can deepen students' understanding and reveal many powerful mathematical ideas. For example, calculator skip counting forward and backward (refer to In the classroom 7.4, 7.5 and 8.3) can reveal patterns, give students practice with important skills, and reinforce important knowledge such as the meaning of multiplication and division. Older children can use calculators to see patterns such as powers of 10 (10, 100, 1000) and decimals (0.1, 0.01, 0.001). Teacher resources (refer Judd 2007; Moss & Grover 2007) are rich in ways to help students use calculators effectively.

In the classroom, teachers can have calculators readily available for use when computational skills are not the main focus of learning. Encouraging exploration with calculators can be helpful in learning those skills. Clearly, the use of calculators should not interfere with the establishment and retention of basic facts, which remain an important goal of mathematics learning in the primary years. Basic facts are natural stepping stones to mental computation and assist with mindful use of written computation to the 4 operations. Basic fact recall aids estimation and further development of number sense; therefore, calculators should not be used for basic facts.

IN THE CLASSROOM 10.1

Calculators versus mental computation — when to use which method

Objective: Help students recognise which types of computation are best done with calculators and which types are best done mentally.

Year level: 3–5 and beyond

- Divide the class into pairs of students. For each pair, 1 student has a calculator and uses it to solve the problem. The other student doesn't have a calculator and does the problem mentally.

- For a whole class approach, prepare problems in advance on an interactive whiteboard and keep them hidden or write computation problems on the board 1 at a time. For a small group approach a mini whiteboard or printed sheet can be used.
- As each answer is found the students (in pairs) compare and record who was able to solve the problem first, keeping a tally.
- After everyone has finished the computations, pairs of students review the answers and determine who was quicker on the majority of questions. The teacher writes up the correct answers and students check against their own.
- Alternatively, groups of 3, with 1 student using a stopwatch and/or observation skills, with questions and answers prepared in advance on cards, can create a more student-centred approach to this activity. The teacher's role is facilitator as he or she rotates around the class supporting groups of students as needed.

Computation	First and correct using a calculator	First and correct using mental computation
1. 4×100	☐	☐
2. $400 + 50 + 8$	☐	☐
3. $99 + 99 + 99 + 99$	☐	☐
4. $1000 - 200$	☐	☐
5. $50 + 50 + 50 + 50 + 50 + 50$	☐	☐
6. $\frac{1}{2} + \frac{3}{4}$	☐	☐
7. $6300 \div 7$	☐	☐
8. $10 \times 10 \times 10 \times 10$	☐	☐
9. $0.75 + 0.25 + 0.50$	☐	☐
10. $\$1.00 - \$0.25$	☐	☐

To conclude:

- Teacher or students record the results with 2 lists — 1 list shows the computations that more students did correctly and faster using a calculator; the other list shows the computations that more students did correctly and faster using mental computation.
- Discuss the differences in these results and hypothesise why. Discuss what skills were used to solve the problems efficiently.

When is calculator use appropriate? Some activities with calculators, once considered motivational are of questionable educational value. For example using the calculator as a toy, by turning it upside down to spell words, adds little to mathematical knowledge. Using calculators to 'check the answer' to written computations can undermine students' motivation for doing written procedures. Both of these situations are likely to communicate the wrong message about calculator use. That is, the calculator is for playing games, or worse, using a calculator is cheating. Research reports that using technology, including calculators increases achievement in higher order thinking skills and helps students focus clearly on mathematical concepts (Sutton & Krueger 2002). The current position on calculators according to the National Council of Teachers of Mathematics supports this view. That is their use needs to be selective and strategic, 'Calculators can promote the higher-order thinking and reasoning needed for problem solving in our information- and technology-based society, and they can also increase students' understanding of and fluency with arithmetic operations, algorithms, and numerical relationships' (NCTM 2011). When calculators are used for specific learning activities and outcomes, then students gain the most benefit according to Ellington (2003). There are 2 main uses for the calculator in the classroom: as a computational tool and as an instructional tool.

A calculator should be used as a *computational tool* when it:

- facilitates problem solving
- eases the burden of doing tedious computations
- focuses attention on meaning
- removes anxiety about doing computations incorrectly
- provides motivation and confidence.

A calculator should be used as an *instructional tool* when it:

- facilitates a search for patterns
- supports concept development
- promotes number sense
- encourages creativity and exploration.

Calculators also provide a safety net for students having particular difficulties in learning computational procedures. For example, if students with special needs find it difficult or impossible to use traditional computational procedures, the calculator can provide equity and allow them to engage in problem solving along with other students (Thompson & Sproule 2005). According to research by Bayetto and Yates (2007), teachers who held particularly traditional views of mathematics learning didn't believe the calculator would free students with learning disabilities from routine work in mathematics. Instead, these teachers tended to endorse the idea that students with learning disabilities should learn to do written pen-and-paper computations before using calculators. However, Australian and international research has shown that students who have ready access to calculators develop more advanced and flexible number understandings than children without continued experience of calculators (Groves, Mousley & Forgasz 2006).

Calculators are clearly effective tools for learning, according to teachers involved in large-scale projects. That said, calculators are not used widely — particularly in the early years — as some parents, teachers and children hold reservations about their use. The benefits to be gained mathematically require new perspectives and changes to teaching and learning approaches that are more open-ended, more child-centred, involve real-world problems and investigations, and allow for flexibility between mental, written and calculator-assisted work (Groves, Mousley & Forgasz 2006).

10.2 Mental computation

LEARNING OUTCOME 10.2 Describing effective strategies for mental computation and guidelines for developing mental computation skills.

Mental computation is computation done 'all in the head' — that is, without tools such as a calculator or paper and pencil. This is certainly a natural way to solve maths problems, as well as by the fact that many young children develop ways of thinking mathematically before they are able to write in symbols and words. In fact, research has documented a wide variety of mental computation techniques that children have created on their own and that make sense to them (Fuson 2003; Kamii 2000, Reys & Barger 1994).

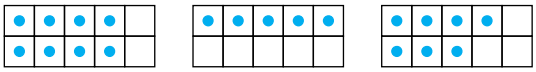
Australian studies of children's mental computation show that competent children have often acquired a range of efficient mental strategies almost 'in spite of' what happens in the classroom. These strategies are flexible, consider the numbers involved and do not replicate written strategies (Groves, Mousley & Forgasz 2006, p. 207).

Strategies and techniques for mental computation

Mental computation builds on the thinking strategies used to develop basic facts (refer to the previous chapter) and naturally extends children's mastery of basic facts. Extending basic facts is an early step in developing more powerful strategies and skills in mental computation. Extending basic facts takes different forms. Students might:

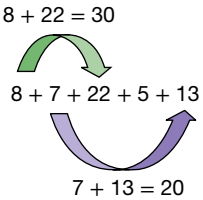
- use place value with basic addition facts and extend $4 + 5 = 9$ to $40 + 50 = 90$ and $400 + 500 = 900$
- use place value in doing a subtraction such as $675 - 200 = 475$ or $675 - 50 = 625$
- combine basic facts with place value and recognise that $47 + 16 = 47 + 10 + 6$ and $47 + 10 = 57$ and $57 + 6 = 63$
- decompose (break up) numbers to make them easier to handle. For example, to solve $18 + 17$ the student could decompose 17 into $2 + 15$; thus, $18 + 17 = 18 + 2 + 15$, and $18 + 2 = 20$ and $20 + 15 = 35$. Alternatively, the student could decompose 18 into $10 + 8$ *and* decompose 17 into $10 + 7$; thus, $18 + 17 = 10 + (8 + 7) + 10$, and $8 + 7 = 15$, and $10 + 15 + 10 = 35$. Both solutions use basic facts and place value while using different ways of decomposing numbers to make them easier to work with mentally.

As you can see from the last 2 examples, mental strategies are often used with compatible or 'friendly' numbers. These are numbers that can be combined to make numbers that are easy to work with, such as 10 (e.g. 8 and 2 are compatible numbers because they add up to 10). Earlier work with the ten-frame will help children see how to link different combinations to make 10:



8 + __ = 10 __+ 5 = 10 7 + __ = 10

Using compatible numbers facilitates mental computation of a string of numbers such as 8 + 7 + 22 + 5 + 13. For example, a student could recognise and reorganise the numbers. Knowing that 8 and 22 are compatible numbers (8 + 22 = 30) and that 7 and 13 are compatible too (7 + 13 = 20) the problem becomes much simpler.



The student could then solve the problem by thinking: 30 + 20 = 50 and 50 + 5 = 55.

Another student could do this same computation differently, by thinking: 8 + 7 = 15, 15 + 5 = 20, 22 + 20 = 42, 13 = 10 + 3, 42 + 10 = 52 and 52 + 3 = 55. There are many other ways of doing this same mental computation, but when working logically, all would reflect good number sense and show the importance of fluency with basic facts and with adding to 10 or multiples of 10. Understanding the different strategies children use and helping children understand each other's strategies will make the development of mental computation exciting.

The first step in developing proficiency with using compatible numbers is learning to recognise them. Figures 10.3 and 10.4 show 1 way the teacher can give children practice in developing this skill.

4	6	1	6	3	8
3	7	6	5	2	2
5	4	2	5	4	1
2	6	2	3	4	9
3	8	2	6	8	2
2	1	7	1	4	6

FIGURE 10.3 Find 2 or more adjacent numbers in a row, column or diagonal with a sum of 10 (try to find at least 8 different combinations).

43	7	16	25	25	16
23	20	34	42	19	34
27	30	21	8	31	11
18	50	29	37	13	39
32	0	47	15	17	40
10	40	3	35	20	10

FIGURE 10.4 Find 2 or more adjacent numbers in a row or column with a sum of 50 (try to find at least 8 different combinations).

Such practice also demonstrates that compatible numbers vary and that they can include combinations that add up to 10, 50 or other totals that are easy to work with mentally.

There are many useful strategies for mental computation. Just as teachers can help children develop a repertoire of problem-solving strategies, they can also help children develop a repertoire of mental strategies. Table 10.2 shows several common strategies for whole number addition. Many of these strategies would be familiar to students and teachers alike.

Mental computation involves flexible thinking, promotes number sense and encourages creative and efficient work with numbers. Consider some of the strategies that children could use to do different types of computation mentally.

- Children could solve a problem in whole-number addition such as 165 + 99 in different ways.
  - 'I subtracted 1 from 165 and added it to 99. Then I added 164 and 100 to get 264.'
  - 'I added 165 plus 100 and got 265, then I subtracted 1 and got 264.'
- For multiplication problems such as 4 × 600 and 4 × 6000, children could extend the basic multiplication fact 4 × 6 = 24 and combine that with their understanding of place value.
  - '4 × 600 is 4 × 6 hundreds, which is 24 hundreds, or 2400.'
  - '4 × 6000 is 4 × 6 thousands, which is 24 thousands, or 24 000.'
- For a division problem such as 1200 ÷ 4, children could combine place value with extending their basic division facts.
  - '1200 is 12 hundreds, so 1200 ÷ 4 is 12 hundreds ÷ 4, or 3 hundreds or 300.'
  - 'Half of 1200 is 600, and half of 600 is 300, 1200 ÷ 4 is 300.'

TABLE 10.2 **Some common strategies for whole-number addition**

Problem	Strategy	How I did it
43 + 48	Adding from the left, using 10s	40 plus 40 is 80, and 3 plus 8 is 11, 80 plus 11 is 91.
43 + 48	Counting on, starting with the biggest number	I'll count by 10s. 48, 58, 68, 78, 88. Then I'll count by 1s. 89, 90, 91.
43 + 48	Making 10s	48 plus 2 is 50, 50 plus 40 is 90, 90 plus 1 more is 91.
43 + 48	Doubling	48 plus 48 is 96. Since 43 is 5 less than 48, 96 minus 5 is 91.
43 + 48	Making compatibles	43 and 7 are compatible which make 50, 50 plus 40 is 90, 90 plus 1 more is 91.
43 + 48	Bridging	I'll break up a number and add the parts. 43 plus 8 is 51, add 40 more is 91.

Can you think of other ways to solve this problem using mental computation?

Combining basic facts with place-value concepts when doing mental computation will suggest different patterns and relationships to students. Further exploration will motivate students to make conjectures and discoveries, leading to reasoning about their findings. For example, students could explore patterns and relationships for multiplication such as 4 × 6, 4 × 60, 4 × 600 and 4 × 6000. They can then explain how the results are alike and how they are different. Another exploration could be used to challenge students to explain why the results of 4 × 600, 40 × 60, 400 × 6 and 4000 × 0.6 are the same. Doing mental computation rewards thoughtful analysis and encourages flexible thinking because students see that they can gain the correct results by using very different techniques, as illustrated in figure 10.5. The students are multiplying together 3 numbers: 20, 8 and 25, the units are centimetres. Each strategy

is meaningful to the student using it, even if it seems strange to others. However, encouraging children to talk with each other about the different ways they do problems in their heads is enlightening for all children and will help them learn to think more freely and flexibly. This sharing and explaining is an important part of learning activities that promote the development of mental computation skills and it often helps children learn new strategies.

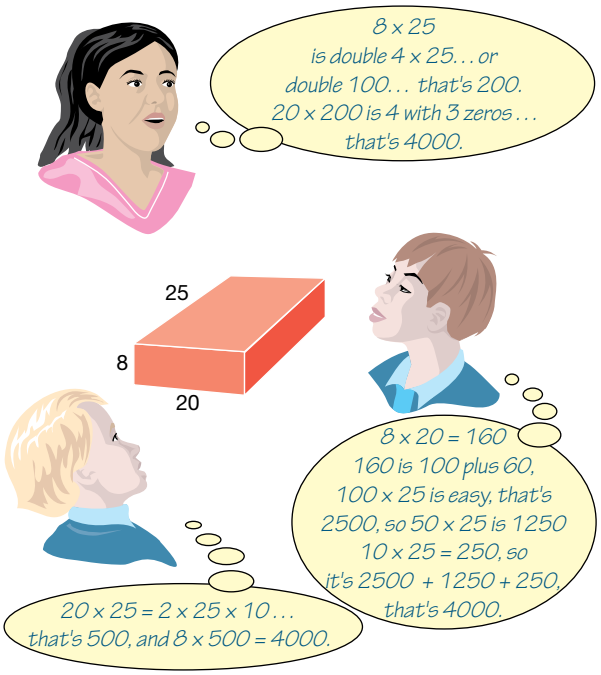


FIGURE 10.5 The volume of the box is 25 cm × 8 cm × 20 cm (each student finds the answer in a different way).

Encouraging mental computation

When doing mental computation, the teacher needs to have available a range of activities that encourage and reward the wide variety of skills that students will develop. Rather than everyone using the same strategy to solve a problem, students can be encouraged to use whichever strategies make sense to them individually. The teacher should encourage children to always try mental computation before using paper and pencil or a calculator. Each of the following examples illustrates 1 or more possible methods.

- Use numbers that are easy to work with:

The problem	*One way to think about it*	*Comment*
$397 \times 4 = ?$	$400 \times 4 = 1600$	Round up using place value and
	$3 \times 4 = 12$	multiply both numbers,
	$1600 - 12 = 1588$	subtract.
$\$6.98 + \$7.98 + \$9.98 = ?$	$\$7 + \$8 + \$10 = \$25$	Round up to the nearest dollar and add
	3×2 cents $= 6$ cents	multiply each difference,
	$\$25 - 6$ cents $= \$24.94$	subtract.

- Look for an easy way:

The problem	*One way to think about it*	*Comment*
$2 \times 3 \times 7 \times 5 = ?$	$2 \times 3 = 6$	Multiply 1 pair of numbers,
	$6 \times 5 = 30$	multiply the easiest number,
	$30 \times 7 = 210$	multiply the final number.

The problem	*One way to think about it*	*Comment*
	OR $2 \times 5 = 10$ $3 \times 7 = 21$ $10 \times 21 = 210$	Multiply the most compatible pairs, use 10 to make an easy factor, combine the results.
$6 \times 8 \times 19 \times 0 = ?$	0 is a factor, which means the product is zero	

- Use logical reasoning:

The problem	*One way to think about it*	*Comment*
$15 \times 120 = ?$	That's halfway between 10×120 and 20×120 Halfway between 1200 and 2400 is 1800	Use the distributive law by breaking 15 into $10 + 5$ so $(10 \times 120) + (5 \times 120)$ is the same as 15×120.
	OR That's 10×120 plus half of 10×120 $1200 + 600 = 1800$	Imagine the problem on a number line and 15 is halfway between 10 and 20, use the idea of multiplying by 2 and dividing by 2.
	OR Double 15 and half 120 $30 \times 60 = 1800$	

- Use knowledge about the number system:

The problem	*One way to think about it*	*Comment*
$56 - 24 = ?$	$50 - 20 = 30$ $6 - 4 = 2$ $30 + 2 = 32$	Subtract the 10s first, subtract the ones in the right order, combine the results.
	OR $54 - 24 = 30$ so $56 - 24 = 32$	Work with the same ones, subtract the extra 2.

Why emphasise mental computation? There are several reasons for helping students develop mental computation skills.

- *Mental computation is very useful.* Adults do more than three-quarters of all their calculations mentally. Teachers need to encourage and reward the early development of this practical skill that students will use for the rest of their lives.
- *Mental computation is the most direct and efficient way of doing many calculations.* For example, research has shown that the computation $200 - 5$ is better done mentally than with paper and pencil. Students are more likely to make sensible judgements about the result and less likely to come up with 205 (a common misconception from recording with symbols). In the middle school years, a computation such as $\frac{3}{4} - \frac{1}{2}$ is easier for children to do mentally than by writing (McIntosh, Reys & Reys 1995).
- *Mental computation is an excellent way to help children develop critical thinking skills and number sense and to reward creative problem solving*. Students must figure out a strategy and use it. In this process, they become aware that there is more than 1 way to perform most calculations mentally and they are encouraged to seek simple and economical methods that make sense to them.
- *Proficiency in mental computation contributes to increased skill in estimation*. Mental computation is the foundation for estimation because it helps children see alternative methods and nonstandard techniques for finding answers.

Mental computation versus written algorithms

As already discussed, there are many efficient techniques for solving problems mentally. Moreover, in the process of becoming proficient at mental computation, children gain mathematical insights and develop skills that will serve them well all their life. Unfortunately, however, once children have learned a written

algorithm they can lose the skills previously gained with mental computation. For example, when children above Year 2 try mental computation for addition, their dominant strategy is to apply written algorithms mentally (McIntosh et al. 1997), which is inefficient and difficult. When asked to solve 165 + 99 mentally, students no longer said 165 + 100 – 1 but a typical response was: 'I added 5 plus 9 and got 14. I carried the 1, and 5 plus 9 plus 1 is 15. I carried that 1 and got 2. It's two six four, or 264.' The answer 264, was found mentally but using a 'written' strategy. Many students think of mental computation this way, as applying written algorithms in their heads. Solving a problem with a mental strategy is completely different to performing a traditional algorithm with pencil and paper. Mental computation should not be confused with thinking about the problem as if it were written because the mind finds that method harder.

There is a serious problem in that practice with written computation increases the likelihood that students will try applying written algorithms mentally. Internationally this is a dilemma. In Canada and the United States, for instance, research showed that the range of strategies used for mental computation was far greater than in Japan, but in all 3 countries the dominant strategy used by students was to apply a written algorithm mentally. The mental application of written algorithms most likely reflects the emphasis given to written algorithms in school. Unfortunately, this strategy appears to inhibit the development of flexible and more efficient strategies for mental computation (Reys & Yang 1998; Shigematsu, lwasaki & Koyama 1994). The teacher's role can be important in increasing an emphasis on flexible, student-generated strategies that enable them to compute mentally with confidence.

Guidelines for developing mental computation skills

Here are guidelines recommended for teachers to follow when helping students develop mental computation skills.

- *Encourage students to do computations mentally*. Make it clear that when mental computation is possible, it is not only acceptable but preferable to written computation. Students report that mental computation is not encouraged and is often even discouraged in school. Educators have suggested that when teachers say things like 'show your working out', students interpret this to mean that mental computation is unacceptable. Consequently time is wasted on unnecessary and inefficient written algorithms. The product of 1000 and 945 can be quickly and easily solved mentally, using place value understanding. The following algorithm is completely disheartening when seen in use:

$$\begin{array}{r} 945 \\ \times\ 1000 \\ \hline 000 \\ 000 \\ 000 \\ 945 \\ \hline 945\,000 \end{array}$$

- *Learn which computations students prefer to do mentally*. Research suggests that, when given the choice, students from Year 4 onwards prefer to use written computations rather than calculators or mental computation. For example, most Year 5 students chose to do 1000×945 with either a calculator or paper and pencil, even though this computation is most easily done mentally (McIntosh & Sparrow 2004). This choice typically reflects the student's lack of experience and confidence in doing computations mentally. In the classroom 10.2 illustrates one way to explore the computational preferences of students with whole numbers, and In the classroom 10.3 does the same with fractions but this assumes a deeper knowledge of working with unlike denominators. If the teacher observes students using a calculator or paper and pencil to do computations that are better done mentally, then time needs to be given for helping children develop mental computation skills that lead them to change their preferences.
- *Find out if students are applying written algorithms mentally*. Asking students to explain how they did a computation in their heads either individually with the teacher or in small groups will make their thinking strategies clearer and support can be given.

IN THE CLASSROOM 10.2

How would you do it?

Objective: Encourage wise use of computational alternatives.

Year level: 4–5

	In your head	With a calculator	With paper and pencil
60×60	☐	☐	☐
945×1000	☐	☐	☐
450×45	☐	☐	☐
$24 \times 5 \times 2$	☐	☐	☐
$2000\overline{)16\,000}$	☐	☐	☐
$45\overline{)450}$	☐	☐	☐
4×15	☐	☐	☐
$50 \times 17 \times 2$	☐	☐	☐

Discuss the situations above as a team and decide why you chose your particular method.
- Write a computation *you* would solve with a calculator.
- Write a computation *you* would solve mentally.
- Write a computation *you* would solve with paper and pencil.

- *Plan to include mental computation systematically and regularly as an integral part of your teaching and learning program*. Systematic attention and practice will help children improve their mental computation skills. The teacher's role should focus on helping students develop strategies and thinking patterns that make sense to them. Students may develop these strategies and patterns on their own or may adopt ones they have learned from others.
- *Keep practice sessions short, perhaps 10 minutes at a time*. Practice sessions at the beginning of a lesson can settle students if done in a non-threatening manner and help set up a class routine for mathematical thinking, regardless of the current topic being studied. Sometimes teachers use an activity such as *follow me* while children are waiting in line. A string of operations is given: '$3 \times 4 + 10 - 4 + 20 \ldots$' where the teacher starts by saying '3 times 4' and the first student in line says '12', then that student creates the next operation, for example 'plus 10' and the next student says '22' continuing down the line. It should be okay for a student to 'pass' if the person in front of them has created something a bit too hard to do mentally. Another useful activity is *today's target*, where the date serves as a target number that students have to 'hit' using specific types of computation (see figure 10.6). The teacher can vary this activity by changing the target number to something besides a date — for example, 'Today's target is 100', and then small groups of students think of different ways to make 100. Alternatively, questions can be varied by asking, for example, 'use addition and subtraction' or 'use multiplication or division' or 'solve it in an unusual way' and for older students 'combine all 4 operations' in your answer. This type of activity gives students practice with mental computation and their solutions help the teacher gain insight into students' thinking — both in terms of the strategies they use and the level of computational complexity at which they are operating.

- *Develop children's confidence*. Pick numbers that are easy to work with at first (e.g. 3 × 99 or 1/2 of 84), and then increase the difficulty (e.g. 5 × 75 or 2/3 of 96). Working with compatible numbers helps children develop confidence at mental computation and improves their number sense.

Today's target is 24 May	
Try to hit this target by	
Adding 3 numbers	**Subtracting 1 number from another number**
Some possible answers 8 + 8 + 8 20 + 2 + 2 10 + 10 + 4	*Some possible answers* 30 – 6 25 – 1 100 – 76
Multiplying 2 numbers	**Using a fraction**
Some possible answers 4 × 6 8 × 3 12 × 2	*Some possible answers* $23 + \frac{9}{10} + \frac{1}{10}$ $48 \times \frac{1}{2}$ $24\frac{1}{2} - \frac{1}{2}$

FIGURE 10.6 Today's target number and some ways to hit it

IN THE CLASSROOM 10.3

How would you do it?

Objective: Encourage wise use of computational alternatives.

Year level: 6–7

	In your head	With a calculator	With paper and pencil
$\frac{1}{2} + \frac{1}{4}$	☐	☐	☐
$1 - \frac{1}{3}$	☐	☐	☐
$\frac{3}{4} + \frac{3}{4}$	☐	☐	☐
$\frac{1}{5} + \frac{1}{6}$	☐	☐	☐
$\frac{1}{2} + \frac{5}{6}$	☐	☐	☐
$1\frac{1}{2} + 2\frac{3}{4}$	☐	☐	☐
$2 - \frac{3}{4}$	☐	☐	☐
$\frac{1}{2} - \frac{1}{3}$	☐	☐	☐

Discuss the situations above as a team and decide why you chose your particular method.
- Write a computation *you* would solve with a calculator.
- Write a computation *you* would solve mentally.
- Write a computation *you* would solve with paper and pencil.

- *Encourage inventiveness*. There is never just 1 right way to do any mental computation, but certain ways may be more efficient and interesting than others. Asking students 'How did you do that?' can reveal highly ingenious mental computation strategies. For example, students who computed 60 × 15 mentally reported the following strategies:
  - '10 times 60 is 600. 5 times 60 is 300. 600 plus 300 is 900.'
  - '60 times 10 is 600 and half of 600 is 300, so it is 600 plus 300, or 900.'
  - '60 is 4 times 15, so that is 4 times 15 times 15. 15 squared is 225, times 4 is 900.' (Wow! An amazing approach, reported by a Year 8 student.)

  As we have noted, children often develop their own strategies for mental computation; the power of such self-developed, out-of-school techniques has been documented (McIntosh & Sparrow 2004).
- *Mental computation or estimation?* Make sure children are aware of the difference between estimation (in which answers are approximate) and mental computation (in which answers are exact).

10.3 Estimation

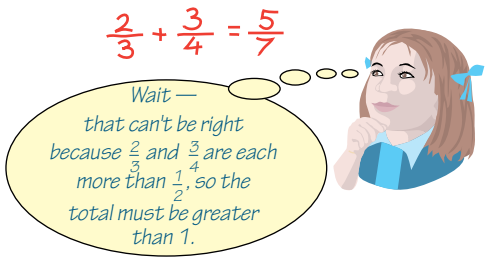
LEARNING OUTCOME 10.3 Explaining different approaches to computational estimation and how teachers can encourage estimation.

Estimation when solving mathematical problems is a valuable process that produces answers close enough to allow for appropriate decisions without performing elaborate or exact computations. Computational estimation is typically done mentally. Figure 10.7 illustrates how students can use estimation to monitor their computations at 3 different places in the process.

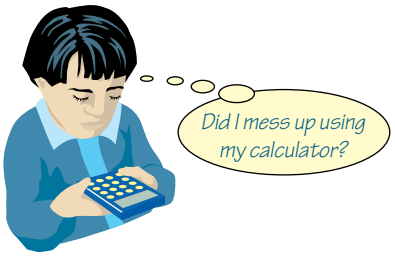
- *Before solving a problem*, students can use estimation to gain a general sense of what to expect.
- *During the problem*, students can use estimation to determine if their solution method looks right.
- *After solving the problem*, students can use estimation to reflect on their answer and decide if it makes sense.



(a) Before computing



(b) During computing



(c) After computing

FIGURE 10.7 Estimation process before, during and after computation

As children become more aware of these different uses of estimation, they develop a greater respect for its power and view it as an essential part of the whole process.

Background for estimating

Students who are proficient at written computations are not necessarily good at estimation. Research suggests that memorising rules and procedures (doing mathematics by rote) can discourage students from using estimation (Alajmi & Reys 2007; Reys & Yang 1998). Acquiring mental computation skills and number sense provide the foundation for successful development of estimation skills. The teacher can significantly improve students' estimation skills by paying attention to using estimation on a regular basis when helping children learn mathematics. Although some dramatic improvements can occur quickly, the development of quality estimation skills is a process that takes time. Therefore, students need to be given repeated opportunities to estimate.

Learning about estimation gives students experience with an area of mathematics that does not focus on exact answers but naturally arises when building number sense. As young children talk about estimation in mathematics, their vocabulary will expand to include words such as *about, almost, just over*, *nearly* and *closer to this than that*. As children grow older, they will add words such as *approximate, reasonable* and *unreasonable* to their vocabulary for describing mathematical answers, as well as informal phrases such as *in between* and *close enough*. Becoming comfortable and confident in using language to describe the inexactness found in the real world contributes not only to developing number sense but also to developing better estimation skills.

The role of the teacher, in the beginning is to help children acquire a tolerance for error. Estimation involves a different perspective from the view that emphasises an exact answer. The teacher will need to encourage children to change their exact-answer mind-set before beginning to teach specific estimation strategies. Change can begin once students recognise that estimation is an essential and practical skill that they might already use effectively in their daily lives. Hence the teacher needs to build on prior experience and emphasise the use of estimation for a range of different problems and alongside procedures for exact computation.

Furthermore, it is important to give students immediate feedback on their estimates but to allow a reasonable margin for error. Initially it is wise not to be overly critical. Asking students to explain how they obtained their estimates in a non-threatening way adds value to their thinking. The discussion generated will help clarify the procedures they used and listening to other students' approaches might suggest new ways to estimate mathematically. With time and practice children will understand that estimation, which produces an approximate answer, is not the same as mental computation, which produces an exact answer. This is an important shift in mathematical thinking because students move away from the right/wrong mentality towards an appropriate solution in context.

One of the keys to helping children develop good estimation strategies is encouraging them to be flexible when thinking about numbers. For example, suppose you want students to estimate 418 + 349. One child might think about place value and upper and lower bands, '400 + 300 is 700 and 49 + 18 is less than 100, so the sum is between 700 and 800'. This approach uses the leading, or front-end, digits. Another child might think, '418 is about 400 and 349 is almost 350, so the sum is about 750'. This child rounded the numbers either up or down so that they are easy to work with and used number sense to make adjustments. Both ideas are excellent.

As with mental computation, it is important that children develop different strategies for estimating different number operations. In other words, they need to think about the problem as a whole and not rely on a set of rules to produce an estimate. Many adults think only of rounding when considering estimation but this is a limited perspective. As with mental computation, the teacher will need to help children develop a range of strategies from which they can choose. In the following sections, we discuss some of these strategies.

Front-end estimation

The front-end strategy for estimation is a basic yet powerful approach that can be used in a variety of situations. Front-end estimation involves firstly, observing the leading, or front-end, digit in a number and then the place value of that digit. Figure 10.8 shows how front-end digits can be used to obtain an initial estimate for addition in the case where the numbers are the same place value. The unique advantage of the front-end strategy is that the original problem, which shows all the numbers that students have to work with, can be saved for a latter adjustment. Focusing on the front-end digits enables students to reach an estimate quickly and easily. This strategy also encourages students to use number sense as they think about the computations. The teacher will need to help students understand however, that a front-end estimate is a lower bound and that students can use the remaining digits to adjust this estimate, as discussed in the next section.

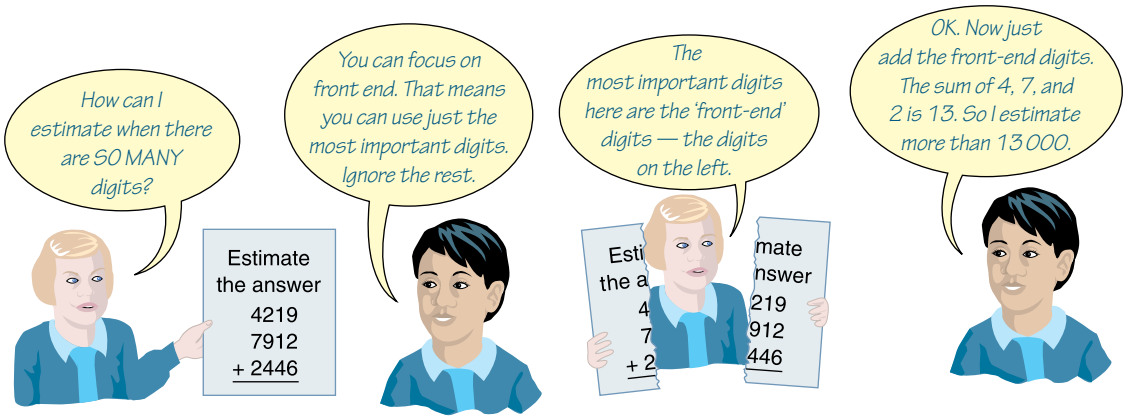


FIGURE 10.8 Front-end estimation in action

Adjusting

Number sense has many dimensions, 1 of which is recognising when an estimate is a little more or a little less than the exact answer would be. As students develop their estimation skills, it becomes natural for them to refine an initial estimate by adjusting it or compensating. Students can do this no matter which strategy they have used to make their initial estimate.

For example, consider how a student could adjust a front-end estimate of 4000 for the sum 1213 + 2926 + 1578, by thinking: 'The answer would be greater than 4000 because the numbers are all greater than zero'. The student would then consider the digits in the hundreds place and raise the estimate accordingly. In this case an adjustment of 200 + 900 + 500 or 600 would make a more accurate estimation. Working with both thousands and hundreds is difficult for the student to compute mentally but an extra 1700, when added to the 4000 from before, gives an adjusted estimation of 5700. In effect the student is looking at the 2 front-end digits in 2 steps.

Figure 10.9 illustrates how this type of adjusting would work in solving a real-world problem such as deciding if you have enough money to buy a particular lunch. For example, the lunch will cost closer to $7 than $5. In the classroom 10.4 presents an activity that will give students practice in using the front-end estimation strategy and then adjusting.

When teaching estimation, the teacher needs to help children understand that adjusting estimates is a worthwhile habit regardless of their initial estimation strategy and regardless of the operations involved in the problem.



FIGURE 10.9 Front-end estimation with adjustment

IN THE CLASSROOM 10.4

Front-end estimation

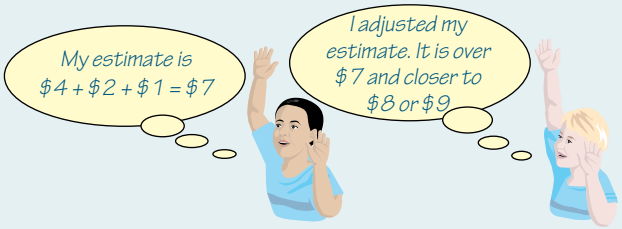
Objective: Using front-end digits and adjusting strategies to estimate sums and products.

Year level: 3–5

- Front-end estimation is quick, reasonable and done in your head.

Estimate

$4.50
$2.20
$1.90



- Try the following:

1. $1.05
   $2.50
   $1.60

   Front-end _____

   Adjust to get closer

   Over $5 _____

   Under $5 _____

2. $8.50
   $6.60
   $3.95

   Front-end _____

   Adjust to get closer

   Over $20 _____

   Under $20 _____

3. $5.20
   $0.40
   $0.60
   $1.45

   Front-end _____

   Adjust to get closer

   Over $7 _____

   Under $7 _____

4. $0.25
   $6.55
   $5.95
   $0.20
   $8.50

   Front-end _____

   Adjust to get closer

   Over $20 _____

   Under $20 _____

5. I have 3 packets of bon-bons that cost $5.50 each, estimate how much will they cost?

   Front-end _____

   Adjust to get closer _____

6. I have 5 pens that cost $3.75 each, estimate how much will they cost?

   Front-end _____

   Adjust to get closer _____

Ask your friend to create a problem for you to solve.

Compatible numbers

In a similar way to mental computation, using compatible numbers — numbers that go together naturally and are easy to work with mentally — is often helpful in estimation. Figure 10.10 illustrates how compatible numbers can make estimation easier with a variety of different operations.

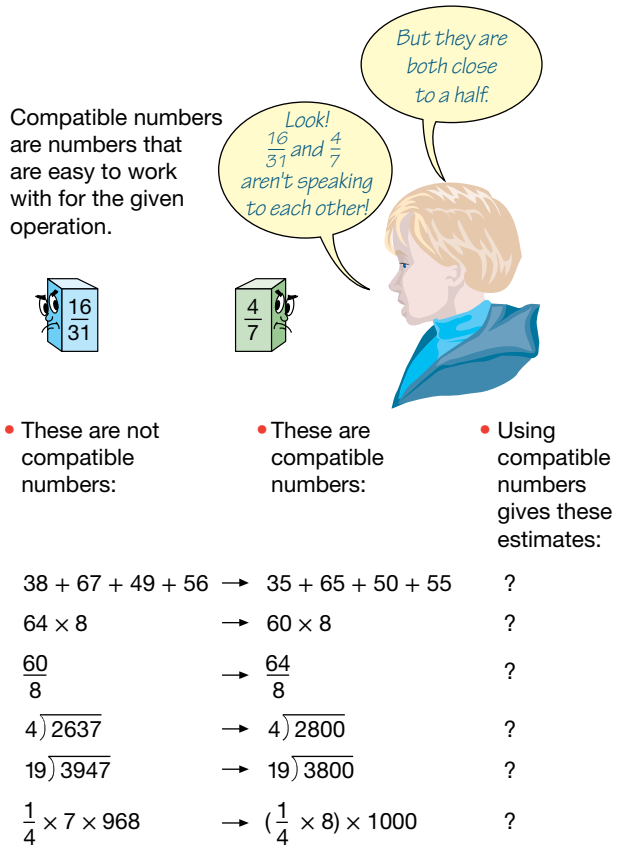


FIGURE 10.10 Compatible numbers for different operations

Using compatible numbers for estimation means changing the numbers in the problem to numbers that are easier to work with, given the operations needed to solve the problem. Notice in figure 10.10 that 64 and 8 are not compatible for multiplication (64×8 is hard to compute mentally) but are compatible for division (64/8 is easy to compute mentally because $8 \times 8 = 64$). Similarly, 60 and 8 are not compatible for division but are compatible for multiplication. Therefore, compatibility depends on the operation as well as on the numbers themselves.

Using a compatible numbers strategy is particularly powerful for division. For example, when working with problems like those shown in figure 10.11, children need to be encouraged to think about number facts. The basic facts $6 \times 4 = 24$ and $6 \times 5 = 30$ are helpful when solving this estimate because 2400 and 3000 are compatible numbers for the problem. Basic facts for multiplication, as discussed earlier, are crucial for students to learn between Year 3 and 5 because estimation and computation with the 4 operations using larger numbers builds on these basic facts. Asking children to think of 'compatible number pairs' for a variety of problems in multiplication and division and explaining why the numbers are compatible will help them come up with well considered ideas. The teacher will be able to observe

students' responses and gain insight into their number sense. For example, if students are asked to think of compatible numbers pairs when estimating the complex problem 7029 divided by 23, they might come up with these or similar ideas.

- 23 doubled is 46, 23 tripled is 69. Using place value, 23 × 30 is 690 and 23 × 300 is 6900, so 23 and 69 are compatible for division. If 6900 divided by 23 is 300, the answer to 7029 ÷ 23 will be close to 300, which is a close estimate.
- 7029 can be rounded up to 8000 and divided evenly by 20, so 20 and 8000 are compatible for division. 8000 divided by 20 is 400, but 8000 is a lot more than 7029 and 20 is less than 23, so the estimate of 400 is going to be too high.
- 7029 can be rounded up to 7500, which is evenly divided by 25, so 75 and 25 are compatible for division. 7500 divided by 25 is 300, and 7500 is fairly close to 7029, and 25 is close to 23, so an estimate of 300 is probably close.
- 7029 can be rounded down to 7000, which can be halved and then divided by 10 to get 350 (a little tricky) but 20 and 7000 are fairly compatible, which will give a reasonable estimate.

Remember that estimation is useful to gain an idea of the answer to a problem before, during or after computation. The student in this case would be looking for an answer between 300 and 400 when solving this problem. Using a calculator and entering the digits 7029 ÷ 23 = the result 305.60869 (to 8 digits) appears. The estimation was definitely close in each case but the second choice. Finally, the answer would need to be interpreted in a way that matches the original problem, for example if it was a money question and involved Australian dollars and cents (5c being the smallest amount), the best answer is $305.60 to the nearest coin value or $305.61 if an electronic transaction were involved.

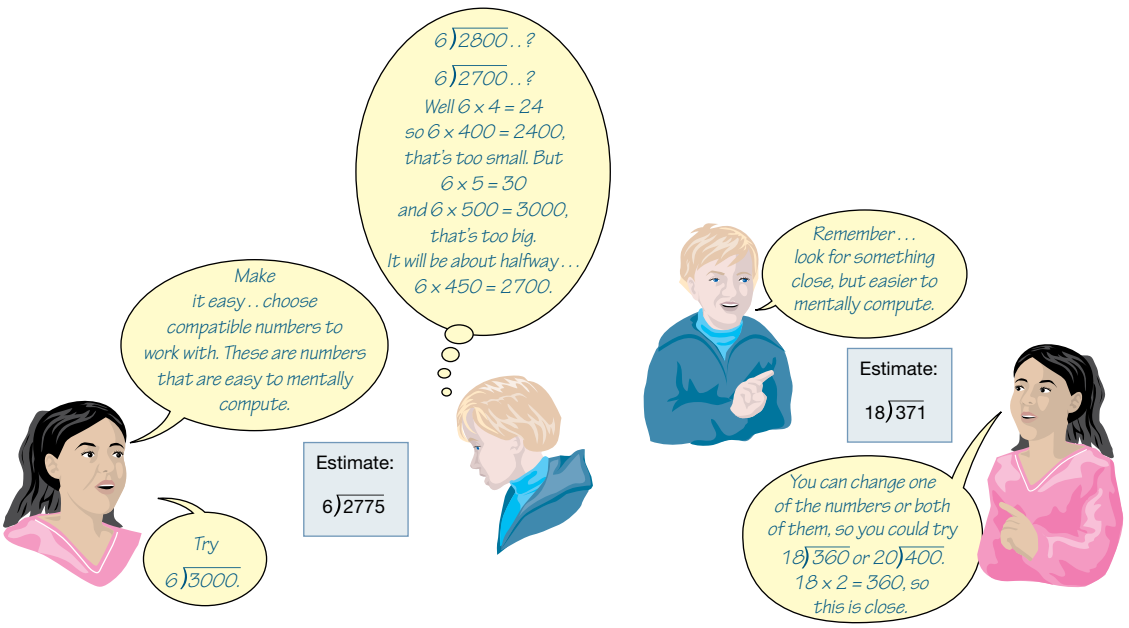
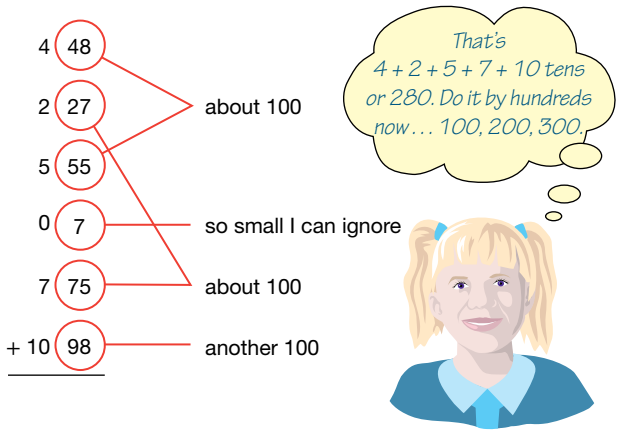


FIGURE 10.11 The compatible numbers strategy for division

Compatible numbers are often used to estimate addition problems, particularly in a large list. In the example below, the student mostly used the front-end strategy for tens and then used compatible numbers to check the initial estimate.



Flexible rounding

Rounding to produce numbers that are easier to work with is a very useful estimation strategy. Rounding is more sophisticated than front-end estimation and adjusting because rounding changes the numbers in the original problem according to what is needed. When using rounding for estimation, students should use *flexible rounding*. Flexible rounding means rounding to numbers that are close but are also compatible. Flexible rounding is appropriate for all 4 operations with all types of numbers, but it is particularly well suited to multiplication. Children should become adept at the process of rounding to compatible numbers so that it becomes second nature. Figure 10.12 illustrates how 3 students started with the same problem but rounded differently.

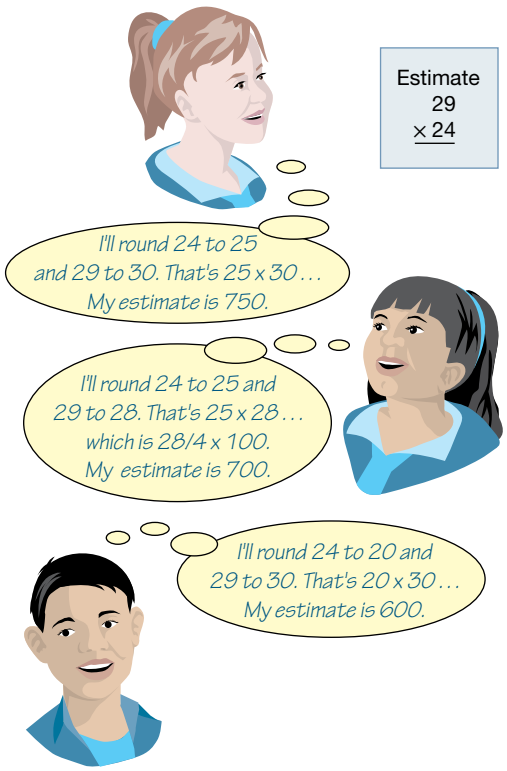


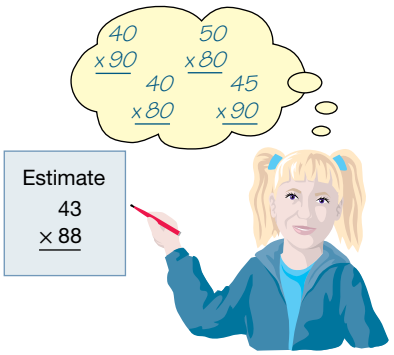
FIGURE 10.12 Flexible rounding and different estimates that result

Notice that each approach produced a different result, but each student gives a reasonable estimate. It is important for students to know they have the freedom to choose different estimation strategies, and that there is no right 1 way of estimating. However, with practice they are likely to realise that different strategies produce different estimates. Accepting a range of reasonable estimates, in this case between 600 and 750, (rather than a single best estimate) fosters number sense and encourages students to choose and use their own strategies.

Problems like the one shown in figure 10.12 need to be accompanied by questions that go beyond asking what the estimate is to encouraging thought about the process they used and why it worked for them.

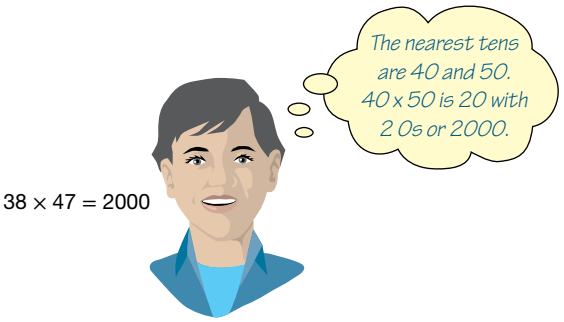
How did you change the numbers?
Why did you change the numbers in this way?

Children should be encouraged to think of possible substitutions for each number and to reason about which substitutions would work best. The choices illustrated in the following example lead naturally to questions like, 'Which pair of numbers would you choose to make the estimate? Why? Should 1 figure go up and 1 go down? Why?'



Note here that the range of estimates is 3200 to 4050.

The teacher can help children to adjust the estimates they get when rounding numbers, in order to compensate for how the numbers were rounded. In the following example, the student makes an initial estimate by rounding both numbers up.



Then the student adjusts their initial estimate knowing that the solution will actually be less.

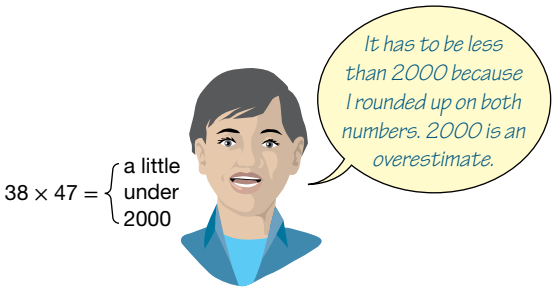


Table 10.3 illustrates some further examples of estimation strategies for multiplication with different types of numbers. In each case a natural strategy has been used based on the size of the numbers involved. For the first 2 examples, the rounding procedure clearly indicates whether the result is an underestimate or an overestimate; however, the last example is not clear in this regard. One number was rounded up and the other rounded down, so the estimate will be close. When real world contexts are given for a problem then students will need to think more deeply about the way in which rounded numbers used to produce their estimate.

TABLE 10.3 **Different ways of estimating products using rounding and then adjusting**

Problem	Estimate with rounding	Adjusted estimate
42 × 61	40 × 60 = 2400	A little more than 2400
39 × 78	40 × 80 = 3200	A little less than 2400
27 × 32	30 × 30 = 900	About 900

As students become more skillful at mental computation, they become more flexible in their use of rounding to make estimates. They learn to rely more on common sense than on traditional rounding rules, so they can round numbers that easily facilitate working mentally. Consequently different people will round numbers in a problem in a variety of ways, depending on what is easiest for them.

Clustering

The *clustering*, or averaging, strategy is used specifically for addition and involves the idea of an average for estimating a total sum. Whenever a group of numbers clusters around a particular value, students can use a 2-step process to estimate the sum of all the numbers in the group. Additionally, thinking about the range of numbers in a group can support this strategy. Looking at the lowest, highest and halfway in between can help with the clustering approach.

1. Estimate the average value of the numbers — that is, the value that the numbers cluster around.
2. Multiply by the number of elements in the group.

Do a quick check of lowest, highest, halfway and then multiply by the amount of numbers.

Figure 10.13 shows 2 examples of this strategy in action. First \$3 is multiplied by 6 to get \$18, then checking \$3.80 – \$2.10 is \$1.70 so halfway is about 80 or 90 cents from the lowest or highest, yes! \$3 is a good estimate for the average. Secondly, 97 is the highest and 82 the lowest so the range is 15. Halfway is 7 or 8, so 90 is a good estimate for the average and 90 × 5 = 450 for the sum.



FIGURE 10.13 The clustering strategy for addition

Questions such as the following can help students understand this strategy:

What value do all the numbers cluster around?
Why is this value the estimated average?
Why is the estimated average multiplied by the number of elements in the group?

Asking what the estimated total is for a range of problems similar to those in figure 10.13 will help children become more proficient at using this strategy.

Clustering is a limited strategy, since it is appropriate only for quickly estimating the sum of a group of numbers that aren't too different from one another (that is, they cluster around some value or have an obvious median). The strength of this strategy is that it eliminates the mental overload involved in adding up a long list of front-end digits or rounded numbers.

Choosing estimation strategies

There are a great variety of estimation strategies and many of these can work for the same problem. The choice of strategy to be used will depend mainly on the student, the specific numbers and operations involved and their level of expertise. For the teacher, the challenge is to help students become aware of various estimation strategies and help them develop confidence in their ability to use any strategy frequently and successfully. Encouraging students to make an estimate before, during or after computation needs to be a priority. It may require a shift in planning to include estimation activities regularly. Here are some useful guidelines for meeting that challenge.

- *Give students problems that encourage and reward estimation*. For example, the teacher can expect students to be able to compute 78 + 83 mentally at Year 6 or 7 and gain an exact answer, but giving them the more complex sum 78 342 + 83 289 in a problem situation where an exact answer is not needed, will encourage them to use estimation. The idea is that the context for the problem and the numbers involved should be realistic enough for students to choose to estimate the answer, rather than compute an exact answer.
- *Make sure students are not computing exact answers and then rounding to produce estimates*. Research has documented that students frequently use this technique (McIntosh & Sparrow 2004) because they aren't confident with estimation strategies. Unfortunately, this problem can go undetected in the classroom. Teachers can create discussion about estimation with individuals or small groups to overcome the 'exact answer first' issue and observe students to see if they are truly estimating.
- *Ask students to explain how they made their estimates*. Individual students often develop unique approaches to estimation. By sharing their different approaches, students develop an appreciation of each other's strategies.
- *Fight the one-right-answer syndrome from the start*. Avoiding the mentality of 'one right' or 'best' estimate is crucial. The teacher can reinforce a variety of quality estimations by helping students realise that several different estimates for the same problem are acceptable. One way to do this is to let students suggest how close to the exact answer an estimate has to be, so all estimates within that range are good estimates. Discuss the value of having a range of acceptable answers. Not only will this counter the idea of one right answer, but it will generally reveal different estimation strategies and help everyone learn about other strategies. These shared experiences help students become more comfortable with the belief that there might be several different but correct estimates.
- *Encourage students to think about real world situations that involve estimation*. Doing this will help students sharpen their critical thinking skills as they decide when to overestimate or underestimate, depending on the situation. The types of mathematics involved might be number, money or measurement. Included in figure 10.14 are situations that might encourage this kind of thinking but children will come up with many and varied examples to share.

Over or under

1. Your car usually gets about 12 kilometres per litre on the highway. Your petrol tank holds 40 litres and is about 1-quarter full and it is 100 kilometres to the next petrol station. Is this a good time to over- or underestimate your fuel economy? Would you fuel up now?
2. You are talking with a car dealer about buying a car, and you ask about the trade in value for your own car. Do you think the car dealer will over- or underestimate the value of your car? Explain why.
3. You have $40 to spend on groceries for a group picnic. As you place each item in the shopping trolley, you estimate the total cost. Would this be a good time to over- or underestimate the cost of each item? What strategy would you use if you made a mistake?
4. You are scientist in charge of forecasting the lava speed of an active volcano. As the lava is moving down the mountainside toward a town, a decision needs to be made when to evacuate the town. Should you over- or underestimate the speed of the lava?
5. Your plane is scheduled to leave at 16:00 hours and it usually takes about 1 hour to get to the airport. Should you over- or underestimate the time needed to get to the airport when deciding what time to leave home and park your car?
6. The pattern for a formal dress recommends 5 metres of fabric. Should you over- or underestimate your purchase?

FIGURE 10.14 Real-world situations requiring underestimation or overestimation

The Australian Curriculum repeatedly highlights the need to use efficient mental and written strategies and appropriate digital technologies from Years 2 to 7. Similarly the National Research Council in the United States underscores the importance of systematically attending to mental computation and estimation in primary school. The NRC recommends that students should have opportunities 'to develop and use techniques for mental arithmetic and estimation as a means of promoting a deeper number sense' (Kilpatrick, Swafford & Findell 2001). To conclude this section, it is important that pre-service and in-service teachers allocate dedicated learning time to a balance of estimation, mental computation, written methods and digital technologies.

MAKING CONNECTIONS

Records of international mathematics assessments over time and Trends in International Mathematics and Science Study (TIMSS 1995–2011) in particular note that students from East Asian countries (Singapore, Hong Kong, South Korea, Taiwan, Japan) have continued to excel in mathematics well above the high international benchmark for Year 4 and Year 8 (Thomson et al. 2012). Lemke et al. (2004) reported that written computation was a key factor in these results and that performance in mental computation was also very strong. In these countries, attention is given to developing mental and written computation skills throughout primary school. There has been little research about estimation in the last decade or more and a real need exists here.

According to Liu (2009), who conducted large-scale research on computational estimation with Years 3 and 5 students in China, students use limited strategies when estimating whole-number multiplications. The most common strategy for Year 3 students was found to be rounding, with little concern for adjusting numbers or results. Liu hypothesised that this strategy might have been the preferred or only method taught in schools and little opportunity was given for students to develop their own. Interview data and previous research found that China's primary teachers supported the '4-down, 5-up' rule for rounding as the dominant strategy. Unfortunately, the implications are that students in this age group may not be thinking about the numbers involved in the problems and be unable to develop their own methods such as those discussed in this chapter.

In Year 5, students who were reluctant to settle for an inexact answer in Liu's study resorted to written algorithms or a mental application of such (which is consistent with previous research). In contrast to

previous international studies focusing on students in Japan, Canada and Taiwan, most of the Chinese students did understand estimation and were able to apply a few strategies.

TABLE 10.4 **Performance of Taiwanese Year 6 students on written computation and parallel estimation items**

Written computation items requiring exact answer		Percentage correct by Year 6 students
$\frac{12}{13} + \frac{7}{8}$		61 61
Estimation items requiring a correct choice		
Without calculating an exact answer, Circle the best estimate for $\frac{12}{13} + \frac{7}{8}$.	A. 1 B. 2* C. 19 D. 21 E. I don't know	10 25 36 16 13
This multiplication has been carried out correctly except for placing the decimal point: $534.6 \times 0.545 = 291\,357$. Place the decimal point using estimation.	A. 29.1357 B. 291.357* C. Other answer	87 11 2

*Denotes correct answer choice.

However, students who are proficient at written computation are not necessarily good at estimating unless teachers prioritise estimation strategies. Taiwanese students who showed considerable skill at tedious written computations performed much more poorly at estimating the identical computations with only 25% giving the correct estimate, compared to 61% (Reys & Yang 1998), as shown in table 10.4. Notice that both of the problems in the table involve numbers that are tedious to compute in writing but that lend themselves well to estimation. That is, 534.6×0.545 is a rather complex computation using a written algorithm, but recognising that 0.545 is about one-half clearly indicates that the reasonable choice for an estimate is 291.357 (one-half of a number between 500 and 600 is between 250 and 300, so 291.357 is the only reasonable answer of the 3 choices). Nevertheless, nearly 90% of the Taiwanese students selected 29.1357, and similar results were found among Year 8 students in Kuwait (Alajmi & Reys 2007). Clearly, 29.1357 is an unreasonable answer, but it is one that might result by applying the rule 'count the digits to the right of the decimal point'. The low performance on estimating the sum of 2 fractions (12/13 + 7/8) was similar to the results of the second national assessment of 13-year-olds in the United States. Because both fractions are slightly less than 1, the sum should be less than 2. However, more than half of the students from both the United States and Taiwan selected either 19 or 21. This is a vivid reminder that teachers need to look beneath correct answers to check on students' understanding and their ability to produce reasonable estimates.

Performance on computational estimation is an issue in many different countries, including Australia, Kuwait, Japan, Sweden, Taiwan and the United States (Reys et al. 1999). Much needs to be done in helping children everywhere become competent at estimation. Students need to have the opportunity to develop their own strategies. This represents a real opportunity for teachers to bring about dramatic change in the way that students view estimation. Additionally a good balance of estimation and computation will certainly improve student performance.

A GLANCE AT WHERE WE'VE BEEN

Teachers need to help students develop confidence and skill in choosing and using different methods of number computation. Finding the right balance of time spent on teaching and learning each method (estimation, MAL, PAL, or CAL) is essential.

The Australian Association of Mathematics Teachers (1996) recommends that appropriate technology should be available to all students as a means to both support and extend students' mathematical learning experiences. Similarly the NCTM (2011) states that the use of technology needs to be selective and strategic. Teachers need to both overcome their own bias (if it exists) and work against myths regarding calculator use. Helping students and their caregivers understand that calculators can be very useful in mathematics, is essential. The important points are these: using a calculator requires thinking; using a calculator can raise student achievement; a calculator is not always the fastest method of computation; the calculator is useful both for computation and as an instructional tool.

Mental computation is a natural way of solving problems. Mental computation techniques include extending basic facts, using place value and decomposing numbers — all aimed at mentally computing with compatible numbers. Children often develop their own efficient techniques for mental computation and it is important for teachers to encourage and reward this. Mental strategies are often the most direct and efficient way of doing calculations; they help children develop their critical thinking skills and number sense, and contribute to the development of estimation skills. It is worth noting that approximately 80% of number computation is done this way by adults in everyday life. Therefore, teachers need to encourage this skill in a variety of ways but be careful that students do not compute mentally by applying written algorithms in their heads.

Estimation produces an approximate rather than an exact answer. Estimation is useful before, during and after exact computations, to get a sense of what kind of answer to expect, to check that the computation is moving in the right direction and to check on the reasonableness of results. Teachers need to help children change their exact-answer perspective so they can see the power and usefulness of estimation. Estimation strategies include front-end estimation, using compatible numbers, flexible rounding and clustering. Regardless of the strategy used, children need to be confident with the degree of error acceptable and decide on a suitable range, adjusting their method to gain a better estimate. Guidelines for helping children choose and use estimation strategies include giving problems that encourage and reward estimation, making sure students are not computing exact answers and then rounding to produce estimates, encouraging students to talk about how they made their estimates, overcoming the one-right-answer syndrome and helping students to think of real-world situations where estimation is needed. Studies across cultures show that students who excel at written computation and mental computation are not necessarily good at estimation.

THINGS TO DO: FROM WHAT YOU'VE READ

1 Here is an actual quote from a Year 4 student expressing what she thinks her teacher wants:

> I think she doesn't like us to use mental (computation) because she can't see the writing. And she doesn't like us to use the calculator a whole lot because then we'll get too used to it and we won't want to learn and stuff . . . And she likes us to use written [computation] because she can see what we're doing, and if we're having problems she can see what we're doing wrong.

Suppose you had an opportunity to talk with the student's teacher. What would you like to discuss about the methods that might be in use in the classroom?

2 How would you respond to the following statement by a parent? 'Calculators should not be used in school because if students use them, they will never have to think.'

3 Look again at figure 10.1. Decide how you think classroom time devoted to computation should be allocated now and in the future. Be prepared to defend your proposal.

4 A Year 5 student described how she solved 7×499 in her head:

> I put the 499 on top and the 7 on bottom, then I get 3 and carry the 6 and 9 carry the 6 and 7 times 4 is 28 plus 6 is 34.

Would you say this student was doing mental computation the best way? Explain why. Describe some other strategies she could have used to do this computation much more easily. Research about cognitive load and discuss the problem above.

5 Discuss why it is important for students to develop a tolerance for an acceptable range of answers when doing computational estimation.

6 Discuss why the following is not a good assignment for students to undertake in class:

> I want you to estimate the answers to these problems, then compute the correct answers, and see how far off your estimate was from the correct answer.

7 Explain why adjusting is an integral part of estimating.

8 Figure 10.2 shows a sample note to parents about calculator use in Year 2. Prepare a letter to parents of Year 6 students, explaining to them that calculators will be used in your class as students find appropriate and can be used for homework. Briefly explain why you are encouraging calculator use.

THINGS TO DO: GOING BEYOND THIS TEXT

IN THE FIELD

1 *Do it mentally*. Ask at least 2 students to compute 99 + 165 mentally. Ask them how they did it. Describe the strategies that they used.

2 *About how much*? Ask several adults to estimate 29 × 24 and explain how they made their estimate. Describe their strategies and results. Did any of them use the rounding strategies shown in figure 10.12?

IN YOUR JOURNAL

3 *Computational methods*. Keep a log of the computational methods (written computation, calculator, mental computation and estimation) that you use for one day. Estimate what percentage of your computations made use of each method. In your journal, summarise your results and describe a plan for improving your use of these different computational methods.

4 *Assessing mental computation and estimation*. Select a standardised achievement test. Review the test and determine how much attention is given to assessing mental computation and computational estimation. In your journal, discuss the strengths and weakness of test items that address mental computation or estimation.

WITH ADDITIONAL RESOURCES

5 Read the article 'Computation in the elementary curriculum: Shifting the emphasis' (Reys & Reys 1998). Do you agree or disagree with the authors' view of change? Explain why. Investigate the following research article, 'Teaching computation in primary school without traditional written algorithms' (Hartnett 2015) and compare and contrast the findings with the first article.

6 Review a mental mathematics strategy book, such as McIntosh, DeNardi and Swan (1994). Select a lesson and highlight the key ideas for your peers.

7 Read the article 'Calculators as learning tools for young children's explorations of number' (Huinker 2002). Summarise this article and describe an activity that uses the calculator to expand young children's opportunities to explore number concepts.

WITH TECHNOLOGY

8 *Calculators and order of operations*. Some calculators use arithmetic logic in performing operations. For example, if you entered 3 + 5 × 4, the calculator would display 32 which is the answer for (3 + 5) × 4. Other calculators use algebraic logic and would display 23 = 3 + (5 × 4). Explain how you would solve these 2 problems using each type of calculator: (a) (253 − 85)/4; (b) 253 − 85/4). Which one of these best aligns to what is often taught as BODMAS?

9 *Go to the nrich maths website* and search by topic, for example mental calculations strategies yields the group work activity www.nrich.maths.org/6901 **How do you do it?** Which encourages students to share calculation strategies and to think about which strategy might be the most efficient. Go through the lesson with a peer/colleague and discuss what worked well for you.

CHILDREN'S LITERATURE CONNECTIONS

Clement, R. (1990). *Counting on Frank*. Sydney: William Collins Pty Ltd.
In this book, a boy and his dog see mathematical ideas in many aspects of their daily life. He estimates quantity from the number of jelly beans in a jar to the number of peas dropped off a plate over time. He uses visual spatial thinking and measurement concepts in fascinating ways.

Goldstone, B. (2006). *Great Estimations*. New York: Henry Holt and Company.
The teacher will see how to present and teach ways to estimate different things in everyday life. By using base ten, the reader can see or the teacher can show how to estimate using base ten with different objects. The book shows everything from money to swimmers and how to go about estimating just how many there could be. A book all classrooms should have to teach estimation.

Neuschwander, C. (1998). *Amanda Bean's Amazing Dream*. New York: Scholastic.
This mental computation book shows the reader that Amanda Bean loves to count. She has a dream that helps her realise that being able to multiply will help her count things faster.

Pinczes, E. J. & MacKain, B. (1993). *One Hundred Hungry Ants*. Boston, MA: Houghton Mifflin.
This is a great book for teaching multiplication, division and grouping strategies. It begins with ants looking for a picnic; however, they discover that spending too much time organising can result in not getting the product! The ants march in 1 group of 100, 2 groups of 50, 4 groups of 25 and so on. Students are encouraged to model the ants, draw new patterns and write a happier ending for the story.

Scieszka, J. & Smith, L. (1998). *Maths Curse*. London: Puffin Books.
The girl in this picture book is amazed by the mathematics everywhere in her daily life. She arrives at myriad mathematical problems as a result of her teacher saying that you can think of almost everything as a mathematics problem.

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CHAPTER 11

Solving problems with written strategies

LEARNING OUTCOMES

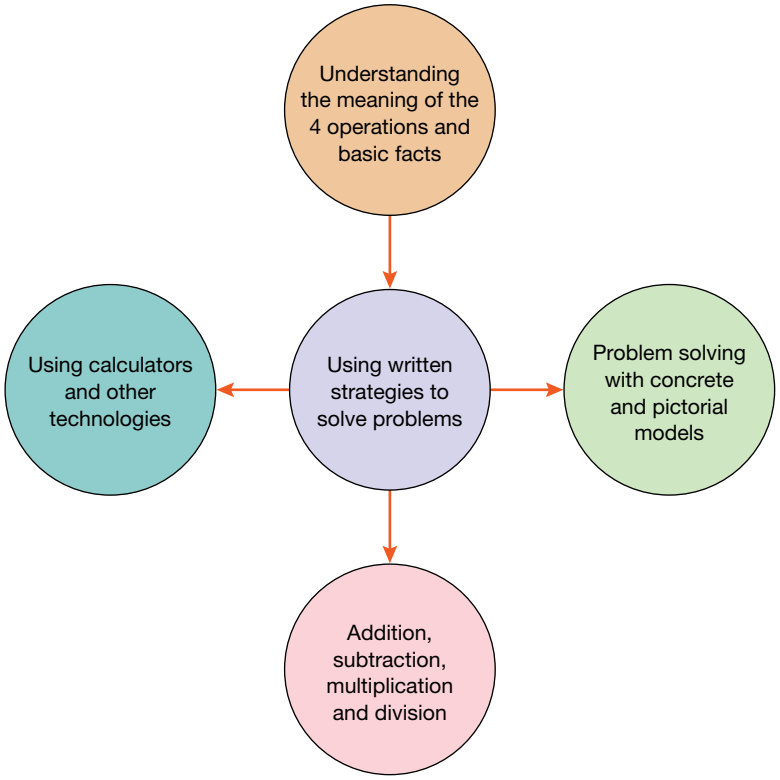
11.1 Articulating written strategies and explaining how and why concrete materials are useful for helping children develop a deeper understanding of written strategies.
11.2 Understanding that children might choose, use and record different addition methods to algorithms.
11.3 Exploring various subtraction algorithms and how they are developed.
11.4 Describing ways that the distributive property supports the development of the multiplication algorithm.
11.5 Explaining why the traditional division algorithm is the most difficult for children to master.
11.6 Making decisions about estimation, checking solutions and the right balance between practice and proficiency.



‘Mathematics is as much an aspect of culture as it is a collection of algorithms.’

Carl Boyer, mathematics historian (1906–1976)

Chapter 11 concept map



Introduction

For hundreds of years, solving mathematical problems with paper-and-pencil procedures (or *computational algorithms*) has been viewed as an essential component of children's mathematical education. In recent years however, the teaching of written strategies has become more open-ended. Instead of teachers merely presenting one preferred algorithm and showing children exactly what to do, step by step by step, the focus has shifted to what children themselves construct or develop. This is followed by meaningful connections with other students' ideas and teachers' knowledge to arrive at methods that are mathematically correct but adaptable. Written computation itself has become a problem-solving process, one in which students are encouraged to reason their way towards solutions, justify their thinking and communicate their findings. The tradition of memorising procedures given by the teacher, without understanding is not the preferred approach.

This change means that children can explore alternative strategies for solving problems in written form. Children who have created their own way of solving a problem, thought about alternative multi-digit calculations, and who have made sense of various approaches, are likely to be more flexible and efficient. This is in contrast to being taught one algorithm that works. The algorithms that have traditionally been presented in mathematics lessons were refined over many centuries. Whilst they are highly efficient, they do not necessarily reflect the way children think mathematically. Accuracy and efficiency are still important today; however, efficiency is of less importance due to digital technology. No one needs to undertake pages of calculations by hand anymore. Instead, it is important that the preferred practice gives serious attention to students' abilities to reason and think about written strategies.

Within the sub-strands of number and place value and money and financial mathematics, the Australian Curriculum identifies processes and content knowledge that are important for the 4 operations and

emphasises mental, written and digital methods of calculation for Years 2 to 7. These are shown in table 11.1 as follows.

TABLE 11.1 **Proficiencies and content descriptions for solving mathematical problems using written strategies in Australia**

Year level	Content description
Year 2 **Problem-solving** includes using number sentences that represent problem situations. **Reasoning** includes using known facts to derive strategies for unfamiliar calculations, and comparing and contrasting related models of operations.	**Number and place value** • Solve simple addition and subtraction problems using a range of efficient mental and written strategies.
Year 3 **Reasoning** includes generalising from number properties and results of calculations.	**Number and place value** • Represent and solve problems involving multiplication using efficient mental and written strategies and appropriate digital technologies.
Year 4 **Reasoning** includes generalising from number properties and results of calculations, and deriving strategies for unfamiliar multiplication and division tasks.	**Number and place value** • Develop efficient mental and written strategies and use appropriate digital technologies for multiplication and division where there is no remainder. **Money and financial mathematics** • Solve problems involving purchases and the calculation of change to the nearest five cents with and without digital technologies.
Year 5 **Problem-solving** includes formulating and solving authentic problems using whole numbers. **Reasoning** includes investigating strategies to perform calculations efficiently.	**Number and place value** • Solve problems involving multiplication of large numbers by 1- or 2-digit numbers using efficient mental, written and appropriate digital technologies. • Solve problems involving division by a 1-digit number including those that result in a remainder. • Use efficient mental and written strategies and apply appropriate digital technologies to solve problems.
Year 6 **Problem-solving** includes formulating and solving authentic problems using fractions, decimals, percentages and measurements.	**Number and place value** • Select and apply efficient mental and written strategies and appropriate digital technologies to solve problems involving all four operations with whole numbers. **Money and financial mathematics** • Investigate and calculate percentage discounts of 10%, 25% and 50% with and without digital technologies.
Year 7 **Fluency** includes investigating best buys. **Problem-solving** includes formulating and solving authentic problems using numbers and measurements. **Reasoning** includes applying the number laws to calculations.	**Number and place value** • Apply the associative, commutative and distributive laws to aid mental and written computation. **Money and financial mathematics** • Investigate and calculate 'best buys' with and without digital technologies.

Source: © Australian Curriculum, Assessment and Reporting Authority 2015.

The information contained in table 11.1 clearly demonstrates that developing skills in choosing appropriate procedures and carrying them out flexibly, accurately and appropriately is a focus of learning

beyond doing pen-and-paper calculations alone. Given the year levels for development of mental and written strategies, and with calculators readily available to relieve the burden of more complex computation, the ability to use paper-and-pencil algorithms efficiently is still considered essential for all students by the end of Year 6. Both the Australian Curriculum: Mathematics (ACARA v8.1) and the Principles and Standards for School Mathematics (National Council of Teachers of Mathematics 2000) stress the need for children to be able to develop and use computational techniques that are appropriate for solving problems, to estimate results and to know that their results are reasonable. As described in the Australian Curriculum, students are building upon previous understanding of numeration, place value and the meaning of the 4 operations.

A balance between understanding, fluency and problem-solving abilities are essential for developing computational proficiency. Such proficiency when learning computation, includes the following:

- fostering a solid understanding of and fluency with simple calculations
- abandoning the teaching of tedious calculations using paper-and-pencil algorithms in favour of exploring more mathematics
- nurturing the use of a wide variety of computation and estimation techniques — ranging from quick mental calculation to paper-and-pencil work and using calculators or computers — suited to different mathematical settings
- developing the skills necessary to use appropriate technology and then translating computed results to the problem setting
- providing students with ways to check the reasonableness of answers (encouraging number and algorithmic sense, together with estimation skills).

Over many years, educators have developed several models for teaching algorithms for the 4 basic operations. Underlying the development of these models is the idea that children need to be actively involved in constructing their own mathematical learning. Children may be more comfortable using a standard algorithm or an alternative variation but ultimately the choice should be theirs. In this chapter, we emphasise the importance of helping children to choose an appropriate procedure when calculating the solution to a problem, depending on whether an exact or approximate answer is needed.

11.1 Learning strategies that lead to understanding

LEARNING OUTCOME 11.1 Articulating written strategies and explaining how and why concrete materials are useful for helping children develop a deeper understanding of written strategies.

Computational fluency with addition, subtraction, multiplication and division includes the ability to use a variety of methods when solving problems and to recognise and reason about relationships between the methods. This ability requires children to develop and learn the procedures for particular algorithms, however, their learning needs to be done with meaning and understanding which supports their procedural knowledge. As teachers focus upon students' learning of written strategies, they need to support children in gaining 'a balance and connection between computational proficiency and conceptual understanding' (NCTM 2000, p. 35). These 2 considerations are important for teachers in ensuring that children do not just learn procedures by rote, but they learn them with understanding.

Using materials

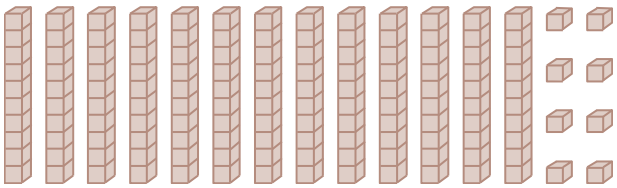
The use of concrete materials in developing understanding of written strategies is essential. Materials form a bridge between the real-life problem situation, the pictorial image or visualisation of the problem, and the abstract algorithm. This helps to strengthen the idea that what is written down represents real objects and actions. Children need to be given sufficient time to handle the materials and make the transition to pictures and symbols. Materials such as unifix cubes, base-ten blocks, counters, pop-sticks, bean sticks and loose beans, buttons and other materials — either derived from the problem situation or representative of it — help children construct an understanding of when and how a written strategy will work.

Using place value

Each of the written strategies for whole number computation is based upon place value ideas, many of which were discussed in a previous chapter. Children need an in depth understanding of these ideas before they can work effectively with the algorithms. Linking place value ideas directly when renaming or regrouping number is a necessary step as children explore and develop written strategies for each operation. Providing numerous trading, regrouping and renaming activities will scaffold students for the 4 operations.

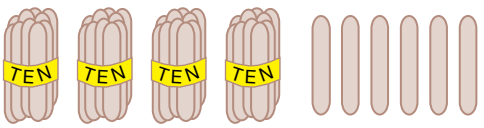
Regroup 138 ones into tens and ones using base-ten blocks:

138 is _____ tens _____ ones.



Show 46 with bundles of tens and ones, using pop-sticks and rubber bands to bundle them:

46 is _____ tens _____ ones.
Rename:



- 7 tens 16 ones = 8 tens _____ ones.

Tens	Ones
7	16

T	O
8	6
~~7~~	~~16~~

- 35 is the same as 2 tens _____ ones.

Tens	Ones
3	5

T	O
2	?
~~3~~	~~5~~

- Regroup 5 tens 3 ones to show more ones:

Tens	Ones
5	3

T	O
4	13
~~5~~	~~3~~

Write:

- 6 tens 5 ones in 3 different ways
- 37 in all the different ways you can.

The last question, which is open-ended, is likely to give rise to many different ideas. Students can frequently handle challenges that might have been thought beyond them. Building naturally on their prior knowledge of place value, students will develop algorithms that work with multi-digit numbers. Helping children build connections between key ideas in place value such as grouping, partitioning and rearranging collections and symbols will help them understand the structure of multi-digit numbers when solving problems in written form.

11.2 Addition

LEARNING OUTCOME 11.2 Understanding that children might choose, use and record different addition methods to algorithms.

Understanding and fluency, with basic addition are goals for the early years, but gaining these proficiencies depends on many and diverse experiences. As children work with different objects, they develop a concrete understanding of addition. They move objects together and use counting and grouping strategies to identify the total number of objects. Initially, children begin by recounting all the objects but later, this strategy becomes more efficient when they count on, particularly from the largest group. A closer look at how children make sense of addition reveals a wide variety of estimation and computation strategies.

Hypothetically, the teacher (below) is working with Year 1 children. After a brief activity to review prior knowledge by renaming numbers (for example, 52 as 5 tens and 2 ones and 4 tens and 12 ones, and so on), the following problem is given:

> Jill and Jeff both collect game cards. Jill has 27 cards and Jeff has 35. How many do they have altogether?

To focus attention on the reasonableness of the answers, it is good to suggest an estimate for the total.

> Would they have more than 50 cards? Less than 100 cards? Why do you think so?

Then the teacher might provide concrete material, counters, blocks, actual cards (or slips of paper) for each small group of children, with the suggestion to work out an answer, record it on paper and be ready to explain their solution. The teacher might encourage students to use a familiar problem-solving strategy, including drawing pictures. At other times, instead of working in groups, the children might be asked to solve the problem individually and to write an explanation of what they did and swap it with a peer. In each case, children need time and repeated opportunities to explore and invent strategies and to make connections between different ways of thinking about or doing mathematics. After each group or individual has reached an answer, the whole class might come together to share the ways in which they solved the problem. The teacher might record the essence of each explanation or encourage students to present their own recording. For example:

- Group 1 counted out 27 cards and counted out 35 cards, and then counted all the cards together, 1 by 1. They recorded in symbols their solution as $27 + 35 = 62$.
- Group 2 made 2 piles of 10 cards, with 7 extras, and 3 piles of 10 cards, with 5 extras. They explained their working with place value language and numbers.

$$\begin{array}{l} 2 \text{ tens} + \ 7 \\ \underline{3 \text{ tens} + \ 5} \\ 5 \text{ tens} + 12, \text{ which is renamed as } 6 \text{ tens} + 2, \text{ or } 62 \end{array}$$

- Group 3 thought about 20 and 30 making 50, and then 7 and 5 making 12. Later they wrote down their ideas.

$$\begin{array}{r} 20 + \ 7 \\ \underline{+30 + \ 5} \\ 50 + 12, \text{ or } 62 \end{array}$$

- Group 4 relied on the written strategy that Kim's older brother had shown her, first adding the 1s $5 + 7$, to get 12 ones and writing the 2 ones and 'carrying' 1 group of 10, then adding 1 to 2 tens and 3 tens to get 6 tens, or 62 altogether. The teacher showed an expanded way to record this way of thinking about the problem.

$$\begin{array}{r} {\scriptstyle 1} \\ 27 \\ \underline{+35} \\ 62 \end{array} \qquad \begin{array}{r} 27 \\ \underline{+35} \\ 12 \\ \underline{50} \\ 62 \end{array}$$

- Group 5 did almost the same thing, but they thought about adding 20 + 30 before adding 7 and 5. The teacher proposed recording their work a little differently.

$$\begin{array}{r} 27 \\ +35 \\ \hline 50 \\ 12 \\ \hline 62 \end{array}$$

- Group 6 decided to work with a multiple of 10 (that is easier to think about) and added 3 to the 27. They compensated by subtracting 3 from the second part of the answer. They checked their answer by doing the problem a different way, too!

$$\begin{array}{r} 27 + 3 = 30 \\ +35 - 3 = 32 \\ \hline 62 \end{array} \qquad \begin{array}{r} 27 - 5 = 22 \\ +35 + 5 = 40 \\ \hline 62 \end{array}$$

- Group 7 added the ones of the second number, 27 + 5 = 32, and then added the tens, 32 + 30 = 62. Some of the students in the class pointed out that you could also think 27 + 30 = 57, and then 57 + 5 = 62.

Everyone was sure they were right and could explain why — and they were sure that the other groups were right, too! Finally the teacher asked if 62 was correct because it was the answer every group reached. A number of children replied 'no' and another estimation with compensation was offered, that 62 was a reasonable answer because 27 is almost 30, and 35 is halfway to 40 so 65 is close but 62 is right. Note that some of the students worked from left to right, as children frequently do when constructing their own algorithms. Not all children used the concrete materials, but those who did were able to describe or write a strategy that modelled what they had done with the materials. Some students used procedures that they had invented or been taught outside of school. All students gained a positive experience from the activity. If children regularly experience opportunities to think about how numbers work together, and how our base-ten system can help with mental strategies they will be more likely to be flexible in their thinking about written strategies. For example, In the classroom 11.1 shows a game that helps students develop the ability to add and subtract up to ten and multiples of 10 from any number.

Standard addition algorithm

The Year 1 children working in groups as described above were using procedures that were as plausible for mental computation as for paper-and-pencil computation. They were asked to use a strategy that worked for them and record on paper. Children will need to recognise that often addition involving 1- and 2-digit numbers can and should be done mentally. Activities which involve regrouping and renaming help give children a good foundation from which to tackle addition in written form. Note that only the group that relied on what Kim's brother had told them came up with the standard addition algorithm for the problem 27 + 35.

	Think
$\overset{1}{}$	7 + 5 = 12
27	Write 2 in the 1s column
+35	and 1 in the 10s column.
62	2 + 3 + 1 = 6 tens

It is not surprising that most students did not use the standard algorithm because that procedure is the result of centuries of refinement. It is commonly taught in schools and used with pen and paper because it is efficient, requiring less writing than some other methods, but it is also less obvious. Someone would need to explain about placing the 1 ten from 12 in the tens column. When students attempt to use the standard algorithm without understanding why it works, they are more likely to make errors

compared to when they do addition in ways that intuitively make sense to them. For this reason it is important that students develop an addition method after using concrete materials such as those in figure 11.1. As the 7 ones and 5 ones are added together the total is 12 ones. It is easy to see that 1 ten is created by trading 10 ones for 1 ten using the base-ten blocks.

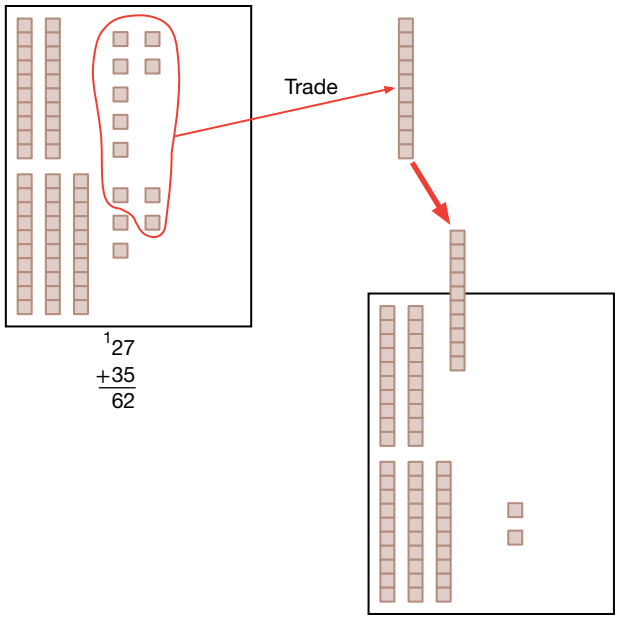


FIGURE 11.1 Base-ten blocks build the ideas of regrouping in the standard addition algorithm.

As groups of students share their different ways of working, they learn from each other. There will become a developmental moment when each student will be ready to see the standard algorithm, know it makes sense, and note that it is shorter than some of their own methods. With scaffolding they might recognise that parents and other adults learned addition this way and continue to use this traditional algorithm. Using coloured counters and a place-value mat, as in figure 11.2, can provide a natural progression in building the notion of regrouping after base-ten blocks and before introducing the standard algorithm.

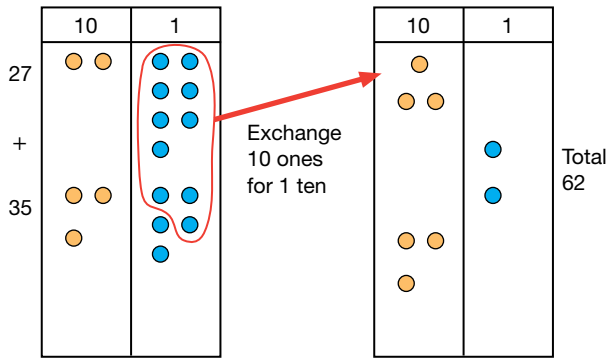


FIGURE 11.2 Use of coloured counters and a place-value mat for modelling the standard addition algorithm

IN THE CLASSROOM 11.1

Capture

Objective: Use a hundred chart and game to help children learn addition and subtraction of multiples of tens and ones.

Year level: 1–3

Materials

- A hundred chart divided into 100 squares numbered from 1 to 100 (see Appendix C).
- A deck of 56 change cards numbered +1, −1, +2, −2, +3, −3, +5, −5, +10, −10, +20, −20, +30, −30 (each of these numbers appears 4 times).
- Blank sheets for students to record moves.
- 12 markers of the same colour.
- 1 game piece per person or group.

How to play

1. Place each marker covering 12 numbers on the chart.
2. Place your game piece on any number without a marker. That is where you will begin playing.
3. Place the shuffled change cards face down.
4. Take 5 cards from the deck of change cards.
5. Spread them on your table in the order in which you want to use them.
6. A number with a minus (–) sign tells you to take that number of steps backwards. A number with a plus (+) sign tells you to go forward.
7. You can use 1 or many cards at a time.
8. If your last calculation is on a marker, capture that marker.
9. You can use your card or cards only once. A mixture of addition and subtraction cards can be used. After you have used them, place them face down on a discard pile and replace them from the change card pile.
10. If your card or cards do not allow a capture, replace them with new ones from the deck and reshuffle the deck.
11. If you replace a card or cards, you lose your chance to play that turn.
12. To practise the game, place markers on the squares shown in the chart below. Just do a few questions until you are confident and then start your own game.

Questions

1. Jane's game piece was on 9 and she captured a marker on 25. She wrote her equation as 9 + 3 +10 + 2 + 1 = 25. What was Jane's total move forward? Explain how you found this out.
   Rewrite Jane's equation to show how far she moved: 9 + ____ = 25.
2. Rowen's game piece was on 52 and he captured a marker on 85. He wrote his equation as 52 + 3 + 20 + 10 = 85. What was Rowen's total move forward? Explain how you found this out.
   Rewrite Rowen's equation to show how far he moved: 52 + ____ = 85.
3. Craig's game piece was on 10 and he captured a marker on 33. He wrote his equation as 10 + 20 + 2 + 1 = 33. What was Craig's total move forward? Explain how you found this out.
   Rewrite Craig's equation to show how far he moved: 10 + ____ = 33.
4. Alana's game piece was on 72 and she captured a marker on 59. She wrote her equation as 72 – 10 – 1 – 2 = 59. How many spaces did Alana move backwards? Explain how you found this out.
   Rewrite Alana's equation to show how far she moved: 72 – ____ = 59.
5. Renae's game piece was on 97 and she captured a marker on 56. She wrote her equation as 97 – 30 – 10 – 3 + 2 = 56. How many spaces did Renae move altogether? Explain how you found this out.
   Rewrite Renae's equation to show how far she moved: 97 – ____ = 56.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Partner with 1 of your classmates. Now, place the 12 markers wherever you want and start a new game of your own.

Source: Reys et al. (2009). *Helping Children Learn Mathematics,* 9th edn. Adapted from Economopoulos, K. & Russell, S. J. (1998). *Investigations in Number, Data, and Space Curriculum.*

Partial-sum addition algorithm

The algorithm used by group 5 previously in the Year 1 example is similar to the standard algorithm but much clearer because all the partial sums are shown separately. In this way, there is less chance of error when regrouping. As group 5 used it, they added from left to right — that is tens first, then ones — which is more natural than working right to left. However, this same partial-sum algorithm works equally well starting with ones, then tens, and so on (as shown by group 4). The partial-sum algorithm can be used as an alternative algorithm for addition or it can be useful as a transitional algorithm — that is, an intermediate step on the way to learning the standard algorithm. For students creating their own method of recording based on the use of non-proportional material and place value mats (as shown in figure 11.2) this method works well and leads to a deeper understanding of the addition algorithm. Ideally children should be encouraged to work with whichever procedure they understand as they build confidence.

Suppose the next problem posed to the children involved the addition of 13 and 54. Until recently, teachers might have introduced this problem, which does not require regrouping, far earlier than the problem involving 27 and 35, which does involve regrouping. Teachers have come to realise however, that controlling the problem situations 'to work with easy numbers first' is artificial and can lead to misconceptions about the operation and the algorithm. It is more realistic and better pedagogically for children to work with addition situations stemming from problems that arise naturally and involve both regrouping and no regrouping.

Working with 3-digit numbers and beyond proceeds easily from addition of numbers less than 100.

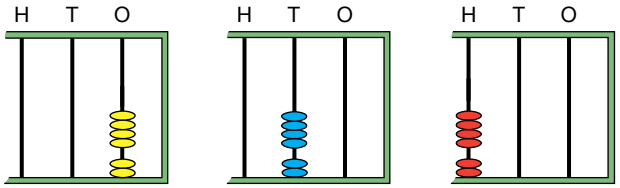
For example, 427 + 235 = ?

$$\begin{array}{rl} 427 & \\ +235 & \\ \hline 12 & \text{ones} \\ 50 & \text{tens} \\ 600 & \text{hundred} \\ \hline 662 & \end{array}$$

This method is an extension of group 4's idea for the written addition algorithm. Before tackling the given or invented problem, children could be encouraged to use number sense and estimation to judge the reasonableness of their answer. Estimating a sum between 600 and 650 using the front-end and adjusting method would help students have some confidence with their written result.

Children need to gain experience with numbers greater than 2 digits so they realise that there are no hidden difficulties. That is, if the problem is realistic and children chose not to use calculators, they would be able to efficiently use the written method. During Year 2 and 3, practice for addition can focus on 2- and 3-digit numbers, which will lead to proficiency with solving problems.

Adding multiples of 10, 100 and so on is an extension of the simpler case. This is an important skill for students to develop for use both in mental computation and estimation and as a building block in performing written computations. The abacus model clearly illustrates this idea:



4 + 2 | 40 + 20 | 400 + 200

As children work with written methods for addition and other operations, they can be encouraged to move from closed questions to more open-ended questions. Creating their own problems leads to a variety of problem solving strategies. Using larger numbers can extend students' ability and encourage estimation, ascertaining whether the answer they reach is approximately correct. Use of *compatible numbers* — numbers that are easy to compute mentally — is a powerful estimation strategy that can be applied. Some written activities to practise mental computation and sharpen estimation skills, as well as deepen understanding of addition algorithms, are found in In the classroom 11.2.

IN THE CLASSROOM 11.2

Starters

Objective: Use compatible numbers to develop mental computation and estimation skills involving addition.

Year level: 2–3

Use what you know about addition to find each missing digit:

52	29	■8	452
+16	+36	+21	+■7
6■	■5	79	489

Use only the digits given in the cloud to create a problem with the sum shown:

$$\begin{array}{r} 4\ 4 \\ +\ 4\ 6 \\ \hline 9\ 0 \end{array}$$ (4, 6) $$\begin{array}{r} \_\ \_ \\ +\ \_\ \_ \\ \hline 8\ 8 \end{array}$$ (3, 5) $$\begin{array}{r} \_\ \_ \\ +\ \_\ \_ \\ \hline 7\ 0 \end{array}$$ (7, 3)

Use 2, 4, 6 and 8 for these problems:

- Use each digit once to make the smallest sum possible.



- Use each digit once to make the largest sum possible.



- Use each digit once to make a sum as near 100 as possible.



Use only these numbers:

24	40	22	15	31	14

- Name 2 numbers whose sum is 64. _____
- Name 2 numbers whose sum is more than 70. _____
- Name 2 numbers whose sum ends in 8. _____
- Name 3 numbers whose sum is 93. _____

Create some new missing digit problems to try with a friend, for example all odd numbers.

Higher-decade addition

Combinations such as 17 + 4 or 47 + 8 or 3 + 28, called *higher-decade combinations*, are used in a strategy that builds on the basic fact, in this case 7 + 4 = 11. Note that the 2-digit number may come either before or after the 1-digit number.

The need for higher-decade addition arises in many real-life problems; for instance, adding $8 GST to a purchase of $80. The strategy of counting on will probably occur to some children, and it is clearly 1 way of solving this type of problem, but counting on is not efficient. In the classroom 11.3 focuses attention on the relationship of 9 + 5, 19 + 5, 29 + 5 and so on. As a result of this activity, children realise that:

- in each example, the sum will have a 4 in the ones place because 9 + 5 = 14, and the tens place will always have 1 more 10
- the sum of 9 + 5 is 4 more than 10, so the sum of 19 + 5 is 4 more than 20 and the sum of 29 + 5 is 4 more than 30
- for 59 + 5, there will be a 4 in the ones place and 6 in the tens place.

Children should learn to perform higher-decade additions automatically, without needing to think about adding ones and then tens and without counting on. Children who have already developed good number sense and facility with basic-fact addition strategies (as described in a previous chapter) should have little difficulty thinking through higher-decade addition problems. Looking for patterns such as those in In the classroom 11.3 and explicitly discussing them with the teacher and peers will encourage mental computation and increases the skill of higher-decade addition.

IN THE CLASSROOM 11.3

Look for patterns

Objective: Use observed patterns to develop mental computational skills with addition.

Year level: 2–3

Complete the sums in these addition problems:

$$3 + 6 = 9$$
$$13 + 6 = 19$$
$$23 + 6 = 29$$
$$53 + 6 = \blacksquare$$
$$83 + 6 = \blacksquare$$

How does the answer for 4 + 8 help solve 14 + 8?

$$\begin{array}{rr} 4 & 14 \\ +8 & +8 \\ \hline 12 & 22 \end{array}$$

First have a go at the addition problems which follow and then create some new problems to try with a friend.

$$\begin{array}{rrrrrr} 9 & 19 & 29 & 39 & 79 & 89 \\ +5 & +5 & +5 & +5 & +5 & +5 \\ \hline 7 & 17 & 27 & 37 & 47 & 87 \\ +3 & +3 & +3 & +3 & +3 & +3 \end{array}$$

11.3 Subtraction

LEARNING OUTCOME 11.3 Exploring various subtraction algorithms and how they are developed.

Like addition, the process of subtraction with multi-digit numbers requires prerequisite knowledge of basic facts and of place value. Experiences with trading or regrouping are important to develop the idea that 10 in any place-value column can be traded up for 1 in the column to the left, and that 1 can be traded down for 10 in the column to the right. Children need to become comfortable with different representations of any number, for example, that 37 can be thought of as 3 tens and 7 ones, or as 2 tens and 17 ones, or as 1 ten and 27 ones or just as no tens and 37 ones. If they can think this way, they have developed the flexibility of thought needed to develop strategies for subtraction with meaning. In a similar way to addition, children benefit from experiences with problem-solving situations involving subtraction (solved in any way they can) prior to learning standard written methods.

Standard subtraction algorithm

The standard subtraction method taught in Australia for the past 30 years or more is the *decomposition algorithm*. The equal addition method (although relatively straightforward to do) is no longer taught in schools because it was always learned by rote and rarely learned meaningfully and is even more complex to describe mathematically. The preferred method, decomposition, involves a logical process of decomposing or renaming the sum (the number you are subtracting from). The other number is not changed in any way. In the following example, 9 tens and 1 one is renamed as 8 tens and 11 ones:

	Think
$\begin{array}{r} {\scriptstyle 8\ 1}\\ 91 \\ -24 \\ \hline 67 \end{array}$	11 – 4 = 7 ones 8 tens – 2 tens = 6 tens

Children often develop different methods for subtraction intuitively, just as they do for addition. They might try to work from left to right, thinking about the number as a whole. To avoid confusion, the child

needs to think carefully about the numbers, and in doing so, might construct interesting alternatives. The following examples use the idea of negative numbers and/or the operation of addition as the inverse of subtraction.

$$\begin{array}{r} 74 \\ -58 \\ \hline 20 \\ -4 \\ \hline 16 \end{array}$$

or

$$\begin{array}{r} 70 + 4 \\ -50 - 8 \\ \hline 20 - 4 = 16 \end{array}$$

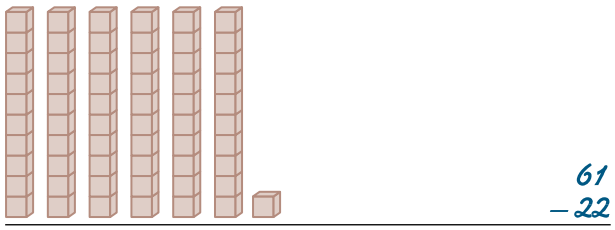
or

$$\begin{array}{r} 74 \\ -58 \end{array} \rightarrow \begin{array}{r} 58 \\ + \\ \hline 74 \end{array} \rightarrow \begin{array}{r} {\scriptstyle 1} \\ 58 \\ +\ 6 \\ \hline 74 \end{array} \rightarrow \begin{array}{r} {\scriptstyle 1} \\ 58 \\ +16 \\ \hline 74 \end{array} \rightarrow \begin{array}{r} 74 \\ -58 \\ \hline 16 \end{array}$$

To learn to use algorithms with understanding, scaffolding with concrete materials is helpful. Children can think about connecting the steps used to solve a problem with concrete materials to the steps used in the written algorithm. Questions that encourage children to focus on realistic situations, concrete and pictorial representation and finally written strategies are important. The following example with dialogue (is scripted in detail), but clearly indicates the process and symbols for decomposition subtraction.

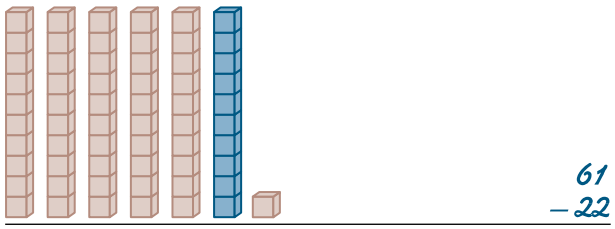
There were 61 children in 3 Years 4 and 5 classes at school. Of these, 22 ordered lunch from the school canteen. The rest brought their lunch from home. How many brought lunch?

What are we trying to find?	*How many brought lunch?*
Do you think it will be more or less than 50?	*Less, because there are 61 children altogether, but more than 10 children ordered their lunch.*
How many are in the 2 year levels?	*61*
How many children ordered lunch?	*22*
How can we show this problem with materials?	*Use cubes to model 61 and then separate out 22 cubes.*



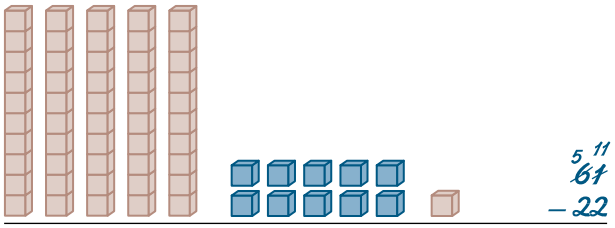
Let's assume we want to subtract 2 ones first. Are there enough ones to subtract 2 ones? *No.*

What can we do to gain more ones?	*We can trade a group of tens.*

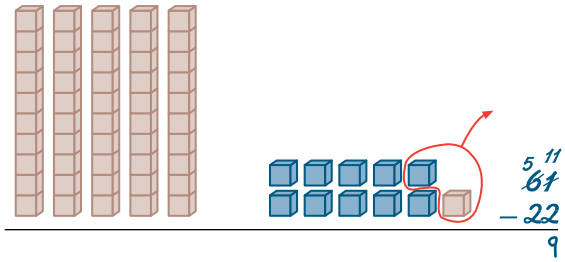


If we trade in 1 ten for 10 ones, how many ones do we get? *10 ones.*

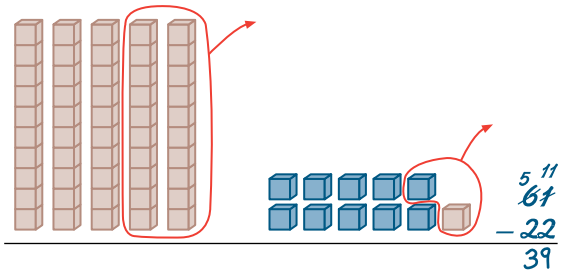
How many ones do we have now?	*11 ones altogether.*
And how many tens remain?	*5 tens.*
Is our number still 61?	*Yes, 5 tens and 11 ones is 61.*



Now are there enough ones to subtract 2? *Yes, we have 11 ones. 11 is more than 2.*
What is 11 – 2? *9.*



Now what do we subtract? *The tens.*
What is 5 tens minus 2 tens? *3 tens.*



The answer has been found. If we have 9 ones and 3 tens, how many children brought lunch from home? *39.*

Does this answer make sense? Why? *Yes, 39 makes sense because it is close to 40. There were just over 60 children altogether. There were about 20 children who ordered lunch. In my mind it is 60 – 20 = 40. Ok, so there should be about 40 children who brought lunch from home. Because 39 is close to 40, it makes sense.*

As shown here and earlier in the chapter, concrete experiences with regrouping for trading up (addition) and trading down (subtraction) are very helpful. In the classroom 11.4 (Race-to-a-flat and Give-away-a-flat) describes 2 worthwhile games played with base-ten blocks that can help develop students' understanding of regrouping for addition and subtraction.

Please note that presence of zeros in the sum demands special attention when the standard decomposition algorithm is used. In this example, with zero in the ones place only, 50 is renamed as 4 tens and 10 ones:

$$\begin{array}{r} {\scriptstyle 4\ 10} \\ 850 \\ -237 \\ \hline \end{array}$$

When 0 appears in the tens place, the problem is slightly more difficult, especially when regrouping in the ones place is also necessary. A couple of ways to approach the regrouping (using either 1 or 2 steps) are illustrated in figure 11.3. A word of caution about the over-emphasis of zeros in the sum; students can incorrectly think that the trading occurs from left to right in all cases, which leads to confusion. Scaffolding with concrete materials and/or adding on strategies can help to overcome this misconception.

IN THE CLASSROOM 11.4

Racing with base-ten blocks

Objective: Develop students' understanding of regrouping for addition or subtraction using base-ten blocks.

Year level: 3–5

Number of players: 2–5

Materials: Base-ten blocks or equivalent, place-value mat for each player (see Appendix C or create A4 paper/card divided into 3 columns, and 1 pair of dice.

1. Race-to-a-hundred

Before the game begins: Choose 1 player to be the base-ten 'banker'.

How to play: Players take turns rolling the dice, adding the numbers on the tops of the 2 dice, and then asking the banker for that number of ones (or ten and ones). For example, if you roll a 4 and a 5, as shown below, ask for 9 ones.



Place the pieces on your place-value mat. A collection of 10 or more pieces in any section of your place-value mat is not allowed. Whenever you need to, you must trade ones for a 10 (or tens for a hundred). When all the trading is complete, it is the next player's turn to roll the dice, collect ones and trading pieces. The first player to trade up to 100 is the winner. They become the new banker in the next game.

2. Give-away-one-hundred

Before the game begins: Choose 1 player to be the base-ten 'banker'. The banker distributes 100 to each player, who places it in the correct column on their place-value mat.

How to play: Players take turns rolling the dice, finding the sum of the numbers and *giving away* that number of ones to the banker. To begin with (and throughout the game), trading down will occur as needed. For example, every player can trade to have 9 tens and 10 ones in the beginning. The first player to return all of his or her base-ten pieces to the banker is the winner. The final turn should be an exact amount.

Questions for Give-away-a-flat

1. Is it possible to give away 1 long with 1 roll of the dice?
2. Is it possible to give away 2 longs with 1 roll of the dice?
3. Can you win the game with 1 roll of the dice?
4. What is the fewest number of rolls required to give away all of the pieces?
5. What is the highest number of rolls required to give away all of the pieces?

Variations: To challenge Years 4, 5 or 6 students, use 2 dodecahedron dice (numbered 1 to 12 on each) which will allow possible sums to 24, and race to a cube block (1000) or give away a cube. Similarly, games involving larger goals can help children practise the basic facts for multiplication by introducing 2 dice and a round of 'prize time'. A further adaptation is to use play money from $1 to $100.

Trading down with 0s in the addend, like give away 100 are a little tricky at first. The following language is helpful in explaining the subtraction process, for example with 39 from 207, as illustrated in figure 11.3.

> There are not enough ones to subtract 9 from 7 so we need to decompose the number 207 (trade down). There are no tens so 100 is traded for 10 tens and 1 of these tens is traded for 10 ones. The 207 is now renamed as 1 hundred, 9 tens and 17 ones so the subtraction can be done. Subtract 9 from 17 ones, leaving 8. Subtract 3 tens, leaving 6.

A subtraction involving numbers with more than 1 zero needs special care as it is even more difficult and prone to errors. Students need to do this concretely first by trading down from left to right before abstract representation. Multiple renaming — renaming from hundreds to tens, then renaming again from tens to ones — is demonstrated below. Note that the amount for 500 changes (decomposes) but remains the same quantity.

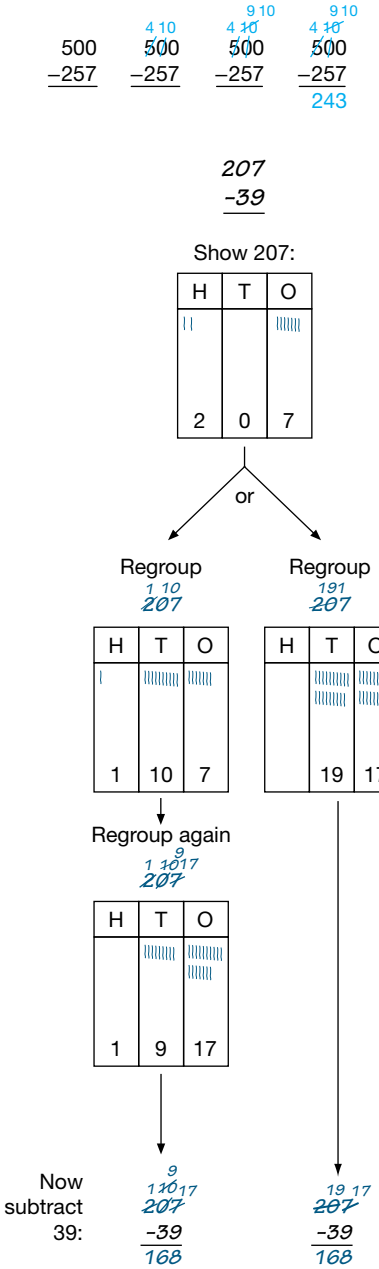


FIGURE 11.3 Place-value representations for subtraction by the decomposition method

It is important that students do not learn a procedure like this by rote, without understanding. If done by rote there will be confusion and it is likely that they will make errors but be unable to notice or

correct them. Place-value experiences are important in preparing children to cope with these problems. They must clearly understand that 500 can be renamed as 4 hundreds and 10 tens or as 4 hundreds, 9 tens and 10 ones. They will then find it easier to recognise the need for multiple regrouping when they see multiple zeros and will be able to do all the renaming at once but with care:

```
  500      4 9 10
 –283      5 0 0    500 = 4 hundreds, 9 tens, 10 ones
           –283
```

Alternately, if they understand that 500 can be thought of as 50 tens, they can think of renaming it directly as 49 tens and 10 ones. In figure 11.3, for the problem 207 – 39, 207 can be renamed as 1 hundred, 9 tens and 17 ones or 19 tens and 17 ones in one step. The need to do this is not readily recognisable by children, therefore renaming all place values should be encouraged.

Partial-difference subtraction algorithm

A common misconception held by children who have learned the subtraction algorithm without understanding is that they believe it is acceptable to subtract the digits in each column individually, regardless of position. That is, they will subtract the smaller number from the larger, without recognising when regrouping is required. For example, a student might incorrectly write a solution of 262 for the following problem:

```
                  523
                 –385
When it should be 138.   262
```

If this sort of error occurs, the teacher could ask the child to check their result with addition first. The sum of 262 and 385 = 647, not 523! Then using either pictorial representation or concrete materials, or both, the child can try doing the problem with both numbers represented. For further scaffolding simpler problems (such as 23 – 19 or 34 – 15) can be given and the child will be able to see why the answers are different to what their idea tells them (23 – 19 ≠ 16, 34 – 15 ≠ 21).

There is however, an interesting variation of this approach which works correctly and has been named the ‘partial-difference subtraction algorithm’ because it bears some resemblance to the previously mentioned partial-sum addition algorithm. Here, the subtractions in each column can be done in any order — left to right — right to left or even out of order. The main thing is to first write down the result of each subtraction, column by column, taking into consideration the positive and negative results. The next step is each place value is combined to give the final answer. The key is that when the bottom number is larger than the top number, the result is written as negative. For example, in the 523 – 385 problem 3 subtract 5 is – 2, 20 subtract 80 is – 60 and 5 subtract 3 is 2. Students beyond Year 6 will have met real world situations involving integers so they will think of this as ‘negative 60’, but younger children are likely to about going beyond zero, that is subtracting 60 more. The second step in the procedure is to combine the partial differences. In the example illustrated below 200 – 60 – 2 is 140 – 2, which results in 138. If a child has developed good number sense for basic subtraction facts and for working with multiples of 10 and 100, it will not be difficult for him or her to do each part of the second step mentally. Some students will find this to be a logical and meaningful method that works easily for them.

```
  523
 –385
  200
–  60
–   2
  140
–   2
  138
```

In the above example, a useful method of describing the answer might be as follows:

> Subtract 300 from 500, leaving 200; 80 from 20 is '60 more than zero', that is, –60; 5 ones from 3 is '2 more than 0' or –2. This leaves 138.

Finally, another point of view is the 'think addition' method. This can be considered in the same way that a grocer might add on to give change for a money transaction, just adding a little extra at the end. In the example given above, think addition would work by adding on from 385 to get to 500 and then adding an extra 23. Using place value such as 385 + 5 = 390, 390 + 10 = 400, 400 + 100 = 500 is helpful. So far, 115 has been added, with just another 23 needed, so it would be 115 + 23 = 138. The method could be written in a number of different ways where the final result is found by a simple addition of 5 + 10 + 100 + 23 = 138.

$$\begin{array}{r} 385 \\ +\ 5 \\ \hline 390 \\ +10 \\ \hline 400 \\ +100 \\ \hline 500 \\ +23 \\ \hline 523 \end{array}$$

11.4 Multiplication

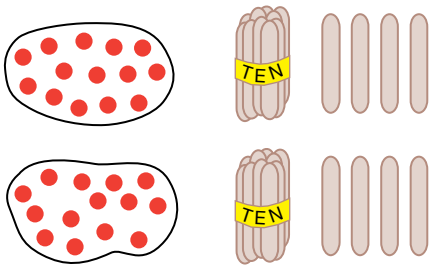
LEARNING OUTCOME 11.4 Describing ways that the distributive property supports the development of the multiplication algorithm.

Before children tackle written strategies for multiplication, they need to have developed a deep understanding of place value, expanded notation, addition methods and the distributive property, as well as the basic facts of multiplication. As with the other operations, it is wise to review each of these prerequisites before beginning work with multiplication algorithms. Children who are unable to cope with the written algorithm will need support to overcome gaps and weaknesses. Mathematical thinking needs to remain a priority when further developing the 4 operations and this is best done by using situations or problems where children can choose to estimate or compute mentally without concern for paper-and-pencil methods.

Multiplication with 1-digit multipliers

Mathematical language associated with any method (concrete, pictorial or written) needs to be encouraged in order for the teacher to gain a perspective on children's thinking. Furthermore, as children talk about their actions the meaning of multiplication is reinforced.

$$\begin{array}{r} 14 \\ \times\ 2 \\ \hline \end{array} \qquad \begin{array}{r} 14 \\ +\ 14 \\ \hline \end{array}$$

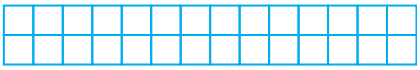


$2 \times 14 = 14 + 14$

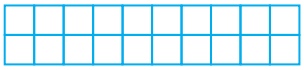
Looking at these models, we can see that 2 groups of 14 can be represented in 2 diverse ways. A bundle of 10 and 4 single pop-sticks is duplicated to represent $14 + 14 = 28$ (the additive view); alternatively, 2 groups of 14 counters are doubled to $2 \times 14 = 28$ (the multiplicative view). These differences point out the use of the distributive property:

$$\begin{aligned} 2 \times 14 = 2 \times (10 + 4) &= (2 \times 10) + (2 \times 4) \\ &= 20 + 8 = 28 \end{aligned}$$

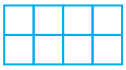
Arrays can be used to further develop meaning and give visual images of multiplication for numbers larger than 9×9.



2 × 14

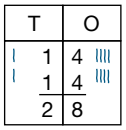


2 × 10

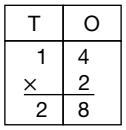


2 × 4

Base-ten materials and symbolic representation of place-value ideas can be shown side by side to support understanding.

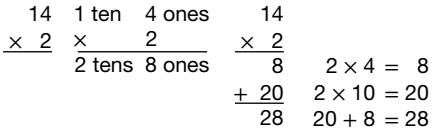


T	O
1	4
1	4
2	8



T	O
1	4
×	2
2	8

Expanded algorithms can be constructed by children, using a mixture of symbols and words, to help make connections with concrete and pictorial and written representations previously used.



14	1 ten	4 ones	14	
× 2	×	2	× 2	
	2 tens	8 ones	8	$2 \times 4 = 8$
			+ 20	$2 \times 10 = 20$
			28	$20 + 8 = 28$

Partial-products multiplication algorithm

The partial-products written strategy for multiplication (refer to figure 11.4) is one form of the expanded algorithm, where all the individual multiplications produced by the problem are recorded on separate lines and then summed. Just as the partial-sums algorithm for addition is a useful alternative or transitional algorithm, the partial-products algorithm can be useful for multiplication.

The standard multiplication algorithm is a streamlined version of the partial-products algorithm. In the traditional (contracted) algorithm, some of the partial products are added mentally before recording their sum. While this traditional method is more efficient, because it requires less writing, it is harder to understand step-by-step and is more susceptible to error since steps are often combined and done mentally. Therefore, children should not be forced to learn and use the contracted method initially as understanding of the process needs to come first. In fact children struggling with any method should be encouraged to use one that works best for them.

The distributive property helps children understand the relationship between the partial-products algorithm and the traditional algorithm (see figure 11.4). Whenever students see a multiplication such as

7×124, the best way to think about the solution is through multiplying each part of the expanded number. For example, $(7 \times 100) + (7 \times 20) + (7 \times 4)$ becomes $700 + 140 + 28$. Similarly, they can think of the multiplication of 4 by 59 as $(4 \times 50) + (4 \times 9)$ which is $200 + 36$. If the multiplication was to be done mentally, another strategy for 4×59 might be $(4 \times 60) - (4 \times 1)$. In the classroom 11.5 includes several types of activity that focus on using the distributive property.

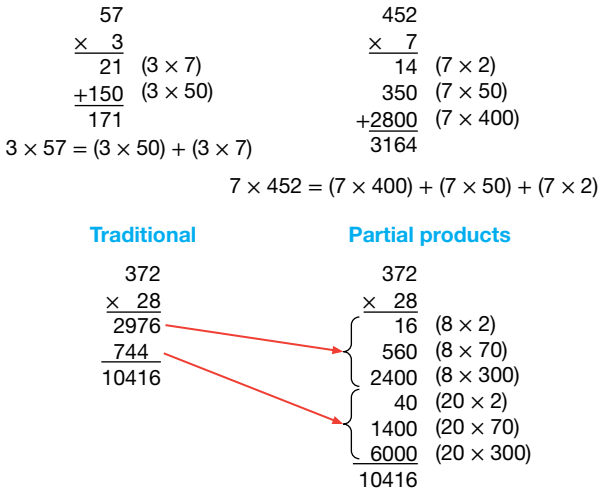


FIGURE 11.4 Multiplication comparing the traditional contracted algorithm with the partial-products algorithm

IN THE CLASSROOM 11.5

What's missing?

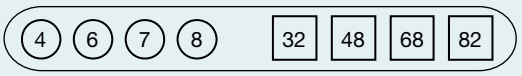
Objective: Use number puzzles to develop estimation skills with multiplication.

Year level: 5–6

- Create a mystery problem for a friend to try first. Use 2-digit by 1-digit multiplication as a beginning.
- Guess the numbers that will go into the circles and boxes. The calculator is a good tool to use here.



Write your number and then solve it on the calculator.
Score 2 points if correct on the first try and 1 point if correct on the second try.

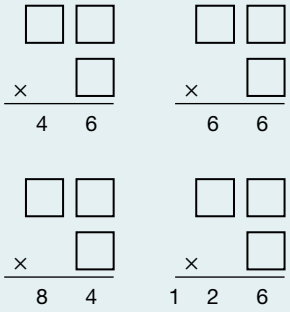


○ × □ = 408 ○ × □ = 272

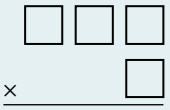
○ × □ = 476 ○ × □ = 336

○ × □ = 384 ○ × □ = 492

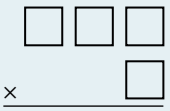
- Try the following multiplications using only the numbers 2, 3 and 4. You may use the same numeral more than once.



- Now use only 4, 6, 8 and 9 to make the largest possible product. Use each numeral only once.



- Repeat to make the smallest product. Use each numeral only once.



Lattice multiplication algorithm

Lattice multiplication is an algorithm very similar to long multiplication. Historians believe it has Hindu origins dating back more than 900 years. It was introduced to Europe by Fibonacci in 1202 and first published in Italy in 1478. Lattice multiplication was also published by Napier, a Scottish mathematician in 1617 in the form of 'Napier's bones' (or rods). The beauty of this method lies in its simplicity, which builds on the basic facts for multiplication and places the value for each number in the cells of the rectangle. The answers are organised diagonally into a rectangular array, without the need for zeros. This method does not require complex recall of separate parts as the tens and ones fit neatly in each cell. It can be used to scaffold students' learning if the traditional algorithm is causing too many errors.

Lattice multiplication

For each multiplication problem, you draw a rectangle the size of the problem (e.g. in figure 11.5 the problem is a 3-digit by 2-digit multiplication). The rectangle is made into a lattice of rows, columns and diagonals. To do the multiplication, you write the numbers to be multiplied on the exterior top and right of the rectangle, 1 digit per square, and within the lattice in each interior square you write the basic facts for each product. The grid organises the digits into place values diagonally and each diagonal represents ones, tens, hundreds, thousands (and more) starting from the bottom right-hand corner and ending in the upper left-hand corner. In figure 11.5 you can see the same multiplication problem (26 × 368) represented in 2 ways. The partial-products algorithm (method A) and the lattice multiplication algorithm (method B) are basically the same except for setting out.

The lattice method eliminates the mental strain and confusion of 'carrying' numbers as each basic fact is placed correctly according to the design of the lattice and addition occurs at the conclusion. The multiplication of 368 × 26 can be done in any order as long as the basic facts are placed in the correct boxes, with tens above the diagonal line and ones below. Looking at the top right-hand square for example,

$8 \times 2 = 16$ (this is actually $8 \times 20 = 160$). The grid places the answer correctly in the diagonals. Below it $8 \times 6 = 48$ is placed and in combination 8×26 has been done in the first column. Those same answers can be found in the partial-products algorithm with the zero included. To complete the problem, digits are added diagonally as shown by colours in figure 11.5.

Method A
(partial-products algorithm)

$$\begin{array}{r} 368 \\ \times\ 26 \\ \hline 48 \\ 360 \\ 1800 \\ 160 \\ 1200 \\ \underline{6000} \\ 9568 \end{array}$$

Method B
(lattice multiplication algorithm)

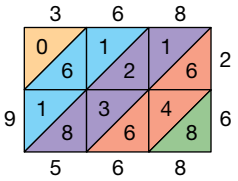


FIGURE 11.5 Comparing partial-products multiplication with lattice multiplication

Starting with the units (green), there are 8. Next, for the 10s (orange), there are 16; $6 + 4 + 6 = 16$, which is 6 tens and 1 hundred (place the 6 and remember or place a stroke for 1 hundred). Adding the hundreds (purple), there are 14 plus 1, carried forward from tens; $8 + 3 + 2 + 1 + 1$ extra $= 15$ hundreds or 5 hundreds and 1 thousand. For thousands (blue), there are 8 plus 1 from before; $1 + 6 + 1 + 1$ extra $= 9$ thousands. The answer is now read: it is 9568.

Comparing the 2 methods leads to a deeper understanding for the pre-service or in-service teacher. Look at the digits in each column in method A and compare them with the digits found along each diagonal in method B. For example, can you see which diagonal in method B corresponds with each column (ones, tens, hundreds, thousands) in method A? Now look at the rows in method A and compare them with each box in method B. Can you see where the basic facts appear — for example, 6×8, 6×6, 6×3, and 2×8, 2×6, 2×3? Note that with each fact the tens digit is recorded above the diagonal and the ones digit below the diagonal. Finally, each digit of the answer is found by the sum of each number in the diagonals (ones, tens, hundreds, thousands) working from bottom right to top left. Can you explain to someone else how method B works? Any size grid for whatever size, small or large can be made.

Multiplication by 10 and multiples of 10

The ability to work flexibly with powers of 10 is an important prerequisite for understanding multiplication methods. Multiplying by 10 comes easily to most children, and is readily extended to multiplying by 100 and 1000 as children gain an understanding of larger numbers. Any time a number is multiplied by 10, all the digits move 1 place to the left — for example, in $10 \times 763 = 7630$, the 700 becomes 7000, the 60 becomes 600 and the 3 becomes 30. Similarly, all the digits move 2 places to the left if a number is multiplied by 100. Notice that it makes much more sense to think about the digits moving to the left, than to think about the decimal point moving to the right (Flores 2007). Children can be shown a series of examples, after which they are asked to discuss and generalise, noting where the digits have moved.

Multiplying by 20, 30, 200, 300 and so on is an extension of multiplying by 10 and 100. Emphasise what happens across examples and generalise from the pattern.

For example, have children consider 3×50:

$3 \times 5 = 15$
3×5 tens $= 15$ tens $= 150$
$3 \times 50 = 150$

Then have them consider 4×50:

$4 \times 5 = 20$
4×5 tens = _____ tens = _____
4×50 = _____

A similar sequence can be followed for examples with hundreds, such as 2×300:

$2 \times 3 = 6$
2×3 hundreds = _____ hundreds = _____
2×300 = _____

Multiplication with zeros

When zeros appear in the factor being multiplied, particular attention needs to be given to the effect on the product or partial product. Many children are prone to ignore the zero. Thus, for 9×306, their answer may (incorrectly) be 324. Number sense lets us know that this answer cannot be correct because approximately 10 by 300 is 3000. When an estimate is made first, children have a way of determining whether their answer is near to the expected one. Writing each partial product separately, rather than attempting to add them mentally, can also help avoid place value errors due to zeros. In the previous problem, a student might write the partial products 54 and 0 and 2700. Using a place value chart and/or expanded notation can help students understand what is the correct procedure.

Estimate

306
× 9

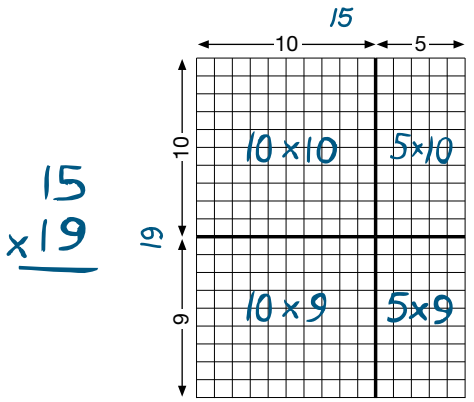
$9 \times 3 = 27$
$9 \times 300 = 2700$

T	H	T	O
	3	0	6
			9
2	7	5	4

$9 \times 306 = 9 \times (300 + 6)$
$= (9 \times 300) + (9 \times 6)$
$= 2700 + 54$
$= 2754$

Multiplication with 2-digit multipliers

When children begin to work with 2-digit multipliers, the use of concrete materials becomes cumbersome. Arrays or grids offer one way to bridge the gap pictorially from concrete materials to symbols and also help illustrate why the partial-products algorithm makes sense. As shown in figure 11.6, a grid can provide entirely new ways of viewing and writing about a multiplication problem. Materials such as base-ten blocks can be used to tie this work to previously learned procedures but increasingly the emphasis will shift to working with symbols, as shown in figure 11.7. This approach is possible when students already have a conceptual foundation building on the use of materials and an understanding of the distributive property.



$15 \times 19 = 15 \times (10 + 9)$
$= 15 \times 10 + 15 \times 9$
$= 10 \times 10 + 5 \times 10 + 10 \times 9 + 5 \times 9$
$= 100 + 50 + 90 + 45$
$= 150 + 135$
$= 285$

FIGURE 11.6 Use of a grid to solve multiplication with a 2-digit multiplier

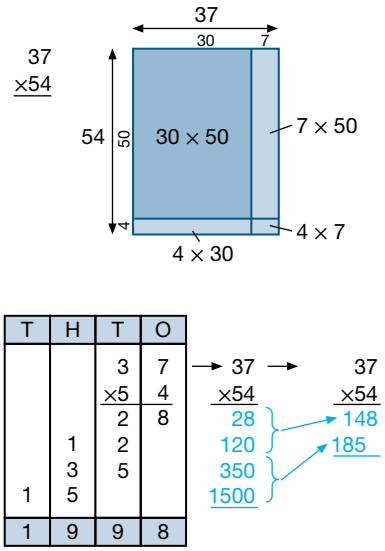


FIGURE 11.7 Multiplication with a 2-digit multiplier

When helping children learn the process of multiplication with 2- or 3-digit numbers it is important to connect new ideas to earlier understanding. Therefore working with larger numbers can be seen as an extension of the method for multiplying by a 1-digit number. The distributive law is applied twice, for example 37 is multiplied by 4, then 37 is multiplied by 50. Illustrating this process by using an array is essentially the same no matter what the multiplier. By the middle years, particularly Year 7, students begin to learn about symbolic manipulation in algebra, and apply the arithmetic laws to solving algebraic expressions. The conceptual understanding of multi-digit multiplication will help students make connections to the new idea of multiplying binomials. In figure 11.8, the illustration of the 'foil' method for multiplying 2 binomials should look familiar because it is the same as the illustration for multiplying 2 2-digit numbers together. Basically, the distributive property is the foundation for both of these procedures.

Multiplication with large numbers

In the upper primary years, students will need to choose and use appropriate strategies for all 4 operations. Multiplication with the calculator, combined with place value understanding can be used to solve problems involving numbers that are very large. As children experiment with using a calculator for multiplication, there will come a time when they overload the calculator. Sometimes the number to be entered contains more digits than the display will allow. At other times the factors can be entered, but the product will be too big for the display. For instance, an error message will result if the following multiplication is entered into a simple 8-digit 4-function calculator:

$$\begin{array}{r} 2345678 \\ \times\ 403 \\ \hline \end{array}$$

When this happens, children should be encouraged to estimate an answer and use the distributive property together with whichever method work best in the context of the problem. It might be written computation, mental computation and the calculator. The following example is one of many possible solution paths.

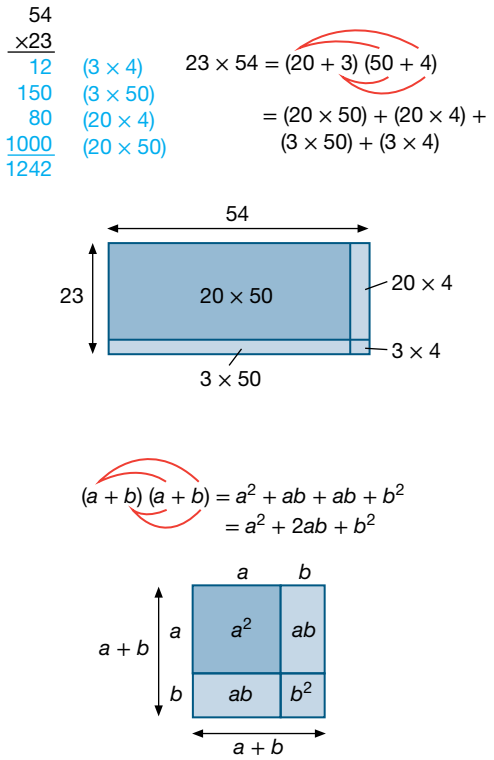


FIGURE 11.8 Extending multiplication ideas to algebra

Estimate

$400 \times 2\,000\,000 = 800\,000\,000$

(4 followed by 2 zeros) times (2 followed by 6 zeros) equals 4×2 or 8 followed by 8 zeros

Split one of the multiplication factors and *calculate*

$403 = 400 + 3 = (4 \times 100) + 3$

$4 \times 2\,345\,678 = 9\,382\,712$ (with the calculator)

$9\,382\,712 \times 100 = 938\,271\,200$ (mentally)

$3 \times 2\,345\,678 = 7\,037\,034$ (with the calculator)

Combine both parts and *calculate*

938 271 200 + 7 037 034 (by pen and paper) equals 945 308 234, or with the calculator by temporarily leaving off the 93 at the front end and replacing it afterwards), resulting in the same answer 945 308 234.

Check against estimate

800 000 000 is close to 945 308 234

There are many other ways of solving this problem, for example splitting the other number

$2\,345 \times 403 \times 1\,000 + 678 \times 403$.

The lattice method works particularly well in this situation, the grid needed is a 6 by 3 in size!

Depending on the degree of accuracy required, scientific notation can also be used with older students.

Estimate $4 \times 10^2 \times 2 \times 10^6 = 8 \times 10^8$

Calculate $4 \times 10^2 \times 2.3 \times 10^6 = 9.2 \times 10^8$ (to 1 decimal place).

These examples show how an understanding of multiplication and problem-solving skills can be used with calculators to reach a solution. Use of the calculator and mental and written strategies may be integrated to solve a single problem. Versatility in the approach supports the idea that the calculator is a useful tool and requires a different way of thinking mathematically when compared to complex paper-and-pencil algorithms.

11.5 Division

LEARNING OUTCOME 11.5 Explaining why the traditional division algorithm is the most difficult for children to master.

The process for solving short division problems with pen and paper is not too complex for children in upper primary years. The long-division algorithm; however, is without doubt the most difficult of all written strategies for children to master. There are a number of issues associated with the division algorithm.

- Place value is turned around — that is, computation begins at the left, rather than at the right compared with the other operations.
- The algorithm involves not only the basic multiplication facts but estimation and subtraction.
- There are several steps to solve in the algorithm, but the pattern moves from 1 place value to another.
- Trial quotients, involving large numbers, need to be used and may not be successful at the first attempt — or even the second.

Many teachers struggle to teach the long division algorithm, and many children struggle to learn it. Since the introduction of the Australian Curriculum (2012) and in many state curriculum documents prior to that, division with 1-digit divisors has been required at Year 5 level, but not long division. Many schools have been slow to accept this change, which has been recommended for more than 30 years. Since handheld calculators became readily available, it was possible for everyone to calculate answers to division problems quickly and easily, with accuracy. Mathematics educators have suggested that too much emphasis on the division algorithm is not productive and in fact can be destructive to children's self-image. There has long been a waste of precious learning time given over to the division algorithm for larger numbers. As *An Agenda for Action* (NCTM 1980, p. 6) indicated, more than 30 years ago:

> For most students, much of a full year of instruction in mathematics is spent on the division of whole numbers — a massive investment with increasingly limited productive return . . . For most complex problems, using the calculator for rapid and accurate computation makes a far greater contribution to functional competence in daily life.

In the NAPLAN assessments, some questions are asked with and without the use of calculators. For example the Year 7, 2010 assessment with calculators asked the following division problem, with 4 possible solutions:

> Helen paid $4465 for some sheep. She paid the same amount of money for each sheep. The cost of each sheep was a whole number of dollars. Which of the following could be the number of sheep Helen bought?

☐43 ☐45 ☐47 ☐49 (ACARA 2010, p. 8).

This kind of problem was not asking for a long-division strategy; instead, it was asking students to understand that the process of division is required and then use logical reasoning, division with a calculator and number sense to interpret their answer. In a similar way the 2012 NAPLAN sample for Year 5, asked 5 427 ÷ 9 = ?

☐63 ☐603 ☐630 ☐6003 (ACARA 2012, p. 5).

In this case the problem is presented as a symbolic question, which students would be best solving by estimation and short division. The options for this question also include a common misconception that relies on the understanding of place value and use of zero as a placeholder in the answer. The correct answer 603 and not 630 could be quickly checked by multiplication.

Helping students understand *when* division is appropriate and *how to estimate* answers to division problems is very important. In terms of actually performing computations, short division with 1-digit divisors should be the focus of teaching and learning, followed by minimal work with 2-digit divisors as an extension, if time allows. Although it might be considered beneficial for students to understand how such division is done, the process is rarely completed in real life by pen and paper and students do not require the mathematics until Year 11 when they learn division of polynomials. Performing more complex division, perhaps anything that takes more than 30 seconds to do with paper and pencil is a thing of the past for most adults. Schools should not demand that children spend countless hours mastering an antiquated skill. Teachers can't afford the teaching time for this task, there is more essential mathematics to be met in the curriculum and mathematical thinking and reasoning is more important than rote learning a dated skill. Instead, estimation skills should be used to define the boundaries of the quotient so that the reasonableness of calculator answers can be determined. The same process of estimation for short division and in fact for all operations (addition, subtraction and multiplication) is a higher priority and an essential skill.

Division with 1-digit divisors

Using real world problems is the best approach for learning about division. The most important thing is for students to develop initially is an understanding of what division is all about. Whilst the curriculum recommends that division problems with no remainders are solved in Year 4 and division problems with remainders are solved in Year 5 both situations can be met from the outset of children's work with division. Concrete materials can be used to model division situations and to explain their thinking. The teacher can scaffold students to record their thinking clearly and efficiently on paper. The written strategy can be meaningfully connected to the concrete experience of division.

Many students will use number sense, logical reasoning based on their knowledge of basic facts, and place value to come up with their own solution methods which will vary a great deal. For example:

58 ÷ 8 = ?

Think: 7 × 8 = 56. 58 divided by 8 is close to 56 divided by 8, which is 7. But there are 2 left over. So the answer includes a remainder, which can be interpreted in different ways, depending on the question. It could be 7, remainder 2, or $7\frac{1}{4}$, or $7.25.

85 ÷ 5 = ?

Think: 5 × 4 = 20 and 20 × 4 = 80, so 5 × 4 × 4 = 80. That is 80 ÷ 5 is 16 plus 1 more 5 is 17.

Or 5 × 10 = 50 and double 5 × 20 is 100, too big. 5 × 6 is 30, 16 × 5 is 80, and 1 more 5 is 85. So the answer is 17.

Or 5 × 20 = 100 and 3 × 5 = 15 since 100 – 15 = 85, and 20 – 3 = 17, the answer is 17.

65 ÷ 4 = ?

Think: 60 + 5. Since 4 × 15 = 60 (60 ÷ 2 = 30 and 30 ÷ 2 = 15), and 5 divided by 4 = 1, with a remainder of 1. So the answer is 16 remainder 1, or $16\frac{1}{4}$ or $16.25.

There are many and varied possibilities for constructing ways to find an answer to division problems. Some of these ideas use mental arithmetic only and some might use a combination of mental and written methods. Most all of them use estimation and place value understanding. Given free reign on these problems, the number of possible strategies will be surprising to the teacher. Encouraging exploration, justification and communication can be a very enjoyable process. Being an explorer right along with the children will help them see that their ideas are valued and there is no one right algorithm. Please note, some short, but informative articles about division algorithms and making sense of division include Boerst 2004; Fuson 2003; Gregg and Gregg 2007; and Hedges, Huinker and Steinmeyer 2005.

Distributive algorithm

Clearly, there are many ways to figure out and record the answer to a division problem and certain methods are more efficient than others. Over time 2 different algorithms have been used for division both of which are effective. The *distributive algorithm* is most common and familiar and is considered the standard division algorithm. It can be written in both short and long form but is best learned by connecting concrete experiences with pictorial and written methods. To scaffold students with any of the written forms, place-value columns for ones, tens, hundreds etc. can be drawn or a laminated place-value mat with erasable markers can be used.

Consider the problem of sharing $954 among 4 people shown in figure 11.9(a).

The outcome of sharing $954 between 4 is $238 for each person, with $2 extra. Working with dollars and cents, the remaining $2 can be shared equally with 50 cents each. The most correct real-world answer to this division is $238.50.

The language used when thinking about and doing the written method works left to right. By using 3 repeated questions (and actions, if done with base-ten blocks) the student can be supported in expressing the method in written form:

Question 1: Can you share 9 hundreds among 4 people?
[*Yes. We can share 8 hundreds by giving 200 to each person (200 × 4 = 800).*]

Record 2 hundred in the answer space above the 9 hundred in the question.

Question 2: How many did you use when sharing?
[*Altogether, we used 8 hundreds or 800.*]

Record with or without zeros and subtract to find the remainder of 1 hundred.

Question 3: How many remained?
[*There are 100 remaining, which will need to be traded for 10 tens. Combined with 5 tens you now have 15 tens.*]

Repeat question 1: Can you share 15 tens among 4 people?
[*Yes. We can share 12 tens by giving 30 to each person 30 × 4 = 120.*]

Record 3 tens in the answer space above the 5 tens in the question.

Repeat question 2: How many did you use when sharing?
[*Altogether we used 120 or 12 tens.*]

Record, with or without 0, 12 tens and subtract to find the remainder of 30.

Repeat question 3: How many remained?
[*There are 30 remaining which will need to be traded for ones and combined with the 4 ones, you now have 34 ones.*]

Repeat question 1: Can you share 34 among 4 people?
[*Yes. We can share 8 each or 32 ones. (8 × 4 = 32).*]

Record 8 ones in the answer space above the 4 in the question.

Repeat question 2: How many did you use when sharing?
[*Altogether we used 32.*]

Record and subtract to find 2 remaining.

Repeat question 3: How many remained?
[*The 2 units remaining can be expressed as a remainder of 2, a fraction of the divisor — $\frac{2}{4}$ or $\frac{1}{2}$ — or can be continued to 2 decimal places depending on the real-world situation (e.g. dollars and cents).*]

In this case the money situation gives a clear result as to how the remainder should be expressed.

Subtractive algorithm

Although not as familiar as the distributive algorithm, the *subtractive algorithm*, provides an intuitive and straightforward method for helping children learn to divide. This method is about repeated subtraction and is easier than the standard algorithm for most children to learn. Furthermore, when using

the subtractive algorithm, a logical guess will bring you closer to your desired answer. The subtractive algorithm can be used to solve \$954 shared among 4 people in a number of ways as illustrated in figures 11.9(b)–(d). Using place value effectively, for example $4 \times 100 = 400$, $4 \times 10 = 40$ and $4 \times 1 = 4$, as starting points gives rise to a sequential subtraction of hundreds, tens and ones. Alternatively, an educated guess, based on acting out the problem with play money is helpful. Give each person \$50 first which is \$200 shared out so far. Subtract and see that you still have \$754 to share. Obviously, you didn't need to start with such a small share. Now choose to give each person \$100, which is \$400 shared. You still have \$354 to share. You might give each person \$50 again, leaving \$154 still to share. You don't have enough to share \$50 again, so maybe give each person \$30. That leaves \$34. Now give each person \$8, leaving \$2 as remainder. The total each person received was \$50 + \$100 + \$50 + \$30 + \$8, or \$238, the same answer obtained above with the standard algorithm. The choices made here in using the subtractive algorithm were arbitrary. You might instead have chosen to repeatedly distribute \$30 over and over, and in that case the problem would take a long time to complete as shown in figure 11.9(c). On the other hand, if you were quite efficient and used place value as your aid, the solution would look very much like the standard algorithm shown in figure 11.9(d).

In using the subtractive algorithm, the child can choose to subtract any multiple of the divisor at each step. Although the process may take more or less time, depending on how efficient the thinking is, the answer will be correct no matter how many steps are required. If children are efficient and use place value as a guide, the subtractive algorithm can look almost exactly like the traditional division algorithm. Thus the subtractive algorithm can serve as a transitional algorithm (a stage on the way to learning to divide using the traditional method), or children may learn it as an alternative algorithm as it is both logical and prone to fewer errors.

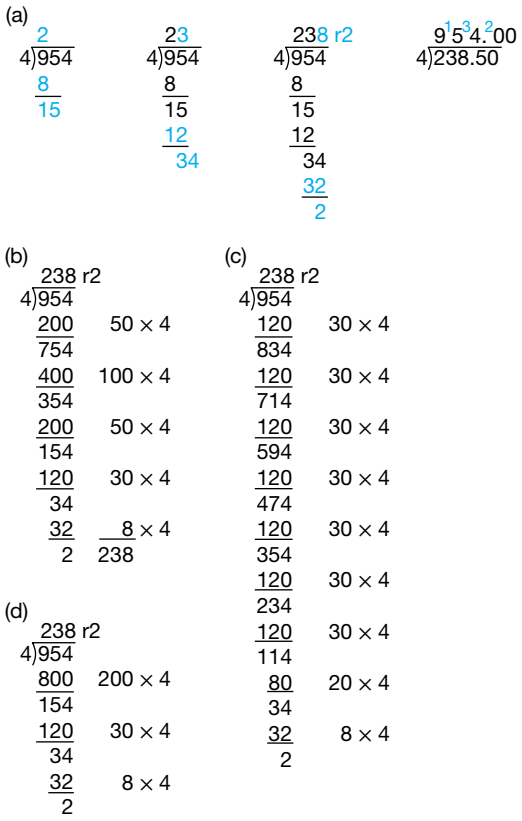


FIGURE 11.9 (a)–(d) Examples of the distributive division – algorithm (a) and the subtractive division algorithm (b–d)

Although teachers can help children understand algorithms such as those described previously, they also need to allow exploration. As with the other operations, children are likely to develop their own algorithms. However, it is the teacher's role to make sure that children do not create (or incompletely learn) incorrect methods for solving the 4 operations. Being on the watch for false algorithms is necessary but needs careful consideration. Asking children to prove that a procedure really works in different situations is more beneficial than 'telling' someone they are wrong. Furthermore, teacher's need to be aware of misconceptions or error patterns that crop up in children's work, which might indicate there is a difficulty with or inadequate knowledge about place value, basic facts or other issues. Fortunately, there are sources of information, such as Ashlock (2006), where the teacher can gain information. This source not only helps the teacher to identify error patterns but also provides suggestions for helping the child get back on track.

As with the other operations, children's learning can be scaffolded with concrete materials and place value charts to further develop understanding if there is lack of clarity about what they are doing. Keeping in mind the problem situation and remembering that the division process has meaning (partition or measurement), is worthwhile. A variety of experiences in creating problems such as those in In the classroom 11.6 and 11.7 could help children develop a deeper understanding of division.

Determining a reasonable answer can be particularly difficult with division, but questioning can help students make sense of the problem. Estimating the size of the quotient using place value is a valuable first step.

Consider $839 \div 6 = ?$

Is there at least 1 6 in 839? [*Yes 1* $\times$ *6 = 6.*]
Are there as many as 10 sixes in 839? [*Yes, 10 sixes are 60.*]
Are there as many as 100 sixes? [*Yes, 100 sixes are 600.*]
Are there as many as 1000 sixes? [*No, 1000 sixes are 6000.*]
Are there as many as 200 sixes? [*No, 200* $\times$ *6 =1200.*]
So the quotient is between 100 and 200 — and probably closer to 100.

Going in reverse order, consider $187 \div 3 = ?$

Are there enough hundreds that we could share them equally 3 times? [*No. We have only have 1 hundred.*]
Are there enough tens? [*Yes, 18 tens can be shared 3 ways. 18 tens* $\div$ *3 = 6 tens.*]
So the quotient has 2 digits [*and there are 6 tens.*]
So you know the answer is between 60 and 70.

Developing the habit of mentally asking these sorts of questions helps students recognise the range for a quotient. Mental questions such as these also help students make sense of the division algorithm while developing valuable estimation skills.

IN THE CLASSROOM 11.6

Creating division problems

Objective: Use number puzzles to develop number sense involving division.

Year level: 5

Design a real-world division example with:
- a dividend of 47 and a divisor of 3
- a dividend of 81 and a divisor of 5.

Now create and solve a division with:
- a quotient of 6 r2
- a quotient of 10 r4
- a quotient of 23 r5.

Design an example where there is:
- a divisor of 6 and a quotient of $15\frac{1}{2}$
- a divisor of 3 and a quotient of $25\frac{2}{3}$
- a dividend of 83 and a quotient of 11 r6.

Division with 2-digit divisors

Working with 2-digit divisors should aim toward helping children understand the procedure involved but not necessarily achieving mastery of a written strategy. Using the calculator, together with estimation skills and understanding the context of each problem, will solve the majority of multi-digit division problems for adults, so there is little reason to have children spend months or years mastering pen and paper algorithms. Other mathematics is of more importance for children to learn.

The development of division with 2-digit divisors proceeds through stages from concrete to pictorial, to abstract, paralleling the work with 1-digit divisor. Practice to a certain stage may be needed with the symbolic form if proficiency is the goal; however, as previously discussed this objective is no longer included in the curriculum. Use of the calculator is interwoven into the activities, as indicated in In the classroom 11.7, where the calculator is used to strengthen understanding of the relationships between numbers.

Making sense of division with remainders

Children should encounter division problems involving remainders from the time they begin to work with division ideas, in the same way that they encountered addition involving regrouping from the outset. It is artificial and unhelpful to begin teaching division with problems that have 'nice numbers' that come out exactly, therefore saving problems with remainders until later. As long as problems remain meaningful and can be understood at a concrete level, then the concept of a remainder can be easily interpreted.

Here are a variety of situations for the same number operation.

- Pass out 17 chocolates to 3 children.
  Each child receives 5 chocolates with 2 chocolates remaining. This is an unequal share unless the chocolates can be cut into pieces, then each child could have 5 chocolates plus $\frac{2}{3}$ of a chocolate.
- To make each Mother's Day card you need 3 pieces of lace. There are 17 pieces of lace. How many cards can be made?
  Five cards can be made. Probably 2 extra pieces of lace will be discarded, or saved for another project.
- If 17 children are going on an excursion and 3 children can ride in each car, how many cars are needed?
  There *will need to be 6 cars. Five cars can seat only 15 children, which means 2 children won't have a ride but with 6 cars, all 17 children have a ride. There will be 1 car with a spare seat.*
- Separate a class of 17 children into 3 teams.
  There can be 5 children on each team, and the 2 children remaining can be given a particular role like the scorekeepers. Alternatively there could be perhaps 2 teams of 6 and 1 team of 5 children.
- Grandma had $17 to give her 3 grandchildren. How much did each child receive (to the nearest coin)?
  Using the calculator the answer is 5.666 666 6 which needs to be rounded to the second decimal place, which is $5.65. There will be 5c remaining which can't be shared.

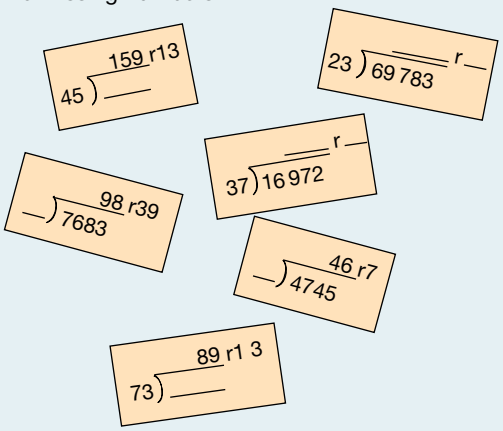
IN THE CLASSROOM 11.7

Easy does it!

Objective: Use calculators to develop number sense involving estimation, place value and division. Recognise that multiplication is the reverse of division.

Year level: 5–6

Use your calculator to find the missing numbers:



Note that the remainder is handled differently in each of these real-world problems. In the chocolate problem, both the whole number 5 and the remainder $\frac{2}{3}$ are reported as parts of the answer. In the card problem, the answer involves only the whole-number part (the remainder must be discarded or ignored). In the car problem, the quotient is rounded up because the remainder cannot be ignored (no matter how small children cannot be left behind). In the team problem, children need to find some other use for the remainder or they may abandon the attempt to form equal-sized groups if they want everyone to be able to play. Finally in the dollars problem the whole number and decimal part are both necessary but a remainder still occurs after the division.

Therefore it is important for children to think about and be able to deal with remainders as they appear in real-world situations. With calculators, the results of division might be expressed in decimal form; therefore children need to learn how to interpret the remainder when it is not an integer. The concept of recurring decimals is a natural consequence of division problems and can be interpreted from Year 4 (ACARA v8.1 2015).

IN THE CLASSROOM 11.8

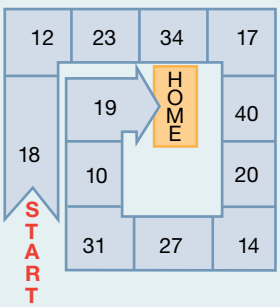
The remainder game (for 2–4 players)

Objective: Use a game to practise mental computation involving division.

Year level: 4–5

Materials

- A copy of the following game board on cardboard.
- 4 cards of each numeral 0 to 9 or a 10-sided die.
- A counter for each player.



Game rules

1. If using cards, place them shuffled and face down in a pile.
2. The first player draws a card or rolls the die.
   For a player's first turn, divide the first number on the board (which is 18) by the number drawn.
3. Move the counter forward by the number of spaces indicated by the remainder. If the remainder is 0, no move is made.
4. For the following turns, the dividend is the number on the space where his or her counter landed on the previous turn.
5. Each player proceeds in turn, starting from 18.
6. To get 'home', a player must be able to move the exact number of spaces left. The first person home wins!

Initially, children are taught to write the remainder in one of the following ways:

```
  2    with 1 left over
6 )13
 12
  1

  5    remainder 2
5 )27  (later shortened to R2 or r2)
 25
  2
```

It is important to emphasise the real-life situations from which examples arise and to decide whether a 'remainder of 2' makes sense in that situation. Activities such as the game in In the classroom 11.8 provide practice with identifying the remainder.

Calculators can be used efficiently to solve problems involving remainders, but the correct answer needs to be interpreted by the child given the situation involved. Consider this problem:

> A bus holds 36 children. If 460 children are being transported to a concert, how many buses are needed?

Research shows this problem to be difficult for children to solve with or without a calculator. An answer of 12 remainder 28 is often reported without a calculator; and 12.777 buses is often reported with a calculator. The children seem more intent on producing an answer than on deciding if their answer makes sense. A sensible answer would be 13 buses, since 12 buses would be full. The last bus would have some empty seats or each bus could include seats for adults (2 or 3 adults per bus) to accompany the children.

11.6 Finding the balance between practice and proficiency

LEARNING OUTCOME 11.6 Making decisions about estimation, checking solutions and the right balance between practice and proficiency.

Using operations to check answers

Just as it is important to estimate before solving problems, it is also important to check after a strategy has been applied. Ordinarily, addition and subtraction are used to check each other, as are multiplication and division. Unfortunately, checking does not always achieve its purpose of ascertaining correctness. In fact, when talking with children as they perform the computational check, it may be found that they frequently force the check by making the results agree. Children need to understand the purpose of

checking as well as what they need to do if the solution found by another method does not agree with the original solution.

While the calculator can serve many functions, it should not be used primarily to check paper-and-pencil computation. It insults students to ask them to spend large amounts of time on a paper-and-pencil computation and then use a machine that does the computation instantly. Encouraging the use of estimation can overcome this issue and increase number sense. Identifying the range of reasonable answers using estimation is more worthwhile for establishing correctness than relying on a calculator answer.

Choosing appropriate methods

As discussed in the previous chapter and clearly identified in the Australian Curriculum, students need to choose an appropriate method for solving mathematical problems. Sometimes an estimate is all that is needed. At other times, mental strategies will efficiently give rise to an accurate solution. In some situations, written strategies are required but for other occasions, a calculator may prove a better choice. Regardless of method used, students should be encouraged to justify their answers as this will yield valuable insights into their thinking. Children need to discuss when each method is appropriate and they need practice in making their choice. Further discussion will help them personalise and rationalise their decision. A problem that one child chooses to do with a calculator may be done with mental computation by another. The skill of choosing between different strategies depending on the numbers involved and the context of the mathematical problem is an essential real life skill, which needs to be developed by everyone.

Building proficiency

Computational fluency with addition, subtraction, multiplication and division is an important part of mathematics education in the primary years. Developing fluency requires a balance between conceptual understanding and appropriately choosing and carrying out procedures (ACARA v8.1, 2015). Extensive practice without understanding can lead to methods being forgotten or remembered incorrectly but timely practice can help students gain computational fluency. Hence, providing opportunities for practice together with the development and maintenance of conceptual understanding is essential. If students use reasoning during their practice, it is more likely to help them develop computational fluency. In their review of Australian research Groves, Mousley and Forgasz (2006, p. 209) found that researchers called for:

- a more holistic approach to the teaching of number, emphasising connections rather than compartmentalised knowledge
- more focus on children's spontaneous, informal computational strategies
- greater prominence in school curricula for mental computation at the expense of standard algorithms.

Many kinds of computer software are available to purchase for education. The challenge is to identify programs that require more than mindless practice with the algorithms. Teachers find that most programs are typically visually appealing and they use a variety of characters in a story format; however, few programs involve problem solving or estimation. Some involve the use of games to motivate the child's participation. The question to ask is whether the software offers advantages over the materials that are already available. The search for useful computer software to supplement children's learning needs to be guided by the quality of their approach to mathematics, and not by the 'bells and whistles' features.

Similarly, resource books with 'worksheets' are available to practise skills with computation; however great care is needed when choosing them. Some worksheets feature a mystery for students to solve using the answers obtained by completing the computations. In the classroom 11.9 illustrates an example of this type of worksheet. In this example, the answers to some of the multiplication problems provide clues to identifying the mystery number. The student is required to organise these clues to help identify the mystery number. While these types of worksheet provide needed practice, overuse can lead to mindless work rather than actually helping students gain computational fluency or deepen their understanding.

Overuse of worksheets can negatively affect children's attitudes towards mathematics and stifle creativity. The implications for teaching and learning are that worksheets (usually mass produced) are not created with the needs of individual students or the diversity of students in mind, nor are they able to actively involve learners.

IN THE CLASSROOM 11.9

What number am I?

Objective: Practise with 2-digit multiplication to solve a mystery number.

Year level: 4–5

What to do

There are 12 multiplication problems to solve and 12 pairings of numbers and clues.
Solve the following multiplication problems and you will find that 5 of the solutions correspond with the numbers in the number clue pairings. Use these clues to determine the 'mystery' number. The other clues are of no value at all and there is only a single correct mystery number.

1. 15 ×9	2. 51 ×21	3. 63 ×24	4. 63 ×21
5. 35 ×31	6. 18 ×15	7. 22 ×14	8. 69 ×66
9. 18 ×9	10. 72 ×9	11. 88 ×18	12. 34 ×31

	Clue
162	I am less than 100.
189	I am more than 100.
1502	I am an even number.
1071	I am an odd number.
1085	I am divisible by 3.
1308	I am divisible by 4.
280	I am divisible by 5.
647	I am a multiple of 3.
4554	I am a multiple of 5.
1607	I am a multiple of 10.
208	The sum of my digits is an even number.
1428	The sum of my digits is an odd number.

Open-ended problems can be better than worksheets as they allow for creativity and multiple solutions.

For example: Using 4 number cards 9, 8, 7 and 6, solve the following 3 types of multiplication problems and justify the similarities and differences between your answers and those of a peer.

1. □□ × □ = ____(2-digit by 1-digit multiplication)
2. □□ × □□ = ____(2-digit by 2-digit multiplication)
3. □□□ × □ = ____(3-digit by 1-digit multiplication).

(Adapted from Sullivan & Lilburn 2004, p. 45)

MAKING CONNECTIONS

There are a variety of written strategies taught in different parts of the world. People learn and use different ways of doing paper-and-pencil computation. There may be children in any given classroom who have attended school in other countries or whose parents were educated elsewhere. If children have been supported with their learning by parents or siblings, confusion might arise when different methods are demonstrated by the teacher. It helps if the teacher is aware of the existence and validity of alternative algorithms. Any method is effective if it works and makes sense to the student and if it can be used efficiently in all situations. Asking students to explain their methods can help the teacher uncover misconceptions and errors that might exist but it can also help the teacher to see and appreciate different cultural perspectives. The teacher might need to do some careful thinking to figure out how an unfamiliar algorithm works and why it is valid.

For example, the division problem 360 ÷ 8 can be taught by the distributive algorithm in long form:

$$
\begin{array}{r}
45 \\
8\overline{)360} \\
\underline{32} \\
40 \\
\underline{40} \\
0
\end{array}
$$

This same computation might be taught in Australia in short form where the number remaining from 36 tens divided by 8 is 4 tens or 40 ones, often recorded in superscript above the next digit but might be remembered without symbols:

$$
\begin{array}{r}
45 \\
8\overline{)360}
\end{array}
$$

Although this notation is used in other cultures, students arriving in schools from some countries might have been taught different methods.

Children schooled in other countries may also use different methods for subtraction. In particular, it is helpful to be familiar with the *equal-additions* or *add-tens-to-both* strategy used in many European and Latin American countries. In fact, before the decomposition algorithm for subtraction gained prominence, the *equal-additions algorithm* was also taught here. The equal-additions algorithm is still taught and used in many other countries around the world therefore students taught by parents or teachers elsewhere might prefer this approach. In the equal-additions subtraction algorithm, both the sum (the number you are subtracting from) and the known addend (the number you are subtracting) are renamed. In this example, 10 is added to each number involved in the subtraction: here, 10 ones have been added to the sum (91) and 1 ten has been added to the known addend (24):

$$
\begin{array}{rl}
{\scriptstyle 1\ \ \ } & \textit{Think} \\
91 & 11 - 4 = 7 \text{ ones} \\
{\scriptstyle 3\ \ \ } & \\
\underline{-\ 24} & 9 \text{ tens} - 3 \text{ tens} = 6 \text{ tens} \\
67 &
\end{array}
$$

This algorithm works because the difference between the numbers (91 and 24) remains the same after both have been increased by the same amount — that is, $91 - 24 = (91 + 10) - (24 + 10)$. It may help to picture a number line. The distance between 91 and 24 on the number line is what you are finding when you subtract. If you move both numbers up the number line the same amount (10), the distance between them remains the same. The reason for adding 10 to each number (but adding it in different forms) is to make the subtraction possible without any borrowing or regrouping. Although this algorithm may seem strange at first, people who have learned to subtract this way generally find it easy and natural to use. What is most important is that individuals understand *what* they are doing and *why* when they learn an algorithm. A classic research study established that both the decomposition algorithm and

the equal-additions algorithm are effective in terms of speed and accuracy when taught meaningfully (Brownell 1947). A third type of subtraction algorithm is used by the Japanese and it also involves doing some addition as part of the procedure for subtracting. To do 531 − 398 using the Japanese method, you first determine the 'nines complement' of 398, then add that number to 531 and make an adjustment to get the final answer. The 'nines complement' of any number is found by subtracting each of its digits from 9, so the 'nines complement' of 398 is 601 (because 999 − 398 = 601). Add 601 to 531 and you get 1132. Remove the 1 from the front end digit (thousands place), add 1 to the ones place to get 133 and this is the answer to the desired computation: 531 − 398 = 133. Why does this work? This question is left for you to ponder. *Hint*: At first you are actually figuring 999 − 398 = 601. What happens next in the algorithm?

Many other interesting algorithms are used throughout the world. Recognising their validity and acknowledging children's rights to use them is important in supporting the cultural backgrounds of all students.

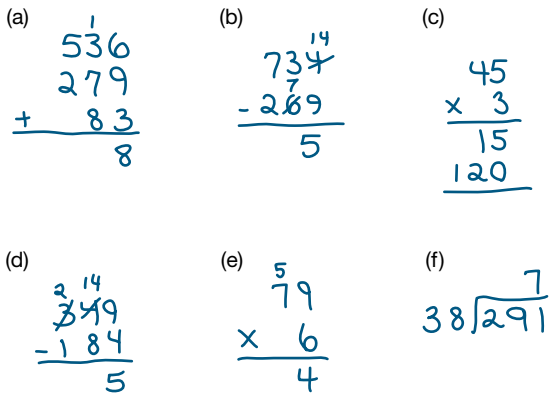
A GLANCE AT WHERE WE'VE BEEN

Although the ability to solve problems in written form is viewed as an essential component of children's mathematical achievement, its role in the curriculum and the methods of teaching continue to evolve over time. Helping children learn written strategies involves encouraging them to reason their own way through problems requiring basic computations. Important prerequisites to developing fluency include knowing basic facts, understanding place value and using mathematical resources such as concrete materials and drawings as models. This chapter has provided suggestions for developing the standard algorithms for each of the 4 operations alongside describing a number of transitional and alternative algorithms. The use of calculators has been interwoven with activities designed around the algorithms as well as activities using other materials.

THINGS TO DO: FROM WHAT YOU'VE READ

1 What prerequisites would be desirable for a student working with a peer?

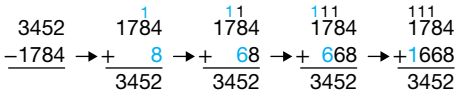
2 Consider each of the following 6 examples of the 4 algorithms. Describe what a child might think or say as he or she worked. What questions might the teacher ask to develop or explain the procedure being used? How could manipulative materials be used?



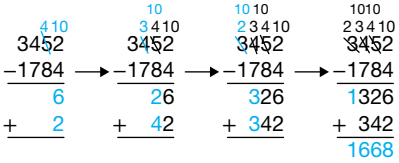
3 Explain why it is not recommended to spend long periods of time working on complex divisors with the long-division algorithm. What level of facility with division would you, as a future teacher, be satisfied with? How would you justify this position when parents question your teaching of written strategies?

4 Analyse each of the following student invented algorithms. Do they work all the time? If so, practise some more with examples using that method. If the method does not work, explain why this is so.

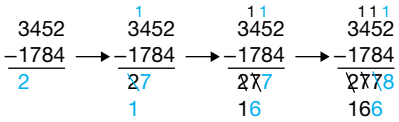
(a) Subtraction — uses missing addend addition:



(b) Subtraction — subtract from 10 when regrouping and then add on:



(c) Subtraction — work left to right, crossing out when regrouping is needed:



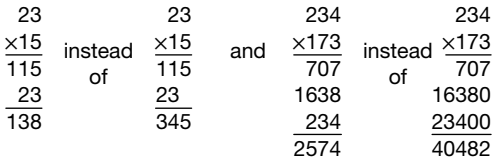
5 Sketch a diagram to show how 12×17 can be modelled with an array on grid paper. Relate the model to the partial-products algorithm for 12×17, showing how each of the partial products in the array relates to the model.

6 *Hands-on modelling of the traditional division algorithm.* Consider a problem such as $798 \div 6$. Model the dividend (798) with base-ten blocks. The divisor (6) tells how many equal-sized groups you want to make. Start by sharing the biggest blocks possible and then make trades if necessary. Next, share the next-biggest blocks. Again, make trades if necessary and continue the process until all blocks possible have been shared. Try several other problems such as $458 \div 3$, $597 \div 2$, $4\,782 \div 4$ and $7\,365 \div 5$. After doing several of these problems using just the materials, do them over again and record your findings step by step. Think about the following questions (repeatedly), after sharing cubes or flats or longs or units.

- How many did you use when sharing? (Write this number below and subtract.)
- How many remained? (Write down this result.)
- Can you share again?

7 Consider problems involving subtraction with regrouping, such as $52 - 25$, $91 - 79$ and $47 - 18$. How would you approach these problems if you were teaching Year 2 or Year 3 students? What do you think children need to understand or be able to do before they can start learning subtraction with regrouping?

8 Assume you are a Year 6 teacher. You notice that several of your students are making the same mistake when multiplying multi-digit numbers. When computing 15×23 or 173×234, they seem to forget to use 0 as a place holder for tens and hundreds on each line. They are writing:



What will you do to help these students? How will you help them correct their mistake?

9 For the problem 4 578 ÷ 7, find the quotient by using the subtractive algorithm. Then do the problem again, using at least 3 more or 3 fewer steps than you used the first time to illustrate the flexible nature of this algorithm.

10 Write 2 story problems involving 35 ÷ 6 where the answers appropriate for the real world would actually not be the exact answer to the computation. For your first story problem involving 35 ÷ 6, the appropriate answer should be 5. For your second story problem involving 35 ÷ 6, the appropriate answer should be 6.

11 Examine the games in In the classroom 11.4 (Race-to-one-hundred and Give-away-one-hundred). Answer the following questions about Race-to-one-hundred, remembering you have a pair of normal dice.

(a) Can you win a 10 with 1 roll of the dice? Can you win 2 tens with 1 roll? Or 3 tens?
(b) Can you win 1 hundred with 1 roll of the dice?
(c) What is the fewest number of rolls you would need to win 1 hundred? What is the highest number of rolls to win 100?
(d) What is the fewest number of ones a player will need to play this game?
(e) How many ones, tens and hundreds should you give each student so that he or she is sure to have enough pieces to play the game no matter what rolls they get?

Answer the following questions about Give-away-one-hundred.

(f) Is it possible to give away a 10 with 1 roll of the dice? Two tens?
(g) Can you win the game with 1 roll of the dice?
(h) What is the fewest number of rolls required to give away all of the blocks?
(i) What is the most number of rolls required to give away all of the blocks?
(j) How many ones, tens and hundreds should you give each student so that he or she is sure to have enough pieces to play the game no matter what rolls they get?

THINGS TO DO: GOING BEYOND THIS TEXT

IN THE FIELD

1 *Computation in the classroom*. Choose a teacher resource book for the year level of a classroom you can visit and analyse a lesson plan on written strategies in the teacher's guide. What stages from concrete to abstract are involved in the lesson? Talk to the teacher about how he or she teaches the 4 operations. Is the textbook approach useful? How?

2 *Addition with regrouping*. Develop 3 evaluation items that would evaluate students' understanding of adding 2 2-digit numbers with regrouping. Explain why you have chosen these examples. Try your items with several children and analyse their responses.

3 *Transitional algorithms*. Ask a classroom teacher if you can spend some time working with a student who is having trouble with written strategies (addition, subtraction, multiplication or division, depending on age and year level). See if you can figure out any misconceptions the child might have. Ask the child to explain their thinking to you so that you might better understand what the problem might be. If possible, help him or her by suggesting the use of one of the transitional algorithms described in this chapter.

4 Choose 2 or 3 Year 3 students and give them a word problem for addition involving 2-digit numbers. Encourage them to use at least 2 different methods to solve it. Evaluate their methods and determine any similarities or differences to the strategies discussed in this chapter.

IN YOUR JOURNAL

5 Suppose you are a Year 4 teacher. Following advice from the Australian Curriculum you have decided to spend less time this year than in the past on multi-digit multiplication and on long division. Instead, you will spend more time on mental computation, number sense, problem-solving situations and developing students' abilities to recognise and work with patterns and functions (early

algebraic thinking). At the upcoming open night for parents, you would like to explain to the parents how you have decided to modify your teaching approach and why. In advance you have done some research about the way mathematics is used in several occupations. You decide that getting parents involved with 3 or 4 problems as examples may be the best way to help them understand the shift in your approach. Identify 3 real-world problems to present at the open night. Write a brief narrative explaining why you chose each problem from a mathematics perspective, why it is relevant from an employment perspective and how you would use it with the parents to convince them of the reasonableness of your new approach to teaching mathematics.

6 Educators know that there is value in encouraging students to invent their own written methods even though students are unlikely to discover the standard algorithms on their own. Similarly, some cultural and historical methods that were in widespread use and are easy for students to understand are no longer preferred. The standard algorithms, which are generally quick and efficient, were refined through centuries of invention and modification. The question some people might ask is, why not just teach the standard (traditional) algorithms directly to students? Answer this question in your journal — either agreeing or disagreeing that standard algorithms should be taught directly — and provide a rationale for your position.

7 Consider the order in which the 4 operations have been introduced in this chapter. Do you think the order in which they are taught to students makes a difference? Why or why not? Analyse and write about the relationship between the written strategies for the 4 operations.

8 The view that 'with the use of calculators, students should learn to estimate, do and check when solving long-division problems' is often argued. Do you agree? Why? Reflect on your own experience of Year 4 or Year 5 mathematics and support your main points taking into consideration your own experience and current research about the topic.

WITH ADDITIONAL RESOURCES

9 Choose a teacher resource appropriate for Year 3 or 4 students. Trace the development of multiplication algorithms. How are they introduced? What steps do children go through? Find out about Russian multiplication and decide whether you like their approach. Would you consider using this method as an extension for more advanced students or as a support method for students struggling with the traditional written strategy for multiplication? Alternatively, choose a teacher resource series and look through the books to determine strategies for the development of division. At what year level do whole number division algorithms first appear? What is the highest year level where whole number division is included? Does this textbook series include opportunity for student-generated methods or short division methods only? Is the long-division algorithm included and what emphasis is given?

10 Read 'Creative arithmetic: Exploring alternative methods' (Ellis & Yeh 2008), which shows examples of some different ways to subtract and multiply. Create a 2-digit subtraction problem and a 3-digit subtraction problem, and show how you would do them using the method described in this article. Create a 2-digit multiplication problem and show how you would do it using the method described in the article. Ask a parent, grandparent, neighbour or friend if they know another way of doing subtraction or multiplication. Ask them to show you an example. Create another problem and ask them to watch you using their method. Explain how their method works to someone else and record it.

11 Read 'Computational fluency, algorithms, and mathematical proficiency: One mathematician's perspective', an article by the noted mathematician, Hyman Bass (2003). Using the 5 qualities that Dr Bass considers important in evaluating the usefulness of computational algorithms, write an analysis of the relative usefulness of 3 multiplication algorithms: the partial-products method, the lattice multiplication method and the traditional method.

12 Choose a concept or skill from this chapter and compile a brief bibliography of recent articles containing research or teaching ideas on this topic. You might choose to include website resources as well as academic articles.

WITH TECHNOLOGY

13 *Australian Curriculum Lessons*. The author Joanne Villis (Inter-tech Education) has put together a sequence of over 15 activities involving multiplication and division for Years 3 and 4 and beyond. There are teaching ideas that build upon patterns, practise basic facts, use concrete materials and games including the internet and iPads. Find it at www.australiancurriculumlessons.com.au/2014/08/23/multiplication-division-teaching-ideas-activities/

14 *Digital technology*. When solving a division problem, the calculator might provide an answer with a decimal remainder — for example, 476 ÷ 23 = 20.695 652 17. Describe 2 different ways to find the fractional remainder using a simple calculator. There are several excellent websites that students can use to further explore division with remainders. When put into the 'CalculatorSoup®' online calculator resource, the example above comes up with 20 R16 or the mixed number 20 16/23.

Step by step descriptions for division can be found at the www.calculatorsoup.com/calculators/math/longdivision.php as well as the following 2 sources.

Just ask Dr. Math for simple and more advanced division problems. (http://mathforum.org/library/drmath/view/58818.html)

Math is Fun demonstrates the process for long division with remainders. (www.mathsisfun.com/long_division2.html).

CHILDREN'S LITERATURE CONNECTIONS

Anno, M. (1999). *Anno's Magic Seeds*. New York: Philomel Books.
Jack meets a wizard who gives him two magic seeds and instructs him to eat one, which will sustain him for a full year, and to plant the other. The following spring, the plant bears two seeds. Jack eats 1 and plants the other, as he does for several years until he decides to plant both seeds. The next year he has two plants each bearing two seeds; he eats one seed and plants the other three. Six seeds! As the years go by, he marries, raises a family, plants many crops, endures a flood and saves enough seeds to feed his family and start planting again. The following link will take you to a reading of this book: https://www.youtube.com/watch?v=otru0y1XwkE

Calvert, P. (2006). *Multiplying Menace: The Revenge of Rumpelstiltskin.* (Illustrated by Wayne Geehan). Watertown, MA: Charlesbridge Publishing Co.
After 10 years of being tricked, Rumpelstiltskin returns to the royal family to wreak vengeance by multiplication (by whole numbers). Fractions are also used in the story.

Dodds, D. A. (1999). *The Great Divide*. Cambridge, MA: Candlewick Press.
Eight people begin the cross-country race using bicycles. As they race, the great divide begins. New challenges occur and each time they do, the number of racers is divided in half.

Murphy, S. J. (1996). *Too Many Kangaroo Things To Do*. New York: HarperCollins.
A kangaroo is getting ready to have a birthday and his friends are planning a surprise party. The reader will see how to add and multiply many different things that they will have at the party. There are activities and games in the back of the book for teachers and parents to use.

Murphy, S. J. (1997). *Divide and Ride*. New York: HarperCollins.
A group of 11 friends go to a carnival. At the carnival, some of the rides take 2 people to a seat and, therefore, they must divide themselves for the ride. The book uses different algorithms to show how they divide themselves for the rides as well as for tickets and other carnival activities. There are activities and games to use in the back of the book for teachers and parents to use.

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ACKNOWLEDGEMENTS

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CHAPTER 12

Fractions and decimals: meanings and operations

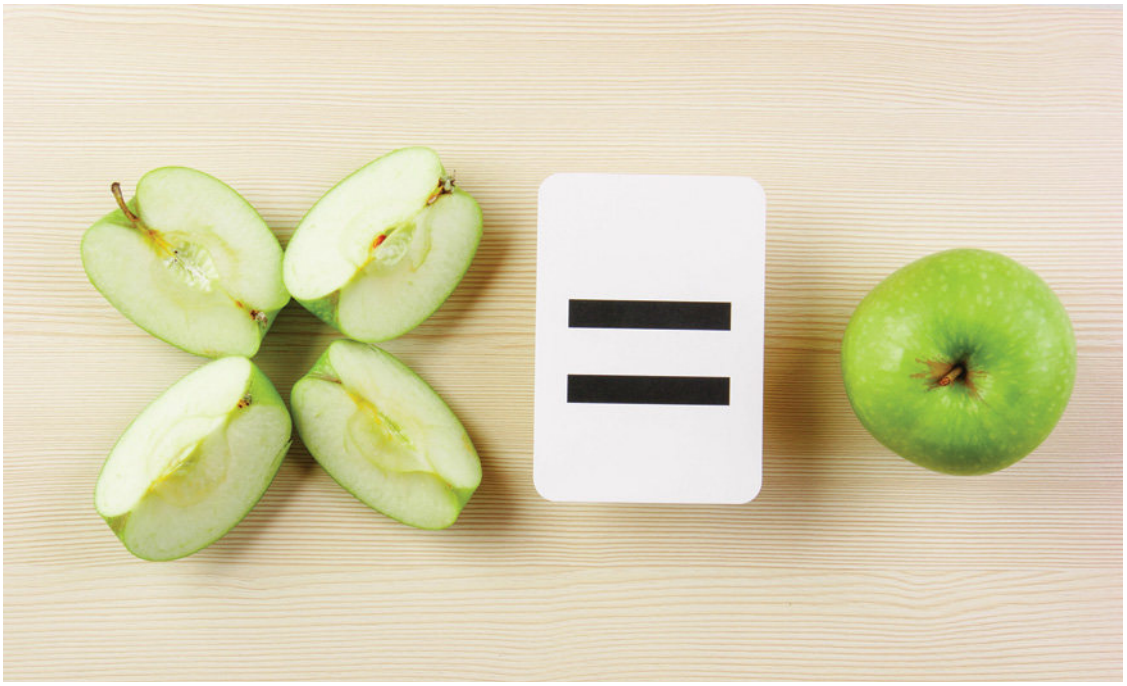
LEARNING OUTCOMES

12.1 Comparing 3 different meanings of fractions and identifying the models of the part–whole meaning.

12.2 Demonstrating ways to make sense of fractions, using concrete and pictorial models to develop understanding of ordering fractions and equivalent fractions.

12.3 Describing how estimation strategies are used for adding and subtracting by rounding to whole numbers and benchmark numbers.

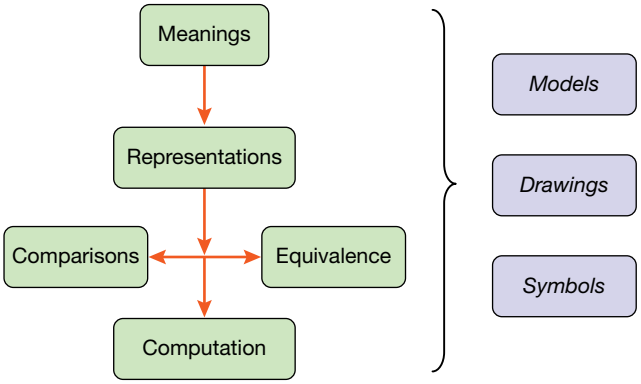
12.4 Explaining how models assist the development of conceptual understanding of adding, subtracting, multiplying and dividing fractions or decimals.



'The advantage of growing up with siblings is that you become very good at fractions.'

Robert Brault

Chapter 12 concept map



Introduction

Fractions and decimals have long been a stumbling block for many students. One reason may be that curricula tend to rush to symbolisation and operations without developing the strong conceptual understandings that children must have for fractions and decimals. Therefore, much of this chapter is devoted to helping children develop the concepts they need for working with fractions and decimals and for the mathematical processes underlying operations with these numbers.

Tables 12.1 and 12.2 are extracted from the Australian Curriculum and identify the expectations in relation to students' understandings of fractions and decimals from the Foundation year of schooling to Year 7. The recommendations clearly indicate that older children should work to develop fluency in computation of fractions and decimals but do so after they have developed relevant conceptual understandings about fractions.

TABLE 12.1 **Expectations for children's learning of fractions and decimals from the Foundation year to Year 7 based on the Australian Curriculum Number and Algebra content strand**

Year level	Content description
Year 1	**Fractions and decimals** • Recognise and describe one-half as one of two equal parts of a whole (ACMNA016).
Year 2	**Fractions and decimals** • Recognise and interpret common uses of halves, quarters and eighths of shapes and collections (ACMNA033).
Year 3	**Fractions and decimals** • Model and represent unit fractions including $\frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}$ and their multiples to a complete whole (ACMNA058). **Money and financial mathematics** • Represent money values in multiple ways and count the change required for simple transactions to the nearest five cents (ACMNA059).
Year 4	**Fractions and decimals** • Investigate equivalent fractions used in contexts (ACMNA077). • Count by quarters, halves and thirds, including with mixed numerals. Locate and represent these fractions on a number line (ACMNA078). • Recognise that the place value system can be extended to tenths and hundredths. Make connections between fractions and decimal notation (ACMNA079). **Money and financial mathematics** • Solve problems involving purchases and the calculation of change to the nearest five cents with and without digital technologies (ACMNA080).

(continued)

TABLE 12.1 *(continued)*

Year level	Content description
Year 5	**Fractions and decimals** • Compare and order common unit fractions and locate and represent them on a number line (ACMNA102). • Investigate strategies to solve problems involving addition and subtraction of fractions with the same denominator (ACMNA103). • Recognise that the place value system can be extended beyond hundredths (ACMNA104). • Compare, order and represent decimals (ACMNA105). **Patterns and algebra** • Describe, continue and create patterns with fractions, decimals and whole numbers resulting from addition and subtraction (ACMNA107).
Year 6	**Fractions and decimals** • Compare fractions with related denominators and locate and represent them on a number line (ACMNA125). • Solve problems involving addition and subtraction of fractions with the same or related denominators (ACMNA126). • Find a simple fraction of a quantity where the result is a whole number, with and without digital technologies (ACMNA127). • Add and subtract decimals, with and without digital technologies, and use estimation and rounding to check the reasonableness of answers (ACMNA128). • Multiply decimals by whole numbers and perform divisions by non-zero whole numbers where the results are terminating decimals, with and without digital technologies (ACMNA129). • Multiply and divide decimals by powers of 10 (ACMNA130). • Make connections between equivalent fractions, decimals and percentages (ACMNA131). **Patterns and algebra** • Continue and create sequences involving whole numbers, fractions and decimals. Describe the rule used to create the sequence (ACMNA133).
Year 7	**Real numbers** • Compare fractions using equivalence. Locate and represent positive and negative fractions and mixed numerals on a number line (ACMNA152). • Solve problems involving addition and subtraction of fractions, including those with unrelated denominators (ACMNA153). • Multiply and divide fractions and decimals using efficient written strategies and digital technologies (ACMNA154). • Express one quantity as a fraction of another, with and without the use of digital technologies (ACMNA155). • Round decimals to a specified number of decimal places (ACMNA156). • Connect fractions, decimals and percentages and carry out simple conversions (ACMNA157).

Source: © Australian Curriculum, Assessment and Reporting Authority 2015.

As you will see, the concepts associated with fractions and decimals are complex; however, 2 rather simple but powerful ideas — *partitioning* and *equivalence* — can help tie many of the concepts together. Partitioning refers to sharing equally (sharing a cake equally among 8 people means partitioning the cake into 8 equal portions; sharing 6 chocolate bars equally among 4 people means partitioning 6 chocolate bars into 4 equal portions). Equivalence focuses on different representations of the same amount (three-quarters of a cake is the same amount as six-eighths of that cake, so three-quarters is equivalent to six-eighths). Keep in mind that equivalence is not just a relationship among fractions but also among decimals and fractions. For example, $\frac{1}{2}$, $\frac{5}{10}$, and 0.5 are all equivalent ways of representing the same amount: one-half. Decimals are just another notation for fractions, but fractions can represent any partitioning whereas decimals can represent partitions of only tenths and powers of tenths (hundredths, thousandths and so on).

In this chapter, the initial concentration on fractions does not imply that the entire study of fractions precedes the study of decimals. Teachers can introduce decimal notation after a beginning foundation has been built with fractions. In fact, many operations with decimals are easier than the corresponding operations with fractions and can be taught meaningfully before children learn all the operations with fractions.

There is some controversy over the importance of studying fractions, since technology uses decimals almost exclusively; however, not only is an understanding of fractions required in order to undertake advanced mathematics study (Fuchs et al. 2013; Lamon 2012; Siegler & Fazio 2010) but 'it is virtually impossible to run a household or to understand magazine or newspaper articles' (Lamon 2012, p. 256) without an understanding of fractions, which provide the foundations of understandings of proportionality. Consequently, the Australian Curriculum continues to emphasise the development of sound conceptual understandings of fractions.

TABLE 12.2 **Further expectations for children's learning of fractions and decimals from the Foundation year to Year 7 based on the Australian Curriculum proficiency strands**

Year	Proficiency strand
Year 3	*Understanding:* representing unit fractions
Year 4	*Understanding:* extending place value to decimals *Fluency:* communicating sequences of simple fractions
Year 5	*Understanding:* using fractions to represent probabilities, comparing and ordering fractions and decimals and representing them in various ways *Reasoning*: continuing patterns involving fractions and decimals
Year 6	*Understanding:* using fractions and decimals to describe probabilities, representing fractions and decimals in various ways and describing connections between them, and making reasonable estimations *Fluency:* converting between fractions and decimals, using operations with fractions, decimals and percentage *Problem-solving:* formulating and solving authentic problems using fractions and decimals
Year 7	*Understanding:* recognising equivalences between fractions, decimals, percentages and ratios *Fluency:* representing fractions and decimals in various ways

Source: © Australian Curriculum, Assessment and Reporting Authority 2015.

12.1 Conceptual development of fractions

LEARNING OUTCOME 12.1 Comparing 3 different meanings of fractions and identifying the models of the part–whole meaning.

The Australian Curriculum emphasises that students should be given the opportunity to develop concepts as well as number sense with fractions and decimals. This continues to be a challenging target. In an Australian study of 323 grade 6 students barely half (50.5%) could correctly nominate which was larger when presented with the pair of fractions $\frac{2}{4}$ and $\frac{4}{2}$ (Clarke and Roche, 2009), while only 30% of students were able to give the correct answer when asked to indicate 'how much each person would get if three pizzas were shared between five people' (Clarke 2006). Many authors draw a distinction between conceptual and procedural knowledge, Siegler & Fazio (2010, p. 6) suggest that a conceptual knowledge of fractions is an understanding of the meaning of fractions, for example:

> their magnitudes and relations to physical quantities, an understanding of why arithmetic procedures with fractions are mathematically justified and why they yield the answers they do. Such conceptual knowledge can be contrasted with procedural knowledge — the ability to execute a series of steps to solve a problem.

A range of researchers report that students who have been given the opportunity to develop stronger conceptual understandings of fractions are more successful remembering and applying procedures (Siegler & Fazio 2010; Gabriel et al. 2012; Petit, Laird & Marsden 2010; Vukovic et al. 2014). Despite this the teaching of fractions is an area where there has been a tendency to resort to teaching procedures; we all remember the catchphrase 'Ours is not to reason why, just invert and multiply'! Australian researcher Dianne Siemon (2010, p. 3) argues that these 'helpful rules' not only lead to more fragile learning, but that they 'serve to reinforce the view that school mathematics is about learning and applying fairly meaningless rules and procedures'. On the other side of the globe French researchers complained that often students 'must accept many things simply on the basis that the teacher says so, and in the long run [they have] no coherent foundation for the concepts' (Brousseau, Brousseau & Warfield 2014, p. 5). While in America, Susan Lamon (2012) links a more general fear of mathematics to the teaching of fractions:

> Many people have a fear of mathematics. In high school, they were reluctant to take more than the minimum required courses. They had feelings of "being lost" or 'in the dark' when it came to mathematics. For most of these people, their relationship with mathematics started downhill early in elementary school, right after they were introduced to fractions. They may have been able to pass courses — perhaps even get good grades — beyond the third and fourth grades by memorizing much of what they were expected to know, but they can remember the anxiety of not understanding what was going on in their mathematics classes.

Helping students develop the necessary conceptual understandings is better done with a curriculum that emphasises many representations — physical, pictorial, verbal, real world and symbolic — than with more traditional approaches that depend only on rote procedures (Lamon 2012; Siegler & Fazio 2010; Spangler 2011; Way 2011). Interacting with a range of representations and models with different attributes, presents different challenges to students, 'causing students to continuously rethink and ultimately generalize the concepts' (Petit, Laird & Marsden 2010, p. 3). This approach will take longer as the concepts are developed, but thereafter less time will be spent reviewing and re-teaching (Spangler 2011) and students will build proficiency more quickly (Lamon 2012).

The Australian Curriculum (see tables 12.1 and 12.2) recommends that during the early and middle years of primary education the focus should be on developing the meaning of fractions — as equal parts of a whole or equal parts of a set — and the investigation of equivalence of fractions, and the comparison and ordering of fractions using a range of models including number lines. In the upper years of primary school, the focus shifts to solving problems with fractions.

Each of these ideas is discussed in this section; however, if you are teaching in the upper primary you will find that many students are still struggling with understanding fractions and consequently trying to apply rules by rote to solve problems. These students will benefit from the opportunity to undertake activities, such as those described here, designed to develop their understanding of the meaning of fractions.

Many materials, such as fraction bars and pattern blocks, are available to assist you in helping children develop concepts about fractions (see the Appendix C for paper versions of these commercial materials). Some of the most effective materials, such as construction paper and counters, are commonly found in the classroom.

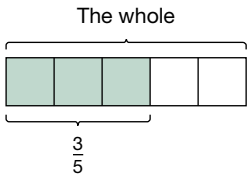
Mathematics literature can also be very useful in helping children develop concepts for fractions. For example, *My Half Day* (Fisher & Sneed 2008) develops the language of fractions. *The Doorbell Rang* (Hutchins 1986) is excellent for introducing the division meaning of fractions.

Three meanings of fractions

There are 3 distinct meanings of fractions — part–whole, quotient and ratio — that are found in most primary mathematics programs, but the focus is usually on the part–whole meaning, with little development of the other 2 meanings. Ignoring these other meanings may be one source of students' difficulty with fractions.

Part–whole

The part–whole meaning of a fraction indicates that a whole has been partitioned into equal parts — for example, the fraction $\frac{3}{5}$ indicates that a whole has been partitioned into 5 equal parts and that 3 of those parts are being considered. This fraction may be shown with a region model:



Quotient

The fraction $\frac{3}{5}$ may also be considered as a quotient, 3 divided by 5. This meaning also indicates partitioning. For example, suppose you had 20 big biscuits to share equally among 5 people. If you gave each person 1 biscuit, then another, and so on until you had distributed all the biscuits, you could represent this process mathematically by the quotient $20 \div 5$. Each person would get 4 biscuits.

However, what if you had just 3 big biscuits and you still wanted to share them equally among 5 people? Now the quotient would be $3 \div 5$. How much would each person get? Would anyone get a whole biscuit? One way to solve this problem would be to use pictures of the biscuits, as shown in figure 12.1. This interpretation of fractions is used when a remainder in a division problem is expressed as a fraction. It is also the interpretation that is needed to change any fraction to a decimal (i.e. 58 is equivalent to 0.625).

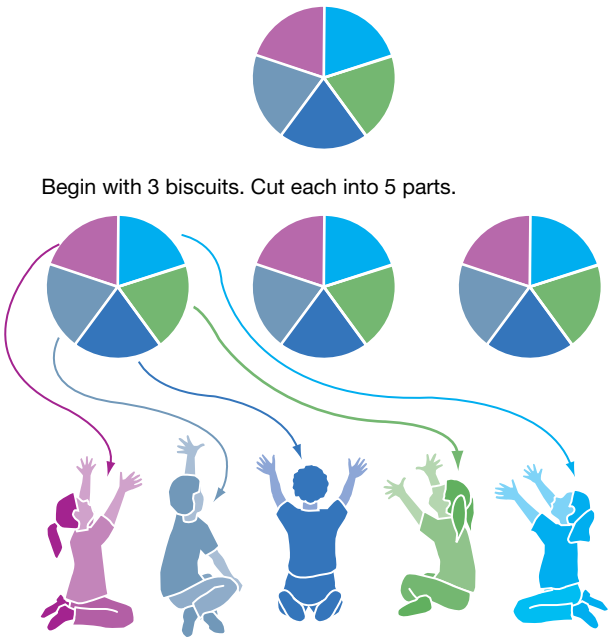


FIGURE 12.1 The quotient interpretation of a fraction

Ratio

The fraction notation $\frac{3}{5}$ may also represent a ratio — for example, in a class of 6 boys and 10 girls, the ratio of boys to girls is 6 to 10, which is equivalent to $\frac{3}{5}$. That is, for every 3 boys there are 5 girls. What else could be the make-up of a class with this ratio?



The ratio interpretation of fractions is conceptually different from the part–whole and quotient interpretations, since it doesn't involve the idea of partitioning. In this chapter, we focus on the part–whole interpretation and bring in the quotient interpretation as relevant (see the next chapter for discussion of the ratio interpretation). You may read about the measurement meaning (Flores, Samson & Yanik 2007), but we have not made a special section on this meaning since it is closely aligned with the measurable attributes discussed here.

SPOTLIGHT ON EARLY CHILDHOOD

Part–whole experiences

Young children are naturally curious about their world and have an interest in how the things in their world fit together or can be taken apart, which leads to an understanding of parts and wholes. They also develop an interest in how things can be shared between people. Ideas such as these underpin the later development of fraction concepts. There will be many opportunities to explore these ideas and develop the appropriate language with young children, both in their daily life and in pre-primary settings.

There are 3 types of part–whole experience that young children will encounter.

1. *Naming parts.* They learn that many of the objects in everyday life consist of different parts: toys, sets of pencils, packs of cards, cars, plants, animals and even their own bodies. Parts can be named and in some circumstances taken apart and put back together (e.g. a pack of cards), although in other situations the parts cannot be taken apart and put back together (e.g. an animal's or a child's own body).
2. *Parts of wholes*. They learn that whole things can be divided into parts: fruit is cut up, a loaf of bread is sliced, juice is poured into several glasses, a piece of paper is cut into pieces and the playdough is divided into several smaller chunks. The size of these pieces can differ. Activities such as these provide opportunities to introduce and practise the language of part–whole: divided, shared, same/different, equal/unequal, more or less and, where appropriate, fraction names (e.g. half, quarter, third, fifth).
3. *Parts of sets*. Children learn that groups of things can be divided: biscuits are shared at morning tea, the pencils are distributed fairly and cards are dealt for a game of snap. Again, there are opportunities for comparing amounts and using the relevant language — divided, shared, equal/unequal, more or less and fraction names.

Initially young children who are given 1 biscuit when another person receives 2 biscuits will be quick to say that this is an unfair distribution. However, if their biscuit is broken into 2 parts, they will be satisfied that it is now a fair distribution. When asked why this is a fair distribution they will say that they each have '2 biscuits'. According to Piaget & Inhelder (1969), young children who think like this are still in the preoperational stage and have not learnt to *conserve quantity.* As a result of further cognitive development these children will learn to conserve quantity and be able to understand that the amount of biscuit they have has not changed simply because their biscuit is broken into 2 pieces and that this is still an unfair distribution.

Models of the part–whole meaning

Attributes such as region, length, set and area are useful in modelling the part–whole meaning of a fraction. The region is the most commonly used model for this and the most easily handled by children.

You can also model the quotient interpretation using any of these attributes, but here again the region model is most often used because it is the simplest. Other attributes, such as capacity, volume and time, also can model the meaning of fractions.

Region

In the region model, which is a special version of the area model, the region is the whole (the unit), and the parts are congruent (same size and shape). The region may be any shape, such as a circle, rectangle, square or triangle. You should use a variety of shapes when presenting the region model so that the children do not think that a fraction is always a particular shape — for instance, 'part of a pie' (if the region is a circle).

A rectangle is the easiest region for children to draw, and as figure 12.2 indicates, it is also the easiest region to partition. (Try partitioning each of the shapes shown into three equal parts to see which is easiest.) The circle has the advantage of being easy to see as a whole, but this does not outweigh the advantages of the rectangle, which we use extensively throughout this chapter.

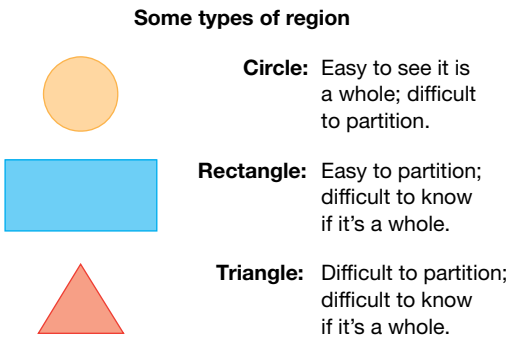


FIGURE 12.2 Types of region for concrete models

Length

Any unit of length can be partitioned into fractional parts of equal length. Children can begin by folding (partitioning) a long, thin strip of paper into halves, quarters and so on. Later, you can use length partitioning to represent fractions as points on a number line. For example, as shown in figure 12.3, you could partition lengths into thirds to help children realise that the number $4\frac{1}{3}$ is $4 + \frac{1}{3}$.

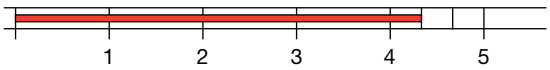


FIGURE 12.3 Length model of $4\frac{1}{3}$

Keep in mind, however, that young children need plenty of prior experience with the number line before they can understand it as a model for fractions. For example, fewer than one-third of Year 4 students could identify the point on a number line that represented a given fraction, even when the number line was partitioned into the same number of parts as the fraction indicated (Kloosterman et al. 2004). Because of the many subtleties involved in using a number line, we will return to it in the discussion of making sense of fractions.

Set

The set model uses a set of objects as a whole (the unit). This model sometimes causes children difficulty, partly because they have not often considered a set of, say, 12 objects as a unit. Consequently,

it is important to explore with children 'what is the unit?' in each new problem, making explicit that we cannot determine the fraction unless we first know what the unit is (Lamon 2012). Another, more obvious reason why children have difficulty with the set model is that they have not had much experience with physically partitioning the objects in a set into thirds, quarters, fifths and so on.

We will revisit the set model in the discussion of making sense of fractions because of its misuse in most curricula.

Without mentioning fractions, you should give children experiences with physically partitioning sets. This will give them a background for both division and fractions. For example, you could ask a student to share 12 toys equally among 4 children. Later, you should focus on whether a given number of objects can be partitioned equally among a given number of people. For example, you can ask whether 15 toys can be partitioned (shared) equally among:

5 people? [*Yes*]
4 people? [*No*]
3 people? [*Yes*]
2 people? [*No*]

Once children develop the understanding to deal with these questions, you can use the set model for fractions (e.g. finding fifths by partitioning the set into 5 equal parts). Figure 12.4 shows a set of 15 marbles that has been partitioned into 5 equal parts. Each part is one-fifth of the whole set. From this modelling, children develop the set meaning of fractions and can answer questions such as these:

What is one-fifth of 15? [*3*]
Two-fifths of 15? [*6*]
Three-fifths of 15? [*9*]

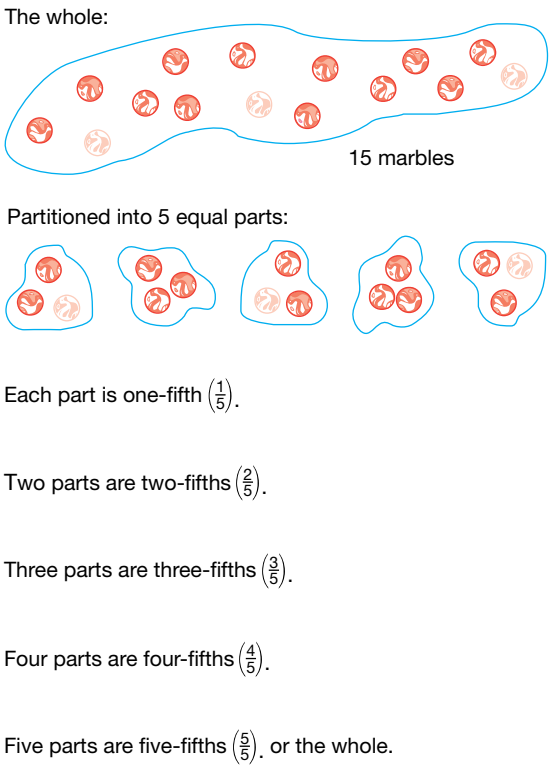


FIGURE 12.4 Working with a set model

Experience of this type allows children to solve many practical problems and prepares them for multiplication of fractions. It also provides background experience for multiplicative reasoning as students determine the number of marbles, for example, in 4 groups of 3 marbles. This is an appropriate activity to include in Year 3 as children are building their concepts both of fractions and of multiplication and division.

Area

The area model is a more general version of the region model, in which the parts must be equal in area but not necessarily congruent. Before using this model, you must be sure that your students have some idea of when 2 different shapes have equal areas. This makes the model more appropriate for older children (Years 4 and 5) than for younger ones.

In the classroom 12.1 shows examples and non-examples of three-quarters. It provides a way to assess children's understanding of the different models — region, length, set and area. Use it first as a self-assessment and then try it with primary students. Which do you expect to be the easiest examples for students?

IN THE CLASSROOM 12.1

Three-quarters

Objective: Assess children's understanding of three-quarters.

Year level: 3–5

- In which of the following models does the shading show three-quarters of the whole? Why or why not?

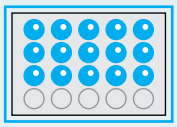
A Whole: large rectangle



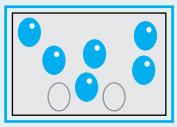
F Whole: large rectangle



B Whole: box of marbles



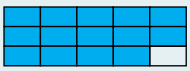
G Whole: box of lollies



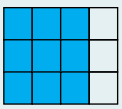
C Whole: large rectangle



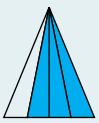
H Whole: large rectangle



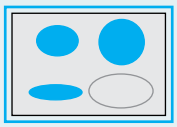
D Whole: large rectangle



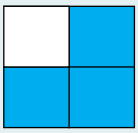
I Whole: large triangle



E Whole: box of shapes



J Whole: large rectangle



Making sense of fractions

Begin to build children's conceptual understanding of fractions by starting with the simplest meaning and the simplest model: the part–whole meaning and the region model. After you introduce this model and the language and symbols associated with fractions, you can show the children other models of the part–whole (length, set and area). Students can develop much of this conceptual background in early childhood, but you should be ready to revisit these ideas with students who do not have this background.

Partitioning

The important idea in partitioning is the idea of making equal shares by separating a whole into equal parts. The whole is whatever is specified as the unit. At first, you should use a whole that will be obvious to children (e.g. a rectangle). If the whole is a region, then the parts must be equal in size and the same shape; if it is a length, then the parts must be equal lengths; if it is a set, then the parts must have the same number of objects in each; and if it is an area, then the parts must be equal in area. Whatever the model, children must learn to partition the whole into equal parts and to describe those parts with fractional names.

Let the children do the partitioning. For example, each child could be given a 'chocolate bar' (a piece of paper the size of a large chocolate bar) to share with a friend. Have them fold the 'chocolate bar' to show how they would share it. Talk about whether a fold like the one sketched here would be a 'fair share' for 2 people.



Move on to sharing equally among 3, 4, 6 and 8 people. For some of the more difficult sharing problems, such as with 5 people, you may want to use a strip of paper that has already been marked. When children become familiar with the process of sharing by partitioning regions into equal parts, describing the parts with words and counting the parts, it is time to move to the other models. Children especially like to share collections of objects (set model) — for example, sharing 12 marbles between 2 friends is appealing to children because they can share fairly, with each friend getting half the marbles.

Words

As children develop the idea of equal parts, introduce the words *halves, thirds, quarters* and so on. Many of the words will be familiar to students since they are usually the plural of the ordinal words: third, fifth, sixth and so on. This connection can help, but you need to make sure that students hear the *s* and that they connect the word with the number of parts of the whole. Help them make that connection by asking such questions as 'How many fifths make a whole? How many eighths? How many twenty-fourths?'

Counting

Once children are familiar with the words for fractional parts, begin counting the parts. This process should not be any more difficult than counting apples, but children need to know what fractional parts they are counting. In the classroom 12.2 has a few ideas for getting children to practise counting fractional parts.

Counting the number of parts can lead to questions about ordering and equivalent fractions — which is more — 3 quarters or 5 eighths? How many halves are equal to 4 eighths? Counting fractional parts can also help children see how much more is needed to make a whole. For example, if children have counted to 3 fifths with a model, they can see that 2 more fifths would make the whole. Counting fractional parts can lead to developing improper fractions and mixed numbers. For example, if 3 children are counting all the quarters they have, the class can hear the third child counting '9 quarters, 10 quarters . . . ' and can see that the 3 children have 2 wholes and 1 quarter, 2 wholes and 2 quarters and so on. When they hear '12 quarters', they will see the 3 wholes. This also is a natural introduction to adding and subtracting fractions with like denominators.

IN THE CLASSROOM 12.2

Fraction bars

Objective: Develop an understanding of equal sharing and counting fractional parts using fraction bars.

Year level: 3–5

Materials: 4 construction paper strips (7 cm × 22 cm) of 4 different colours for each child

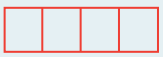
Preparation of fraction bars: Each child should fold a green strip into 'halves', mark the fold with a dark line and write halves on the back.

Make fraction bars for thirds, quarters and sixths in other colours in the same way.



Activities

- Ask each child to count the parts on a quarters bar: [*One-quarter, two-quarters*]



1, 2, 3, 4 quarters

- Ask each to count the parts of another bar.
- Ask a pair of children to count the quarters in 2 bars:



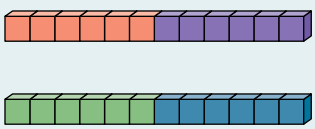
1, 2, 3, 4 5, 6, 7, 8 quarters

- Make sure the class sees that eight-quarters is 2 wholes, six-quarters is 1 whole and two-quarters and so on.
- Ask a group of 4 students to count all the sixths in their group's sixths bars.
- Challenge all the children to state how many strips it would take to show 11 sixths or 23 sixths. Let them experiment in groups of 4.

After developing the initial ideas of partitioning, the words for fractional parts and counting parts, you can introduce other fractional models. For example, In the classroom 12.3 uses a length model and connects it to the set model while still using only the fractional words without symbols.

IN THE CLASSROOM 12.3

Sharing rods



Objective: Investigate sharing rods equally and naming the fractional parts and the amount in each equal part.

Year level: 2–3

Materials: Unifix cubes or other connecting cubes.

Launch: Have each child make a Unifix rod of 12 cubes and name it the 12-rod. Tell the children that they are going to investigate making halves, thirds, quarters and sixths from the 12-rod.

Questions

1. Have the children 'break' their 12-rod into 2 rods of equal length. (Make 2 shares.) Ask the children: 'What is each share (fractional part) called?' [*Halves*] 'How many halves make the 12-rod?' [*2*] 'How many cubes in each half?' [*6*] 'What is one-half of a 12?' [*6*]
2. Have the children make 3 equal shares from their 12-rod. Ask: 'What fractional part is 1 share?' [thirds] 'How many thirds make the 12-rod?' [*3*] 'How many cubes in each third?' [*4*] 'What is one-third of 12?' [*4*]
3. Have the children make 6 equal shares from their 12-rod. Ask: 'What fractional part is 1 share?' [*Sixths*] 'How many sixths make the 12-rod?' [*6*] 'How many cubes in each sixth?' [*2*] 'What is one-sixth of 12?' [*2*]
4. Have the children make 4 equal shares from their 12-rod. Ask: 'What fractional part is 1 share?' [*Quarters*] 'How many quarters make the 12-rod?' [*4*] 'How many cubes in each quarter?' [*3*] 'What is one-quarter of 12?' [*3*]

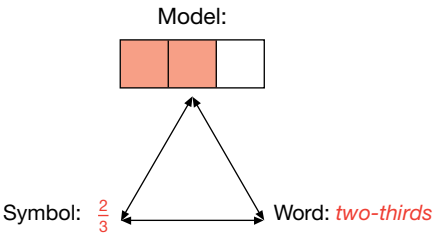
Extend

1. Ask the children how many cubes in two-thirds and three-thirds of their 12-rod. Continue with other fractional parts, such as two-quarters and five-sixths.
2. Use other rods, such as a 10-rod (halves and fifths) or a 16-rod (halves, quarters, eighths).

Symbols

Symbols and written words should be used together or alternately, and linked to concrete materials and visual representations, until students understand the meaning of the symbols.

Connections can be depicted as follows:



When children can match the words with the model, it is time to tell them that $\frac{2}{3}$ is the symbol for *two-thirds*. Then they need many opportunities to make all 3 kinds of connections between the different representations:

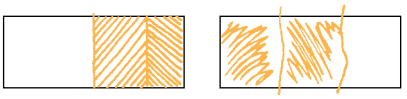
Connection 1. Model and words
Connection 2. Model and symbol
Connection 3. Words and symbol

Textbooks usually concentrate on part of connection 2 (going from a model to a symbol), so you will find plenty of examples of these. However, they often do not go from a symbol to having the students produce the model. Connection 3 requires oral work, so you need to give children opportunities to solve problems and describe the solutions. When you read fractions, be sure to use the fraction words (e.g. *three-fifths*). Children who hear only '3 over 5' often think of 2 numbers rather than a fractional part.

Drawing a model

Children who have modelled fractions by folding paper or by choosing a picture should also be able to draw a picture of fractional parts. The rectangle is probably the easiest shape to use to show a good approximation of a fractional part. Encourage the children to be as accurate as possible, and do not

worry too much about other aspects of their drawings. For example, which of these 2 drawings would you prefer as a picture of two-thirds?



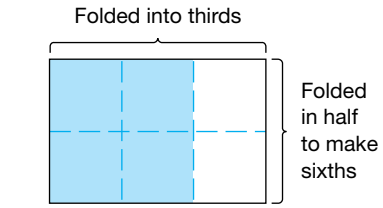
Steve's work Marilyn's work

Steve's work is neater than Marilyn's, but he seems to have missed the point that the 3 parts must be equal. You might help Marilyn be a little neater, but she does seem to have the idea of two-thirds.

Extending the model

Table 12.3 shows a question on the Second Mathematics Assessment of NAEP and how children responded to it (Carpenter et al. 1981). Little has changed in the over 30 years since this item was given (Petit, Laird & Marsden 2010). There is no doubt that this model is more complicated than those described in the previous section, since the number of parts (12) is greater than the denominator (3) of the simplest fraction corresponding to it, $\frac{1}{3}$. However, this model is useful for introducing equivalent fractions and for teaching children how to order fractions.

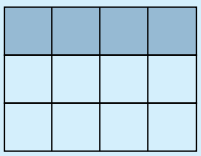
You can also use paper folding to introduce the model in table 12.3 as well as to introduce equivalent fractions. Give each child a sheet of plain paper to fold into thirds and to shade two-thirds. Then fold the paper in half the other way. Ask how many parts and what kind of parts [*6, sixths*]. Then ask what part is shaded, and make it clear that both $\frac{2}{3}$ and $\frac{4}{6}$ are correct answers. Tell the children that $\frac{2}{3}$ and $\frac{4}{6}$ are called *equivalent fractions* because they represent the same amount.



Shaded to show $\frac{2}{3}$ is the same as $\frac{4}{6}$

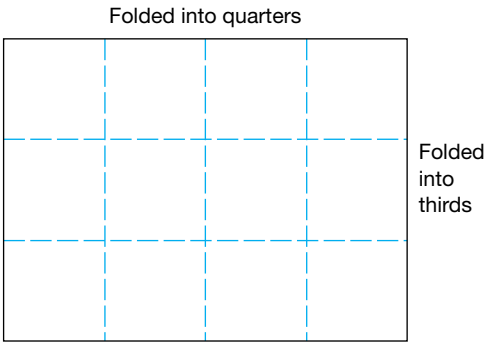
TABLE 12.3 **Results of national assessment question using the region model**

What fractional part of the figure is shaded?



	Per cent responding	
Responses	**Age 9**	**Age 13**
Acceptable responses $\frac{1}{3}$, $\frac{4}{12}$, 0.33	20	82
Unacceptable responses $\frac{1}{4}$, 0.25	5	4
Top 4 or $\frac{1}{4}$	36	6
Other	15	6
I don't know	17	1
No response	7	1

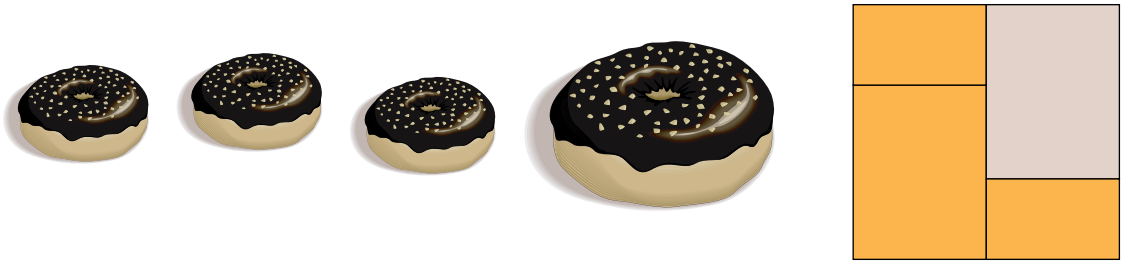
Move to rectangle models such as the one below. Make certain the students can identify the way the paper was folded in both directions. Have them show thirds and quarters. Then have them identify $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{1}{3}$ and $\frac{2}{3}$, as well as $\frac{4}{12}$ and $\frac{8}{12}$. Some children will need to colour in the parts on different models to focus their attention on the specific part.



After doing some more examples and getting more practice with folding paper, children should be ready for drawing. Reinforce the connection between folding and drawing by having the children make a step-by-step record of the folding process on a picture of a rectangle.

A closer look at the set model

Look at the 2 examples below:

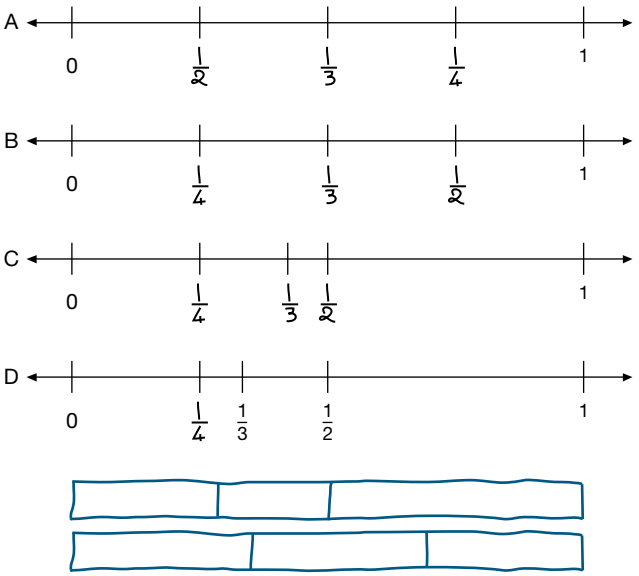


The child that responds that $\frac{3}{4}$ of the doughnuts are small is correct but the child that says $\frac{3}{4}$ of the rectangle is yellow is incorrect. In the doughnut example the child has learned that '3 out of the 4' doughnuts are small; thus, the fraction is $\frac{3}{4}$. Isn't it logical that the child could also reason that '3 out of the 4' pieces are yellow?

We unwittingly encourage this thinking by overusing *out of*. The *out of* encourages the above actions as well as thinking of the fraction only as 2 whole numbers. It certainly does not help children understand the set model and how partitioning is related to it. It also does not help students focus on what attribute is being considered. In fact, the child who says $\frac{3}{4}$ of the rectangle is yellow is correct if the attribute is not a region but is a set of 4 pieces. Jennifer Way (2011, p. 155) identifies this problem and argues that 'care needs to be taken when "reading" fractions'. She concludes that it would be 'better' to say three-quarters than *3 over 4* or *3 out of 4*. If we could make one change in curricula and assessment, it would be to throw out all the early references to *over* or *out of*.

A closer look at the number line

Many children use whole-number ideas when working with fractions or they think of a fraction as 2 numbers, the numerator and the denominator. When they do not know what to do, they revert to thinking of only whole numbers. A common error when children were asked to place unit fractions on a number line is shown in the example below.



Child A has relied on his or her knowledge of whole numbers and put the denominators in order without regard for the fractions. Child B has the correct order but does not have them placed reasonably on the number line. Child C realises the placement of one-half as being halfway between 0 and 1. It is not clear whether he or she knows that one-quarter is halfway between 0 and one-half. It is evident that he or she knows that $\frac{1}{3}$ is more than $\frac{1}{4}$ and less than $\frac{1}{2}$ but does not realise that $\frac{1}{3}$ is nearer to $\frac{1}{4}$. Child D relied on a length model to place the fractions. It is very difficult to place fractions on a number line that is not subdivided. Children can get more accurate with the drawings, but one has to question how much accuracy is needed.

It is also clear what Child D is thinking. Look at the drawing D. The bottom left rectangle represents one-third. The child has successfully transferred this model to the length (distance) model, realising the end mark is one-third of the distance from 0 to 1.

We have only shown a number line between 0 and 1, but students also need to see the number line including negative integers and numbers greater than 1. Often when children see a number line from 0 to 2 and are asked to mark $\frac{1}{2}$, they put the mark at 1. They are answering another question — the halfway point between 0 and 2, not where is $\frac{1}{2}$ located. They need to realise that the unit on a number line is 1, or the distance from 0 to 1. When negative numbers are used, students often locate fractions to the left of 0 and reason (incorrectly), 'A fraction is little so it must be less than 0.'

For more information about children's understanding of the number line and ways to support students' learning, read the chapter by Saxe et al. (2007). It clearly shows that it takes skilful teaching and time for students to learn to use this model of fractions.

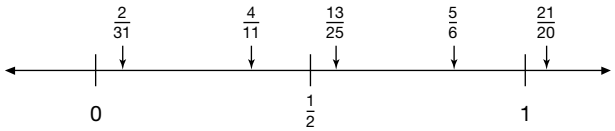
Benchmarks

Initially, children need to develop facility with friendly fractions such as halves, thirds and quarters so that they can relate these to fractions that look more intimidating, such as sevenths, elevenths and twenty-fifths. If you help children build benchmarks, they will be able to reason with these more difficult fractions. For example, when you consider the following fractions, you can easily see that they are all near $\frac{1}{2}$:

$$\frac{6}{11} \quad \frac{11}{23} \quad \frac{19}{35} \quad \frac{44}{90} \quad \frac{354}{700} \quad \frac{800}{1605} \quad \frac{100\,001}{200\,000}$$

Moreover, you can quickly say which are more than $\frac{1}{2}$ and which are less than $\frac{1}{2}$. Children can build this same intuitive feel for fractions near the benchmark $\frac{1}{2}$ by beginning with smaller numerators and denominators.

Students also need to be able to tell if a fraction is near 0 or near 1. This will let them use the benchmarks 0, $\frac{1}{2}$ and 1 to put in order a set of fractions such as $\frac{13}{25}$, $\frac{2}{31}$, $\frac{5}{6}$, $\frac{4}{11}$ and $\frac{21}{20}$, which would be a time-consuming task if they tried to find a common denominator. Using benchmarks as shown on the following number line makes the task rather easy: $\frac{13}{25}$ is a little more than $\frac{1}{2}$, $\frac{4}{11}$ is a little less than $\frac{1}{2}$, $\frac{2}{31}$ is near zero, $\frac{5}{6}$ is a little less than 1 and $\frac{21}{20}$ is a little more than 1. Thus the correct order is $\frac{2}{31}$, $\frac{4}{11}$, $\frac{13}{25}$, $\frac{5}{6}$, $\frac{31}{20}$.



Children can also use benchmarks to check the reasonableness of computations with fractions and to estimate answers. For example, if they know that $\frac{3}{5}$ and $\frac{6}{7}$ are each more than $\frac{1}{2}$, then they know that $\frac{3}{5} + \frac{7}{6}$ is not $\frac{9}{12}$ — the sum must be more than 1.

Going from a part to a whole

Most students can recognise a simple representation of a fractional part of a whole. However, asking students to figure out a whole when they are shown a fractional part will often reveal that children's understanding of fractions is limited. The following exchange from a Year 5 classroom illustrates the kind of question you should ask (Mrs May is the teacher):

Mrs May: If I told you that this picture represents three-fifths of a cake, could you draw a picture of the whole cake?



$\frac{3}{5}$ of a cake

Rose: It sure would be a skinny cake.

Camille: That depends on where the two-fifths was cut off.

Mrs May: You are both right; you won't be able to tell exactly what the shape of the cake was before it was cut. Let's pretend you can see that it was cut here. [*pointing to the right side*] What do you know?

Gerard: You have 3 of the 5 equal pieces. If we divide this into 3 equal parts, we know how large one-fifth is.



Three parts of the cake

Olav: Now all you have to do is add the missing two-fifths.

Rose, it *is* a skinny cake!



Whole cake

Mrs May continued the classroom discussion by giving the children other tasks — for example: Given $\frac{2}{7}$ of a stick of licorice (to represent a length) or $\frac{3}{5}$ (12 pieces) of a box of chocolates (set model), find the whole. Children should work independently on several problems and then share their solutions. These types of task should also include an improper fraction (e.g. $\frac{8}{5}$) and a mixed number (e.g.$1\frac{2}{3}$), so you can see which students have made sense of fractions greater than 1.

The importance of understanding equivalence

As we have discussed, making sense of fractions begins with partitioning and continues with the gradual introduction of words, symbols and more complicated models and meanings. Before moving to

operations with fractions, you must make sure that your students have a firm understanding of equivalence. Children who have worked only with whole numbers are used to describing the number of objects with only one number — for example, if they have counted 8 objects, they will say 'eight' or write '8' to describe those objects. Fractional parts, however, can be described in many equivalent ways, depending on the denominator — for example, one-half of a pie can be described with $\frac{1}{2}$ or $\frac{4}{8}$ (depending on the problem being solved, one name may be more appropriate than another).

Ordering fractions and equivalent fractions

Part of understanding fractions is realising that they are numbers and therefore can be ordered, added, subtracted, multiplied and divided. The goal is to have children order fractions and find equivalent fractions symbolically, but you can help children connect the concrete to the symbolic. Not only will this connection help children realise what they are doing when they are ordering and finding equivalent fractions, but it also will give them another context in which to practise relating fractions and the models. Many problems involving ordering capture children's interest because they want to know which is more, which is shorter, which is larger and so on. Problems about equivalent fractions are central to understanding fractions and being able to operate (add, subtract, multiply and divide) with them.

Concrete models

Children can make concrete models and then use them to order fractions or to find equivalent fractions. For example, fraction bars (see In the classroom 12.2) are concrete models that children can make and then use to find out which is larger, $\frac{2}{3}$ or $\frac{3}{4}$. They can fold the thirds bar so that it is $\frac{2}{3}$ long and the quarters bar so that it is $\frac{3}{4}$ long and then compare the models to see which is longer.

Fraction bars can also be used as concrete models to find equivalent fractions. For example, by comparing a thirds bar and a sixths bar, children can readily see that $\frac{2}{3}$ is the same as $\frac{4}{6}$. For simplicity, our examples have focused on the rectangle and on paper models. Keep in mind, however, that children need to work with multiple representations. Pattern blocks (see Appendix C) are another model accessible to young children. In the classroom 12.4 shows how to plan a lesson using pattern blocks for ordering fractions and finding equivalent fractions.

IN THE CLASSROOM 12.4

Comparing models

Objective: Use concrete models to represent fractional parts of a whole.

Year level: 3–5

Materials: Pattern blocks

The 4 basic shapes of the pattern blocks that you will use are the hexagon, triangle, rhombus and trapezium. Each child or pair of children will need multiple copies of each shape.



Let children have a chance to explore by making designs and seeing how they could make the hexagon with the other pieces.

Idea 1

Use the hexagon as the whole. What is the size of each of the other pieces? Use the pieces to show that:

$\frac{2}{6} = \frac{1}{3}$ $\frac{3}{3} = 1$ $\frac{3}{6} = \frac{1}{2}$ $\frac{2}{3} > \frac{1}{2}$ $2\frac{2}{3} = \frac{8}{3}$

Idea 2

Use the figure made from 2 hexagons as the whole.

Use  to cover the yellow shape. How many does it take? What fractional part of the whole is the rhombus? What fractional part of the whole is the trapezium?

Use the shape to show that $\frac{3}{4}$ is greater than $\frac{2}{3}$.

Idea 3

Give the shape of a fractional part and have children construct the whole.

If  is $\frac{2}{3}$, what is the whole?

If  is $\frac{3}{4}$, make the whole.

If  is $\frac{1}{5}$, make the whole.

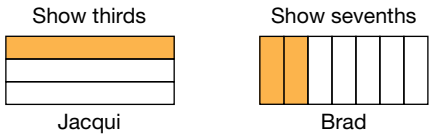
If  is $\frac{2}{5}$, draw the whole.

Pictorial models

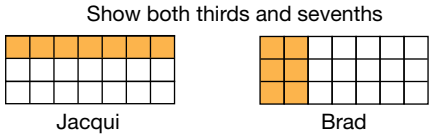
Children can also use pictorial models to order fractions. For example, the accurate scale of the fraction bars in Appendix C lets children compare lengths to decide which is larger, $\frac{7}{9}$ or $\frac{8}{10}$. These bars can also be used to identify equivalent fractions — for instance, children can see that $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$, $\frac{4}{8}$ and $\frac{6}{12}$ are all the same length and therefore represent the same amount.

You should also make sure that children are able to draw their own pictorial models to represent fractions so that they can solve problems for which no ready-made models are available. For example, suppose you give students this problem: 'Jacqui ate $\frac{1}{3}$ of the cake. Brad ate $\frac{2}{7}$ of the cake. Who ate more?' You could have students approach this problem as follows:

1. Have the children draw 2 equal-sized pictures of the cake, partitioning one into thirds horizontally and the other into sevenths vertically and then shading $\frac{1}{3}$ of one and $\frac{2}{7}$ of the other, as shown below:



2. Have the children recognise that it is difficult to tell which shaded part is larger.
3. Have the children draw 2 more equal-sized pictures of the cake, but this time partitioning both into thirds horizontally and into sevenths vertically, as shown below:



4. Now the children can see that Jacqui ate 7 pieces and Brad ate 6 pieces, so Jacqui ate more (i.e. $\frac{1}{3}$ is more than $\frac{2}{7}$).

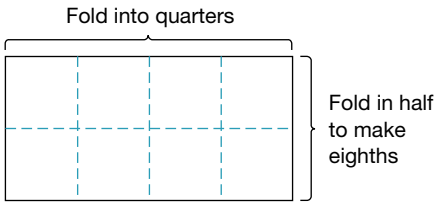
Your students may find other ways to solve the problem, but this is one approach you could have in your repertoire. Also, you can use the picture of the cake partitioned into thirds horizontally and into sevenths vertically to help children understand a symbolic way of comparing fractions as well as to help them understand a way of adding and subtracting fractions.

Symbolic representation

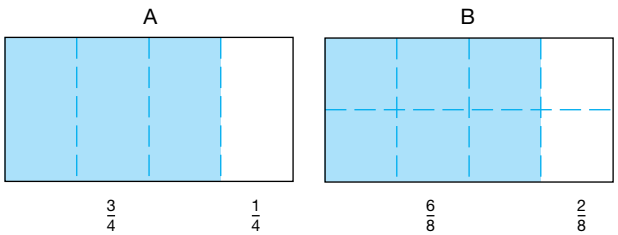
It is easier to compare 2 measurements given in the same unit (e.g. 78 metres and 20 metres) than 2 measurements given in different units (e.g. 83 metres and 231 feet). Similarly, it is easier to compare 2 fractions that are symbolically represented by the same partitioning (e.g. $\frac{3}{5}$ and $\frac{2}{5}$; the denominators are the same) than 2 fractions that are represented by different partitioning (e.g. $\frac{3}{7}$ and $\frac{2}{5}$; the denominators are different). In mathematics, we often try to deal with difficult cases by changing them into simpler ones. With fractions, making the difficult case with different denominators simpler means expressing each fraction as an equivalent fraction so that both fractions represent the same partitioning — that is, so they have a common denominator.

Now we can see how to use the cake partitioning to compare $\frac{1}{3}$ and $\frac{2}{7}$ by expressing them as equivalent fractions with a common denominator. The rectangle that has been partitioned into both thirds and sevenths is partitioned into 21 parts — that is, into twenty-firsts. Children can easily recognise that $\frac{1}{3}$ is 7 of the 21 parts, or $\frac{7}{21}$, and that $\frac{2}{7}$ is 6 of the 21 parts, or $\frac{6}{21}$. Since $\frac{7}{21}$ is greater than $\frac{6}{21}$, $\frac{1}{3}$ is greater than $\frac{2}{7}$.

Students who have considered various models showing equivalent fractional parts such $\frac{2}{3}$ and $\frac{4}{6}$ will be familiar with the concept of equivalent fractions, but they may not have developed many of the skills associated with finding equivalent fractions symbolically. Finding an equivalent fraction rests on the generalisation that both the numerator and denominator of a fraction may be multiplied (or divided) by the same number without changing the value of the fraction. One way to develop this generalisation is to start with a paper-folding model and symbolically describe what is happening.



1. Make a model of $\frac{3}{4}$ by folding a piece of paper in quarters and shading three of the quarters. Write down how many quarters there are (4) and how many are shaded (3). Then fold the paper in half the other way to make eighths.
2. Have the children look at the quarters. Ask what happened to the quarters when the paper was folded in half. You created twice as many equal parts ($2 \times 4 = 8$).



3. Notice that there are also twice as many shaded parts in B ($2 \times 3 = 6$).
4. Express the model symbolically:

$$\frac{2 \times 3}{2 \times 4} = \frac{6}{8}$$

After investigating more examples like this one, children should be able to make the generalisation that multiplying the numerator and the denominator by the same number results in an equivalent fraction.

Later when they have learned about multiplying fractions, they will realise they were multiplying the original fraction by 1. Thus, they have used a version of the multiplicative identity property ($a \times 1 = a$).

Note that you could also begin with the model of 8 parts and describe how to get to 4 parts by dividing the 8 parts by 2 (8 ÷ 2) and dividing the 6 shaded parts by 2 (6 ÷ 2). In this case, you would express the model symbolically as:

$$\frac{6}{8} = \frac{6 \div 2}{8 \div 2} = \frac{3}{4}$$

Examples of this type should lead children to make the generalisation that dividing the numerator and denominator by the same number also results in an equivalent fraction. The activity described in In the classroom 12.5 is an initial step in helping children decide whether or not a fraction can be written as an equivalent fraction with a specified denominator. For example, can quarters be expressed as eighths, as tenths or as twelfths?

IN THE CLASSROOM 12.5

Can you divide it?

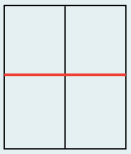
Objective: Use an area model to learn how to compare fractions.

Year level: 4–5

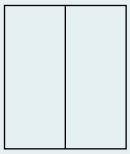
Here are a few ideas to start your discussion. After children understand these, have them investigate other fractional parts.

- All of these regions have been divided into parts with vertical (up and down) lines. Use horizontal (across) lines to see if you can change the model to the fraction given. The first one has been done.

- To quarters:



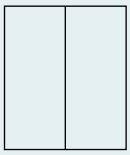
- To tenths:



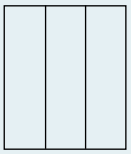
- To sixths:



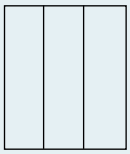
- To eighths:



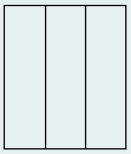
- To twelfths:



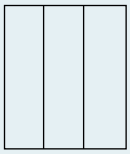
- To tenths:



- To ninths:



- To sixths:



- Draw a region divided into fifths by vertical lines. Can you draw horizontal lines to change it to other fractional parts? Which ones? Write a conjecture and explain why you think it is true.

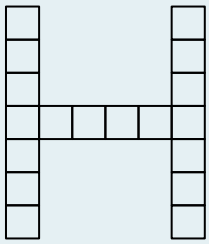
Children who can order fractions symbolically may enjoy the quick and easy game from In the classroom 12.6. If they have difficulty playing, let them use a concrete or pictorial representation of the fractions. If the game is too easy, use a gameboard with 48 squares and fractions less than $\frac{1}{3}$ that may be expressed as equivalent fractions with a denominator of 48 (e.g. $\frac{1}{4} = \frac{12}{18}$ or $\frac{3}{16} = \frac{9}{48}$) .

IN THE CLASSROOM 12.6

Whole hog

Objective: Compare fractions.

Year level: 3–5





Materials: A copy of the H grid, squares of paper, markers

- Cut 10 squares of paper. Write one of these fractions on each square:

$$\frac{1}{2}\ \ \frac{1}{3}\ \ \frac{1}{6}\ \ \frac{2}{6}\ \ \frac{1}{9}\ \ \frac{2}{9}\ \ \frac{3}{9}\ \ \frac{1}{18}\ \ \frac{2}{18}\ \ \frac{3}{18}$$

Game rules

1. Shuffle the fraction cards and put them in a pile face down.
2. Each player picks a card from the top of the pile and turns it face up.
3. The player with the smaller fraction colours in that fractional part of her or his H.
4. Put both cards at the bottom of the pile.
5. If the player with the smaller fraction cannot colour the fractional part shown on the card (because not enough of the H is left uncoloured), both players put their cards at the bottom of the pile and pick again.
6. Continue playing until one player colours the whole H. That person is the first to go 'the whole hog' and loses the game.

- Make a new H and play again.

Comparing fractions using various reasoning strategies

Although we have looked at various models (concrete, pictorial and symbolic) for comparing fractions, children with a firm foundation are able to use a range of strategies to compare fractions. For example, children could be asked 'How would you tell which is the larger fraction in each of the following pairs?'

A. $\frac{3}{7}$ and $\frac{5}{7}$ C. $\frac{4}{5}$ and $\frac{6}{7}$ E. $\frac{9}{11}$ and $\frac{2}{13}$

B. $\frac{1}{5}$ and $\frac{1}{7}$ D. $\frac{3}{8}$ and $\frac{3}{5}$ F. $\frac{18}{25}$ and $\frac{9}{13}$

Students in primary school reported using a range of reasoning strategies (these are reasoning strategies from students answering other pairs in this way — the names and pairs have been changed).

Ali: That is easy — you know that 5 of something is more than 3 of that same thing — the thing is sevenths.

Rob: I thought of a pie. Fifths means that 5 people share and sevenths mean that 7 people share — so those pieces are smaller. So the fifths are bigger and so one-fifth is larger than one-seventh.

Carl: It is like the last one except someone ate a piece out of each pie. The person that ate the one-fifth ate more — so more is left in the sevenths pie. Six-sevenths is more. (Some students still needed to see a sketch to see what Carl was saying.)

Debbie: The numerators are the same. So 3 of the smaller pieces (eighths) is less than 3 of the larger pieces (fifths).

Emma: Nine-elevenths is way over one-half and two-thirteenths is way under. So it's nine-elevenths.

Faye: I used the 'get the same numerator' — so I had 18 of each — twenty-fifths and twenty-sixths. Twenty-fifths is larger, so 18 of them would be more.

Each of these strategies is linked to the proficiency strands of Understanding and Reasoning in the Australian Curriculum.

Finding fractions between two given fractions

Between any 2 whole numbers, there may or may not be another whole number. For example, between 3 and 8 there are 4 whole numbers (4, 5, 6 and 7), but between 5 and 6 there is no whole number. Between any 2 fractions, there is an infinite number of fractions. This is called the *density property* of fractions.

You should begin working with students so they see that between any 2 fractions, there is another fraction. Eventually, they will see that there are many fractions between any 2 given fractions. You may want to connect this to the number line and 2 given points, say A and B. Students can see that there are many points between these 2 given points.



It is not quite as clear to students how to find the fraction names of these points. Suppose A is $\frac{3}{8}$ and B is $\frac{6}{8}$. Ask students what fractions are between these 2 fractions. It is rather clear that $\frac{4}{8}$ and $\frac{5}{8}$ are between $\frac{3}{8}$ and $\frac{4}{8}$. See if they can find another fraction between $\frac{4}{8}$ and $\frac{5}{8}$. Continuing this line of questions and showing the associated points on the number line will help students build the idea that there are many fractions between any 2 fractions.

Students who have a firm conceptual understanding of fractions will find many different strategies for finding fractions between 2 given fractions depending on the pair. Have students try to find a fraction between the following pairs. See if the students use different strategies such as concrete or pictorial models, decimal equivalents or equivalent fractions.

Pair A: $\frac{3}{5}$ and $\frac{4}{5}$

Pair B: $\frac{1}{4}$ and $\frac{1}{2}$

Pair C: $\frac{3}{7}$ and $\frac{4}{7}$

Pair D: $\frac{3}{8}$ and $\frac{3}{7}$

Clearly, there are points between A and B. Suppose A is $\frac{3}{8}$ and B is $\frac{6}{8}$. What fraction do you know is between? What other fractions are between? (You do not have to stick with eighths.)

With your classmates, see what strategies you can use to find 2 fractions between each of the following pairs: $\frac{3}{5}$ and $\frac{4}{5}$, $\frac{1}{4}$ and $\frac{1}{2}$, $\frac{3}{7}$ and $\frac{3}{8}$. Did you use concrete or pictorial models? Did you use decimal equivalents? Did you change each of the 2 given fractions to equivalent fractions?

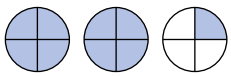
Mixed numbers and improper fractions

Through models, you can lead children naturally into mixed numbers and improper fractions, even as they are learning the initial concepts of fractions. We began the ideas when we were counting fractional parts — one-quarter, two-quarters, three-quarters, four-quarters, five-quarters (oh! we have a whole and

another quarter). This leads naturally to examining the equivalence between proper fractions (a fraction greater than 1) and a mixed number (a whole number and fraction). A mixed number is a natural symbolic representation of the following model:



You can add partitions in the model to show all the quarters, so children can see that the initial counting is 9 quarters, or the improper fraction $\frac{9}{4}$.



This helps them understand that $2\frac{1}{4}$ and $\frac{9}{4}$ are equivalent representations. To help children gain experience with mixed numbers and improper fractions, use the models as much as possible and ask them to write *both* types of number to represent the models. After children understand this process, they need practice in changing from one form to the other without the use of models. Do not rush to teach them a routine, however, but encourage them to think problems through. In the following example, see if you can follow Claire's thinking as she considers how to change $8\frac{2}{3}$ to an improper fraction. Note that her thinking depends on understanding that a whole can be equivalently expressed in terms of fractional parts (e.g. 5 fifths and 6 sixths each make a whole).



Claire: I know that 1 is 3 thirds. So I need to figure how many thirds in 8 wholes. That is 8 groups of 3 thirds, which is 24 thirds. Now I have to add on the 2 thirds, so it is 26 thirds, or $\frac{26}{3}$.

Children also must be able to think through the opposite process — changing an improper fraction to a mixed number — for example, changing $\frac{17}{6}$ to $2\frac{5}{6}$. In this case, their thinking could go like this: 6 sixths make a whole, so 17 would make 2 wholes and there would be 5 sixths more. So $\frac{17}{6}$ is $2\frac{5}{6}$. If children understand the quotient meaning of fractions, then students can also think of $\frac{17}{6}$ as $17 \div 6$, which they can calculate as $2\frac{5}{6}$.

12.2 Operations with fractions

LEARNING OUTCOME 12.2 Demonstrating ways to make sense of fractions, using concrete and pictorial models to develop understanding of ordering fractions and equivalent fractions.

The key to helping children understand operations with fractions is to make sure they understand fractions, especially the idea of equivalent fractions. When children are solving problems that involve operations with fractions, be sure the children can use what they know to make sense of the operations. Whenever possible, they should be able to extend to fractions what they know about operations with whole numbers. Keep in mind, however, that some operations with whole numbers do not extend directly to fractions. For example, multiplication of 2 fractions is not repeated addition as it is with whole numbers. There are other differences, too — for instance, when you multiply 2 positive whole numbers, the product is always larger than either factor; but when you multiply 2 proper fractions, the product is always less than either factor. Children often have the idea that if you divide, you make something smaller. However, if you divide a whole number by a proper fraction, the quotient will be larger than the whole number.

The Australian Curriculum recommends that students begin to solve problems involving addition and subtraction of fractions with the same denominator in Year 5 (ACMNA103) (ACARA 2015). This foundation is built on during the following 2 years so that by Year 7, students are expected to be able to do the following.

- Solve problems involving addition and subtraction of fractions, including those with unrelated denominators (ACMNA153).
- Multiply and divide fractions and decimals using efficient written strategies and digital technologies (ACMNA154) (ACARA 2015).

Children gain a better understanding of operations with fractions if they learn to estimate answers by using whole numbers and benchmarks such as $\frac{1}{2}$. For example, before computing the answer to $3\frac{2}{3} + 4\frac{5}{6}$, they should estimate the answer by considering, first, that the sum must be more than 7 (3 + 4) and, second, that $\frac{2}{3}$ and $\frac{5}{6}$ are each more than $\frac{1}{2}$, so the answer must be more than 8. Developing this type of number and operation sense makes it easier to figure out whether answers to problems are reasonable. Having the conceptual understanding to check answers for reasonableness is also important because many calculators can perform operations with fractions.

Your goal is to help children make sense of adding and subtracting fractions. One of the most famous examples from NAEP (Carpenter et al. 1981, p. 36) was a multiple-choice item given to students at ages 13 and 17:

Estimate the answer to $\frac{12}{13} + \frac{7}{8}$.

You will not have to time to solve the problem using paper and pencil.

The choices for answers were 1, 2, 19, 21 and 'I don't know.' At age 13, a greater percentage of students chose 19 (28%) and 21 (27%) than the correct answer of 2 (24%). Students actually did better when asked to add fractions with paper and pencil, which shows they had little understanding of reasonable answers.

In a more recent study that built on this example (Vermont Mathematics Partnership 2006), students in primary school were asked the following question:

> The sum of $\frac{1}{12}$ and $\frac{7}{8}$ is closest to:
> (a) 20
> (b) 8
> (c) $\frac{1}{2}$
> (d) 1
> Use words, pictures or diagrams to explain your answer.

Many students in the upper primary years could add the 2 fractions, but a large number said the sum was closest to 20 and many could not explain their reasoning. Would you say these students were developing an understanding of adding fractions?

Addition and subtraction

To help with building understanding, we are suggesting that instead of beginning addition and subtraction of fractions with a symbolic sentence such as $\frac{2}{3} + \frac{1}{4}$, begin with problems that involve joining and separating. Such problems, together with pictorial models, can help children:

- see that adding and subtracting fractions can solve problems similar to problems involving whole numbers
- develop an idea of a reasonable answer
- see why a common denominator is necessary when adding or subtracting fractions with unlike denominators.

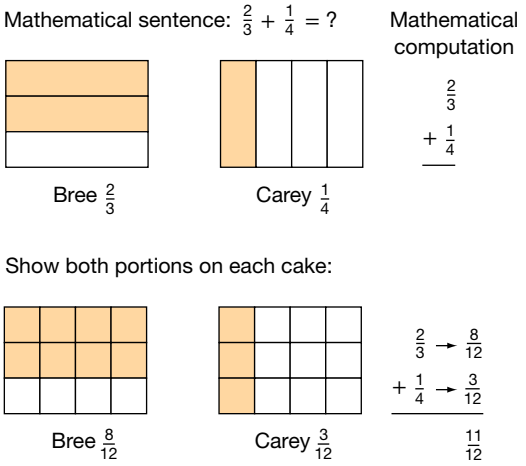
Use what children know about solving whole-number addition and subtraction word problems to begin examining how to add and subtract fractions. If children understand how to count fractional parts (e.g. 3 sevenths and 2 more sevenths is 5 sevenths) and how to compare 2 fractions using a pictorial model, they may find their own ways to solve joining and separating problems with fractions.

'Look at the word problems in figure 12.5. Even very young children have no difficulty with A, and those with a strong conceptual background in fractions should be able to solve B without being taught a rule. Have them show how they solved each one, encouraging them to tell how the 2 problems are alike (e.g. both problems involve adding 2 and 1; in problem A, you have to add Bree's 2 apples to Carey's 1 apple, and in problem B, you have to add Bree's 2 fifths of the pie to Carey's 1 fifth). Also, have children draw a picture to illustrate the first 2 problems, even if they find it quite simple to do so. Then let them try problem C. Discuss how problem C is like the other 2 problems and how it is different. If the children need help, encourage them to draw a rectangle to represent the cake.

A
Bree ate 2 apples and Carey ate 1 apple. How many apples did they eat altogether?
B
Bree ate $\frac{2}{5}$ of a pie and Carey ate $\frac{1}{5}$ of a pie. How much pie did they eat altogether?
C
Bree ate $\frac{2}{3}$ of a cake and Carey ate $\frac{1}{4}$ of the cake. How much of the cake did they eat altogether?

FIGURE 12.5 Three joining problems

Children may come up with various ways to approach the problem. If they have drawn pictures to compare fractions, they may draw pictures like these:



Note how the picture above could also be used to see the result if Carey had eaten $\frac{3}{4}$ of a cake and Bree had still eaten $\frac{2}{3}$. The picture shaded differently could show that Carey had eaten $\frac{9}{12}$ of a cake. Together, therefore, they would have eaten $\frac{17}{12}$ or one whole cake and $\frac{5}{12}$ of another.

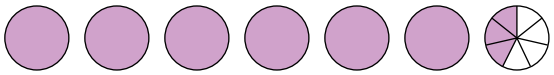
This highlights the point that, when adding fractions, the whole (the unit) is assumed to be the same. That is, in this example, where Bree ate $\frac{2}{3}$ of a cake and Carey ate $\frac{3}{4}$ of a cake, not only would there have to be two cakes, but the cakes are assumed to be the same size. If Bree ate $\frac{2}{3}$ of a very small cake and Carey ate $\frac{3}{4}$ of a large cake, you would need to know the relative sizes of the cakes before you could determine how much they ate altogether or who ate more.

Adding mixed numbers is no more difficult than adding proper fractions. Children who have made sense of adding proper fractions can put this together with what they know about adding whole numbers to handle a computation such as $5\frac{2}{3} + 3\frac{3}{4}$. Adding the whole numbers gives 8 and adding the fractions gives $\frac{17}{12}$, as we saw above. Putting these together gives $8\frac{17}{12}$, and children can then use their knowledge that $\frac{17}{12}$ is $1\frac{5}{12}$ to express the total as $9\frac{5}{12}$.

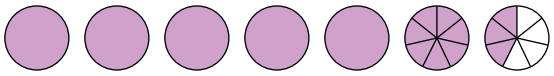
You can approach subtracting two proper fractions in the same way that you approached adding. Begin with subtraction problems that involve comparison. For example, how much more cake did Bree eat than Carey in problem C (figure 12.5)? You could use the same model to find the answer. Note that knowing how to order fractions is important in real-life situations. In problem C, it would not make sense to ask how much more cake Carey ate than Bree. Subtracting improper fractions or mixed numbers is often more difficult, partly because children have not worked on changing a mixed number to another mixed number and partly because they lack understanding of regrouping. Using a model such as the fraction bars in the online appendix can help children think about how fraction equivalents are useful for subtracting when regrouping is necessary. For example, the symbolic subtraction problem $6\frac{3}{7} - 2\frac{4}{7}$ requires regrouping $6\frac{3}{7}$ to express it as $5\frac{10}{7}$. You can help children understand this regrouping by getting them to see that 1 is $\frac{7}{7}$ so 6 is $5+\frac{7}{7}$.

Thus, $6\frac{3}{7}$ is $5 + \frac{7}{7} + \frac{3}{7}$, or $5\frac{10}{7}$.

If children are having difficulty with regrouping fractions, a model like the following can help them see, for example, that $6\frac{3}{7} = 5\frac{10}{7}$.



$6\frac{3}{7}$ is 6 wholes plus $\frac{3}{7}$



$6\frac{3}{7}$ is 5 wholes plus $\frac{7}{7}$ plus $\frac{3}{7}$ or $5\frac{10}{7}$

Students often find subtraction more difficult than addition because subtraction often involves more steps. As in addition, students must express the fractions as equivalent fractions with a common denominator. However, in subtraction, they may also have to regroup if the fractional part being subtracted is larger than the other fractional part. The previous example ($6\frac{3}{7}$ - $2\frac{4}{7}$) requires children to regroup $6\frac{3}{7}$ to $5\frac{10}{7}$ since they cannot subtract $\frac{4}{7}$ from $\frac{3}{7}$.

Students who approach subtraction problems by trying to follow a rule have a great deal of difficulty with problems like $2 - 1\frac{3}{5}$. In contrast, students who have a mental picture of this subtraction and who see that $\frac{2}{5} + \frac{3}{5}$ makes a whole will be able to think through this problem rather than compute mindlessly. Children following a rule lack the flexibility of procedural fluency, an essential part of being mathematically proficient. Children who have a firm understanding of fractions will find ways to add and subtract fractions without having to rely solely on the traditional algorithms.

Multiplication

The algorithm for multiplication of fractions is one of the simplest: multiplying the numerators gives the numerator of the product, and multiplying the denominators gives the denominator of the product. Students can learn this algorithm in minutes (and forget it in seconds unless they practise it a great deal); however, learning this algorithm does not give students insight into why it works or when to use it. That is why we suggest teaching multiplication of fractions in a way that helps students understand the meaning of multiplication, helps them get an idea of the size of an answer and helps them see why the algorithm works before they are presented with the algorithm itself.

In the following sections, we examine 3 different cases of multiplication: a whole number times a fraction, a fraction times a whole number and a fraction times a fraction. In each case, we tie multiplication with fractions to a meaning of multiplication with whole numbers — for example, using the knowledge that 3×4 means 3 groups of 4 and that 3×4 is the area of a 3-by-4 rectangle. It is not the purpose here to teach algorithms, as it will be in early secondary school, but to build understanding of the operations.

Whole number times a fraction

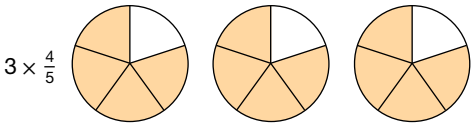
Begin with a problem like this one:

> You have 3 pans, each with $\frac{4}{5}$ of a pizza. How much pizza do you have?

Ask your students how this problem is like having 3 bags, each with 4 marbles. See if this helps them understand the similarity between multiplying whole numbers and multiplying a whole number times a fraction. How do we find 3 groups of 4? We could put out 3 groups of 4 marbles and count them. Similarly, to find 3 groups of $\frac{4}{5}$, we could put out 3 pans of pizza, each with $\frac{4}{5}$ of a pizza, and count the number of fifths. This approach is shown by a model like the following:



3 groups of 4 is 12



3 groups of 4 fifths is 12 fifths

If we symbolise 3 groups of 4 as 3×4, it makes sense to symbolise 3 groups of $\frac{4}{5}$ as $3 \times \frac{4}{5}$.

Then we can consider 3×4 as repeated addition: $4 + 4 + 4 = 12$. We can approach $3 \times \frac{4}{5}$ in the same way:

$$\frac{4}{5} + \frac{4}{5} + \frac{4}{5} = \frac{12}{5}$$

After students have solved problems like this one with pictures, see if they can solve them without pictures, maybe with repeated addition. Be sure to place a strong emphasis on the words. For example, in discussing the problem $5 \times \frac{2}{3}$, have them listen carefully as you read:

> $5 \times \frac{2}{3}$ is 5 groups of two-thirds, which is 5
>
> groups of 2 *thirds* or 10 *thirds* — which is $\frac{10}{3}$.

Fraction times a whole number

Again, begin with a problem — for example:

> You have $\frac{3}{4}$ of a case of 24 bottles. How many bottles do you have?

If children have worked with the set model for fractions, they have the background to solve this problem using physical objects. Now they must move to solving it symbolically and tying it to multiplication.

First, help the children look at why it makes sense to consider this problem a multiplication problem. Ask them: 'If you had 5 cases with 24 bottles in each, what would you do to find out how many bottles you have?' Get them to see that they would multiply 5×24. Similarly, for 20 cases they would multiply 20×24, and for 53 cases they would multiply 53×24. Thus, for $\frac{3}{4}$ of a case it is logical that they should multiply $\frac{3}{4} \times 24$.

Then have children describe how they would find $\frac{3}{4}$ of 24. Lead them to see that they would first partition (or divide) the set into 4 equal parts (each part would have 6); in other words, first find $\frac{1}{4}$ of 24, which is 6, and then look at 3 of the parts — that is, $\frac{3}{4}$ is 3 times as many as $\frac{1}{4}$, or $3 \times 6 = 18$. Thus $\frac{3}{4} \times 24 = 18$. This thinking is summarised below:

$\frac{3}{4} \times 24 = \square$ Think
one-quarter of 24 is 6 so
three-quarters is 3 times as many,
or 3×6, and
$3 \times 6 = 18$

At first glance, this procedure looks different from the denominators. First you divide the number in the set (24) by 4 and then multiply that by 3. When you perform the standard algorithm, you multiply first (3×24) and then divide by 4.

Another way to approach this type of problem (fraction times a whole number) is to use the idea of commutativity. That is, point out to the children that $\frac{3}{4} \times 24$ is the same as $24 \times \frac{3}{4}$, so they can multiply using the procedure discussed above (whole number times a fraction). However, children do not readily understand that $\frac{3}{4}$ of a case of 24 is the same as having 24 groups of $\frac{3}{4}$. Although this approach may be easier, the opportunity to present the *of* meaning of multiplication is lost.

Fraction times a fraction

Consider this problem:

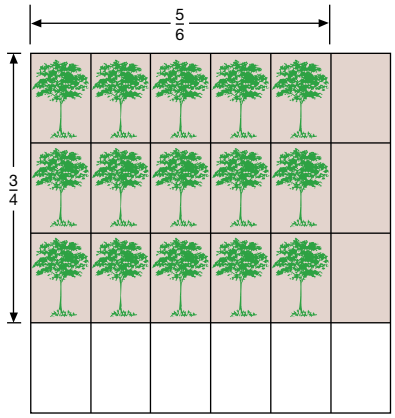
> You own $\frac{3}{4}$ of an acre of land, and $\frac{5}{6}$ of this is planted in trees. What part of the acre is planted in trees?

Children need to understand why this is a multiplication problem. Showing them pictures like the following ones may help them see how this problem is related to the area of a rectangle and thus to multiplication. In the first picture, the acre is partitioned into quarters and the amount you own ($\frac{3}{4}$) is shaded.



$\frac{3}{4}$ of an acre (the amount you own)

In the next picture, the acre is partitioned into sixths, with little drawings of trees marking the $\frac{5}{6}$ of your land that is planted in trees. Help the children see that they need to identify the size of each of the small rectangles in order to find what part of the whole acre is planted in trees. Since there are 24 small rectangles, each is $\frac{1}{24}$ of an acre. And since 15 of the small rectangles are planted in trees, $\frac{15}{24}$ of the acre is planted in trees.



$\frac{5}{6}$ of what you own

Pictures like these can help the children see that the tree-planted rectangle is $\frac{3}{4}$ by $\frac{5}{6}$, so its area is found by multiplying $\frac{3}{4}$ by $\frac{5}{6}$.

You need to develop this model slowly and only after children have had experience with finding areas of rectangles. After children use a model for problems like this, make a list of fraction times fraction products (do not simplify the products). See if the children notice a pattern for multiplying 2 fractions.

This model can be used with children who have had experience with finding area to help them understand the algorithm for multiplying fractions (multiply the numerators, multiply the denominators). You can refer to the diagrams to show why the procedure works. The acre is partitioned into quarters one way and sixths the other way, so there are $4 \times 6 = 24$ equal parts (24 is the denominator). Trees were planted in 3 rows of 5 of these parts, so the numerator is $3 \times 5 = 15$. Symbolically, write:

$$\frac{3}{4} \times \frac{5}{6} = \frac{3 \times 5}{4 \times 6} = \frac{15}{24}$$

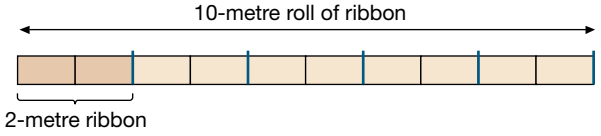
At this point, return to earlier examples and have the children verify that this algorithm works for those examples also. Make sure the children understand that any whole number can be represented by a fraction (e.g. $6 = \frac{6}{1}$); this will help them see that they can use the algorithm of multiplying numerators and denominators when one of the factors is a whole number.

Division

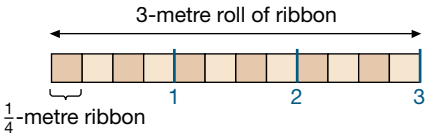
Consider this problem:

> How many 2-metre ribbons can be made from a 10-metre roll of ribbon?

Students should understand that they are being asked to find how many 2-metre pieces (the size of the group) are in the whole, or how many 2s are in 10. A picture like this would show them how they might do this in real life:

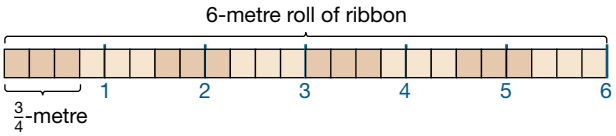


Students can readily see that the 10-metre roll contains 5 2-metre ribbons. This is symbolised by $10 \div 2 = 5$. Now you could modify this picture by marking off $\frac{1}{2}$-metre sections instead of 1-metre sections to show students how to find the number of $\frac{1}{2}$-metre ribbons they could make from this same 10-metre roll (i.e. what is $10 \div \frac{1}{2}$?). The picture would look similar to the one Susie drew when she was finding the number of $\frac{1}{4}$-metre ribbons she could make from a 3-metre roll:



In this problem, we want to know how many one-quarters are in 3, or $3 \div \frac{1}{4}$. Students can see that there are twelve $\frac{1}{4}$-metre ribbons in a 3-metre roll. You should also have students notice that the picture shows that there are 3 groups of 4 (3×4) hair ribbons.

Now ask students how they would draw a picture to find out how many $\frac{3}{4}$-metre pieces there are in a 6-metre roll of ribbon? They should see that, in this case, they would mark off $\frac{3}{4}$-metre lengths, as shown here:



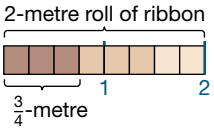
Point out that quarters are marked off first, and then the quarters are grouped by 3s. That is, we multiplied by 4 and divided by 3 (or multiplied by $\frac{1}{3}$). When you were in school, you probably learned this as inverting the divisor and then multiplying, as indicated by this symbolic representation:

$$6 \div \frac{3}{4} = \frac{6}{1} \times \frac{4}{3} = \frac{6 \times 4}{1 \times 3} = \frac{24}{3}, \text{ or } 8$$

In these types of problem, students (even adult students!) often have difficulty with ribbon being left over after they cut as many pieces of the specified length as possible. You can help students understand such cases by posing a problem like this:

> Suppose that you have a 2-metre roll of ribbon and you want to cut as many $\frac{3}{4}$-metre ribbons as you can. You know that you can cut at least 2 because you can get 1 from each metre of ribbon. Can you cut 3?

Students can approach this problem by looking at this picture.



Students can see that they can cut two $\frac{3}{4}$-metre ribbons (indicated by the two shaded parts) but that the leftover (unshaded) part is not enough for another $\frac{3}{4}$-metre ribbon. Ask them what part of another they can make. They should see that the unshaded part is $\frac{2}{3}$ of a $\frac{3}{4}$-metre ribbon. From this, children should be able to conclude that they can make $2\frac{2}{3}$ hair ribbons (each $\frac{3}{4}$-metre long).

Doing examples like the last one ($2 \div \frac{3}{4}$) will help children reason through problems in which the divisor is larger than the dividend. For example, how many $\frac{1}{2}$-metre ribbons could be cut from a $\frac{1}{3}$-metre

strip of ribbon ($\frac{1}{3} \div \frac{1}{2}$)? Children should recognise that $\frac{1}{2}$ is larger than $\frac{1}{3}$, so they could not cut even one $\frac{1}{2}$-metre ribbon. The question is, what part of a $\frac{1}{2}$-metre ribbon could they cut? Have the children draw a picture to convince themselves that they could cut $\frac{2}{3}$ of a $\frac{1}{2}$-metre ribbon.

In the classroom 12.7 consists of a few rhymes that incorporate questions about fractions. Even older children like their silliness. Be sure, however, that you balance fanciful activities like this with activities involving situations from students' everyday lives. Use the diversity in your class as a resource by making up problems that relate to students' different customs.

IN THE CLASSROOM 12.7

A new twist on old rhymes

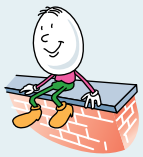
Objective: Verify whether or not computational statements about fractions are true.

Year level: 4–6

- Have the students decide if the following rhymes are true and show how they know by using models or drawing pictures.

Humpty Dumpty sat on a wall.
'Wow, I have 45 balloons in all.'
The wind blew $\frac{2}{5}$ of them to the sky.
'Now, I have 30 balloons, oh my!'

Little Miss Muffet
Sat on a tuffet
And ate $\frac{1}{2}$ of a chocolate bar.
Along came a spider
And ate $\frac{1}{5}$ right beside her.
The remaining $\frac{1}{10}$ won't go far.



Jack and Jill went up the hill
To fetch a pail of water.
Jack filled $\frac{2}{3}$ without fail.
Jill added $\frac{1}{4}$ to the pail.
And $\frac{11}{12}$ spilled as Jack caught her.

Little Jack Horner
Sat in the corner
Eating a pizza pie.
He ate $\frac{1}{2}$ and $\frac{1}{2}$ of the rest
'None left,' said he, 'I am the best.'



- Have the students make up their own rhymes using fractions or change these rhymes with other fractions.

12.3 Conceptual development of decimals

LEARNING OUTCOME 12.3 Describing how estimation strategies are used for adding and subtracting by rounding to whole numbers and benchmark numbers.

When you develop decimal concepts, you should relate decimals to what students already know about common fractions and place value. For ease of discussion, we consider these relationships separately, but you should keep in mind that you will need to weave them together in your teaching. You should also keep in mind that it is important to relate decimals to everyday experiences familiar to the students, since contextual situations often help students understand the mathematics. The Australian Curriculum recommends concentrating on decimal concepts in Years 4 and 5. This includes all the

concepts and skills discussed in this section. Operations with decimals are emphasised in Year 6 and beyond.

Relationship to common fractions

As we mentioned near the beginning of this chapter, decimals are just another notation for fractions — specifically, for tenths, hundredths and other powers often parts of a unit. The basics to understanding decimals is understanding these fractional parts — an understanding that you should help students build when you are developing the meaning of fractions.

Tenths

Before introducing the decimal notation for tenths, review what students should know about tenths from their prior study of fractions. They should know that partitioning a unit into tenths results in 10 equal parts. They also should be able to make the connection between a model of a fraction, the word for the fraction and the fraction written symbolically. They should also know that 10 tenths make a whole, that 7 tenths is less than a whole and that 27 tenths is more than a whole.

With this background, children should be ready to learn that, for example, 0.3 is a symbol for $\frac{3}{10}$. You should help children link their knowledge about place value to this new notation (see the discussion in chapter 8 on place value). You also need to stress that 0.3 and $\frac{3}{10}$ are both said as 'three-tenths'. Then have them look at a number such as $2\frac{7}{10}$. Show them that this is written as 2.7 in the new notation and said as 'two and seven-tenths', rather than '2 point 7' or '2 decimal 7'. Point out that the word *and* is said for the decimal point. It is important as always to link the new language to the concrete representations and the written notation, in order to establish 'a more accurate association between the three elements' (Caswell, p. 26, 2006). Saying decimals aloud in this way also helps students relate what they know about fractions to decimals. In the classroom 12.8 shows a quick game that children can play to practise writing in decimal and fraction notation.

IN THE CLASSROOM 12.8

Can you beat the toss?

Objective: Write numbers as fractions or decimals.

Year level: 3–5

Materials: Paper and pencil for children, a dollar coin for the teacher

Directions

- The teacher reads a number.
- Each child writes the number in either decimal or fraction notation.
- Each child receives 1 point if he or she writes the number correctly.
- The teacher tosses a coin.
  - If it's *heads*, each child who wrote a correct decimal receives one point.
  - If it's *tails*, each child who wrote a correct fraction receives one point.

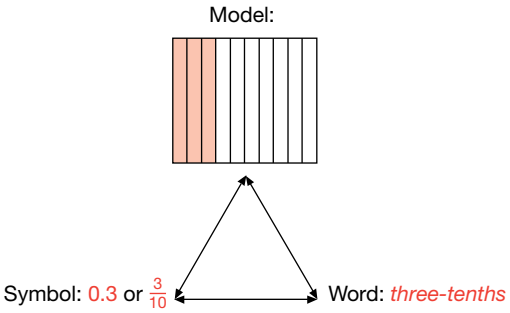
Suggested numbers

two-tenths	one and three-tenths
seven-tenths	twenty-two and five-tenths
five and seven-tenths	thirty-four and no-tenths
eleven and four-tenths	

Challenge numbers

sixteen-tenths	five and eleven-tenths

Before introducing hundredths, you should make certain that students have made all 3 connections shown by this triangle (model and symbol, symbol and word, word and model).



Hundredths

Give each child multiple copies of the model of hundredths. (Appendix C contains decimal or percentage paper that you can use for this model.) Make certain the children know that the unit (the entire square grid) is one and that each part (each small square) is one-hundredth. Ask them to shade $\frac{3}{100}$, $\frac{7}{100}$, $\frac{10}{100}$ and $\frac{21}{100}$ on the different copies of the model.

Then have the children shade the first and second columns on the hundredth model and three more small squares. Ask what part of the whole is covered. Elicit both responses: $\frac{2}{10} + \frac{3}{100}$ as well as $\frac{23}{100}$. Then write the following:

$$\frac{23}{100} = \frac{2}{10} = \frac{3}{100}$$

Now connect the place-value interpretation and the decimal notation 0.23. You should also use the model to show that $0.2 = 0.20$. Continue having children make the connections shown in the triangle (i.e. by connecting the model, the symbol and the word). These examples are only a brief introduction to hundredths. You will need to return to models as you continue helping students make sense of hundredths and how they relate to tenths. (See chapter 8 for other place-value models. You can change these models and use them with decimals.) The dollar is also a natural model for hundredths. However, it should not be the only model, since children do not necessarily relate the cents (whole numbers) to a fractional part of a dollar. Furthermore, in the Australian context there are severe limitations in using dollars and cents as a model for hundredths, as the absence of 1-cent and 2-cent coins makes it impossible to represent most actual amounts using our currency.

Thousandths and other decimals

Older students often do not understand small decimals such as 0.234 or 0.0003, partly because there is less emphasis on these decimals but also because teachers often expect children to generalise from hundredths to all the other places. Most of your work with smaller decimals should be done through the place-value interpretation because the fractions and models become unwieldy. However, you should make sure children understand that a thousandth is one-tenth of a hundredth, and you should develop this understanding by using a model. Students often have difficulty hearing the difference between 'thousands' and 'thousandths'— careful pronunciation of all the fractional names (tenths, hundredths, thousandths etc.) is essential.

Writing fractions as decimals

If decimals have been introduced carefully, students should be able to write the fraction notation for any decimal and the decimal notation for any fraction expressed in tenths, hundredths and so on. For example, they should be able to write $\frac{2}{10}$ and $\frac{23}{100}$ as 0.2 and 0.23, no matter whether they read the written fractions or hear them read as 'two-tenths' and 'twenty-three hundredths.'

However, the fact that students can write fractions as decimals when the fractions are expressed in tenths, hundredths, and so on does not mean that they will be able to write other fractions as decimals. Students try to make sense of the relationship between fractions and decimals, but they often do not have

a firm understanding of the quotient meaning of fractions. For example, they do not understand that $\frac{4}{5}$ means 4 divided by 5. They should realise that the quotient is less than 1. In context, this is evident. For example, if you have 4 biscuits to divide among 5 children, each will receive less than 1 biscuit. To divide 4 by 5, children must know that 4 is equivalent to 4.0 and must be able to divide decimals. If you can help students gain these understandings, they can use a calculator or the traditional algorithm to see that $\frac{4}{5}$ is 0.8. Children who also understand equivalent fractions should know that $\frac{4}{5}$ is $\frac{8}{10}$ or 0.8. Having children see that both methods lead to the same decimal is important. They can use the method of division to change any fraction to a decimal (or a decimal approximation, if the division doesn't terminate in zeros).

Children can explore many interesting patterns. Some can be investigated with a calculator — for example, the decimal approximations for ninths:

$$\frac{1}{9} = 0.11111$$

$$\frac{2}{9} = 0.22222$$

$$\frac{3}{9} = 0.33333$$

You should do a few of these calculations by hand to show that the pattern continues forever. If you only use a calculator, students may think that $\frac{1}{9} = 0.11111111$ exactly (i.e. the number of places on their calculator display).

Similarly, you should be looking at the meaning of $\frac{1}{3}$ to help students see that $\frac{1}{3}$ does not exactly equal 0.33, 0.333 or 0.3333. For instance, you can point out that if $\frac{1}{3} = 0.33$, then each of three equal parts of a whole would be 0.33, and thus the whole would be 0.33 + 0.33 + 0.33, or 0.99, which is not 1. Similarly, you can show students that $\frac{1}{3}$ does not equal 0.3333 (although this is a closer approximation to $\frac{1}{3}$).

Relationship to place value

The place-value interpretation of decimals is most useful for helping children understand computation with decimals. Let's look now at a way to develop this interpretation and consider how to use it in teaching children how to order and round decimals.

What do children know about place value? Consider the number 2463. By Year 3, children can identify the places (ones, tens, hundreds and thousands) and the number in each place (3, 6, 4 and 2). They know, for example, that the 4 means 4 hundreds. They also have learned how the places are filled: beginning with 1 as a unit, grouping 10 of these to form a new unit (tens), grouping 10 of these to form a new unit (hundreds) and so forth.

In introducing place-value ideas with decimals, again begin with one as the unit. But instead of grouping by tens, group by one-tenth of the one to form the new unit of tenths. To indicate this new unit in the place-value system, use a decimal point after the ones place. Help students understand that 10 of the tenths make a one (just as 10 of any unit make the next larger unit). They should also be able to identify the tenths place in a number. Be sure to integrate the place-value interpretation of decimals with the interpretation of decimals as fractions.

When introducing hundredths, you should again focus on the place-value interpretation. Given a number such as 51.63, a child should be able to tell what number is in the tenths place and what number is in the hundredths place, as well as the relationships between the places (a hundredth is one-tenth of a tenth, or 10 hundredths is one-tenth). After you introduce thousandths in a similar way, the children should be able to generalise to any decimal place.

A common misconception that students develop during this introductory phase is that the place value names are symmetrical around the decimal point, resulting in a column for the 'oneths' (MacDonald 2008). It is important when working with decimals to emphasise that the place value names are symmetrical around the ones column – and not the decimal point, which simply identifies which digit is in the ones column. For this reason some teachers choose to place the decimal point within the ones column, rather than on the line between the ones and tenths.

Be sure to read decimals aloud in a way that emphasises the tie between decimals and fractions. For example, read 24.09 as 'twenty-four and nine hundredths', not as 'two four point zero nine' or 'twenty four point oh nine.' The words *tenths, hundredths* and so on help students maintain that tie to the meaning.

Often, you can use a place-value grid to help students having difficulty with decimals. Consider, for example, this grid for the number 32.43:

T	O	t^{th}	h^{th}
3	2 •	4	3

Point out that the decimal can be seen both as *32 and 43 hundredths* and as *32 and 4 tenths and 3 hundredths*. It can also be read as 3243 hundredths. What other ways could you express this number in words?

These children will also benefit from being asked questions like the ones in figure 12.6 and organising their thoughts on a decimal grid.

Write 8 hundredths:

T	O	t^{th}	h^{th}
	•	0	8

Children should realise there are 0 tenths.

What is 29 hundredths?

T	O	t^{th}	h^{th}
	•	2	9

Children should realise there are 2 tenths and 9 hundredths.

What is 29 tenths?

T	O	t^{th}	h^{th}
	2 •	9	

Children should realise this is 2 ones and 9 tenths.

Now look at 4.3 on the grid. How many tens?

T	O	t^{th}	h^{th}
	4 •	3	

Children should realise there are no tens.

Tell the children that we could write a 0 in the tens place, but this is not customary.

How many hundredths?

T	O	t^{th}	h^{th}
	4 •	3	0

Children should realise that 4.3 is equivalent to 4.30.

FIGURE 12.6 Using a decimal grid with place value

Children can use a grid as a way of rote learning a rule unless you continue to help them explain their thinking and relate the grid to the model. It is certainly not enough to fill in zeros on the grid anywhere there is not a number. It is more important to discuss why, for example, 4.3 is equivalent to 4.30.

Ordering and rounding decimals

Children should be able to understand the ordering and rounding of decimals based directly on their understanding of decimals and their ability to order and round whole numbers. For decimals, this understanding must include being able to interpret the decimals in terms of place value and being able to think of, for example, 0.2 as 0.20 or 0.200.

Here is an example of 2 children discussing which is larger, 23.61 or 23.9.

Carter: I think that 23.61 is larger because it has more places, just like 2361 is larger than 239.

Hayley: But both of these (23.61 and 23.9) are 23 and some more, so all we have to do is compare the decimal parts.

Carter: Okay, it's not like whole numbers.

Hayley: Well, sort of. We begin with the largest place — the tens are the same, the ones are the same, so we need to look at the tenths. The 9 is larger than the 6, so 23.9 is larger than 23.61.

Carter: I follow you until that last part. Now you have a 9 and a 61, and 61 is larger.

Hayley: But the 9 is 9 tenths and the 61 is 61 hundredths. Want to look at a model?

Carter: I get it — I remember the model. Nine tenths is 9 strips or 90 little squares. It is 90 hundredths, so it is larger.

As you see from this conversation, if children really understand the notation and meaning of decimals, they can figure out ways to compare decimals. However, many students do not have these understandings. Australian research found that when Year 7 students were given the question, 30.1×0.97 is closest to which of the following answers: 0.003, 3, 30, 3000, almost 40% did not choose the correct answer of 30, with more than half of these students selecting 3000. This choice of 3000 as the correct answer suggests that these students completely ignored the decimal point and multiplied 30×97, or used estimation, 30×100.

In rounding a decimal such as 24.78 to the nearest tenth, children need to ask themselves the same types of question as with whole numbers. Children must also understand that $24.7 = 24.70$.

Question	Expected response
What 'tenths' is 24.78 between?	It's between 24.7 and 24.8 (or 24.70 and 24.80).
Is it nearer 24.7 or 24.8?	It's nearer to 24.80.
How will you round it?	To 24.8.

12.4 Operations with decimals

LEARNING OUTCOME 12.4 Explaining how models assist the development of conceptual understanding of adding, subtracting, multiplying and dividing fractions or decimals.

One advantage of decimals over fractions is that computation is much easier, since it basically follows the same rules as for whole numbers. In teaching the algorithms for decimals, you should build on the place-value interpretation of decimals and the corresponding whole-number algorithms. (Adapt the ideas in chapters 10 and 11 for use with decimals operations.) Given the wide availability of calculators, it is important that you spend as much time seeing whether answers are reasonable as you spend on the algorithms. Thus, estimation skills are crucial (see an earlier chapter).

In the classroom 12.9 shows an activity that requires estimation skills. Encourage students to use whole numbers that are compatible when trying to answer the questions. You can present questions of this type to students before they learn about decimal computation. For example, England's population is about 50 million and Scotland's is about 5 million, so England has about 10 times as many people as Scotland.

IN THE CLASSROOM 12.9

What's your answer?

Objective: Use estimations about decimal quantities to answer questions.

Year level: 5–6



- Use the data in this table. These are data about the parts of the United Kingdom.

Parts	*Area (thousands of km²)*	*Population (millions)*
England	130.422	50.714
Northern Ireland	14.144	1.733
Scotland	78.772	5.103
Wales	20.761	2.977

- Answer these questions:
  - About how much larger in area is England than Scotland? _____
  - About how large in area is the United Kingdom? _____
  - About what is the population of the United Kingdom? _____
  - Do twice as many people live in Wales than live in Northern Ireland? _____
  - Is England more than 6 times larger in area than Wales? _____

Addition and subtraction

Children who have a good concept of decimals have little trouble extending the whole-number algorithms for addition and subtraction to decimals. They realise that they need to:

- add or subtract like units (tenths with tenths, hundredths with hundredths etc.)
- regroup in the decimal places as they did with whole numbers.

Difficulty with adding or subtracting decimals arises mainly when the values are given in horizontal format or in terms of a story problem and the decimals are expressed to a different number of places (e.g. 51.23 + 434.7). To deal with this difficulty, have the children first estimate by looking at the wholes (about 480). Some children may need help in lining up the like units and so may benefit from using a grid:

	H	T	O	t^{th}	h^{th}
		5	1 •	2	3
+	4	3	4 •	7	
	4	8	5 •	9	3

Multiplication and division

Multiplying and dividing a decimal by a whole number are conceptually easier than multiplying two decimals or dividing a decimal by a decimal. Children can build on this to understand the more sophisticated computation. Begin by having children consider a problem like this:

There are 6 lined up end to end. Each table is 2.3 metres long. How long is the line of tables?

From their previous work with multiplication, students should realise that this is a multiplication problem.

Of course, they should also see that they can solve the problem by adding decimals. That is: $6 \times 2.3 = 2.3 + 2.3 + 2.3 + 2.3 + 2.3 + 2.3$, which equals 13.8.

However, just as children moved away from repeated addition to find the product of 2 whole numbers, they need to do so when multiplying a decimal by a whole number.

Children with a firm understanding of the place-value interpretation of decimals can think about multiplication with decimals in another way. For example, 2.37 is 237 hundredths, so 2.37×6 is:

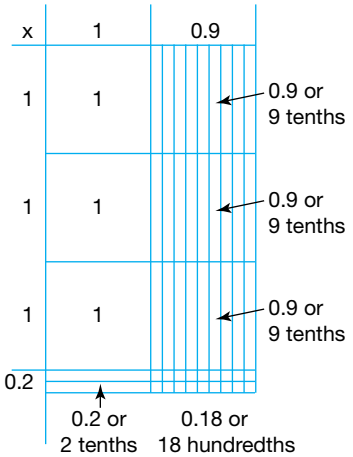
23 tenths
$\times$ 6
138 tenths, which is 13.8.

Similarly, for the problem 2.37×6, children could realise that 2.37 is 237 hundredths, so:

237 hundredths
$\times$ 6
1422 hundredths, which is 14.22.

Working with a decimal grid can help students remember that 138 tenths is 13.8. In using this method, you should first have the students decide on a reasonable answer. Ask whether 6×2.37 is more than 12 and whether it as much as 142.

To help students make sense of multiplying a decimal by a decimal, rather than only remembering a rule about counting decimal places, see if they can model this problem: 'What is the area of a porch that is 3.2 metres by 1.9 metres?'



Add the parts:
1 + 1 + 1 = 3
9 tenths + 9 tenths + 9 tenths =
27 tenths or 2.7
Now find the total:

3.
2.7
0.2
0.18
6.08

Students need to discuss whether 6.08 square metres is a reasonable answer. Ask them: 'What would be the area of a porch that is 3 metres by 2 metres?'

Children can use the distributive algorithm for dividing whole numbers to divide a decimal by a whole number. Consider this problem:

A vinegar company distributed 123.2 million litres of vinegar equally among 8 customers. How much vinegar did each customer receive?

First, ask for reasonable answers.

Did each customer get more than 10 million litres?
[Yes, that would be only 80 million litres.]
Did each get more than 20 million litres? [No, that would be 160 million litres.]
What is an estimate for the answer? [Between 10 and 20 million litres.]

Then, talk through the division, as in figure 12.7.

For students in middle school, you will need to develop division of a decimal by a decimal as well as extending the work on multiplying decimals. The development of fluency with multiplication and division of decimals takes more practice than with addition and subtraction, partly because children's skills with multiplication and division of whole numbers is not as firm as their skills with addition and subtraction. You might want to begin collecting some challenging activities such as in In the classroom 12.10, as well as games and practice materials at other levels that will help children develop computational fluency with decimals.

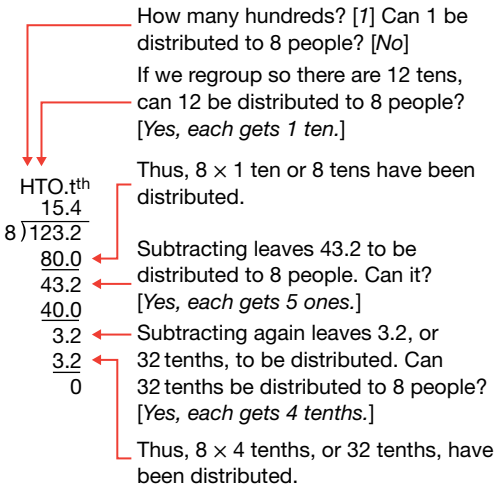


FIGURE 12.7 Talking through division of a decimal by a whole number

IN THE CLASSROOM 12.10

Plug-in puzzles

Objective: Develop decimal number sense.

Year level: 4–5

- Use these decimals to fill in the blanks.
  8.3 4.2 5.5 3.1 7.6 6.7

Do each multiplication or division. In the division problems, divide to the hundredths place. Sum the 4 answers.

_____ × _____ = □
_____ ÷ 5 = □
_____ × _____ = □
_____ ÷ 3 = □
Total □

- Can you get a total greater than 50?
- Can you get one greater than 100?
- What is the largest total you can get?

- Arrange these 4 decimals in the blanks so that the sentence is true. (Remember to do the parts in the parentheses first.)
  5.13 4.24 3.84 3.16
  (_____ × 5) + _____ = (4 × _____) + _____
- Pick the 2 decimals from those listed that will make the sentence true.
  21.21 42.42 36.36 63.63 27.27
  (_____ ÷ 7) × 8 = (_____ ÷ 9) × 6

MAKING CONNECTIONS

The Egyptians as early as 3500 BC were one of the first cultures in recorded history to make use of fractions. However, in their hieroglyphs, they wrote all fractions as the sum of unit fractions (i.e. fractions with 1 as the numerator). For example, $\frac{3}{4}$ would have been written as $\frac{1}{2} + \frac{1}{4}$, and $\frac{2}{3}$ as $\frac{1}{2} + \frac{1}{6}$. They could do all 4 operations with fractions using this notation. (And you thought our system is cumbersome!)

Decimal fractions are found in the *Book of Chapters on Indian Arithmetic*, written by Abul Hassa Al-Uqlidisi in 952. The author said the answer to successively halving 19 5 times was $\dot{0}\,59375$. The vertical mark over 0 indicates that the decimal fraction begins to the right. In Australia, America and Britain, this would be written 0.59375. However, other European countries use a comma instead of a full stop, and it would be written 0,59375. We often think of numbers as being an international language, but there are differences in notation that we use of which we should be cognisant.

At Year 4 there are considerable differences among countries with regard to the emphasis placed on fractions (Mullis et al. 2004). In Australia, Year 4 students are expected to be consolidating their conceptual understandings of fractions, including equivalent fractions, but they are not expected to begin operating with fractions until Year 5. Similarly, Japan expects only half of its Year 4 students to have a firm background in concepts involving fractions; students do little with fractions until about Year 5, when they are expected to add and subtract fractions with like denominators. The United States expects most of Year 4 students to have some conceptual background in fractions as well as experience in adding and subtracting like fractions. Other countries, such as Singapore, expect all students to have experience in both of these areas. If you have students from other countries, you can expect a wide variety of skills and experiences in dealing with fractions.

Australian students begin their study of fractions in Year 1, when they are expected to recognise and describe one-half. This is gradually extended to include other fractions during Years 2 and 3 through the use of a range of models including fractions of areas and sets of objects. In Year 4, they begin to investigate equivalent fractions and locate fractions including mixed numerals on a number line. The use of these models emphasises that a fraction is an amount. In Year 5, they focus on equivalent fractions, comparison of fractions, fractions as quotients and the relation of fractions to decimals. Operation work is left until Year 6. Japanese students begin the study of fractions in Year 4 (Watanabe 2006) and focus on developing concepts of proper fractions, improper fractions and mixed numbers. Students begin developing fractions in a measurement context, which also emphasises that a fraction is an amount. This contrasts with the view of many students in the United States that a fraction is two numbers, which often leads to whole-number thinking rather than fractional thinking.

Some countries teach operations with decimals before operations with fractions, while others do the opposite. Many countries use a comma instead of a full stop for the decimal point, and this, along with the way numbers are displayed on a calculator, is one reason you often see whole numbers written without commas. Some countries say the denominator before they say the numerator. Be alert to these differences.

A GLANCE AT WHERE WE'VE BEEN

Three meanings of fractions are part–whole, quotient and ratio (this chapter focuses on the part–whole meaning). Useful models of the part–whole meaning include such attributes as region, length, set and area. The region model is the easiest for children to understand and the rectangle is the easiest type of region. To help children make sense of fractions, begin with the important idea of partitioning (making equal shares) and then introduce words for fractional parts and counting the parts. Help children make connections among words, symbols and models. Make sure children can figure out a whole from a fractional part and that they understand the idea of equivalent fractions. Use concrete models, pictorial models and symbolic representations to build children's ability to order fractions and find equivalent fractions. Make sure children understand how to use benchmarks for estimating and checking the reasonableness of answers. Use models to develop children's understanding of mixed numbers and improper fractions.

To teach addition and subtraction of fractions, begin with problems and pictorial models that help children (1) see how adding and subtracting fractions can solve problems similar to problems with whole numbers, (2) develop an idea of a reasonable answer and (3) see why a common denominator is necessary. Teach multiplication of fractions in a way that helps children (1) understand the meaning of multiplication and (2) estimate the size of an answer. Later, this will help them see why the algorithm for multiplication works (multiply numerators and denominators). Begin with a whole number times a fraction, then a fraction times a whole number, and then a fraction times a fraction. For division, relate the way partitioning works with whole numbers to the way it works with fractions, using pictorial models.

Help children develop the concept of decimals by relating decimals to common fractions. Emphasise the place–value interpretation of decimals to help children in ordering and rounding decimals and to prepare them for operations with decimals.

Students in different countries learn about fractions and decimals at different times in their primary schooling; they also have different ways of writing and saying decimals. Teachers should be aware of these differences so they can deal appropriately with diverse students.

THINGS TO DO: FROM WHAT YOU'VE READ

1 The part–whole meaning of a fraction can be developed with different attributes, such as length and sets. Draw part–whole models for four different attributes to show $\frac{3}{4}$.

2 Illustrate 3 different meanings of $\frac{3}{4}$.

3 Describe how you can develop the concept of equivalent fractions.

4 What is partitioning? How is it related to division and to fractions?

5 Illustrate how 2 fractions can be ordered using the pictorial model shown in In the classroom 12.5.

6 With pictures, show why $\frac{23}{5}$ is equivalent to $\frac{13}{5}$; show why $\frac{23}{5}$ is equivalent to $\frac{18}{5}$.

7 What is a benchmark for fractions? How are benchmarks used? Propose a problem ordering fractions that could be solved using benchmarks for fractions.

8 How is multiplication of fractions different from multiplication of whole numbers?

9 Describe how decimals and common fractions are alike and different.

10 Give 3 different exercises in which students are asked to order decimals. Explain why you chose each. What would you expect to learn about children who did these exercises? What would you do if the children encountered difficulties?

11 How would you help children who have difficulty with this: 25.03−8.459?

THINGS TO DO: GOING BEYOND THIS TEXT

IN THE FIELD

1. Choose one of the books from the Children's literature connections and develop an activity that links to the story and uses manipulatives to explore fractions
2. Halves. Prepare a template that includes a large circle, a large square, a small circle and a small square. Give each student one shape and ask them to divide it in half by folding and then cut out one half. Then put them in groups and ask them to compare their 'halves' and decide if are they the same size. If they are how do you know? If not, why not?
3. Modelling number lines. Do the activity In the classroom 12.3 with students and when they break the rod into fractions have them record those fractions on a number line before they move onto the next fraction
4. Making equivalent fractions. Do the activity In the classroom 12.5 with a group of students. Are they able to generalise from the activity and predict what other fractions they could make from the fifths? What questions could you use to help them towards making that generalisation?
5. Improper fractions. Using circles divided into quarters, leave some intact, cut some into halves, some into three-quarters and quarters. Give each student one piece and provide a space that they can walk around in. Play music and when the music stops call out a number, students need to form groups of that number. Once in their group they have to put their circles together and work out the total—how many whole circles and what fraction left over, how many halves in total, how many quarters in total. Have students record their results in a diagram and report back before starting the music again.

IN YOUR JOURNAL

6. Examine a primary mathematics textbook series to see what models for fractions they use and whether they introduce set models through partitioning.
7. Describe how you were taught fractions in primary school. Did those experiences help you make sense of fractions?

WITH ADDITIONAL RESOURCES

8. Read 'When getting the right answer is not always enough: Connecting how students order fractions and estimate sums and differences' (Cramer & Wyberg 2007). Answer the first question on page 219 in the article.
9. Read one of the articles about fractions or decimals from *Making Sense of Fractions, Ratios, and Proportions* (Litwiller & Bright 2002). Briefly summarise the article; then tell why you selected it and what you learned.
10. Read 'Low performing students and teaching fractions for understanding: An international analysis' (Epsom 2003). Describe one of the interaction transcripts of the teacher and the 2 children. What did you learn from thinking about the 2 children and this interaction?
11. Read the 'Whole number knowledge and number lines help to develop fraction concepts' (Pearn & Stephens 2007) and 'Partitioning — The missing link in building fraction knowledge and confidence' (Siemon 2003). Comment on the thinking that children need to develop fraction understanding.
12. Read 'But what about the oneths? A Year 7 student's misconception about decimal place value' (MacDonald 2008) and 'Confusions between decimals, fractions and negative numbers: A consequence of the mirror as a conceptual metaphor in 3 different ways' (Stacey, Helme & Steinle 2001). These articles identify some of the difficulties that students encounter when learning about fractions and decimals. Plan an activity that would assist students to develop sound conceptual understandings and overcome the difficulties described in these articles.

WITH TECHNOLOGY

13 Scootle is a library of digital learning objects developed by a combined Australian and New Zealand government project. Most universities have agreements to use these resources. Visit the Scootle website at www.scootle.edu.au/ec/p/home and register for free using your university email. Use the search function to identify learning objects that focus on fractions – you will discover that there are a large number, have a look around. Then search for Fraction Fiddle – Reach the Target. The task this learning object sets you is to identify 2 fractions that will add up to the target fraction. Your challenge is to see how many combinations you can identify that reach your target. In a paragraph or 2 identify any prior knowledge students would need, an appropriate learning objective for this activity and explain how you could use it with students. Remember that virtual manipulatives such as the fraction bars in this learning object, should only be introduced after students have had a wide range of experiences with concrete materials including building fraction bars from unifix or other similar materials and folding paper fraction bars.

14 Go to the NCTM Illuminations site at http://illuminations.nctm.org/Default.aspx. This is a site auspiced by the National Council of Teachers of Mathematics in the United States and like the Scootle site above includes a wide range of learning objects. Again explore the wealth of learning objects that are available here. Then search for Concentration; once you have opened the game you will need to select percentages and fractions (40%, $\frac{4}{10}$, $\frac{2}{5}$) under levels and then select either 1 or 2 players and blinds up (less concentration and more recognising the matching pairs) or blinds down (for the familiar game of concentration where you have to remember where the numbers or pictorial representations are). The aim as in any good game of concentration is to select matching pairs. Once a matching pair is selected they are removed and recording down the left hand side of the screen. Identify which learning outcome in the Australian Curriculum this would be relevant for. How might you use this learning object in your classroom?

CHILDREN'S LITERATURE CONNECTIONS

Hutchins, P. (1987). *The Doorbell Rang*. New York: Greenwillow Books.
This book is an excellent source to use when introducing the concept of partitioning. Children are challenged to share biscuits fairly in what can be an interactive story. Ideas for lesson activities that focus on fractions can be found at http://hwdsbmathassessment.wikispaces.com.

Murphy, S. J. (1999). *Jump, Kangaroo, Jump!* New York: HarperCollins.
This book is about Australian animals planning a field day of events at a camp. Kangaroo and his friends must divide themselves up into different groups for the events that are taking place. There are activities and games in the back of the book for teachers and parents to use.

McElligott, M. (2009). *The Lion's Share*. Newtown, NSW: Walker Books Australia.
This time the food is being shared by the animals. It all begins when Lion bakes a cake to share with his friends. The author's website includes a range of teaching ideas, http://matthewmcelligott.com/lionsshare/projects.php.

Fisher, D. & Sneed, D. (2008). *My Half Day*. Mount Pleasant, SC: Arbordale Publishing.
In this story the protagonist wakes up to find that half his hair has been shaved off and that is just the start of the fraction fun. There are a wealth of teaching ideas on the publisher's website www.arbordalepublishing.com/documents/TeachingActivities/Half_TA.pdf.

Dodds, D. A. (2007). *Full House: An Invitation to Fractions*. Somerville, MA: Candlewick Press.
This book is full of rhymes, repetition and engaging characters. It is an enjoyable introduction to fractions, as Miss Bloom who runs the Strawberry Inn divides the rooms and the dessert for her guests.

Pinczes, E. (2003). *Inchworm and a Half*. New York, NY: Scholastic Inc.
An inchworm is measuring all the vegetables in the garden by looping along the length, but then she when she is measuring a bean and there is a bit left over. She is joined by an inchworm half her size and then another a third her size and another a quarter her size. Lots of rhymes keep the story rolling along. An accessible introduction to measuring with informal units as well as fractions.

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CHAPTER 13

Ratio, proportion and percentages: meanings and applications

LEARNING OUTCOMES

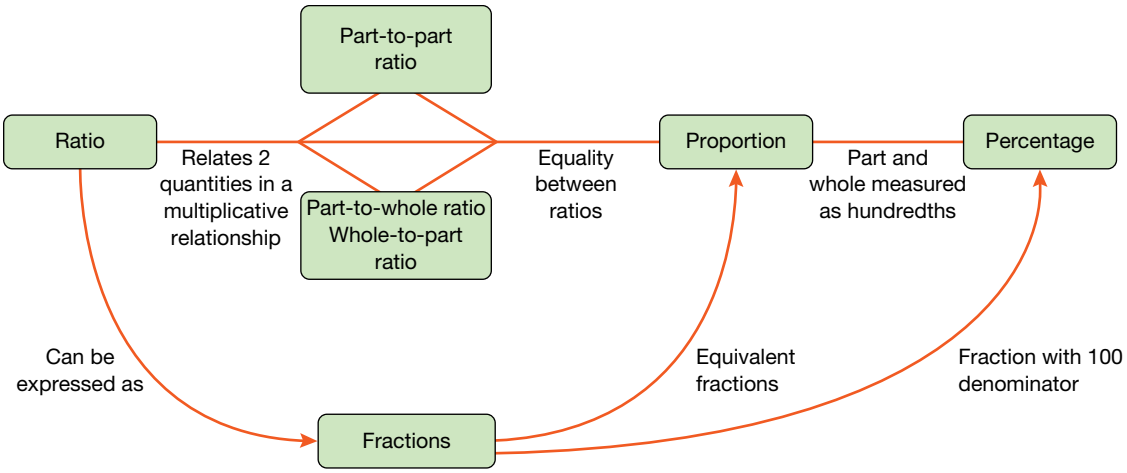
13.1 Understanding how ratio and proportion are related and how they are different.
13.2 Developing ideas that use student intuition and reasoning to progress their thinking about proportions.
13.3 Generating teaching ideas around models that help develop the concept of percentages.



'Pick up a sunflower and count the florets running into its centre . . . you will find an extraordinary truth: recurring numbers, ratios and proportions.'

Charles Jencks

Chapter 13 concept map



Introduction

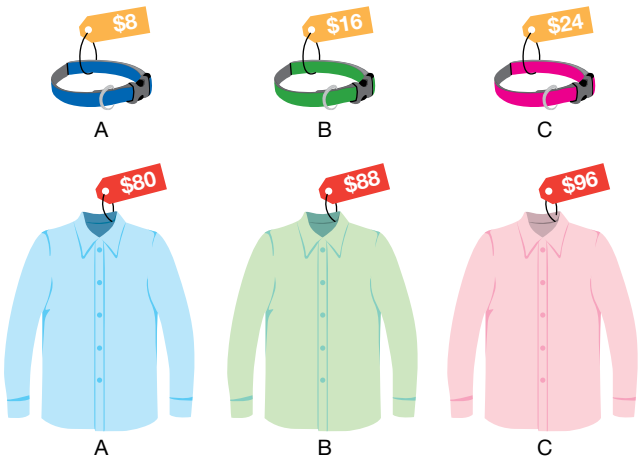
'Twice around your thumb is equivalent to once around your wrist.'
'Family income this year increased by 10%.'
'Ian did only half the work Angela did.'
'Her salary is 3 times my salary.'
'The cost of living tripled during the last 8 years.'
'Your chances of winning the lottery are less than 1 in a million.'
'I can purchase a 360 mL bottle of water for $1 or a 400 mL bottle for $1.50. Which bottle is the better buy?'

These statements show ratio, proportion and percentage in action. They demonstrate that much of quantitative thinking is relational. In such thinking, what is important is the multiplicative relationship between numbers, rather than the actual numbers themselves. Multiplicative thinking is described by Hurst and Hurrell (2014, p. 13) as possibly 'the 'biggest' of the 'big number ideas'. Unfortunately, children can struggle with developing multiplicative thinking and this can impact on their development of other mathematical understandings such as those involving place value. Hurst and Hurrell state:

> As children learn to think additively, they understand and can partition numbers into the hundreds. However, many children initially experience difficulty in moving beyond that, particularly beyond the thousands and a common misconception is that millions follow thousands. This part of place value understanding coincides with the development of multiplicative thinking, specifically that the cyclical pattern in reading and writing numbers continues and that there is a ten times multiplicative relationship between the places in the number system. (2014, p. 13)

By emphasising multiplicative relationships, you will help your students take advantage of the many opportunities to use these relationships in real-world situations and to use real-world situations to develop understandings. The Australian Curriculum (ACARA 2015) states that students in primary and middle school develop concepts and skills to solve a range of problems involving ratios and percentages, with and without digital technologies (see table 13.1). Note that in the content descriptions in table 13.1, the development of these concepts and skills relies on initially making connections between number representations (fractions, decimals and percentages) at Year 6 level. To then solve problems, students at Years 7 and 8 levels use these connections as well as making further connections between ratios, proportions and percentages.

Here is an example to get you thinking about ratios, proportions and percentages. Consider the prices of 3 dog collars and 3 shirts:



The difference in price between collar A and collar B is $8, and the difference between collar B and collar C is also $8. Similarly, the difference in price between shirt A and shirt B is $8, and the difference in price between shirt B and shirt C is also $8. For the dog collars, collar B is twice as expensive as collar A (which could also be stated as 200% more expensive) while collar C is 3 times as expensive as collar A and 50% more expensive than collar B. However, the difference of $8 between the shirts is not as large as the collars if considered from a multiplicative relationship perspective. From a multiplicative perspective, shirt B is not twice (or 200%) as expensive as shirt A — it is 10% more expensive. Shirt C is not 3 times more expensive than shirt A — it is 20% more expensive than shirt A and just under 10% more expensive than shirt B. Consideration of the multiplicative relationship between numbers, rather than the difference in an additive sense, is essential for ratios and proportional thinking.

TABLE 13.1 **Ratio, proportion and percentage: Number and Algebra strand content descriptions for Year 6 to Year 8 in Australia**

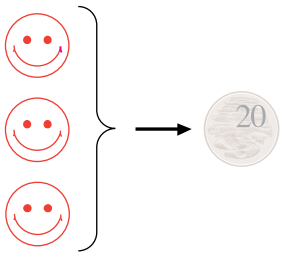
Year level	Content description
Year 6	**Fractions and decimals** • Make connections between equivalent fractions, decimals and percentages (ACMNA131). **Money and financial mathematics** • Investigate and calculate percentage discounts of 10%, 25% and 50% on sale items, with and without digital technologies (ACMNA132).
Year 7	**Real numbers** • Find percentages of quantities and express one quantity as a percentage of another, with and without digital technologies (ACMNA158). • Recognise and solve problems using simple ratios (ACMNA173). **Money and financial mathematics** • Investigate and calculate 'best buys', with and without digital technologies (ACMNA174).
Year 8	**Real numbers** • Solve problems involving the use of percentages, including percentage increases and decreases, with and without digital technologies (ACMNA187). • Solve a range of problems involving rates and ratios, with and without digital technologies (ACMNA188).

Source: © Australian Curriculum, Assessment and Reporting Authority 2015.

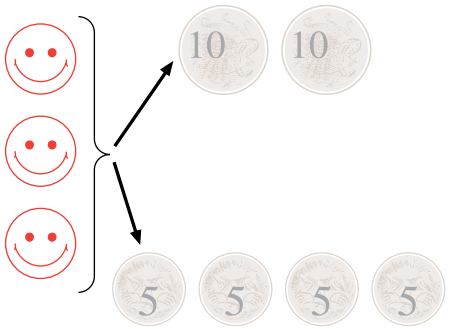
13.1 Ratios

LEARNING OUTCOME 13.1 Understanding how ratio and proportion are related and how they are different.

Ratios involve comparing things. Some of children's earliest experiences with comparisons involve ratios or rates. For example, if a child pays 1 20-cent coin for 3 stickers, the rate is 3 stickers for 1 20-cent coin:



This ratio may be read as 'three to one' and recorded as 3 to 1, 3:1, or $\frac{3}{1}$. Any of these forms is acceptable. We can also form ratios to report the number of stickers to 10-cent coins as 3 to 2 or stickers to 5-cent coins as 3 to 4:



Note that this ratio could also be expressed in reverse, as 4 5-cent coins for every 3 stickers, which we could write as 4 to 3 or as a quotient, $\frac{4}{3}$. You could use the quotient form to help children see that the cost per sticker is more than 1 5-cent coin, which illustrates a powerful application of ratios — knowing the cost of multiple items lets us calculate the cost of a single item.

Early experiences with ratios should enable children to think of 2 related numbers simultaneously. As children use manipulatives such as coins, or draw pictures, they should be encouraged to think about ordered pairs of numbers, such as (3 stickers, 4 5-cent coins). The models help link the operation of multiplication directly to ratios.

Working with facts about money (the number of cents in a 5-cent coin, 5-cent coins in 25 cents and so on) gives children natural and meaningful experiences with ratios. In the classroom 13.1 shows how organising such facts in a ratio table not only visually displays many ratios but also helps students realise that a ratio is a multiplicative relationship between 2 or more numbers in a given order. The ratio table also contains patterns that encourage students to explore relationships, generate formulas and engage in algebraic thinking.

Prices — for instance, 2 cans for 99 cents, 3 kilograms for $1.99 and 88 cents per dozen — provide meaningful contexts for ratios. The money model also lets you naturally extend ratios to more than 2 numbers. Other real-life contexts with data to generate statements about ratios can be found in *Counting on Frank* (Clement 1990) and *How Big is Big?* (1–2W Curl Curl North Public School & Watson 2009).

IN THE CLASSROOM 13.1

Know your coins

Objective: Use the value of coins in a ratio table to develop patterns and proportions and to examine algebraic relationships.

Year level: 4–6

Use a ratio table to help complete these patterns:

Number of 50-cent coins	1	2		4	5	6
Number of 10-cent coins	5	10	15	20		30
Number of 5-cent coins	10	20	30		50	

Answer these questions and fill in the blanks:

- Describe the pattern you found in each row. _____
- Write a ratio for the number of 50-cent coins to 5-cent coins. _____
- Write a ratio for the number of 10-cent coins to 5-cent coins. _____
- Write a ratio for the number of 50-cent coins to 10-cent coins. _____

Try these:

- How many 10-cent coins will be needed for 8 50-cent coins? _____

Show 2 different ways of doing this.

- How many 5-cent coins will be needed for 10 50-cent coins? _____

Show 2 different ways of doing this.

- Give 3 numbers (not shown in the table) that could go:
  - in the 10-cent coins row _____
  - in the 5-cent coins row _____
- Give 3 numbers that could *not* go:
  - in the 10-cent coins row _____
  - in the 5-cent coins row _____
- How many 50-cent coins would you have when the number of 50-cent coins plus the number of 10-cent coins plus the number of 5-cent coins (all in the 1 column of the ratio table) first exceeds $30? _____

Patterns often lead to ratios. For example, in figure 13.1 the ratio of red links to green links in chain A is 2 to 3. Extending the chain would reveal a pattern where 2 red links are followed by 3 green links, followed by 2 red links and so on. Extending chain B would show a different pattern — 4 red links, followed by 6 green links, followed by 4 red links, and so on. In chain B, the ratio of red to green links is 4 to 6. Chain C shows how doubling chain A forms a chain with the same ratio of red to green links as chain B (4 to 6). Chains B and C are the same length (10 links), and each chain has the same number of red links (4) and green links (6). This shows that 2:3 and 4:6 are equivalent ratios (though they make a different pattern). Children may better connect the representation by writing '3 green links to 2 red links' when expressing the ratio 3 to 2, $\frac{3}{2}$ or 3:2.

So far, we've been discussing the chains in figure 13.1 from a part-to-part perspective — that is, we've been discussing the ratio of 1 part of the chain (the red links) to another part of the chain (the green links). Now let's examine these chains from a part-to-whole perspective. For example, consider the ratio between the number of red links (the part) and the total number of links (the whole) in each chain. In chain A, this ratio is 2 to 5, whereas it is 4 to 10 in both chains B and C, which shows that these 2 ratios are also equivalent. Note that we have used the same model to show different ratios, depending on the perspective (part-to-part was 2 to 3 and part-to-whole was 2 to 5). You must be very clear about what is being compared when discussing ratios.



Chain A — with 2 red links and 3 green links — RRGGG



Chain B — with 4 red links and 6 green links — RRRRGGGGGG



Chain C — with 2 red links and 3 green links — RRGGGRRGGG

FIGURE 13.1 Chains showing equivalent ratios

Children encounter ratios in many different forms: '3 raffle tickets for a dollar', 'twice around', 'half as much' and so on. Showing children real-world examples of ratios helps them develop a greater awareness and understanding of ratios and how they work. For instance, adjusting recipes to serve more or fewer people provides an opportunity to work with ratios involving multiple values. Other examples might include a can of mixed nuts with a ratio of cashews to almonds to peanuts of 3:5:10 or a bag of lawn fertiliser with '8–12–20' printed on it, indicating the ratio of the percentages of phosphorus, nitrogen and potash in the fertiliser. You can also use examples like these to remind students of the order of the numbers in a ratio and the importance of understanding what each number represents. Ratios have little meaning unless you know what the numbers represent. Using physical models such as those shown in In the classroom 13.1 and figure 13.1 helps children learn to make sense of ratios.

Figure 13.2 shows a ratio table with the ratios between the numbers of wheels on different types of vehicle, as well as between the number of vehicles and the number of wheels on those vehicles. The ratio table contains many different patterns and gives children a visual way of understanding different ratios. For example, the table lets children see that the ratio of wheels on bicycles to wheels on tricycles could be expressed as 2:3, 6:9 or 10:15; it also lets them see that the ratio of the number of wagons to the number of wheels on those wagons could be expressed as 1:4, 4:16 or 6:24.

	Number of vehicles					
	1	**2**	**3**	**4**	**5**	**6**
Number of wheels on bicycles	2	4	6	8	10	12
Number of wheels on tricycles	3	6	9	12	15	18
Number of wheels on wagons	4	8	12	16	20	24

FIGURE 13.2 Ratio table comparing the number of vehicles with number of wheels for bicycles, tricycles and wagons

Students are sometimes confused by the different symbols used to write ratios. Linking the symbols to models and promoting class discussion can help minimise this confusion. For example, round chips or cut-out pictures of wheels could be used to model the wheels in figure 13.2 before children try writing the ratios. Ratio tables help organise the information. After children record the ratios in writing, you can help them talk about the mathematics by guiding them to discuss the information in the table and the situation represented in the ratios. For the ratios in figure 13.2, for example, you could give children early practice in verbalising and describing some of the mathematics surrounding the concept of ratio by asking them questions such as the following:

If I have 3 bicycles and 6 wheels, is the ratio 3 to 6 or 6 to 3?
If the ratio of bicycles to wheels is 4 to 8, how many bicycles do I have?
If I have 6 bicycles, how many wheels are there? What is the ratio of bicycles to wheels?
If, instead of 6 bicycles, I have half as many bicycles, how many wheels are there? What is the ratio of bicycles to wheels?

In addition to understanding that ratios can be constructed in different ways (e.g. part-to-part and part-to-whole), children need to understand that combining ratios is not the same as adding fractions. For example, in figure 13.3 chain A has 2 blue links and 3 orange links (a ratio of $\frac{2}{3}$), and chain B has 4 blue links and 5 orange links (a ratio of $\frac{4}{5}$). We could join together chains A and B to form chain C, with 6 blue and 8 orange (a ratio of $\frac{6}{8}$). However, the sum of the fractions $\frac{2}{3} + \frac{4}{5}$ is not $\frac{6}{8}$ (it's $1\frac{7}{15}$). Remember that operations, such as addition, can be performed on fractions but not on ratios. Of course, it is important that you help students understand the equivalence of different representations of ratios (e.g. 2 to 3, 2:3 and $\frac{2}{3}$) and that you help them link these representations to real-world models. However, far more important is helping students understand the relationships signified by ratios. Young children with a good understanding of numbers often use expressions such as 'twice as much' or 'half as much'. When children use these expressions, you should take advantage of the opportunity to help them make connections and develop reversible thinking.

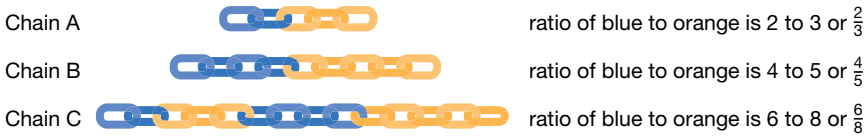


FIGURE 13.3 Combining ratios is different from adding fractions

13.2 Proportions

LEARNING OUTCOME 13.2 Developing ideas that use student intuition and reasoning to progress their thinking about proportions.

A proportion is a statement that 2 or more ratios are equal or 'the same' — for example, that the ratio 2:3 is the same as the ratio 4:6. Alternatively, you could say that 2 ratios are proportional (e.g. 2:3 and 4:6) or not proportional (e.g. 2:3 and 4:8). Understanding the relationships in proportions and working with these relationships is termed *proportional reasoning* and has been called the 'capstone' of [primary] school mathematics (Cramer, Post & Currier 1993). It is called the capstone because proportional reasoning requires a high level of thinking, and it provides a direct link to many algebraic topics, including direct and inverse relationships. To get a sense of what proportional reasoning means, consider the following problem, which is illustrated in figure 13.4.

> A month ago, plant A was 12 cm tall and plant B was 15 cm tall. Now plant A is 18 cm tall and plant B is 21 cm tall. Which plant grew the most?

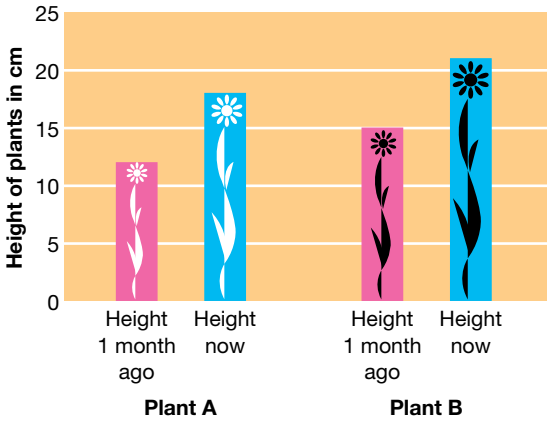
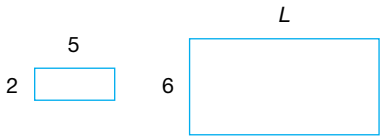


FIGURE 13.4 Which plant grew the most?

You could compare the absolute amounts that the plants grew, which might lead you to conclude that the 2 plants grew equally, since both plants grew 6 cm. Or you could make a multiplicative, or relative, comparison by looking at how much each plant grew in relation to its original height. From this perspective, plant A grew $\frac{6}{12}$ of its original height, whereas plant B grew just $\frac{6}{15}$ of its original height, which might lead you to conclude that plant A grew more. Of course, both conclusions are correct — they just reflect a different basis of comparison, absolute versus relative. However, it is the ability to make comparisons in relative rather than absolute terms that characterises proportional reasoning, and it is this ability that is the hallmark of the formal operational level of cognitive development in Piaget's system (see an earlier chapter).

Now let's look at another example. The 2 rectangles shown below are similar (i.e. have the same shape) but are of different sizes.



When asked to find the length of the side L, students frequently come up with 9, instead of 15, because they add when they should multiply. That is, they see that the height of the large rectangle is 4 units more than the height of the small rectangle, so they add the same 4 units to the length of the small rectangle to get $L = 9$. Instead, they should find the ratio of the height of the large rectangle to the height of the small rectangle $\frac{6}{2} = 3$ and then multiply the length of the small rectangle by that same ratio to find L ($5 \times 3 = 15$). The fact that the 2 rectangles are the same shape means that their corresponding sides are in the same ratio — that is, they are in proportion.

Research shows that this type of error reflects students' misconceptions about ratio and proportion and is not simply the result of carelessness (Lamon 2007; Lesh, Post & Behr 1998).

These 2 problems are similar because they both involve comparison of like quantities — plant height to plant height and length of sides to length of sides. However, problems where proportional reasoning is needed can also involve comparison of unlike quantities, such as the price comparison shown in figure 13.5.

With a problem like this, your first task is to help students clarify what the problem is asking. In this case, the question is, which buns are the best buy, or which cost less per bun? Next, you have to help students determine a method for approaching the problem. For instance, you could encourage them to construct ratio tables like these:

Number of buns	1	3
Cost		50 c

Number of buns	1	4
Cost		60 c

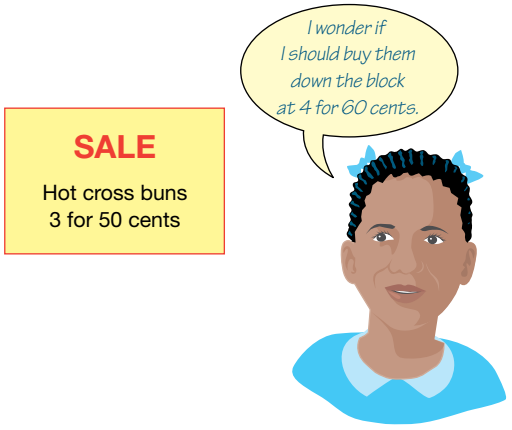


FIGURE 13.5 Comparing prices of hot cross buns in 2 places

Alternatively, you could suggest that double number lines (also called strip or tape diagrams) would be a powerful model:



With either of these models, your goal would be to help students find equivalent ways of representing the cost of a bun at the 2 stores. Both models let students come up with these equivalent representations:

- At shop A: 50 cents:3 = C:1 so C = $16\frac{2}{3}$ cents, which rounds to 17 cents
- At shop B: 60 cents:4 = C:1 so C = 15 cents

Thus, at shop A you would pay 17 cents for a single hot cross bun, while at shop B you would pay 16 cents. This means that the best buy is at shop B. Of course, if you were to define 'best buy' as other than 'cheapest', then the answer might take into consideration quality or the time it takes to travel to a particular shop.

Students will frequently encounter situations involving comparisons of like and unlike quantities, both in problems they are given at school and in their everyday lives. Generally speaking, children have an easier time working with proportions involving unlike quantities, as they are less likely to make the error of adding instead of multiplying.

You can also solve the hot cross buns problem by comparing fractions with common denominators. That is, you can represent the ratios as the quotients $\frac{1}{17}$ and $\frac{1}{15}$, which can be converted into quotients with the common denominator 255 ($15 \times 17 = 255$):

$$\frac{1}{17} = \frac{15}{255} \text{ compared to } \frac{1}{15} = \frac{17}{255}$$

This shows that shop B offers the better buy because you can get 17 buns for \$2.55, compared to 15 for the same cost at shop A.

Now let's examine another problem that requires proportional reasoning:

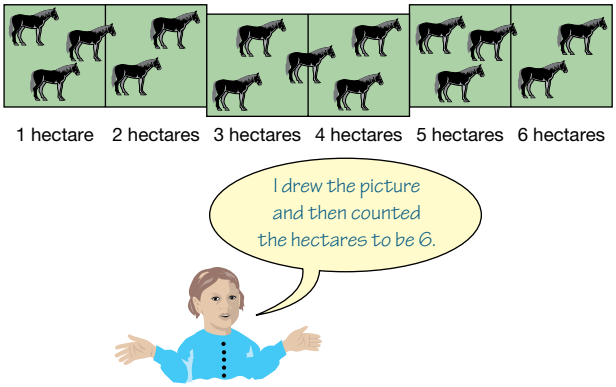
> Suppose 10 horses need 4 hectares of pasture to live comfortably. How many hectares would 15 horses need to live comfortably?

Rather than teach students an algorithm for solving a problem like this, you want them to approach it using their intuition or their existing skills. Figure 13.6 shows 5 student responses to this problem. Many students, like Karen, draw pictures (or use manipulatives); others, like Jeremy, make a list showing a pattern; some make a ratio table and use their knowledge of equivalent fractions, as Carrie did; and some find a unit rate, as Fadi did. Still others may use an incorrect strategy, like Craig.

Note that Craig's approach reflects a misconception that students commonly bring to solving proportion problems: 'When in doubt, add' (Miller & Fey 2000). The concept of proportion is closely linked to multiplication, yet it is often difficult for students to make the necessary connections. Having students share their solutions — both correct and incorrect — may help them grasp this important idea.

We have considered 3 different types of problem where proportional reasoning is needed (Kilpatrick, Swafford & Findell 2001; Miller & Fey 2000). The hot cross buns problem (see figure 13.5) illustrates *numerical-comparison problems*; the horses per hectare problem (see figure 13.6) illustrates *missing-number problems*; and the problem with similar rectangles illustrates a *scaling* problem, as does figure 13.7, which shows a collection of similar triangles and gives children opportunities to examine different patterns. Help students understand that triangle A is similar to triangle B because the ratios of the sides are equal: 1:2:2.5 = 2:4:5. Likewise, triangles B and C are similar. What about triangles D and E? Ask students: If D is similar to A, what is the length of the longest side of D? Here you have to help students see that a proportion needs to be established, and since there are several similar triangles, there are multiple solutions, each of which produces the same result. For example, you could compare corresponding sides of triangles A and D, where L indicates the longest side of D:

$$\frac{1}{4} = \frac{2}{8} = \frac{2.5}{L}, \text{ so } L \text{ must be } 4 \times 2.5, \text{ or } 10 \text{ units}$$



Karen : *Using a picture or a diagram* I drew the picture and then counted the hectares to be 6.

Jeremy : *Using patterns* I noticed:

- 10 horses need 4 hectares of pasture ... so
- 20 horses need 8 hectares of pasture.

Then I split the difference and got 15 horses need 6 hectares of pasture.

Carrie : *Using a ratio table* I did it this way:

Number of horses	10	15
Number of hectares	4	?

I thought 10 is to 4 as 5 is to 2 and 15 is to 6. So they need 6 hectares of land.

Fadi : *Using a unit rate or density* I thought, 4 divides into 10 two times with a remainder of $\frac{2}{4}$ or $\frac{1}{2}$. So there are $2\frac{1}{2}$ horses per hectare. Then I saw how many hectares it would take for 15 horses. I divided 15 by $2\frac{1}{2}$, or $15 \div \frac{5}{2}$ which is the same as $15 \times \frac{2}{5}$. That means that it would be 6 hectares.

Craig : *Using an incorrect addition/subtraction strategy* I figured there are 6 more horses than hectares (10 – 4). So I subtracted 6 from 15 and the number of hectares is 9.

FIGURE 13.6 Methods children use to solve the horse and pasture problem

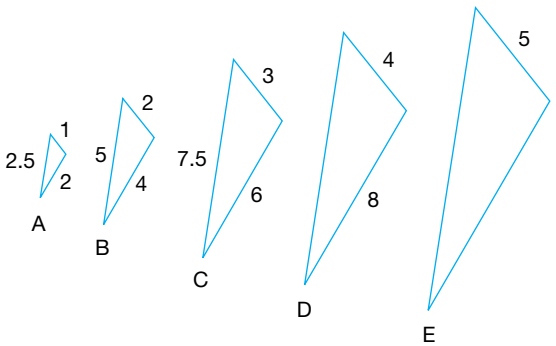


FIGURE 13.7 Some similar triangles

If you compared triangles B and D, the proportion would be:

$$\frac{2}{4} = \frac{4}{8} = \frac{5}{L}, \text{ so } L \text{ must be double 5, or 10 units}$$

Now you could ask students: If triangle E is also similar to triangle A, what proportions can be established to find the lengths of its other 2 sides, where 1 side is S_1 and the other side is S_2? At this point, students should recognise that:

$$\frac{1}{5} = \frac{2}{S_1} = \frac{2.5}{S_2}, \text{ so } S_1 = 5 \times 2 = 10 \text{ units and}$$
$$S_2 = 5 \times 2.5 = 12.5 \text{ units}$$

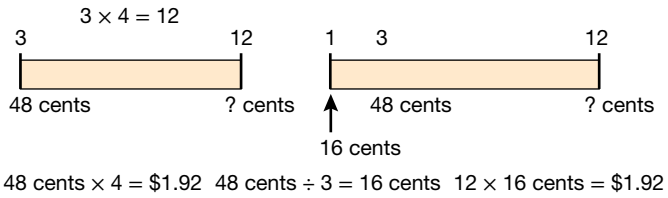
Once again, comparing different similar triangles would have involved different proportions but the same result. For example, comparing E and B gives this:

$$\frac{5}{2} = \frac{S_1}{4} = \frac{S_2}{5}, \text{ so } S_1 = \frac{(4 \times 5)}{2} = 10 \text{ units and}$$
$$S_2 = \frac{(5 \times 5)}{2} = 12.5 \text{ units}$$

Once students have made sense of proportions by solving many problems in a variety of ways, you can introduce them to a more algebraic method of solving missing-number problems. For example, consider the following problem:

> How much will a dozen balloons cost if 3 balloons cost 48 cents?

Here are visual models with the double number line for 2 popular thinking strategies upper primary children use to solve this problem:



By package-per-price strategy:
3 balloons cost $0.48
so 12 balloons is 4 × 3
so 4 × $0.48 is $1.92.
By unit-price strategy:
3 balloons cost $0.48
so each balloon is $0.16
so 12 balloons would be 12 × $0.16 or cost $1.92.

Middle school students will be able to write an algebraic equation in a variety of ways:

$\frac{3}{48} = \frac{12}{\square}$ Thinking $\frac{3 \text{ ballons}}{\$0.48}$ as $\frac{12 \text{ ballons}}{\square}$

(where the symbol □ is the cost of the balloons

$\frac{3}{12} = \frac{48}{\square}$ Thinking $\frac{3 \text{ ballons}}{12 \text{ ballons}}$ as $0.48 is to □.

$\frac{48}{3} = \frac{\square}{12}$ What would students be thinking in this case?

$\frac{12}{3} = \frac{\square}{48}$ What would students be thinking in this case?

Although students should recognise the equivalence of these statements, they should feel free to use the model or form of their choice.

TABLE 13.2 Comparing the number of horses (x) to number of horseshoes (y)

x	y
1	4
2	8
3	12
4	16
5	20
6	24
7	28

The comparisons shown in In the classroom 13.1 contain many equivalent ratios (1:5 = 2:10, 2:10 = 3:15 and so on). The concept of equivalent ratios is important and can be anchored in different ways. For example, table 13.2 shows a table displaying the number of horseshoes needed to shoe specific numbers of horses (assuming each horse is to be shod on all 4 hooves) as a relationship $y = 4x$. These tabular data provide opportunities to discover patterns, explore an algebraic relationship and observe that the ratio of all of these proportions is constant. Another approach to help students make sense of proportions is to use estimation skills. Think about the following problem.

> A trout swam 5.8 metres in 3 seconds. At this rate, how far could it travel in 19 seconds?

Students could think:

> That's about 6 metres in 3 seconds or 2 metres per second. And 2 metres per second times about 20 seconds would give 40 metres.

Another student could say:

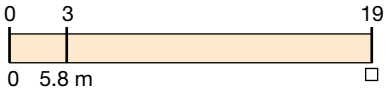
> 19 seconds is about 6 times the 3 seconds, so approximately 6 metres times 6 would be about 36 metres.

Although each estimate is different, each of them is reasonable. This type of estimation thinking is very productive. It should be both encouraged and rewarded. It uses estimation along with ratios to produce ballpark answers. Frequent experiences similar to this one will improve students' judgement, making them less likely to fall victim to unreasonable answers resulting from indiscriminate number crunching. Suppose the student mistakenly set up the proportion:

$$\frac{5.8}{3} = \frac{19}{\square}$$

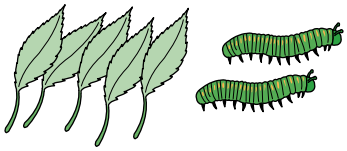
and then correctly computed that $n = 9.8$.

What mistake did the student make in this proportion? How could he determine that he was making an error in his thinking? What would a correct proportion be? Using estimation to produce a ballpark result might help the student realise that 9.8 is an unreasonable answer and encourage the student to reflect on the procedures used.



Of course, using a double number line model provides a model for a correct ratio, $\frac{3}{5.8} = \frac{19}{\square}$, but also allows the rejection of 9.8 as a reasonable answer. Figure 13.8 shows an item given to Year 4 students on a national assessment (Kenney, Lindquist & Heffernan 2002). Examples of 2 correct and incorrect responses are also shown.

The incorrect answers demonstrate how some children considered this an additive rather than multiplicative problem. The overall performance was low, as only 6% of the students gave a correct answer with a correct explanation, and another 7% gave a correct answer without an explanation or made an error using a correct method. The majority (86%) of the Year 4 students gave an incorrect response.

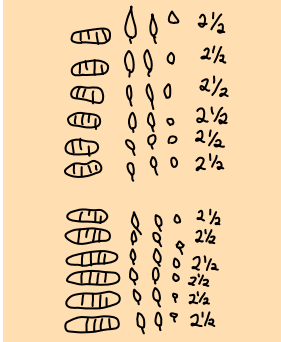


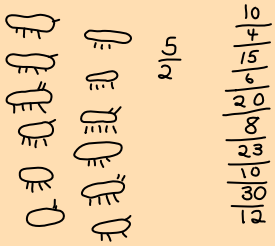
A Year 4 class needs 5 leaves each day to feed its 2 caterpillars. How many leaves would they need each day for 12 caterpillars?

Answer _____

Use drawings, words or numbers to show how you got your answer.

Correct





Incorrect

They added 10 caterpillars, and so I added 10 leaves.

leaves	caterpillars
5	2
6	3
7	4
8	5
9	6
10	7
11	8
12	9
13	10
14	11
15	12

FIGURE 13.8 Samples of students' solutions to a problem involving proportion

Consider using other real-life applications of proportions with which children are familiar (Reeder 2007). This is illustrated by the comparisons between various colour mixtures as seen in this chapter's Snapshot of a lesson. Other real-world applications include recipes, sports data, animal size comparisons, maps and scale drawings or models. Many of these contexts fall within the Australian Curriculum under the mathematics Measurement and Geometry strand (see table 13.3), rather than the Number and Algebra strand that has been the focus of many of the examples so far in this chapter.

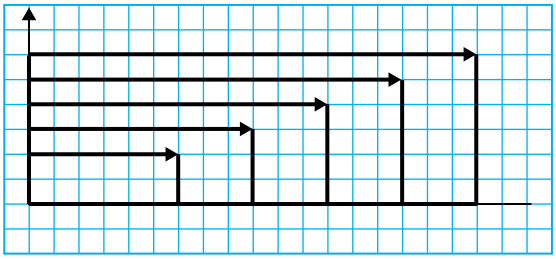
Using a zoom in or out on a computer screen or when photocopying both illustrate proportions between the original and its new image. The concepts of ratio and proportion can be naturally connected to geometry problems, as similarity is based on proportions. Two figures are similar if their respective sides are in the same ratio (i.e. proportional). Thus, all squares are similar, but all rectangles are not. Figure 13.9 shows drawings of similar rectangles. Notice that in each of the 5 rectangles, the ratio between the vertical side and the horizontal size is constant. For example, in the smallest rectangle, the

ratio is 2:6 and in the largest rectangle the ratio is 6:18. The rectangles in figure 13.9(b) show that a single line passes through the vertex of each of these rectangles, and this demonstrates that the slope of the diagonals (i.e. the ratio of the vertical height to the horizontal length) of each of these rectangles is the same. This is visible from figures 13.9(a) and 13.9(b). The ratio table shown in figure 13.9(c) shows these lengths and the equivalence of each of the ordered pairs shown. The pattern can be extended in the graphs as well as in the ratio table. An examination of these ratios, such as 2:6, 3:9 or 5:15, confirms that each of them is equivalent, and the resulting slope is $\frac{1}{3}$.

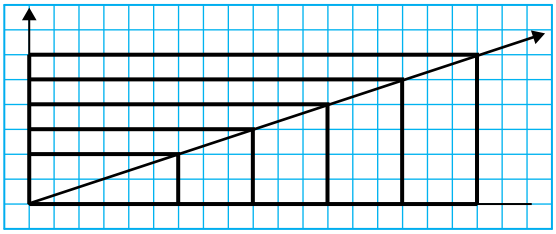
TABLE 13.3 **Ratio and proportion: Measurement and Geometry strand content descriptions for Year 4 to Year 8 in Australia**

Year level	Content description
Year 4	**Location and transformation** • Use simple scales, legends and directions to interpret information contained in basic maps (ACMMG090).
Year 5	**Location and transformation** • Apply the enlargement transformation to familiar two dimensional shapes and explore the properties of the resulting image compared with the original (ACMMG115).
Year 6	**Using units of measurement** • Solve problems involving the comparison of lengths and areas using appropriate units (ACMMG137).
Year 8	**Using units of measurement** • Investigate the relationship between features of circles such as circumference, area, radius and diameter. Use formulas to solve problems involving circumference and area (ACMMG197).

Source: © Australian Curriculum, Assessment and Reporting Authority 2015 (pp. 55–7, 65).



(a)



(b)

Length of vertical side	2	3	4	5	6
Length of horizontal side	6	9	12	15	18

(c)

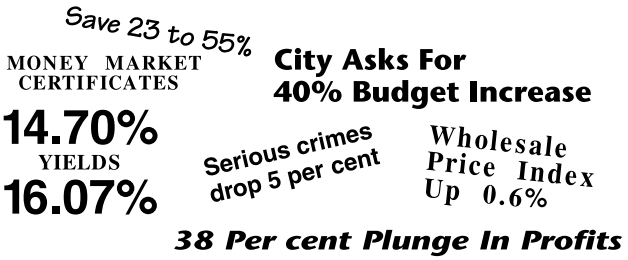
FIGURE 13.9 Drawing similar rectangles and ratio table

Children's books, such as *If the World Were a Village* (Smith 2008), provide different yet interesting contexts to explore ratios and proportions and connect them to percentages. Other excellent ideas on using literature to explore proportional reasoning are available (Beckmann, Thompson & Austin, 2004; Martinie & Bay-Williams 2003).

13.3 Percentages

LEARNING OUTCOME 13.3 Generating teaching ideas around models that help develop the concept of percentages.

You only need to read a newspaper or watch television to be reminded that percentage is one of the most widely used mathematical concepts.



Understanding of percentages is taken for granted, although there is plenty of evidence to the contrary. Incorrect usage of percentages is common among secondary students and adults. Flagrant errors abound, suggesting that often the most basic ideas are unclear (Gay & Aichele 1997).

Misconceptions, distortions and confusion surrounding percentages are surprisingly easy to find. Here are some examples.

- 'Prices reduced 100%.' If this advertisement were correct, the items would be free. Probably, the prices were reduced 50%. If an item that originally cost $400 was on sale for $200, then the ad based the 100% on the sale price, when it should have been based on the original price.
- 'Of all doctors interviewed, 75% recommended our product.' This type of claim could be an effective advertisement for a company. If, however, the ad said '3 out of the 4 doctors we interviewed recommended our product', the consumer reaction might be different. Percentages can often be used to disguise the number involved. Thus they can be misused. Percentages allow for easy comparisons because of the common base of 100, but they may appear to represent a larger sample than actually exists.

Discussing the following questions and providing real-life examples can help students develop a number sense for percentages.

- Can you eat 50% of a cake?
- Can you eat 100% of a cake?
- Can you eat 150% of a cake?
- Can a price increase 50%?
- Can a price increase 100%?
- Can a price increase 150%?
- Can a price decrease 50%?
- Can a price decrease 100%?
- Can a price decrease 150%?

Ironically, the understanding of percentages requires no new skills or concepts beyond those used in mastering fractions, decimals, ratios and proportions. In fact, percentage is not really a mathematical topic, but rather the application of a particular type of notational system. The justification for teaching percentage in school mathematics programs rests solely on its social utility. Luckily, this provides great opportunities for providing context for the mathematics that is completed with the use of real world problems that utilise percentage.

As is true with decimals and fractions, percentages express a relationship between 2 numbers. Percentages are special ratios based on 100 and without a doubt are the most widely used of all ratios. Per cent is derived from the Latin words *per centum*, which mean 'out of a hundred' or 'for every hundred'. Thus, the origin of percentage and its major uses are closely associated with ratio, fractions and decimals. Thus 25% is the ratio 25:100 or $\frac{25}{100}$, which then connects to the decimal (0.25) and the fraction $\frac{1}{4}$.

When is percentage understood? Students understand percentage when they can use it in many different ways. For instance, if a child understands 25%, he or she can do the following tasks.

- Find 25% in various contexts.
  - Cover 25% of a floor with tiles.
  - Determine 25% off the price of a given item.
  - Survey 25% of the students in class.

In many such situations, estimates of 25% are not only appropriate but essential.

- Identify characteristics of 25%.
  - 25% of the milk in a glass is less than half.
  - If 25% of the milk in a glass is spilled, then 75% remains.
- Compare and contrast 25% with a range of other percentages and numbers such as 5%, 50%, 100%, one-quarter, one-half and 0.25.
  - 25% is half as much as 50%, one-quarter as much as 100%, 5 times as much as 5%, less than one-half, and the same as one-quarter and 25 hundredths.

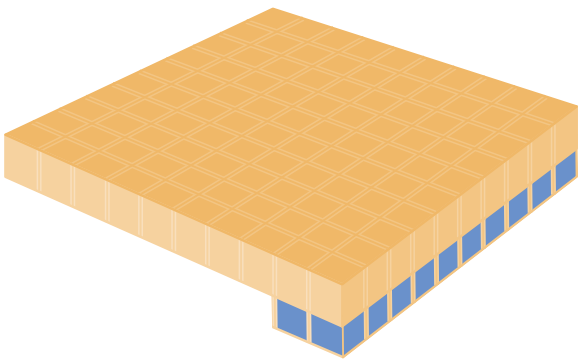
Early development of percentage can be facilitated with literature connections. For example, *If the World Were a Village* (Smith 2008) provides data and a relevant context to consider the meaning of percentage.

Understanding percentages

Percentages should be introduced only after students thoroughly understand fractions and decimals. Percentages are not studied extensively in primary school, although they can typically be introduced in real-world examples in Year 5 or Year 6 or even earlier.

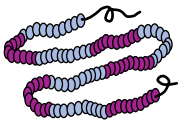
Initially, students need a variety of experiences with the fundamental concepts of percentage, and these experiences should be connected to various concrete models and real-world contexts. Computation applying percentages in problem-solving situations is generally reserved for later. Students who understand that per cent means parts out of 100 and have a good pictorial representation of percentage are more successful in solving percentage problems than those who do not. Research has found that students naturally use benchmarks to make initial judgements about percentage situations (Lembke & Reys 1994). Helping students develop the concepts for common percentage benchmarks and their fraction and decimal equivalents — such as 10%, 25%, 33%, 50% and 75% — helps them apply their percentage number sense to problem-solving situations.

Initial instruction should build on familiar models (Gay & Achiele 1997). Students will be familiar with base-ten blocks for place value, fractions and decimals, and in particular that a 'flat' is made up of 100 units. Hence, base-ten blocks provide a context for connecting percentages to fractions and decimals. For example, 2 'longs' are 20% of a flat.

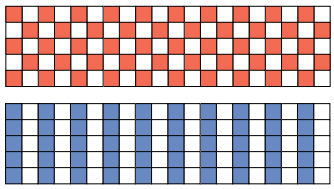


Using this model, a wide range of percentages can be easily illustrated, including 10%, 1%, 99%, 25%, 50% and 200%.

These percentages should be illustrated with a variety of different situations and models. For example, a rope with 100 discs arranged in multiples of 10 in alternating colours shows 50% of each colour.



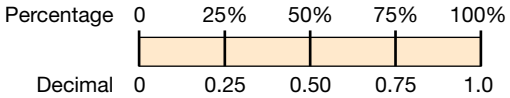
These 2 rectangular grids also show 50%:



Percentage bars can be used to show percentage. This bar shows 75%:



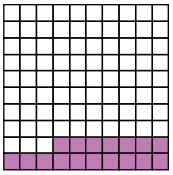
A metre ruler provides another easily accessible and effective model. Cover part of the metre ruler with paper and ask children to estimate the percentage of the metre ruler that is covered. Also, to remind them of the notion of 100%, be sure to ask: 'About what percentage of the metre ruler is not covered with paper?' This model allows many different situations (25%, 50%, 1% and so forth) to be presented and discussed quickly. Patterns may also emerge as students realise that the sum of the covered and uncovered portions always totals 100%. The percentage bars can also be used to show connections between percentages and decimals:



Shaded fraction circles (see Appendix C) can also be used to provide multiple representations and help students see connections between fractions and percentages.

These early experiences with percentages should be followed by activities that centre on direct translation experiences involving 100. Having students use percentage bars and base-ten blocks or share decimal paper provides for concrete models for percentages and helps students make connections between decimals and percentages.

Figure 13.10 illustrates how the same diagram can be represented symbolically by a fraction, a decimal and a percentage. It should be emphasised that each small square represents 1% and the large square represents 100%. Practice activities using this model to convert percentages to fractions and decimals, and vice versa, should be plentiful. For example, figure 13.10 provides a visual reminder that 17% can also be thought of as $17 \times \frac{1}{100}$. Also, 17% can be thought of as $\frac{17}{100}$, so the concept of ratio is reinforced. This approach helps students feel comfortable with different interpretations of percentages. Calculators that convert fractions, decimals and percentages are useful when students are discovering patterns and relationships.



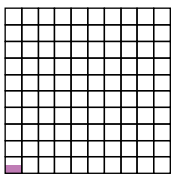
Percentage: 17% Fraction: $\frac{17}{100}$ Decimal: 0.17

FIGURE 13.10 Model and symbolisations of 17%

The importance of establishing 100 as the base for percentages cannot be overemphasised. Also, 50% should be recognised as the fraction $\frac{50}{100}$ or the product $50 \times \frac{1}{100}$. It is also important that students know that an infinite number of equivalent fractions ($\frac{1}{2}$, $\frac{2}{4}$) also represent 50%.

In the classroom 13.2 provides a natural means of developing some important ideas in an informal and yet meaningfully structured way. Each of the 4 activities should further develop children's concept of percentage. Activities B, C and D require some collection and recording of data before reporting the percentages. The use of 3 different base numbers (10 ice-cream sticks, 100 coins and 20 counters) helps strengthen the link between ratio and percentage. Even though answers for each activity depend on the data recorded, some patterns will emerge. A few questions from the teacher should trigger some stimulating discussion. For example, do the percentages in each item total 100%? Why does this happen? Can you think of a time when it would not?

One particularly troublesome aspect of percentages involves small percentages between 0% and 1%. For example, per cent, as in 'per cent milk fat', is not well understood. A visual representation, as in figure 13.11, can help show that percentage may indeed be less than 1%. Understanding rests on the earlier agreement that each small square represents 1%. As is true with all percentages, this percentage also can be shown symbolically as a fraction or a decimal. As Thomas Edison once said, 'We don't know a millionth of 1 per cent about anything.' Clearly this is a small value, and connecting this percentage to a fraction or decimal produces the example in figure 13.11:



Percentage: $\frac{1}{2}$% Fraction: $\frac{\frac{1}{2}}{100} = \frac{1}{200}$ Decimal: 0.005

FIGURE 13.11 Model and symbolisations of $\frac{1}{2}$%

IN THE CLASSROOM 13.2

Using percentages

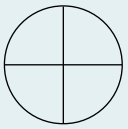
Objective: Use different models to relate quantities to percentages.

Year level: 4–5

A. Colour this circle:

- 25% red
- 50% blue

What percentage is uncoloured? _____



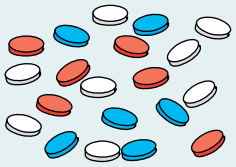
B. Use 3 colours:

- blue
- green
- yellow

Colour each ice-cream stick with only 1 colour. Colour all the ice-cream sticks.

C. Put 100 coins in a box. Shake the box. Count the number of heads, and fill in the blanks.

D. Here are 20 counters. Count the number of each colour.



	Number	*Percentage*
Red	_____	_____
Blue	_____	_____
White	_____	_____

While this computation provides a nice exercise, it is more important in primary school to establish the intuitive notion of relative size of small percentages than to devote extensive time to the algebraic gymnastics of showing the fraction and decimal equivalents.

Development of percentages greater than 100% also is challenging and should be illustrated with models. Once the idea is established that a given region represents 100%, more than one such region can be used to represent percentages greater than 100%. For example, 234% could be represented by 2 completely shaded large squares and a partially shaded one (see figure 13.12). Using every opportunity to show equivalence of percentages, fractions and decimals helps establish and maintain these relationships.

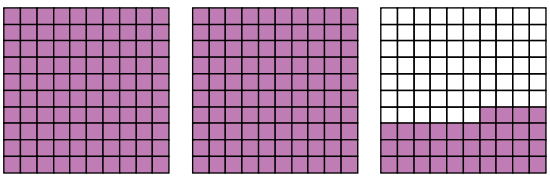


FIGURE 13.12 Model and symbolisations of 234%

Applying percentages

Early experiences with percentages should help children establish important benchmarks, such as 0%, 50% and 100%. Then other benchmarks, such as 10% and 90%, should also be examined. Familiarity with benchmarks provides conceptual anchors to use when percentages are encountered. Children come to realise that 10% is small, 50% is one-half and 90% is nearly the entire unit. In the classroom 13.3 provides opportunities to apply these percentages by making visual estimates. In In the classroom 13.3A, Bree's stack of counters serves as 100% and provides a gauge to estimate the others. The notion of 100% is implicit when the glass is full, as shown in In the classroom 13.3B, whereas a relative comparison is required in In the classroom 13.3C. In all of these activities, estimation rather than exact counts should be used to make decisions.

In primary school, students should solve percentage problems meaningfully and avoid rushing toward symbolic methods. Whenever possible, students should be asked to discuss how they mentally solve percentage problems, using what they know about common percentage benchmarks. Even when a formal method is required, using informal methods first to obtain an estimate promotes sense making and will help students focus on the reasonableness of formal results.

Although percentages are regularly encountered in many real-life problem-solving situations, only 3 basic types of problem involve percentages. Several different formal methods can be used to solve percentage problems; 2 methods typically found in primary and middle school textbooks are equation and ratio, both of which utilise algebraic thinking. The effective use of these methods requires a firm understanding of the concepts of percentage and ratio as well as the ability to solve simple equations and proportions. Such skills are developed over a period of several years and need not be rushed.

IN THE CLASSROOM 13.3

Estimating percentages

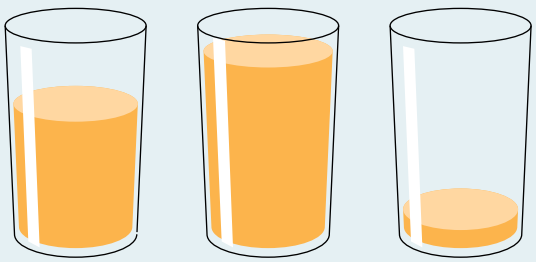
Objective: Connect benchmarks of 10%, 50% and 90% to physical quantities.

Year level: 4–5

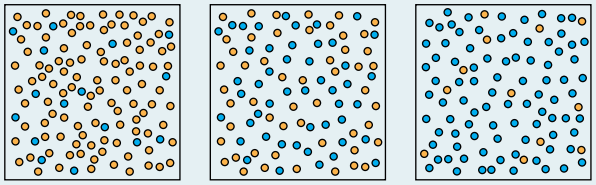
A. Bree has 100 counters. Estimate to decide who has about 90 counters, 50 counters and 10 counters.



B. Estimate to decide which of these glasses are closest to 10%, 50% and 90% full.



C. Estimate to decide in which of these sets are the blue balls closest to 10%, 50% or 90% of all the balls.



In the equation method, the following equation is used:

$$\text{percentage} \times \text{total} = \text{part}$$

The 2 known values are placed in the equation and then students solve the equation for the third, unknown value.

In the ratio method, a (part) is b (per cent) of c (total). The variables are set up as equal ratios or a proportion:

$$\frac{b \text{ (per cent)}}{100} = \frac{a \text{ (part)}}{c \text{ (total)}}$$

Students place the 3 known values into the proportion and solve for the fourth, unknown value.

Now let's take a brief look at each of the 3 types of percentage problem and how they may be solved using both informal and formal methods.

1. *Finding the percentage of a given number.* Lucas receives $60 per month for a paper route. Next month he will get a 10% raise. How much will his raise be?

The context of this problem suggests that the raise will be something considerably less than $60. The situation might be solved mentally or modelled as shown in figure 13.13.

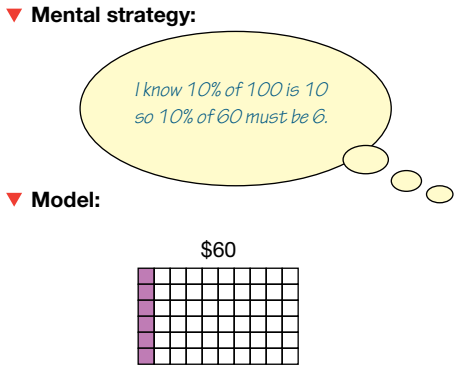
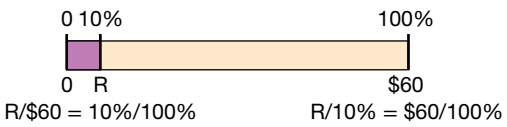


FIGURE 13.13 'How big a raise is 10% of $60?' Use a mental strategy or a model to find a percentage of a number.

The rectangular array shown in figure 13.13 is very similar to the percentage bar model, which can be used to construct several different proportions:



Either of these proportions leads to $R = \$6$. The computation is simple and may disguise the level of difficulty this type of problem presents. For example consider the following football data:

2. *Finding what percentage one number is of another number.* The Bulldogs won 15 of their 20 games. What percentage did they win?

Intuitively, it is clear that the Bulldogs did not win all of their games, so the answer must be less than 100%. Similarly, they won more than half of their games, so it must be more than 50%. The situation could be solved mentally or modelled as shown in figure 13.14.

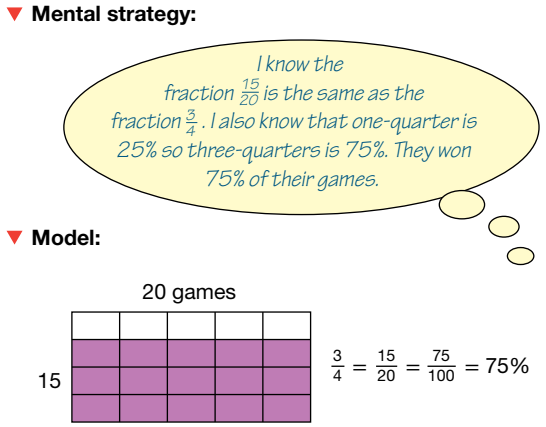
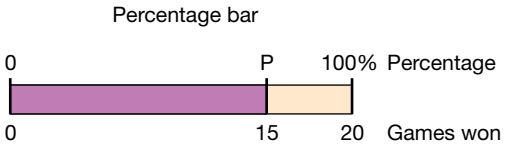


FIGURE 13.14 'The Bulldogs won 15 of their 20 games. What percentage did they win?' Use a mental strategy or model to find what percentage one number is of another number.

Here are 2 other ways to find the solution:



P/100% = 15/20 or P/15 = 100%/20

Equation method: $P \times 20 = 15$

$P = 15/20 = 0.75 = 75\%$

The percentage bar provides a model to help children construct different proportions that lead to the same result. Once again, the computation is easy, but it can be difficult for students. Similarly this next type of problem is often difficult for students:

3. *Finding the total (100%) when only a percentage is known.* The sale price on a coat was \$40 and it was marked down 50%. What was its original price?

Common sense suggests that the original price should be more than \$40. Guess-and-test is often a very effective strategy in solving this type of problem. For example, if children guess an original price of \$60, then the sale price of \$30 is too low. Still, they are on the right track, and if they continue this approach, it will eventually lead to the correct price of \$80. The problem could be solved mentally or modelled as shown in figure 13.15.

The percentage bar model provides a base for either of these solutions, where *OP* is the original price:

$$\text{Ratio method: } \frac{50\%}{100\%} = \frac{\$40}{OP} \quad OP = \$80$$

$$\text{Equation method: } 50\% \text{ of } OP = \$40$$

$$OP = \frac{\$40}{50} = \$80$$

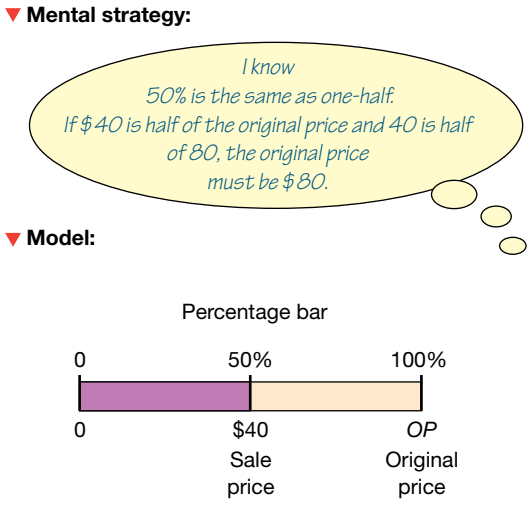


FIGURE 13.15 'A coat is on sale for $40 after being marked down 50%. What was the original price?' Use a mental strategy or model to find the total when only a percentage is known.

Research has shown that this type of problem is typically more difficult to solve (Silver & Kenny 2000). The consistently poor performance on percentage problems means that instruction must become more meaningful. If emphasis is placed on a particular method before the problem is thought through and well understood, the result will probably be confusion and poor performance. Instructional emphasis in primary school must be on thinking aloud and talking about what should be done and what a reasonable answer would be. These teacher-led discussions should occur before any serious efforts are made to solve the problem with pencil and paper. Early emphasis on writing a solution to a percentage problem forces many students to operate mechanically (without any conscious thinking) on the numbers to produce an answer. Students should be encouraged to think quantitatively in solving problems involving percentages. Research does not support the teaching of a single method. You should present a variety of problems involving percentages and then follow student leads flexibly toward solutions. Encourage students to verbalise their solutions as they engage in the actual problem-solving process. This verbalisation helps clarify what was done (right or wrong). It also encourages them to reflect on the reasonableness of an answer.

This less formal, intuitive approach lacks the structure and security of emphasising a particular method, but it has several important advantages. In particular, it encourages students to understand the problem in their own minds, along with possible solutions, and decreases the likelihood of applying a method blindly.

SPOTLIGHT ON EARLY CHILDHOOD

Making comparisons

Young children regularly compare things in mathematical ways in everyday situations by using their senses, comparative language, and possibly counting or knowledge of the relative size of numbers. Consideration of the mathematical activities that young children can engage in is detailed by Bishop (1991) in a seminal work that investigated and described the mathematical activities that he proposed are similar in all cultures. Specifically, these mathematical activities are counting, locating, measuring, designing, playing, and explaining. Bishop (1991, p. 23) described these mathematical activities as:

> motivated by, and in their turn help to motivate, some environmental need. All of them are significant, both separately and in interaction, for the development of mathematical ideas in any culture. Moreover all of them involve special kinds of language and representation. They all help to develop the *symbolic technology* which we call 'mathematics'.

As discussed throughout this chapter, children experience ideas related to mathematical thinking in their everyday lives and from a very young age. These experiences often relate to making comparisons as part of a mathematical experience that would be meeting an environmental need, which positions it beautifully within the description of mathematical activities outlined by Bishop (1991). Examples include: 'James is shorter than Li', 'the jug of milk is heavier than the box of cereal', 'the red square is bigger across than the blue square', 'there are fewer hours of daylight in the winter than in the summer', 'there are more girls than boys in our class', and 'it rained 10 millimetres today and 5 millimetres yesterday, so it rained twice as much today'. The first 5 examples make qualitative comparisons of more and less for particular attributes of people, objects or events. In contrast, the last example is a quantitative comparison of 2 specific numeric values. It would also be correct to make the numerical comparison in an additive context by saying it rained 5 more millimetres today than yesterday. However, it is a multiplicative comparison that is at the heart of the concept of ratio.

Children need opportunities to make sense of the multiplicative, as opposed to additive, nature of comparisons. At a young age they can begin to interpret and calculate within multiplicative contexts, as demonstrated in the following activities featuring 2 children, Hannah and Oscar. However, the challenge of consistently using a multiplicative relationship is also demonstrated in these excerpts.

Interview with Hannah (5 years old)

Hannah is a 'master chef' making a pancake lunch by tripling the pancake recipe. The original recipe has 1 egg, 1 cup flour and 300 mL milk.

Hannah: 3 cups of flour. [*Hannah holds up 3 fingers*.]

Teacher: How did you know that?

Hannah: Because I put in 3 eggs.

Teacher: If we normally do 300 millilitres of milk, how are we going to triple that?

Hannah: 400?

Teacher: 400. Why do you think that?

Hannah: Because we're doing a bit more.

Teacher: A bit more. How much more? If you triple things how much more is that?

Hannah: 1 more. [*Takes time to think in her head*] 2 more.

Here, Hannah correctly multiplies 1 by 3 for the eggs and flour, but turns to an additive answer for the amount of milk.

Interview with Oscar (7 years old)

Oscar is making a red cordial drink, using a tablespoon as the unit of measurement. He has mixed up 5 different concentrations of cordial into small glasses and then begins to explain some of his reasoning in comparing the relative strength of the different mixes.

Comparing 3 red cordial and 1 water (3r + 1w) and 3 red cordial and 5 water (3r + 5w)

Oscar: Yes [*Stronger*]. This one [*pointing to 3r + 1w mixture*] has more red cordial than water . . . but this one's [pointing to 3r + 5w mixture] got more in it. Just because we've got bigger numbers doesn't mean that it can be stronger if one of them is weaker. If the cordial is weaker . . . it has got bigger numbers, but it's got the same amount of red, but more water. So that's why it is weaker.

Comparing 2r + 8w and 1r + 4w

Oscar looks at the 2 glasses and has another taste from each, and then switches the order.

Oscar: This one's [*pointing to 2r + 8w mixture*] a bit weaker. And I think that because it's got 6 more waters and this one's only got 3 more waters.

In the first comparison, Oscar recognises that the strength is related to both numbers and cannot be determined merely by the total quantity, which is an essential aspect of understanding ratio and proportion as part–part and part–whole comparisons. In the second comparison, the dominance of an additive interpretation to situations is evident, indicating how children need to be exposed to ratio situations early in their education so that they can develop the related concepts.

MAKING CONNECTIONS

Ratio, proportion and percentage lend themselves easily to a wide range of practical uses in cross-curricular learning and everyday tasks — including financial literacy, issues pertaining to sustainability, maps and scale drawings, and scientific investigations.

In financial literacy, ratios, proportions and percentages are essential in daily activities in which money features (e.g. shopping, reading a newspaper, listening to a news report, paying bills, interpreting advertisements or understanding documentary text or video). Teachers and students only need to be attentive to their daily environments to find authentic data and information that require ratio, proportion or percentage for analysis. Examples include cost comparisons for 'best buys', profit and loss analysis, GST calculations, changes in utility costs, fluctuations in oil and petrol costs, and changes in the cost of technology for items such as computers, mobile phone plans and video games.

Ratio, proportion and percentage figure greatly in any critical consideration of sustainability issues concerning demographics, population changes (human or animal) or consumption of resources. Specific examples include reporting of percentages of people who use various modes of transport to travel to school or work (e.g. car, public transport, walking or cycling); changes in populations of endangered species; home, business, industrial and agricultural use of water; ratios of crop or animal land area needs in different regions; and cost comparisons between different forms of electricity production. The Australian Bureau of Statistics makes available online very comprehensive data pertaining to some of these examples.

Scales are a key component of maps and are also essential in architecture and design, art, sewing and some scientific endeavours. Specific examples that teachers can use to develop meaningful classroom mathematics activities include examination of different map scales used for different needs such as topographical maps, city map books, state maps, country or continental maps, or world maps; scale drawings of animals or plants within science texts (e.g. insects, flowers, snakes and birds); floor plans for a new school library and technology centre; enlarging patterns or designs for art or sewing; or the design of a garden. Digital technologies are valuable tools for use in many of these examples.

In addition to the use of scale drawings, science requires the understanding and use of ratio, proportion and percentage in a wide range of contexts. Some examples may be accessible as well as interesting to children of all ages — for example, size comparisons of various whales or between whales and other animals; size comparisons of dinosaurs; distance and size comparisons for planets in the solar system; or body part ratios when using archaeological bone discoveries to construct models of prehistoric men or animals.

A GLANCE AT WHERE WE'VE BEEN

Ratios compare 2 or more numbers. They take different forms and have many applications; money, measurements, consumer purchases, scale drawings and blueprints are but a few. Ratio tables provide an excellent model for exploring patterns involving ratios.

A proportion provides a relationship between 2 or more ratios. Together with proportion, ratios provide an opportunity to practise many computational skills as well as strengthen problem-solving skills. Proportions provide a way to find answers to problems where the numbers are relational. These relationships may be considered additive (absolute) or multiplicative (relative). Children often apply additive procedures to proportions that require multiplicative thinking, particularly in dealing with like quantities. The ability of children to apply multiplicative thinking or proportional reasoning is an important instructional goal and is one of the reasons proportional reasoning has been called the capstone of primary school mathematics. The use of the double number line is a powerful model for helping children engage in proportional reasoning.

Proportional reasoning is complex, both in terms of the mathematics and of the developmental experiences it requires. Yet it is an important skill for students to gain because it facilitates algebraic thinking. Ratios and proportions also provide a natural means of studying percentages, which has a comparison base of 100. Because few mathematical topics have a more practical usage than percentages, it is essential that meaningful and systematic development of percentages be provided. Instruction should use concrete models to support foundational percentage concepts. Key among those concepts are benchmarks of 100% and 50% as well as 90%, 10% and 1%. These benchmarks provide anchors for students to solve a wide range of problems involving percentages and to gauge the reasonableness of their answer.

THINGS TO DO: FROM WHAT YOU'VE READ

1. Give a real-life example of a ratio and a proportion. How could you help children distinguish between them?
2. Describe how estimating the number of heartbeats in an hour or in a lifetime uses ratios. How could you use a calculator or a spreadsheet to investigate this context?
3. Describe how proportions could be used to compare 2 products: If a 400-gram can of pears sells for $1.90 and a 600-gram can sells for $2.65, which one is the better buy?
4. Describe how you could use a metre ruler, graduated in millimetres, to illustrate each of the following percentages:
   (a) 35%
   (b) 3.5%
   (c) 0.35%
5. If an electricity bill for a 6-month period is $480 and electricity costs are due to increase by 15% for the next 6-month period, how would you estimate the quantity of the 15% increase? Explain how knowing a way to find 10% of a number mentally would be useful.
6. Rose was making $45 000 per year. She received a 10% raise. Later in the year, the company started losing money and reduced all salaries by 10%. Rose said, 'I'm making less money than last year.' Is her thinking correct? Why or why not?
7. Describe how you would think through a solution to this problem: 'The population of a city increased from 200 000 to 220 000. What is the percentage increase?' Demonstrate 2 different ways to solve the problem.

THINGS TO DO: GOING BEYOND THIS TEXT

IN THE FIELD

1. *Paper ratios*. Explore the paper sizes from A0 to A5. Take the 6 paper sizes into the classroom and let students handle them. Discuss how they are related to each other and how their sizes could be represented. Get students to share how they see the paper sizes as relating to each other.
2. *Making change*. In the classroom 13.1 provides a structured activity for exploring patterns with coins and connecting them to ratio. Use a shop corner setup in your classroom to encourage students to 'make change' for each other. Post questions that students can ask and have students contribute ways of swapping different coin denominations for others. Ask them to include how they thought about the values of the coins they were swapping to and from.
3. *Multiplicative thinking and place value*. Focus on place value and get students to explain how they move between the place value columns. Suggest students represent their ideas and strategies with objects and diagrams. Ask students to share how they can use their objects or diagram to help explain place value. What other mathematical words did they find useful and why might this be? As an educator, how can you use this experience to shape children's mathematical language?

IN YOUR JOURNAL

4 Ratios may be used to determine the best buy. Use local newspaper advertisements to find information and set up ratios to compare prices for different sizes of the same product. Respond in your journal to this question: Can you refute the claim 'the larger the quantity, the lower the unit price'?

WITH ADDITIONAL RESOURCES

5 Read the articles 'Make your own paint chart: A realistic context for developing proportional reasoning with ratios' (Beswick 2011), 'Ratio tables to promote proportional reasoning in the primary classroom' (Dole 2008) and 'Investigating mathematical thinking and discourse with ratio triplets' (Canada, Gilbert & Adolphson 2008). Describe the mathematical learning principles and instructional strategies used in these articles to help students understand and solve problems using ratio and proportion within numerical contexts.

6 Read the articles 'My favourite ratio — An inquiry about pi' (Brown, Watson & Wright 2011), 'Similar shapes and ratios' (Frost & Dornoo 2006) and 'The golden ratio: Real-life math' (Bellomo 2011). Describe how you could use ideas in these articles to develop lessons that examine ratio and proportion within geometric contexts.

WITH TECHNOLOGY

7 The golden ratio is used to describe nature, architecture, and human physique. MathsisFun (www.mathsisfun.com/numbers/golden-ratio.html), the article by Mario Livio on the golden ratio and aesthetics (www.plus.maths.org/content/golden-ratio-and-aesthetics), and NRICH (www.nrich.maths.org/7668) all explore the golden ratio. Review several websites, then create activities that can help children explore the golden ratio, focusing on how technology could aid in their investigation.

CHILDREN'S LITERATURE RESOURCES

Ratio, proportion, and percentage connect children's mathematical experiences with the real world. Ideas about ratio are evident in many children's stories, which enable children to engage with these ideas. Review your favourite children's books and look for ideas about ratio, proportion, and percentage, then share your passion for this book with the children you will work with. Websites are invaluable and quality websites will have a variety of resources – suggestions for experiences, activities, and online games. Look through your favourite reputable educational websites and consider how they can be used in mathematical experiences. Additional examples of books and website resources are provided below.

Clement, R. (1990). *Counting on Frank*. Sydney: William Collins.
Frank is a dog and his owner, a boy, is the book's narrator. This boy has imaginative, unique ways of looking at the world that are rich in mathematical concepts concerning counting and measurement. Many of the examples are ideal for exploring ratio and proportion in the context of enlargements — for example, if a mosquito was 4 million times bigger or if Frank was a humpback whale. The Illuminations website has activities based on the ideas behind this book, see www.illuminations.nctm.org. Think about how you could create a learning experience using these resources.

Math Playground
The website Math Playground has a range of resources that include videos and online activities that children can use to help develop understanding of ratio, proportion, and percentage. Look through these resources (specifically, www.mathplayground.com/index_ratio_proportion_percent.html) and consider how you could use them with students in a classroom situation.

The ratio of Pi.
The ratio of the circumference of a circle to the diameter is a constant called Pi. The Illuminations website has a circle tool (www.illuminations.nctm.org/Activity.aspx?id=3547) enables children to

manipulate a circle size and change the values of the radius, the diameter, the circumference and the area. The tool is effective in showing the ratio as it completes the calculations for all of the circles to enable students to see the ratio.

STEM explorations.
The Teaching Engineering website provides a range of classroom experiences that apply mathematical skills and knowledge. Several activities are related to the ideas behind ratio and proportion. Although the website uses imperial measurement, it is still of benefit for Australian and New Zealand students. Activities include investigating 20/20 vision, discovering Phi, and exploring gears. Have a look through the activities (www.teachengineering.org/browse_activities.php) and consider how you could use these in the classroom. Identify the relevant mathematical skills that students will use when engaging with these experiences.

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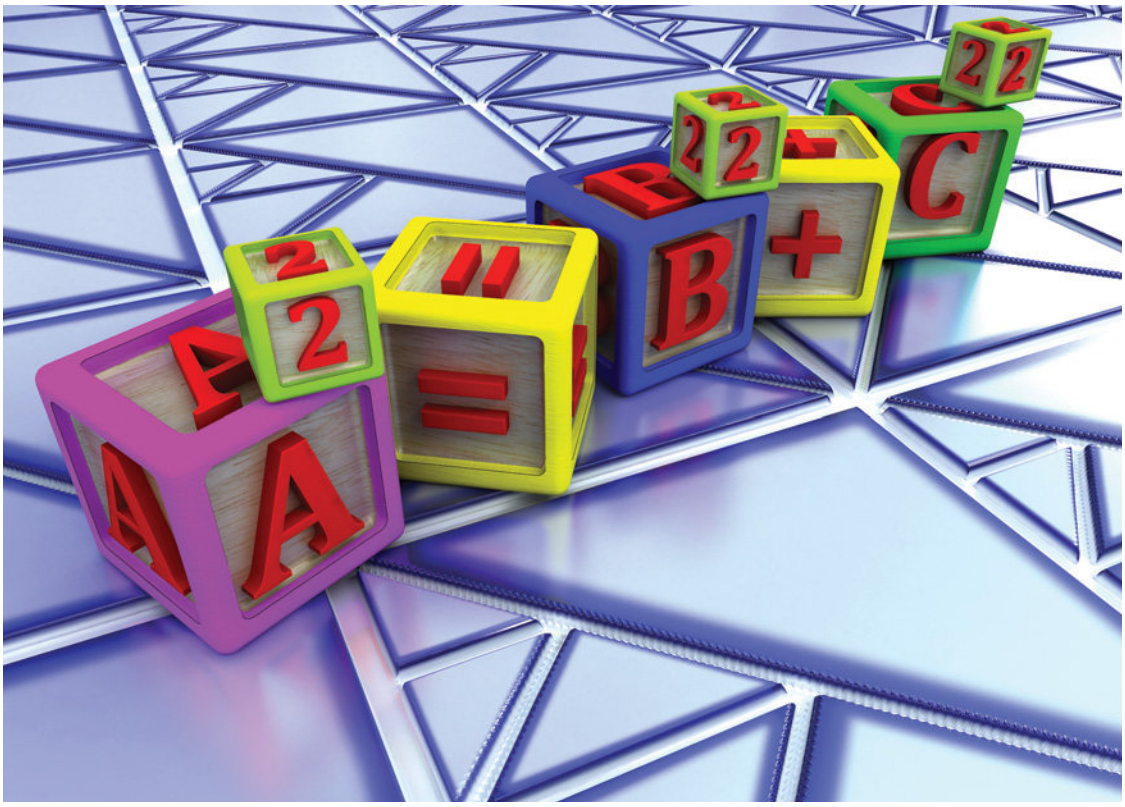
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CHAPTER 14

Extending students with number theory

LEARNING OUTCOMES

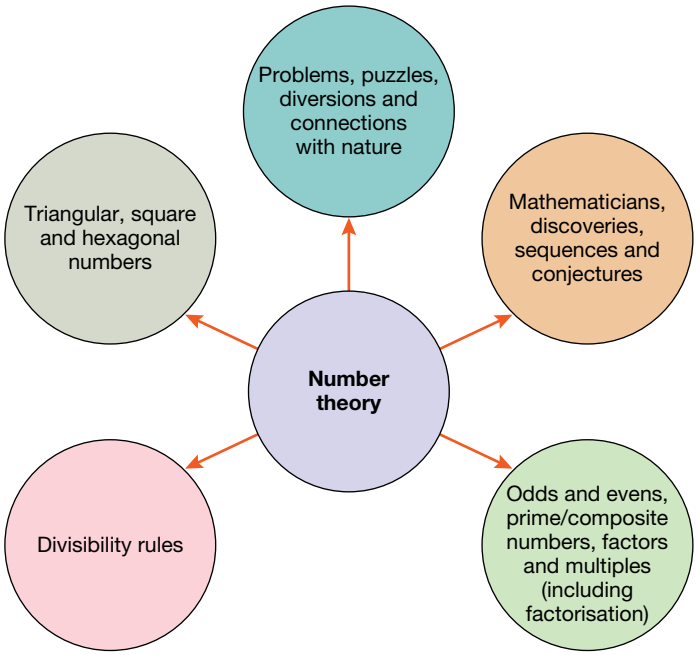
14.1 Establishing why students should study number theory topics in primary school.
14.2 Identifying what number theory topics are appropriate for students in primary school.
14.3 Describing additional number theory topics that extend and complement the teaching and learning of mathematics.



'Pure mathematics is, in its way, the poetry of logical ideas.'

Albert Einstein

Chapter 14 concept map



Introduction

Number theory is a branch of mathematics, mainly concerned with the integers, that has been a topic of study for centuries. For many years its application to practical situations was limited, but with the power of computers it has become important to cryptography, random number generation and coding theory. These applications are beyond most primary students; however, there are many topics that are appropriate for students to investigate. In exploring these topics, students can be given the opportunity to view mathematics in a way that will help make mathematics enjoyable and a greater part of their own lives.

Number theory allows the opportunity to extend and connect mathematical ideas met in other chapters. It is seen as an essential sub-strand in the Australian Curriculum as illustrated in table 14.1.

TABLE 14.1 **Number theory: Number and Algebra content descriptions in the Australian Curriculum**

Year level	Content description
Year 3	• Investigate the conditions required for a number to be odd or even and identify odd and even numbers.
Year 4	• Investigate and use the properties of odd and even numbers.
Year 5	• Identify and describe factors and multiples of whole numbers and use them to solve problems.
Year 6	• Identify and describe properties of prime, composite, square and triangular numbers.
Year 7	• Investigate index notation and represent whole numbers as products of powers of prime numbers. • Investigate and use square roots of perfect square numbers.

Source: © Australian Curriculum, Assessment and Reporting Authority 2015.

14.1 Number theory in primary school mathematics

LEARNING OUTCOME 14.1 Establishing why students should study number theory topics in primary school.

There are 4 reasons we think number theory topics provide the opportunity for students to enjoy mathematics and see how it connects to their lives. Each reason why is briefly discussed and illustrated with an activity. As a future teacher it is worthwhile taking the time to do at least 1 of these activities before reading the remainder of the chapter. If there are terms that you do not understand in these activities, look up the term in a mathematics dictionary or online search.

Number theory is a prime source to show that numbers can be fascinating. From the time of ancient civilisations, people have been fascinated with numbers and number patterns. Ancient peoples often thought numbers had mystical qualities. *Numerology*, which dates back to Pythagorean times, deals with people's birth numbers, year cycles and life stages, has been studied by many over time. Number theory, however, looks at the mathematical properties of numbers. Have you thought about why we have 60 minutes in an hour? What are the properties of 60 that make this appropriate? Would 40 have worked as well?

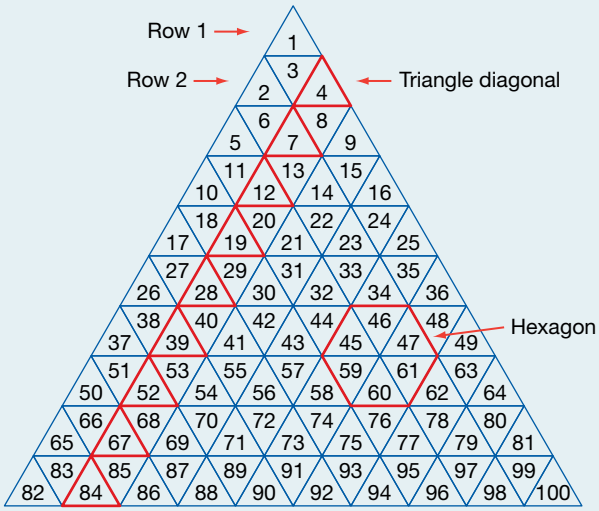
We will look at several number-theory sequences in this chapter. To begin your thinking about how a different arrangement of a number sequence can bring out various patterns, consider the hundred chart arranged in the triangle in In the classroom 14.1 (see Appendix C for extra copies). In the classroom 14.1 asks a few questions about patterns that show up in this arrangement of the numbers.

IN THE CLASSROOM 14.1

What do you see in me?

Objective: Explore patterns in a special hundred chart.

Year level: 5 and above



1. Where are the perfect squares?
2. Where are the odd numbers?
3. Find the sum of the upright diamonds (such as 1 and 3, 2 and 6). If you do this in order, what is the pattern of the sums?
4. Find the sum of the numbers in each row. Find a shortcut for finding the sum.
5. Choose a hexagon. What is the sum of the numbers? Find a shortcut.
6. Take any 2 adjacent numbers in a triangle diagonal — for example, 4 and 7. Where is the product? Does this always work?
7. Find another pattern and describe it.

Number theory opens the doors to many mathematical conjectures. Many conjectures in number theory are easy to state but may be difficult to prove. For example, the famous conjecture by Goldbach is easy enough for Years 4 and 5 students to understand. In 1742, Goldbach conjectured that *any even number greater than 2 could be written as the sum of 2 primes*. For example, 48 is the sum of the 2 primes, 19 and 29. Some numbers, such as 10, have more than one way to express the sum. This conjecture has never been fully proved or disproved, so don't give up on reasonable conjectures that students make but cannot prove.

Children in upper primary can gain quality number practice in finding primes and seeing patterns if they test the conjecture for the first hundred even numbers. Teachers who ask children to do this investigation might put up a chart like the one below, with the numbers in order, and let children write the sum they find.

Number	Sum of 2 primes
4	2 + 2
6	3 + 3
8	3 + 5
10	5 + 5, 3 + 7
12	5 + 7
14	7 + 7
16	

Watch for children who write 7 + 9 for 16 because they see a pattern. What is wrong with that sum? Does it satisfy the conjecture? No, an important part of the conjecture is that the addends must be primes. Nine is, of course, a square number.

In the classroom 14.2 presents 5 conjectures with made-up names that are not true. As will be discussed in the next chapter, it only takes one example to disprove a conjecture. Can you find such an example? This activity is mainly for the pre-service teacher; however, you will have some students who enjoy the challenge of working on such problems. You could use a chart like that suggested for Goldbach's conjecture to engage students in exploring these conjectures.

IN THE CLASSROOM 14.2

Do you believe that?

Objective: Work with number properties.

Year level: 6 and above

Investigate one of these conjectures about *positive whole numbers*. See if you can find an example that disproves it. The examples show that each conjecture works for some numbers.

1. Tinbach conjecture:

Every number can be expressed as the difference of 2 primes. Examples:

$14 = 17 - 3$ $\quad$ $5 = 7 - 2$

$28 = 31 - 3$ $\quad$ $11 = 13 - 2$

2. Zincbach conjecture:

Every number can be expressed as the sum of 3 squares (0 permitted). Examples:

$3 = 1^2 + 1^2 + 1^2$ $\quad$ $14 = 1^2 + 2^2 + 3^2$

$9 = 3^2 + 0^2 + 0^2$

3. Aluminumbach conjecture:
Every odd number can be expressed as the sum of 3 primes. Examples:

$15 = 5 + 5 + 5$ $\quad$ $11 = 3 + 3 + 5$
$21 = 7 + 7 + 7$

4. Brassbach conjecture:
Every square number has exactly 3 divisors.

Examples: 1, 2 and 4 are the 3 divisors of 4;

1, 5 and 25 are the 3 divisors of 25

5. Copperbach conjecture:
The product of any number of primes is odd. Examples:

$5 \times 3 \times 7 = 105$ (3 primes)
$7 \times 3 \times 3 \times 5 = 315$ (4 primes)

With a minor change in the wording, some of these conjectures are true. See if you can find which ones and make the change.

Throughout this chapter, you will find many conjectures that are well within the ability of primary students to explore and to make convincing arguments about their validity.

Number theory provides an avenue to extend and practise mathematical skills. We often need ways to help differentiate students' learning within a topic. For example, it is the beginning of the topic 'Factors and multiples' in a Year 5 class and students are finding factors of whole numbers. Although all children may benefit from some review, they will not need the same amount. Number theory often provides a way to both review finding factors and challenge or extend students in an interesting fashion. In this case, you might have them investigate abundant, deficient and perfect numbers (see In the classroom 14.3).

IN THE CLASSROOM 14.3

Am I abundant, deficient or perfect?

Objective: Examine the sum of factors.

Year level: 5 and above

How to solve the problem

1. Find all the factors of the number.
   Add all the factors except the number itself.
2. If the sum is:
   - greater than the number, then the number is abundant.
   - less than the number, then the number is deficient.
   - equal to the number, then the number is perfect.

Examples

- For 18, $1 + 2 + 3 + 6 + 9 = 21$.
  $21 > 18$, so 18 is abundant.
- For 35, $1 + 5 + 7 = 13$.
  $13 < 35$, so 13 is deficient.
- For 28, $1 + 2 + 4 + 7 + 14 = 28$.
  $28 = 28$, so 28 is perfect.

Tasks

1. With your classmates, classify the first 100 numbers as abundant, deficient or perfect.
2. Are more numbers abundant or deficient?
3. Show that 496 and 8128 are perfect.
4. Explore the conjecture: If you multiply 2 abundant numbers, the product is abundant.
5. Explore the conjecture: If you multiply 2 deficient numbers, the product is deficient.
6. Explore the conjecture: If you multiply 2 perfect numbers, the product is perfect.

Before finding the sum of factors, students will need to go through the numbers from 1 to the nearest square root of the number to test for divisibility. For example, finding the factors for 36 can be done as follows:

1 goes into 36 ($1 \times 36 = 36$).
2 goes into 36, 18 times ($2 \times 18 = 36$).
3 goes into 36, 12 times ($3 \times 12 = 36$).
4 goes into 36, 9 times ($4 \times 9 = 36$).
5 doesn't divide evenly into 36.
6 is the square root of 36 ($6 \times 6 = 36$).

Now we have the factors 1, 2, 3, 4, 6, 9, 12, 18 and 36, we can find out whether 36 is abundant, deficient or perfect by adding together its proper factors (that is, excluding 36 itself). As it turns out, 36 is abundant! Note that, in this case, students would work mostly with basic facts and simple division. However, if more complex numbers were being investigated, a calculator would be a useful tool.

You will see other examples of ways to practise skills in this chapter as students investigate different number theory topics.

Number theory offers a source of recreation. Did you ever think of mathematics as a recreation? Number theory provides many puzzle-type activities that children enjoy. It is one more way to individualise your curriculum. Just as different children enjoy different sports or games, different children will find different puzzles recreational. The puzzle needs to be challenging yet in reach of the student.

In solving such puzzles, students are practising skills, developing number sense and using problem-solving strategies. For example, in the magic square (In the classroom 14.4) children soon learn that the 2 largest numbers cannot go in the same row, column or diagonal. Although this activity may be done with paper and pencil, it is better to use concrete materials such as number cards. First, the cards make the activity seem more like a puzzle to many children. Second, it saves erasing incorrect attempts. The disadvantage of the cards is that children have no record of their attempts, so they may repeat the same mistakes without realising it. This may lead to the need to record attempts on an A4 sheet of paper drawn up with many small squares of 3×3. You can increase the size of the square in the puzzle to any size or use fractions or decimals. For more information about magic squares, many online resources can be found (see 'Things to do' and 'With technology' at the end of this chapter).

The topics that we will present in the next section are only a sampling of number theory topics but include those that are in most primary mathematics resource books for students. This is followed by a brief introduction to other number theory topics. After discussing the topic, examples of activities as well as further resources for explorations are given.

IN THE CLASSROOM 14.4

Magic squares

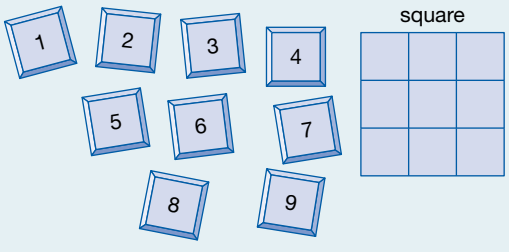
Objective: Explore magic squares.

Year level: 3–7

Materials: Number tiles or paper squares (1–9) and a 3-by-3 square.

A magic square is a square in which the sum of the numbers in each row, each column and each main diagonal is equal.

For younger students, provide this hint: In the magic square below, arrange the number tiles so the sum is 15.



To support students who are struggling let them know the middle number is 5.

Place some of the tiles in the proper places on the 3×3 square. The more numbers you specify, the easier the puzzle will be.

Extend students by using different numbers.

- Use the first 9 even numbers. The sum will be 30.
- Use the first 9 multiples of 5 (including 5). The sum will be 75.
- Use 4, 7, 10, 13, 16, 19, 22, 25 and 28. The sum will be 48.
- Use decimals (0.2, 0.5, 0.8, 1.1, 1.4, 1.7, 2.0, 2.3 and 2.6). The sum will be 4.2.
- Use fractions $\frac{1}{5}$, $\frac{3}{5}$, 1, $\frac{7}{5}$, $\frac{9}{5}$, $\frac{11}{5}$, $\frac{13}{5}$, 3, $\frac{17}{5}$. The sum will be $5\frac{2}{5}$.

Extend students by asking them to create a 4×4 magic square using the numbers 1 to 16.

You should reflect on what you have learned in this chapter and show your interest in learning more mathematics. The 'Things to do' section at the end of the chapter contains questions that will challenge you with a mathematical activity or with a question that is specific to helping children learn mathematics.

14.2 Number theory topics for primary school students

LEARNING OUTCOME 14.2 Identifying what number theory topics are appropriate for students in primary school.

Odds and evens

Classifying numbers as odd or even is one of the first number theory topics that children encounter. As children count by 2 (e.g. 2, 4, 6, 8, 10) they learn that there is something special about these numbers; these are the even numbers, and all the other positive whole numbers are odd except for 0. Young children can explore this idea by using concrete materials such as square tiles to make numbers from 1 to 20. Even numbers will pair up or make rectangles. More precisely, even numbers are those that are divisible by 2 (i.e. the remainder is 0 when divided by 2). See if you can use this definition to justify that 0 is even.

Children often rely on counting to justify which numbers are even — every other number beginning from 0 is even (and every other number beginning from 1 is odd). However, children develop other ways to determine whether a large number is even or odd. They will see that all even numbers end in 0, 2, 4, 6 or 8 and are divisible by 2. On the other hand, the odds will end in 1, 3, 5, 7 or 9 and give uneven pairs or a remainder of 1.

While this seems to be a rather simple idea, only 65% of students in Year 4 could classify a set of numbers involving 2 or 3 digits as even or odd; fewer students could apply their knowledge of even and odd to a word problem about students walking in pairs. They had a better grasp of which numbers less than 50 were even or odd (Warfield & Meier 2007, p. 45).

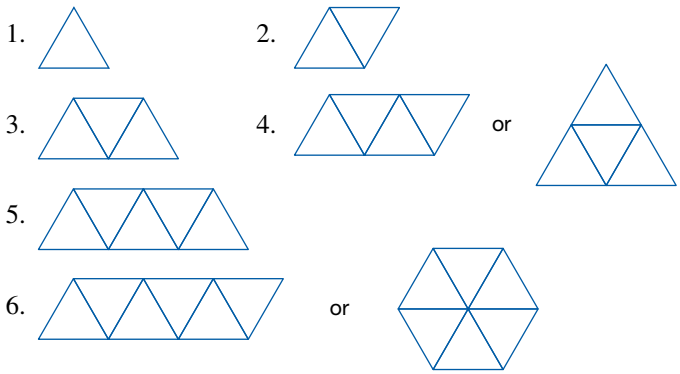
In chapter 5, there is an illustration of a student justifying the conjecture that the sum of 2 odd numbers is even (see figure 5.3). It uses ideas closely associated with pairing by 2s and pictorial representations.

The following conjectures are well within the reach of Year 3 or 4 students especially given concrete materials or grid paper and coloured pencils. You might show them the justification by the student in chapter 5 and have them decide whether these are true and justify them — or disprove them if they are false (only one is false). You can make the conjectures more appropriate for older students by using similar statements about products instead of sums or have children make and check their own conjectures.

- The sum of 2 odd numbers is even.
- The sum of 2 even numbers is even.
- The sum of 3 odd numbers is odd.
- The sum of any number of odd numbers is odd.

- The sum of any number of even numbers is even.
- The sum of 2 odd numbers and an even number is even.

Exploring the numbers from 1 to 24 and beyond with equilateral triangles (plastic or card) is an excellent way of extending the model of odd and even numbers for older students and making connections to other 2-dimensional shapes besides squares and rectangles. As shown in the following diagram, the rhombus, trapezium, parallelogram and hexagon are created from equilateral triangles and are shapes with number properties (even, odd, square, triangle and hexagonal numbers).



Returning to the example of a magic square (see In the classroom 14.4), it is possible to investigate how odds and evens can help solve the magic square. If we use the numbers 1–9, how many odd numbers are we using (and how many even numbers)? The arrangement shown below is one of the arrangements of odd (o) and even (e) numbers. Check it out. Does it have the correct number of odds and evens? Do the sums of each row, each column and the 2 diagonals produce an odd number? (Remember, the sum must be 15 for the magic square involving 1–9.) Why can't all the odds be placed on the 2 diagonals? Use the conjectures about sums of 3 numbers to explain your answer.

e	o	e
o	o	o
e	o	e

See if using this helps you complete the magic square. Look at the row and column with odd numbers. What do you notice? What 3 odd numbers add to 15? [*(3, 5 and 7) and (1, 5 and 9)*]. Place those in the puzzle. Note that there are several ways to do this; one way is shown below. Now see if you can place 2, 4, 6 and 8. You must use some number sense. Does it make sense to place 8 in the same row as 9 or 2 in the column with 1?

e	3	e
1	5	9
e	7	e

In school, using evens and odds naturally arises when grouping students for classwork. Teachers might ask students to do all odd-numbered exercises for practice. Mental arithmetic can be encouraged through

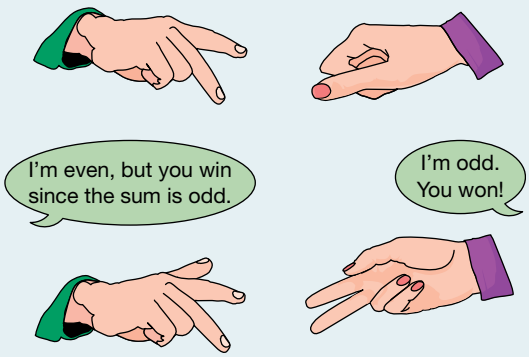
a simple game involving odds and evens as explained in In the classroom 14.5. Older students might find the sum of the first 50 odd numbers (see figure 5.5 for a hint about adding these using a staircase approach). Students could find the probability of throwing an even sum with 2 dice (see the final chapter).

Factors and multiples

While learning about multiplication and division, children have also begun learning about factors and multiples. In this section, we examine how to extend and use these ideas.

IN THE CLASSROOM 14.5

Morra: A game of odds and evens



Objective: Decide if a sum is even or odd.

Year level: 3 and above

Morra is a game from ancient times. The Greeks and the Romans had variations of it. Children enjoy it because it is similar to Rock, Paper, Scissors. The game described here is for 2 players.

Rules

One player is designated the 'odd' and the other player is 'even'.

Each player puts out 0 to 5 fingers on 'go'. (Have the players count '1, 2, 3, go'.)

If the sum is odd, the odd player gets 1 point. If the sum is even, the even player gets 1 point.

Play until one player gets the designated target number (say 10) and wins.

Variations

1. Play with 2 pairs of players as teams. One team is odd and one is even. Find the sum of the fingers of all 4 players and assign points as to odd or even. This is the same game but allows for larger numbers and more practice in adding 4 numbers.
2. Play with 3 players. Designate one player as 0, the second player as 1 and the third player as 2. Follow the same rules of putting out 0 to 5 fingers. Divide the sum by 3. The player whose number is the remainder receives the point.

Factors

A *factor* of a number divides that number with no remainder. For example, 3 is a factor of 15, since 15 divided by 3 is 5 (no remainder). Another way to look at factors is to look at which 2 numbers have a product of 15. Since $3 \times 5 = 15$, both 3 and 5 are factors of 15.

Young children can begin exploring factors by using materials. Figure 14.1 shows a story written by a Year 3 student. Her teacher gave each child a number and asked them to explore its factors. Pam decided to write about it as well as draw the groupings she had made with wrapped lollies.

The 12 Story

I wanted to share 12 lollies with friends. No one could get more. I drew pictures to show my shares.

12 of us 

we each get one. 12 = 12 groups of 1
12 = 12 x 1

11, 10, 9, 8, 7 wouldn't work, BUT
6 of us. We each get 2. 6 groups of 2
12 = 6 x 2

5 – NO

4  12 = 4 x 3

3  12 = 3 x 4

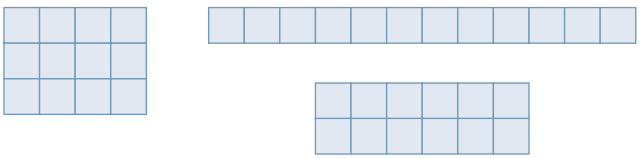
2  12 = 2 x 6

1 That's me! I get all 12, 12 = 1 x 2

Glad they are choclate by PAM

FIGURE 14.1 The story of 12 by Pam

Encouraging children to use square tiles or square graph paper to explore the factors helps them to make the connection to area. For example, have children investigate the factors of 12 by making rectangles with 12 squares.



This shows that 12 is 3 × 4, 1 × 12 and 2 × 6. Note that there are actually 6 rectangles (if you turn each of the 3 models to show 4 × 3, 12 × 1 and 6 × 2). There are 6 factors of 12 in all: 1, 2, 3, 4, 6 and 12. These are the same numbers that Pam found for the number of friends that could share 12 objects equally.

Students in upper primary need other methods of finding all the factors of larger numbers. For example, what are all the factors of 84? Read through this discussion that Mr Thomas is having with a small group of Year 6 students.

Mr Thomas: How can we find all the factors of 84?

Brian: We can divide by all the numbers . . . 1, 2, up to 84?

Damian: No way, that's 84 division problems; it's got to be simpler.

Madeline: Well, I know 1 divides everything. So 1 is a factor — that means 84 is a factor, too.

Mr Thomas: Let's record those 2 like this. You will see why.

1 84

Rosie: 2 is easy. 2 × 42 is 84.

Damian: I'll check 3. Zach, you check 4. Madeline, you check 5 and Rosie can check 6.

Mr Thomas: Good, and I will record what you find. I have to put 2 and 42 on the chart.

Damian reports that 3 and 28 are factors, Brian says 4 × 21, Madeline says 5 won't work and Rosie says 6 is a factor and that means 14 is, too. Mr Thomas records their findings, putting the pairs in colours:

1 2 3 4 6 14 21 28 42 84

Damian: We still have a lot of work. Let's see, 7, 8, 9, 10 — we have 4 more to try now.

Madeline: I tried 7 when 5 wouldn't work. It is 7 × 12.

The students work on 8, 9 and 10 and find they do not divide 84 evenly.

1 2 3 4 6 7 12 14 21 28 42 84

Rosie: When I tried 9 on the calculator, I got 9.333. Then I tried 11 and got 7 and some decimal. The other number just gets smaller — and we tried all the small ones. I think we can stop.

Brian: I see why you recorded that way; you come into sort of the middle. When the numbers almost bump, you are done.

Mr Thomas explored with the group what Rosie and Brian said; they seemed to understand that at some point you have tried enough numbers. Later, he will return to this in other examples to discuss 'when the numbers bump'.

Damian: Thank goodness. It's not as bad as I thought.

The game, Factor me out (see In the classroom 14.6), is a classic game that requires finding factors for the numbers 2–36. As children discover a good strategy, they are beginning to explore prime and composite numbers.

Multiples

A *multiple* of a number is the product of that number and any other whole number. For example, 36 is a multiple of 4 since $4 \times 9 = 36$. In looking for a multiple of a number, we usually begin with the number and generate multiples of it. The positive multiples of 4 are

$$4, 8, 12, 16, 20, 24, 28, 32, 36\ldots$$

Multiplying 4 by 1, 2, 3, 4 and so on generates this sequence.

The concept of multiple is not difficult; a multiple of a number is just the product of that number and another. However, there is often confusion between the 2 words *multiple* and *factor* when the answer is given. For example, 'What is 36 a multiple of?' is actually asking for a factor (36 is a multiple of 4, of 6 and of 7 other numbers). Similarly, 'What is 4 a factor of?' is asking for a multiple of 4. We need to be able to think in both directions: to list all the multiples of 7 and find what 42 is a multiple of. However, when first beginning the study of multiples and factors, the language is best kept to finding the factors and to finding multiples. Gradually, build the reverse questions when students are confident in their understanding.

Greatest common factor and least common multiple

We have looked at factors and multiples of individual numbers; now we will look at pairs of numbers and ask:

'What is the largest number that is a factor of both numbers (greatest *common factor*)?'
'What is the smallest number that is a multiple of both the numbers (*least common multiple*)?'

IN THE CLASSROOM 14.6

Factor me out

Objective: Explore prime and composite numbers.

Year level: 4 and above

Choose a partner and make or copy this chart:

•	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

Rules

- Player 1 chooses a number. He or she gets that many points. The opponent gets points equal to the sum of all the factors other than the number itself.
- Make a table to record scores:

Player 1	Player 2
10	8 (2 + 5 + 1)

- Mark out the number and the factors; these cannot be used again. However, 1 can be used in every turn.
- Repeat with player 2 choosing the number — for example, 31.
- Player 1 only scores 1 point as 31 is prime. The running scores are 11 for player 1 and 39 for player 2.

Player 1	Player 2
10	8
+1	+31
11	39

- Play continues until no numbers remain.

  Extend students by using a larger chart — for example, a 10 × 10 or hundred chart. Allow the use of calculators for factors larger than 10.

  The player with the most points wins.

Understanding these concepts depends mainly on knowing how to find factors and multiples and keeping straight which is which. Algorithms for finding the greatest common factor (GCF) and least

common multiple (LCM) are very similar. While this has an advantage, it can confuse students who do not understand factors and multiples or why they are finding the least or the greatest. The LCM is most often used when finding the least common denominator of fractions, and the GCF is used to simplify fractions — but with the prevalence of calculators, neither of these skills is as important as it once was.

The simplest algorithm for each is just to list the factors (or multiples) of the 2 numbers and identify the greatest (smallest) number that is common to both lists. At the primary level, it provides a good source of problem solving. In the classroom 14.7 has problems from an old textbook with corresponding simpler versions of each problem. Some students will be able to solve the more complicated version after solving the simpler one.

IN THE CLASSROOM 14.7

Problem of the day from olden days

Objective: Compare problems involving multiples and factors.

Year level: 5 and above

The first problem in each pair was taken from an 1887 arithmetic book. The second problem uses smaller numbers and the language of today. Solve one of each pair.

Cover it/Enclose it

Three rooms are 14, 16 and 19 feet wide, respectively. What is the widest carpeting that can be contained exactly in each room?

Biscuits are sold in 3 different-sized boxes: a small box contains 16 biscuits; a medium one, 24; and a large container has 36 biscuits. What is the largest common dimension that can be used for every size box — either small, medium or large?

When will we meet again?

Four men start at the same place to walk around a garden. A can go around in 9 minutes, B in 10 minutes, C in 12 minutes and D in 15 minutes. In what time will they all meet at the starting place?

Ann, Bella, Con and Dan walk around and around a path. They begin at the same place at the same time. Ann can go around once in 4 minutes, Bella in 6 minutes, Con in 8 minutes and Dan in 9 minutes. How many minutes will it be until they are all back at the starting place at the same time?

Digging

A can dig 14 rows of ditch in a week, B can dig 18 rows, C 22 rows and D 24 rows. What is the least number of rows that would afford an exact number of a week's work for each one of them.

Andy can dig 4 ditches in a day, Bev can dig 5 ditches in a day, Cal can dig 6 ditches in a day and Donna can dig 7 ditches in a day. Mr B says, 'I have enough ditches to dig so that any one of you could work all day for a number of days. Of course, some of you would have to work more days than others.' What is the least number of ditches Mr B could have?

Primes and composites

A *prime number* is a whole number greater than 1 that has exactly 2 factors: 1 and itself. Any number with more than 2 factors is a *composite number*. For example, 13 is prime since 1 and 13 are its only factors; 10 is composite since it has factors of 1, 2, 5 and 10.

The Australian Curriculum (ACARA 2015) states that prime and composite numbers be studied in Year 6. However, these topics can be explored by younger students and are another way to address the needs of some of your students.

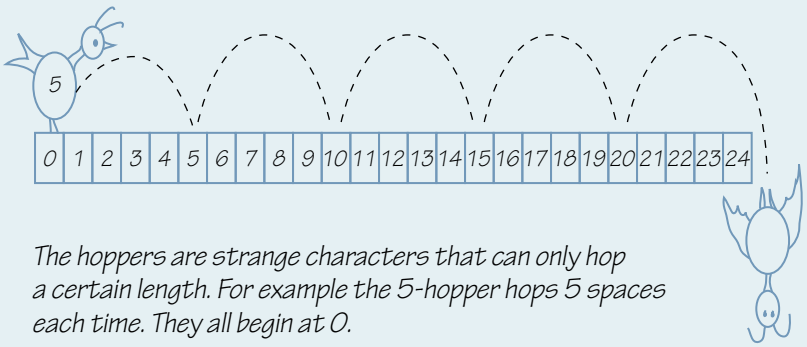
The activity about hopping on the number line (see In the classroom 14.8) introduces 1 model for determining which numbers are primes. Arranging squares to make rectangles, discussed in the section on finding factors, is another model. If only 2 rectangles can be made with a given number of squares, then the number is prime. For example, 13 is prime because only 2 rectangles can be made (1×13 and 13×1) which are the same according to the commutative law.

IN THE CLASSROOM 14.8

The hoppers

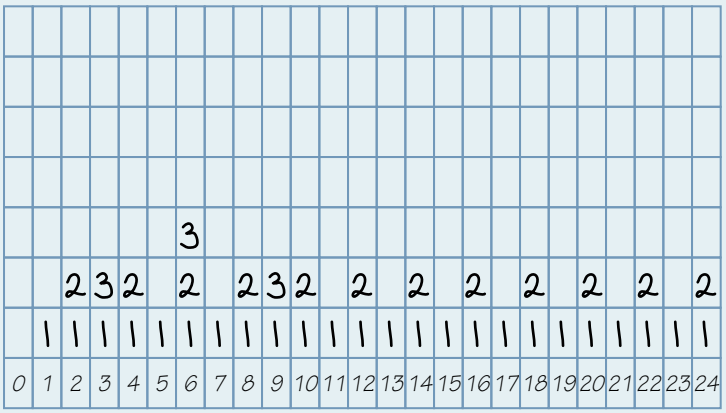
Objective: Find prime and composite numbers.

Year level: 5 and above



The hoppers are strange characters that can only hop a certain length. For example the 5-hopper hops 5 spaces each time. They all begin at 0.

See where each hopper will land. The 1-hopper and the 2-hopper have been done for you. The 3-hopper has only just started.



Be a 3-hopper. You will land at 3 (write a 3 above the 3) and you will land at 6 (write a 3 above the 6) . . . Keep going. (You should have 3s above 9, 12, 15, 18, 21 and 24.)

Do the same for the 4-hopper, the 5-hopper, all the way to a 24-hopper.

Some of the questions have been started for you:

1. What numbers had 2 hopper stops? _____2, 3_____

2. What type of number are these? _____

3. What numbers had more than 2 hopper stops? _____6_____

4. What type of number are these? _____

5. What about a 1-hopper stop?_____

Another way to find the primes is to use the Sieve of Eratosthenes (pronounced 'air-a-toss-the-knees'). The numbers are arranged in order in rows of 10, like the hundred charts in Appendix C. Beginning with 2, the first prime, you circle it and mark out all the multiples of 2. You proceed to the next prime (the

next number not marked out) and circle it, then mark out all its multiples. If you continue this process, only the primes will remain unmarked — that is, they have not dropped through the sieve.

In the classroom 14.9 has a variation of the Sieve of Eratosthenes in which the numbers are arranged in rows of 6.

IN THE CLASSROOM 14.9

Sieve: The primes remain

Objective: Generate the prime numbers from 2 to 102.

Year level: 6 and above

Directions

1. Circle the 2 and then cross out *every multiple* of 2. We have begun this below.
2. Circle the 3 and then cross out every multiple of 3. We have begun this below.
3. Continue with the next number that is not circled or crossed out. Circle it and cross out all its multiples.
4. Continue to 102. Then, answer the questions below.

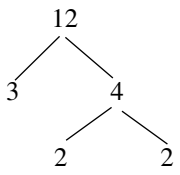
	②	③	~~4~~	5	~~6~~
7	~~8~~	~~9~~	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60
61	62	63	64	65	66
67	68	69	70	71	72
73	74	75	76	77	78
79	80	81	82	83	84
85	86	87	88	89	90
91	92	93	94	95	96
97	98	99	100	101	102

Questions

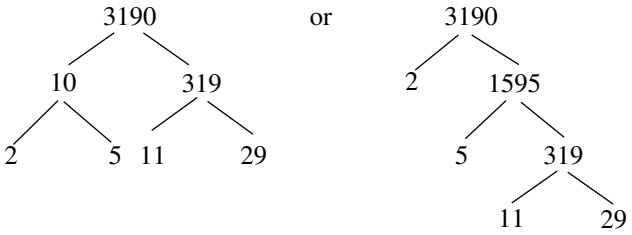
1. What type of number is not crossed out?
2. In which columns are the primes?
3. Which numbers that are not prime appear in these columns?
4. Are all the primes (except 2 and 3) in the following algebraic form?
   $(6 \times n) + 1$ or $(6 \times m) - 1$, for some whole number n or m.
   For example, $13 = (6 \times 2) + 1$ and $89 = (6 \times 15) - 1$.

Prime factorisation

The *fundamental theorem of arithmetic* is the following: Every composite number may be uniquely expressed as a product of primes if the order is ignored. For example, 12 may be written as the product: $2 \times 2 \times 3$. This is called the prime factorisation of 12. The most commonly used method in primary school to find the prime factorisation of a number is called the factor tree.



Two factor trees for 3190 are shown below. Both show that the prime factorisation of 3190 is $2 \times 5 \times 11 \times 29$.



Another method, the division method, involves repeated division by primes. All the divisors are primes. How do you know when you reach the end? What type of number is the last quotient?

$$\begin{array}{r|l} 2 & 3190 \\ 5 & 1595 \\ 11 & 319 \\ & 29 \end{array}$$

The quotient of 3190 divided by 2

Both of these methods are analogous as a prime factor must be used for each branch of the tree. No matter which method your students use, make certain that they write the prime factorisation as $3190 = 2 \times 5 \times 11 \times 29$. Other factorisations, such as $3190 = 10 \times 319$ or $3190 = 2 \times 5 \times 319$, are true statements, but they are not the *prime* factorisation as in each case they are incomplete.

Divisibility

We will now look at 2 aspects of divisibility: divisibility rules and conjectures about divisibility. Any number can be divided by any other number; however, that number is *divisible* by another number if there is no remainder; for example, 84 is divisible by 2, but it is not divisible by 5.

Divisibility rules (tests or procedures) were important as a check for computation when all arithmetic was done mentally or with paper and pencil. At one time, students had to learn the divisibility rules for 2–12. One of the many divisibility rules for 7 is:

> Double the last digit (begin with the ones) and subtract it from the remaining digits. Repeat as many times as needed until the remainder is easily checked. If it is divisible by 7, then the original number is divisible by 7. For example, check 3941. Double the 1 and subtract 2 from 394. This leaves 392. Double the 2 and subtract it from 39, leaving 35. Since 35 is divisible by 7, so too is the original number, 3941.

Wouldn't you rather just divide? These rote rules meant little to students then, but they are useful for extension ideas now. Finding out a rule for yourself or why a rule works is at the heart of mathematics today.

Many children discover the rules for 2 and 5 by observing the patterns in the multiplication facts for 2 and 5. If using a hundred chart coloured in different colours for each 'divisibility rule', students will easily see the pattern of vertical columns ending in 2, 4, 6, 8, 0 for divisibility by 2 and columns ending in 5 and 0 for divisibility by 5. The 0 column shaded by both colours is divisible by both 2 and 5, hence 10.

Again, if using the hundred chart with new colours, students will be likely to discover that the 3 pattern is diagonal and starts at 3, 6, 9 — or 12 and 15 if extrapolated. By asking if there is a rule, students might answer that the sum of the digits in each number is divisible by 3 and that each diagonal matches that sum. For example, 39 is coloured ($3 + 9 = 12$) and appears in the same diagonal as 48 ($4 + 8 = 12$).

In a similar way, the 9 diagonal has sums of 9 for the digits, but also the units number decreases (9, 8, 7, 6, 5, 4, 3, 2, 1, 0) while the tens number increases for each value in the diagonal.

IN THE CLASSROOM 14.10

Divisibility discovery

Objective: Explore divisibility rules for 3 and 9.

Year level: 5 and above

Use your  then fill in the chart and look for patterns.

	Divisible by			Sum divisible	
Number	3?	9?	Sum of digits	by 3?	by 9?
456	yes	no	4 + 5 + 6 = 15	yes	no
891					
892					
514					
37					
78					
79					
1357					
1358					
1359					
1360					
1361					
1362					

Questions

Try any number of your choosing.

- Is a number divisible by 3 always divisible by 9?
- Is a number divisible by 9 always divisible by 3?
- What does the sum of the digits tell you?
- What did you notice about the sequence of numbers 1357, 1358, 1359 . . . 1362?

The rules for other numbers are not quite as evident. In the classroom 14.10 provides a guided discovery of the rules for 3 and 9. The students should conjecture the following 2 rules.

- If the sum of the digits of a number is divisible by 3, then the number is divisible by 3.
- If the sum of the digits of a number is divisible by 9, then the number is divisible by 9.

It is important to discuss these results with the class and to give more examples if the students do not answer the questions clearly. Good questions to raise would be: Does this divisibility test work for larger numbers? Is this a test only for 3 and 9? Can you see other tests for 4, 6 and 8?

Using a colourful approach, students might see that multiples of 4, 6 and 8 relate to a pattern of L-shapes, as well as patterns related to the diagonals and columns.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Multiples of 4
Multiples of 6
Multiples of 8

By examining a model of base-ten blocks and the process of division, students can be helped to make connections about the rules of divisibility.

For example, 243 is divisible by 9 as shown in figure 14.2.

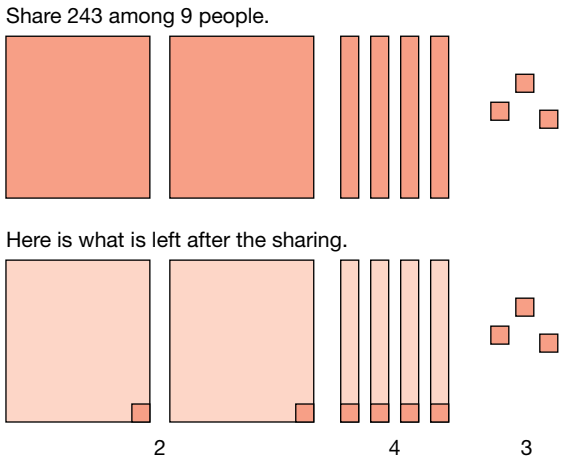


FIGURE 14.2 Is 243 divisible by 9?

Thinking of division by 9 as sharing with 9 people, we can see that if the 9 people shared 1 of the 100-flats, 99 would be used and 1 unit block would be left over. Similarly, 1 unit block would be left over from the other 100-flat. If 9 people shared a 10-long, 1 unit block would be left over from each long. Therefore, if you divided 243 by 9, 2 unit blocks would be left over from the 100-flats, 4 unit blocks would be left over from the 10-longs, and the 3 original unit blocks would also be left over. This makes a total of 9 unit blocks left over, which you can clearly share among 9 people. Thus, 243 is divisible by 9.

Students can investigate other divisibility tests in a similar manner with concrete materials. Using the view of sharing each of the hundreds, tens and ones, do you see why 243 is not divisible by 2? By 5?

More generally, since we can share the 100-flat and the 10-long with 5 people (or 2 people), the only thing left to check is the number of ones. So, we only need to consider the ones place when deciding whether or not a number is divisible by 2 or by 5.

The divisibility test for 4 can also be justified for 3-digit numbers this way. Try it. Older children should also explore larger numbers and write a justification. Returning to place value, what would happen if we have a number in the thousands and tried to share the thousands with 9 people? How about hundred thousands?

Divisibility also provides a wonderful opportunity for students to investigate conjectures. Students can investigate the following conjectures (true or false) about divisibility:

1. If a number is divisible by 2 and 3, it is divisible by 6.
2. If a number is divisible by 2 and 5, it is divisible by 10.
3. If a number is divisible by 12, it is divisible by 3 and 4.
4. If a number is divisible by 2 and 4, it is divisible by 8.
5. If a number is divisible by 3 and 6, it is divisible by 12.

Students (and adults) usually begin to understand the conjecture by trying numbers. For example, in the first conjecture, they may say:

> I'll try 72. It is divisible by 2 and by 3 — and yes, it is divisible by 6.

Some students may list all the beginning multiples of 2, 3 and 6.

> Multiples of 2: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30 . . .
> Multiples of 3: 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42 . . .
> Multiples of 6: 6, 12, 18, 24, 30, 36, 42, 48 . . .

Notice that if a number is on the 2 list and on the 3 list, then it is on the 6 list. Try this for the fourth conjecture. What happens?

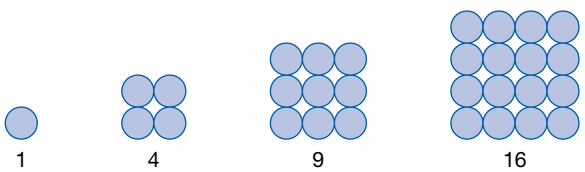
14.3 Other number theory topics

LEARNING OUTCOME 14.3 Describing additional number theory topics that extend and complement the teaching and learning of mathematics.

In this section, other number theory topics are briefly described and just a few of the many possible activities are presented. Many more activities can be found in websites for further exploration by you or by your students.

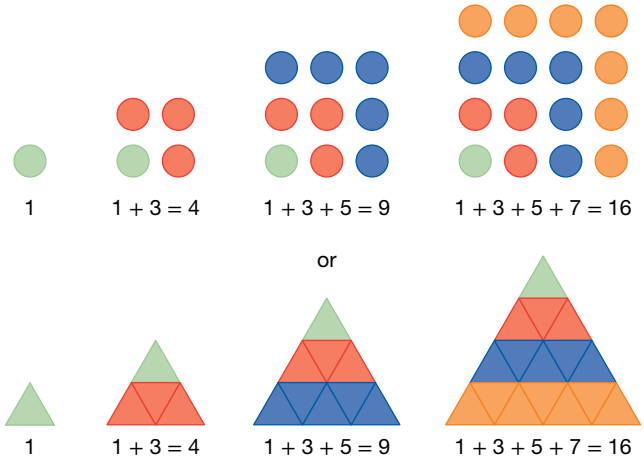
Polygonal numbers

Polygonal or figurate numbers are numbers related to geometric shapes. The most useful for primary students are square, triangular, pentagonal and hexagonal numbers. Students are most familiar with square numbers such as 4, 9 and 16. However, some people only know they are called square numbers because they are 1^2, 2^2, 3^2 and 4^2. They are also called square numbers because they can be related to squares, as shown in this illustration.



Square numbers have many interesting and special mathematical properties. Here are just a few.

- Square numbers are the sum of consecutive odd numbers, as can be seen pictorially and abstractly:



- If you subtract a square number from the next one, you will always get an odd number — for example, $144 - 121 = 23$ and $25 - 16 = 9$.
- All square numbers have an odd number of factors, for example: 16 has five factors: 1, 2, 4, 8, 16 and 36 has nine factors: 1, 2, 3, 4, 6, 9, 12, 18, 36. This is because one factor is repeated and can't 'pair up' with another factor.
- All square numbers end in the digits 0, 1, 4, 5, 6 or 9. So, if you were asked if 4392 is a perfect square, the answer is no! Square numbers cannot end in 2, 3, 7 or 8.
- Square numbers appear as the diagonal in the multiplication chart (see figure 9.13) and exactly divide it into 2 equal parts (hence the law of commutativity).
- Two square numbers can add to give another square — for example, $9 + 16 = 25$, $36 + 64 = 100$ and $81 + 144 = 225$
- Three square numbers can add to give another square — for example, $4 + 9 + 36 = 49$, $1 + 16 + 64 = 81$ and $16 + 25 + 400 = 441$
- Square numbers can be palindromes — for example, 121 (11×11), 484 (22×22) and 676 (26×26)
- A square number can also be a cubic number — for example, 64, ($8 \times 8 = 64$ and $4 \times 4 \times 4 = 64$).
- Square numbers can help solve classic problems like 'how many squares in a chessboard?' or the 'gaoler's dilemma' below:

> A gaoler was in charge of 40 prison cells. One night he was feeling a little under the weather and decided to release all the prisoners. He went around the cells and turned the key in all of the locks, unlocking all the doors. The prisoners slept on, unaware of this event. Later the gaoler had second thoughts and decided to turn the key in every second door, locking these prisoners back in. Feeling further remorse he went and turned the key in every third lock; locking in some prisoners and unlocking others. He continued, turning the key in all the fourth locks, all the fifth locks, all the sixth locks and so on through all the numbers. The last lock to be turned was lock 40. Prisoners from which cells had escaped in the morning?

Triangular numbers as seen in In the classroom 14.11 are also very interesting. In Pascal's triangle (see In the classroom 14.14) they appear in 2 diagonals commencing from row 3. The triangular number 36, which appears in row 9 of Pascal's triangle, is also a square number. Note also that 36 is half of 72 and $(9 \times 8)/2 = 36$ or $n(n - 1)/2$ which is the formula for diagonals in a polygon. Triangular numbers can also help solve classic problems like the following 'how many handshakes?' one:

> If your mathematics group were to meet for the first time, and everyone in the group shook hands with everyone else, how many handshakes would there be?

Students can also explore pentagonal and hexagonal numbers.

Relatively prime pairs of numbers

Two numbers are *relatively prime* if they have no common factors other than 1. For example, 8 and 15 are relatively prime since 1 is the only common factor, but 8 and 18 are not relatively prime since 2 is a common factor of both.

One fun exploration (Bennett 1978) that younger students can begin to explore is described in In the classroom 14.12. Although there are many conjectures about the star patterns, we focus on those that deal with relatively prime and not relatively prime pairs of numbers as well as connect the stars to geometry terms.

IN THE CLASSROOM 14.11

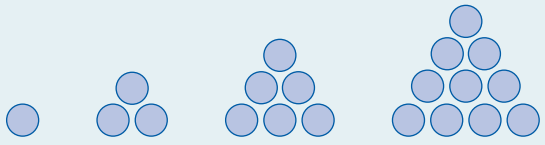
Triangular numbers

Objective: Explore triangular numbers.

Year level: 6 and above

A number is said to be triangular if that number of dots can be arranged in a triangle like the ones below. The sequence of numbers is found in nature and is used in solving many problems. We will use the symbol T with a subscript, such as T_3, to indicate the third triangular number.

1. Using and extending the pictures, find the first 6 triangular numbers.



T_1 T_2 T_3 T_4 Draw T_5 Draw T_6

2. Fill in the chart below:

Triangular number	T1	T2	T3	T4	T5	T6	T7	T8	T9
Number of dots	1			10					

Questions

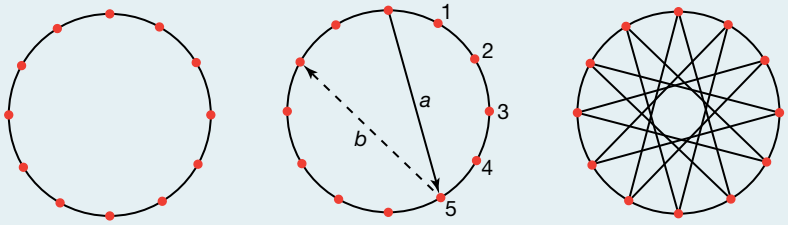
1. What 5 numbers are you adding when you find T_5? What 6 numbers for T_6?
2. If you know T_4, explain a quick way to find T_5. If you know T_6, how do you find T_7?
3. If T_{10} is 55, what is T_{11}?
4. What is T_{100}? Or how can you find the sum of the numbers 1–100?

IN THE CLASSROOM 14.12

Objective: Investigate star patterns.

Year level: 6 and above

A star pattern (*n*, *s*) is created from placing *n* equally spaced points on a circle and connecting them by steps of length *s*. For example, a (12, 5) star is drawn by placing 12 points on a circle. Connect the top point (12 o'clock) to the fifth point, clockwise (*a* below). Continue connecting every 5 points with a straight line.



Using an empty 12-point circle like the one previously, make the following stars with your classmates.

(a) (12, 1)
(b) (12, 2)
(c) (12, 3)
(d) (12, 4)
(e) (12, 5)
(f) (12, 6)
(g) (12, 7)
(h) (12, 8)
(i) (12, 9)
(j) (12, 10)
(k) (12, 11)

If you get back to the starting point before all points are touched, go to the next point (clockwise) and continue.

Questions

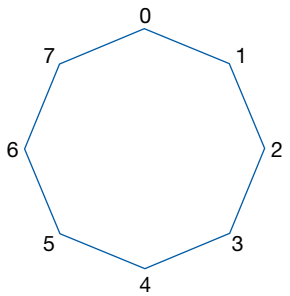
1. Which stars are the same?
2. Which stars make one path as the (12,5) star did? (You did not get back to the starting point before having to move to the next star.)
3. How would you describe the (12, 1), (12, 2), (12, 3), (12,4) and (12, 6) stars in geometric terms?
4. Try some other stars such as one with 18 points and various steps. See if you can tell when the stars will make only one path.

For this activity, use circle paper that has points spaced with 9, 10, 12, 24, 30 and 36. Circles with other numbers of points can be made from these by selecting only some points. For example, an 8-point circle could be made from the 24-point circle by selecting every third point.

Modular arithmetic

In some primary mathematics resources, *modular arithmetic* is called *clock arithmetic*. It is an arithmetic based on a limited number of integers, just like the clock is based on the integers 1–12. This is arithmetic of 12 called mod 12. However, in mod 12, 0 is used instead of 12 because it is arithmetic of remainders. You will see what we mean as we explore mod 8.

In mod 8 (see the 'clock' below), we use the numbers 0, 1, 2 . . . 7. The numbers mean the same as they do in our number system; however, these are all the numbers we have. What do you think the sum of 6 and 7 would be? Try it on the clock by starting at 6 o'clock and adding 7 hours. What time would it be?



If you do enough sums on the clock, you will find a shortcut. In our number system, $6 + 7 = 13$. Thirteen is once around the 8-hour clock and then 5 more. Thus, $6 + 7 = 5 \bmod 8$. What is the remainder after subtracting 8 from 13? How many complete revolutions would you make if you had a sum of 35? How many 'hours' after the last revolution?

Although there are many explorations with modular arithmetic, we have chosen one that connects to art and craft, making quilts based on modular 8 arithmetic. In the classroom 14.13 requires use of an addition or multiplication table for mod 8 and a 2-colour technique. We have begun both tables; students would need to complete the tables before colouring their quilt.

IN THE CLASSROOM 14.13

Modular quilt

Objective: Use modular addition or a multiplication table up to 7 × 7 to make a quilt.

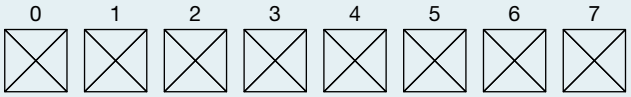
Year level: 4 and above

Materials: Two coloured markers and a sheet of centimetre grid paper (see Appendix C).

Instructions

These are instructions for a mod 8 quilt.

1. Colour each square below with 2 or 3 colours. Use the same set of colours for each square.



2. Number 8 squares across the top of the graph paper with the numbers 0–7. Do the same numbering down the grid as shown below.

3. Complete either the addition or multiplication table for modular 8.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1								
2								
3		4			7	0	1	
4								
5			7				3	
6								
7		0						6

×	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2		4			7
2			4		0		4	
3		3		1	4	7	2	
4								
5	0	5	2	7	4	1	6	3
6								
7								

The 0 in the yellow square is the sum of 3 and 5 mod 8. The 7 in the green square is the product of 5 and 3 mod 8.

4. Colour 8 × 8 squares of the geometric design paper (see appendix C) according to its sum (or product). For example, any sum (or product) that is 0 would be coloured like you coloured the 0 square in step 1.

It is possible for students to generate many different quilts with these instructions by using other modular numbers. For less patient students, try mod 4 and use only a 4 × 4 square.

Pascal's triangle

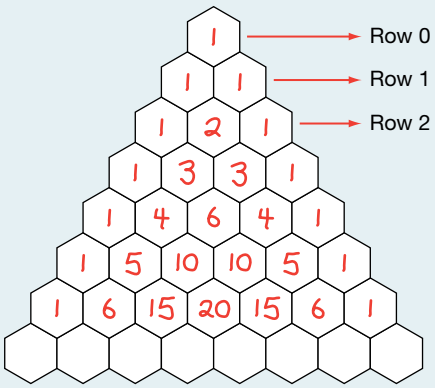
Pascal's triangle is closely associated with probability. However, there are many number patterns in it and other surprising results, some of which are suggested in In the classroom 14.14. Make certain that students know how to generate the rows before exploring the different patterns. That is, the number in a hexagon is the sum of the 2 numbers in the 2 hexagons immediately above it. Students might also observe that the sum of each row is double the previous row. Row 0 is 2^0, row 1 is 2^1, row 2 is $2^2 = 4$ and so on.

IN THE CLASSROOM 14.14

Pascal's triangle

Objective: Explore number patterns within Pascal's triangle.

Year level: 4 and above



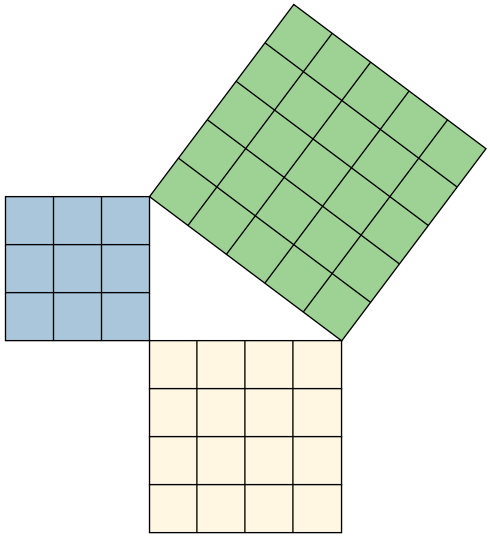
Questions

1. What would be the numbers in row 7? Row 8? Row 9?
2. What patterns do you see?
3. Can you find the triangular numbers?
4. Find the sum of the numbers in each row beginning with row 1 and ending with row 6. (Hint: A chart with each row number and its sum may be helpful.) What do you think would be the sum of row 7? Row 8? Row 20?
5. Look at the numbers in any odd row (ignore the 1s). What do you notice? Do you think that would be true in row 9?
6. Choose any number. Then, going left in that row, alternate subtracting and adding the numbers to the left. For example, choose the second 10 in row 5. Then subtract and add: 10 − 10 + 5 − 1. The result is 4. Where is the nearest 4 in the triangle? Try more numbers. What is your conjecture?

Pythagorean triples

A *Pythagorean triple* is a triple of numbers (a, b, c) such that $a^2 + b^2 = c^2$. You may remember the (3, 4, 5) triple from middle years geometry. (*The sum of the squares of 2 sides of a triangle equals the square of the hypotenuse*.) Check to see that $3^2 + 4^2$ does equal 5^2.

The geometric picture of the Pythagorean relationship is shown below. Note that the square of a side gives the area of the square with that length. For example, the side that is 3 units has an area of 9 square units. The sum of the squares of the blue and yellow squares (the sides) will make the green square (the square on the hypotenuse).



There are many ways to generate Pythagorean triples and many patterns in the triples. In the classroom 14.15 shows one way to generate triples that is an extension idea for upper primary students.

A calculator will make the computation less tedious and allow students to have fun exploring the patterns. The activity also ties the triples to geometry and measurement, as students draw the sides of the right triangles and check to see if the hypotenuse (or long side) is the measure they found in the pattern.

IN THE CLASSROOM 14.15

Pythagorean triples

Objective: Generate and explore Pythagorean triples.

Year level: 8 and above

Here is a table from which you can generate Pythagorean triples.

triple	#1	#2	#3	#4	#5	#6	#7
A	3	5	7	9			
B	4	12	24	40			
C	5	13	25	41			

1. Use a calculator to check if (3, 4, 5), (5, 12, 13) and (7, 24, 25) are Pythagorean triples. That is, see if $3^2 + 4^2 = 5^2$.
2. What would be the A in triple #5? _____
3. What is the relation between B and C in each triple? _____
4. What would be B in triple #5? *Hint:* Look at the 3 numbers of the same colour.
5. Fill in the remaining triples in the table. Check to make certain they are Pythagorean triples.
6. Draw a right triangle with sides 3 cm and 4 cm. Check to see that the third side is 5 cm.
7. Check 1 more triangle and use millimetres.

Fibonacci sequence

Some sequences of numbers are famous enough to be named. One of these is the *Fibonacci sequence of numbers*:

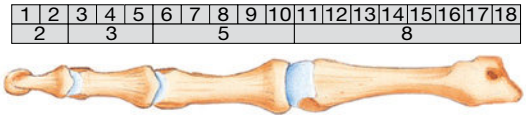
1 1 2 3 5 8 13 21 34 55 . . .

Do you see how it is generated? The first 2 terms are ones; thereafter, any term is the sum of the previous 2 terms.

This sequence is rich — with examples of this pattern in nature, art, history and the human body. Many books have been written about this topic alone. As a beginning place to take the topic in any direction you could consult the website www.maths.surrey.ac.uk and look for the work of R. Knott on Fibonacci.

The original problem that Fibonacci studied was about a pair of rabbits but the sequence can be found in seashells, branching plants and flowers, seeds, leaves and petals as well as in pine cones and pineapples. A great resource for this topic is the *Problem Pictures* CD by Richard Phillips (2002), available through the Australian Association of Mathematics Teachers and online. Upper primary students will find some part of this topic interesting to investigate.

Take a look at the following diagram of a finger. The first section shows the tip of the finger to the first knuckle and so forth to your wrist. Have you seen these numbers before? Check your own hand. You may need to make your own scale, with 2 being the length of the tip to the first knuckle.



The Fibonacci sequence is also related to the Golden ratio — another topic that is rich to investigate. The picture of the hand and many other relations involving the Fibonacci numbers and the Golden ratio may be accessed from the website http://goldennumber.net/.

The golden spiral and examples like the nautilus shell are also fascinating. Students might begin to see maths in new ways through these explorations. Do not stop now — keep looking for the wonders in mathematics all around us.

MAKING CONNECTIONS

The roots of number theory can be found thousands of years ago throughout the ancient world. Looking only at the names we have included in this chapter, you can find influences from many countries. For example, the game involving evens and odds (Morra) comes from the Greeks and Romans.

Pythagoras, the person given credit for the Pythagorean theorem ($a^2 + b^2 = c^2$), lived in Greece in about 500 BC. The Pythagorean triples explored in this chapter are built on this theorem. A thousand years later a Frenchman, Fermat, claimed he had examined an extension of the Pythagorean triples. Fermat's Last Theorem stated that there was no solution involving only nonzero integers to the equation $a^n + b^n = c^n$ for any integer n greater than 2. For example, there are no integers that satisfy $a^3 + b^3 = c^3$. Although Fermat called this a theorem and claimed he had a proof (which was never found), it took over 350 years before this conjecture was proven. Two American mathematicians, Wiles and Taylor, were successful in completing a proof in 1994. So don't give up if your students cannot prove their conjectures!

Fibonacci was an Italian who lived in the thirteenth century. His famous sequence also was known early in India. Pascal's triangle is named for a Frenchman who lived in the seventeenth century. Gauss, a German whose life spanned the eighteenth and nineteenth centuries, is called the father of modular arithmetic. As a young child, he amazed his teacher when he quickly added the first 100 integers.

The famous Lo-Shu or 3×3 magic square, was believed to have appeared as a pattern on the shell of a tortoise seen in the Lo River, China about 4000 years ago. The 4×4 magic square depicted in Albrecht Durer's engraving (1514) *Melancholia* has 72 combinations of the magic number 34 hidden within the design. Benjamin Franklin of the United States is well known for his 8×8 magic square.

Asian mathematics is filled with many interesting connections to number theory. In reading the historical accounts, it is difficult to know which ideas were discovered or adapted by Europeans over time. It can be said, however, that many mathematicians played with and studied numbers and devised new theories and connections in numbers. It is not surprising then that primary students might find this topic fascinating. Giving them the opportunity to explore will help them learn more about numbers.

A GLANCE AT WHERE WE'VE BEEN

After examining reasons for including number theory topics in primary school, we have taken a quick journey through some of those topics. The suggested activities and references to others should give you many ideas about how to extend your planning and help to meet the needs of students. We have connected activities to logical reasoning, art, nature, geometry, measurement and number ideas.

Most importantly, we have chosen activities that we know different children enjoy. We hope that this brief background will give you enough information or inspiration to learn more so that you can engage your students in these topics.

THINGS TO DO: FROM WHAT YOU'VE READ

1 List the 4 reasons for including number theory in primary school. Explain each in your own words.

2 Choose the reason for including number theory that resonates with you. Give one activity, other than the one suggested with it, that you think illustrates that reason. (It can be another activity in the chapter, one you find in other chapters in the book or from other sources.)

3 Justify that 15 is the sum of each row of a 3×3 magic square using the numbers 0–9. *Hint:* Find the sum of all the numbers. What is the sum of each row in a 4×4 magic square that uses the numbers 1–16?

4 How do you think the teacher (see figure 14.1) structured the lesson on finding all the factors of 12? What did she ask the children to do? Looking at the lesson plan outlined in chapter 3, what type of lesson would this be?
5 In the discussion with Mr Thomas about finding all the factors of a number, Brian said there was a point at which 'the numbers almost bump'. What did he mean? How do you know that you have come to that 'bump'? What numbers would be the 'bump' for 144, for 200 or for 900?
6 Using the ways to assess students from chapter 4, explain how you would assess students' understanding of odds and evens or of prime numbers.
7 Design a guided discovery lesson for divisibility by 8 (or 11).

THINGS TO DO: GOING BEYOND THE TEXT

IN THE FIELD

1 *A game of odds and evens.* Play this game (In the classroom 14.5) with a friend and then with a young child. Do you notice any differences in their strategies?
2 *Factor me out*. Play this game (In the classroom 14.6) several times with a child in Years 3–5. What do you notice about the ability of the child to choose a number that will maximise his or her score for that round?
3 *The hoppers*. Do this activity (In the classroom 14.8) with a small group of children. What questions will you ask students? What mathematical outcome do you expect?
4 Interview children about odds and evens (or prime numbers). What are the different conceptions that they hold about these numbers?

IN YOUR JOURNAL

5 Explain how number theory can help you differentiate instruction at a given year level. Give specific examples of activities you could use for that year level.
6 Jasper, a Year 6 student, completed the sieve (In the classroom 14.9). He said, 'if a number is prime, it must be 1 more or 1 less than a multiple of 6'. Other than the primes 2 and 3, this is a true statement. Explain in your own words, after completing the sieve yourself, what led Jasper to this conjecture. Then, justify the conjecture. (You may want to examine some of the algebraic conjectures in chapter 15.) *Hint:* Represent each column algebraically. For example, all of the numbers in column 2 are of the form $6n + 2$ for some integer n.
7 Write up your exploration of one of the conjectures from In the classroom 14.2.
8 Choose your favourite number. Describe it in terms of all the number theory topics discussed in this chapter. (For example: Is it prime? What are its factors? Is it in Pascal's triangle?)

WITH ADDITIONAL RESOURCES

9 *The Case of the Mystery Bone: A Unit of Work on Measurement for Grades 5 to 8* (Clarke 1996) is an excellent resource that will help you understand the mathematics of the human body. Design a series of 3 lessons for a primary class of students. Alternatively, create a series of lessons based on *Professor Stewart's Cabinet of Mathematical Curiosities* (Stewart 2008), *Numbers: Facts, Figures and Fiction* (Phillips 2004) or similar resources.

WITH TECHNOLOGY

10 A fun java script that allows you to make factor trees for single numbers and to explore using factor trees and Venn diagrams for finding the greatest common factor and the least common multiple can be found at http://nlvm.usu.edu/en/nav/frames_asid_202_g_3_t_1.html. Find a method for determining the LCM using this resource. *Hint:* Do some small numbers first and see if you can find a relationship between the product of the 2 numbers and the LGM and GCF.
11 An interactive website that helps students to look at patterns created by colouring multiples of a given number in Pascal's triangle and a more sophisticated exploration of Pascal's triangle involving

prime factorisation can be found at http://www.shodor.org/interactive/activities/ColoringMultiples/ and at http://mathforum.org/workshops/usi/pascal/petals_pascal.html. What patterns can be found?

CHILDREN'S LITERATURE CONNECTIONS

Bessinger, J. & Pless, V. (2006). *The Cryptoclub: Using Mathematics to Make or Break Secret Codes.* Wellesley, MA: A. K. Peters.
This is a challenging book, appropriate for gifted students or mathematics clubs. Boys and girls create and decode ciphers based on number theory topics such as modular arithmetic and prime numbers.

Goennel, H. (1994). *Odds and Evens.* New York: Harper Collins.
This book for young children shows 13 familiar expressions in counting, from a one-horse town to a baker's dozen. Even and odd numbers are emphasised in the pictures associated with the numbers.

Gravett E. (2009). *The Rabbit Problem*. London: Macmillan Children's Books.
This is a thought-provoking book that will appeal to adults and children alike. Based on the Fibonacci sequence, it is presented as a calendar with beautiful illustrations, activities and a famous mathematical problem.

Hulme, J. N. & Schwartz, C. (2005). *Wild Fibonacci: Nature's Secret Code Revealed.* Berkeley, CA: Trycycle Press.
This book shows different Fibonacci patterns in recurring sequences using numbers, plants and animals. The author gives a brief history of Fibonacci numbers and how the reader can find these sequences in nature.

Turner, P. (2001). *Among the Odds and Evens: A Tale of Adventure.* New York: Scholastic.
The Kingdom of Wontoo is visited by 2 adventurers who soon learn there are 2 types of number. The Odds are eccentric and the Evens are orderly and predictable. The reader will see how odd and even numbers are represented in a variety of pictures and will also learn about adding evens and odds. This book is a great start for algebraic reasoning for young students.

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ACKNOWLEDGEMENTS

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CHAPTER 15

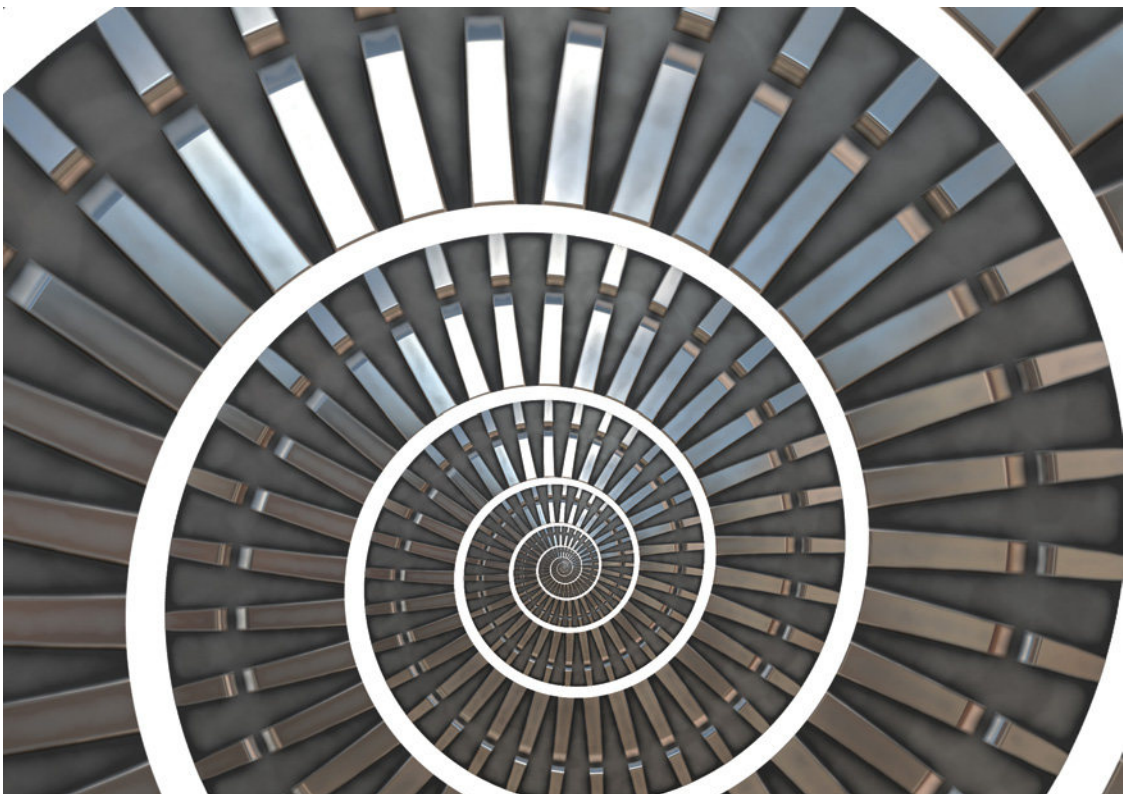
Algebraic thinking

LEARNING OUTCOMES

15.1 Describing how problems, patterns and relations can be used to develop algebraic thinking.

15.2 Identifying the language and symbols associated with algebra that are central to the algebra appropriate for primary school students.

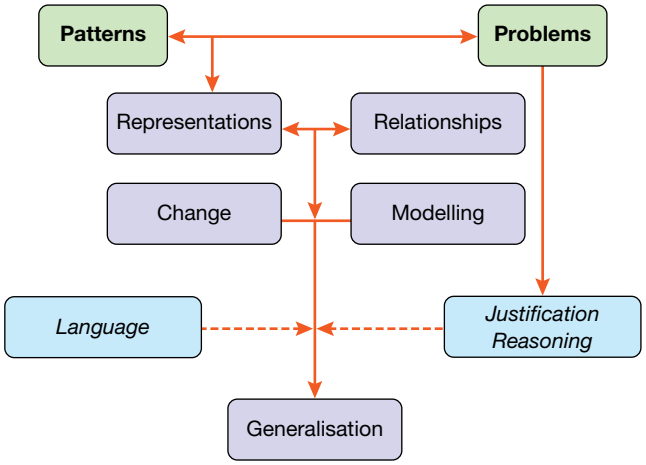
15.3 Describing and using the interconnection between modelling, generalising and justifying to enhance students' algebraic thinking.



'Numbers have life; they're not just symbols on paper.'

Shakuntala Devi (1929–2013)

Chapter 15 concept map



Introduction

Developing an understanding of the nature and prevalence of patterns in the world is an important part of teaching mathematics to primary school students. In studying this chapter you will see how algebraic thinking can help children clarify much of their thinking about arithmetic.

Traditionally, children do not begin to study algebra until they have a solid foundation in arithmetic. Thus, many students who have not done well with arithmetic never have the opportunity to learn algebra. Research in the latter part of the twentieth century indicated that students in primary school should be prepared to study algebra.

The Australian Curriculum: Mathematics reflects this research: Number and algebra are developed together, as each enriches the study of the other. (ACARA 2015, p. 6).

The Australian Curriculum's Number and Algebra strand enables students in the primary years and early years of secondary school to recognise patterns and come to understand concepts of variable and function. By building on their understanding of the number system, students learn to describe relationships and formulate generalisations. These are expressed firstly in informal language and, in the early secondary years, in the symbolic notation and syntax of formal algebra. Throughout these years students learn to apply their number and algebra skills to conduct investigations, solve problems and communicate their reasoning. The content that students will learn is summarised in table 15.1.

TABLE 15.1 **Algebraic thinking: Number and Algebra content descriptions in the Australian Curriculum**

Year level	Content description
Foundation	**Patterns and algebra** • Sort and classify familiar objects and explain the basis for these classifications. Copy, continue and create patterns with objects and drawings.
Year 1	**Patterns and algebra** • Investigate and describe number patterns formed by skip counting and patterns with objects.
Year 2	**Patterns and algebra** • Describe patterns with numbers and identify missing elements. • Solve problems by using number sentences for addition or subtraction.

Year 3	**Patterns and algebra** • Describe, continue and create number patterns resulting from performing addition or subtraction.
Year 4	**Patterns and algebra** • Explore and describe number patterns resulting from performing multiplication. • Solve word problems by using number sentences involving multiplication or division where there is no remainder. • Find unknown quantities in number sentences involving addition and subtraction and identify equivalent number sentences involving addition and subtraction.
Year 5	**Patterns and algebra** • Describe, continue and create patterns with fractions, decimals and whole numbers resulting from addition and subtraction. • Find unknown quantities in number sentences involving multiplication and division and identify equivalent number sentences involving multiplication and division.
Year 6	**Patterns and algebra** • Continue and create sequences involving whole numbers, fractions and decimals. Describe the rule used to create the sequence. • Explore the use of brackets and order of operations to write number sentences.
Year 7	**Patterns and algebra** • Introduce the concept of variables as a way of representing numbers using letters. • Create algebraic expressions and evaluate them by substituting a given value for each variable. • Extend and apply the laws and properties of arithmetic to algebraic terms and expressions. **Linear and non-linear relationships** • Given co-ordinates, plot points on the Cartesian plane and find co-ordinates for a given point. • Solve simple linear equations. • Investigate, interpret and analyse graphs from authentic data.

Source: © Australian Curriculum, Assessment and Reporting Authority 2015.

Substitute 'algebra' for 'mathematics' in the 5 views of mathematics in chapter 1 and you will see reasons and ways to include algebra in primary school.

1. *Algebra is a study of patterns and relationships*. Patterns and relationships are important in their own right and studying them leads naturally into algebraic ideas.
2. *Algebra is a way of thinking*. Algebra provides strategies for analysing representations, modelling situations, generalising ideas and justifying statements.
3. *Algebra is an art, characterised by order and internal consistency*. Children who are involved in algebraic thinking gain a better understanding of the underlying structure and properties of mathematics (Carpenter & Levi 2000).
4. *Algebra is a language that uses carefully defined terms and symbols*. These terms and symbols enhance children's ability to communicate about real-life situations and mathematics itself.
5. *Algebra is a tool*. In the past, many algebra courses presented algebra as a tool. However, too often, students only learned the intricacies of the tool, without learning how to use it in any practical or meaningful way. Algebra is a tool, but one that should be meaningful and useful.

Bringing algebra to primary school does not mean just adding 1 more topic to a crowded curriculum. It does mean using the mathematics in the curriculum to help children develop algebraic thinking. It can be seen as a way of 'tightly interweaving existing topics of early mathematics' (Carraher, Schliemann & Schwartz, 2008) to develop the understandings required for more formal approach to algebra in later years. Algebraic thinking develops through 'investigating patterns in number, measurement and geometry' (Booker, Bond, Sparrow & Swan, 2014). This chapter begins with a discussion of topics — problems,

patterns and relations — that will be used in the remainder of the chapter to build algebraic thinking. After discussing the algebraic language and symbols appropriate for primary students, we will look at ways to help children develop algebraic thinking through modelling, generalising and justifying. Exploring and developing algebraic thinking through a problem-solving approach is widely supported by research (Radford & Puig, 2007; Windsor, 2010) and is useful in building connections for students.

Mason, Graham and Johnston-Wilder (2005) and Kaput, Carraher and Blanton (2008) adopt a constructivist, developmental perspective on the teaching of algebra as the development of algebraic thinking from the earliest years. Their ideas for teaching and the discussion of student understanding complement the perspectives taken in this chapter.

15.1 Problems, patterns and relations

LEARNING OUTCOME 15.1 Describing how problems, patterns and relations can be used to develop algebraic thinking.

The teaching of algebra in primary school should build on ideas that are an essential part of the curriculum. Problems, patterns and relations are each an essential part of primary school mathematics. Throughout this chapter, we will develop these 3 topics to illustrate ways they tie to algebraic thinking. As you read this section, you should work the problems, extend the patterns, answer the questions and ask yourself how these topics can be used to develop algebraic thinking. In subsequent sections, we will discuss the connections of these 3 topics to algebraic thinking.

Problems

Both routine and nonroutine types of problem provide good opportunities for developing algebraic concepts and thinking. We list a few such problems, which we will view from an algebraic perspective later in this chapter.

Routine problems

Routine problems are often considered exercises for practising computation; however, they can also be used to build algebraic understanding. You will find routine problems like the following in every textbook:

> Before her birthday, Jane had 8 toy trucks. At her birthday party, she was given some more trucks. That night she counted all her trucks and found she had 15. How many trucks was she given at her party?
>
> Deepak had 24 pencils to put in 4 boxes. He put the same number of pencils in each box. How many pencils did he put in each box?
>
> Pat took 4 buckets to the beach. She had 3 shovels and 2 rakes in each bucket. How many tools did she take to the beach?

Nonroutine problems

Many nonroutine problems, especially *look-for-a-pattern problems* and *number puzzles*, lend themselves well to an algebraic approach.

Here are the look-for-a-pattern problems that we will use in this chapter:

> *Marbles*. Problems can be solved by examining the numerical pattern of the number of marbles in any V-formation. It is the type of problem that can lead to generalisation in a very natural way. Young children can solve this problem for a given V-formation by making a table, but algebra is needed to represent the general solution for all V-formations.
>
> *Garden path*. A family is building a rectangular garden that is always 1 metre wide surrounded by a path made of 1-metre by 1-metre weed mats. For a 1-metre by 1-metre garden, they need 8 mats. For a 1-metre by 2-metre garden, they need 10 mats. Draw a picture to verify the number of mats needed for these 2 gardens. How many mats do they need for a 1-metre by 8-metre garden?

Cube painting. A large cube is made of small cubes. The large cube is 4 small cubes high, 4 small cubes wide and 4 small cubes deep. The surface of the large cube is to be painted. How many of the small cubes are painted on 6 faces, 5 faces, 4 faces, 3 faces, 2 faces, 1 face and no face?

Here are the number puzzles we will use in this chapter. Try them. Can you explain why each works?

Same number. Pick a number. Add 4. Double the sum. Subtract 6. Divide by 2. Subtract 1. What did you get? Why does it work?

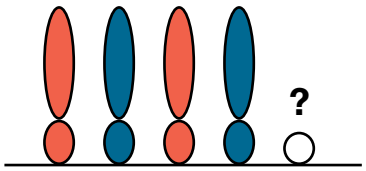
Calendar. Choose any 3 consecutive days (e.g. Tuesday, Wednesday and Thursday) for any 3 consecutive weeks in a month (e.g. 8–10, 15–17 and 22–24 July). How can you easily find the sum of all 9 dates?

Patterns

Patterns are an important part of mathematics because they help children organise their world and understand mathematics. In a previous chapter, we discussed how you can use patterns to help children think about numbers. Here we will use 2 types of pattern, repeating and growing, for building algebraic ideas.

Repeating patterns

A repeating pattern has a core element that is repeated over and over. One of the simplest repeating patterns is an alternating pattern such as the following:



This simple pattern raises many questions about how children work with patterns. What colour should come next? The answer to this question is not as simple as it appears, since there are many ways to think about the pattern. Some children may say that the next colour should be red because the pattern is just alternating red and blue (i.e. the core element is a red followed by a blue). Others might say that the next colour should be blue because the pattern is a red with 1 blue, followed by a red with 2 blues, followed by a red with 3 blues and so on (i.e. the core grows by 1 blue each time it repeats). It is often difficult to get into the minds of children, so having them explain their thinking is crucial. Also, sharing ideas helps children consider alternative ways of looking at patterns. As children become familiar with patterns like this one, they can extend other patterns or make their own patterns.

Research has shown that children should begin recognising and thinking about patterns by describing and extending patterns with pictorial models, as illustrated by In the classroom 15.1. As children become able to identify the missing pieces in the patterns, encourage them to look at the patterns in different ways that would lead to different ways of extending the patterns. Ask questions such as 'How can you describe this pattern?' or 'How can you extend this pattern?' Questions like these help children begin to think algebraically.

IN THE CLASSROOM 15.1

Thinking about patterns

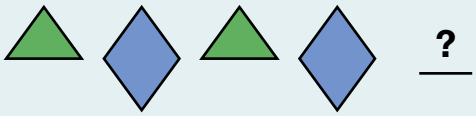
Objective: Recognise, describe and extend patterns.

Year level: 1–2

- Provide children with a variety of patterns like those shown in questions 1–3 below. Ask them questions like:
  - How do you describe this pattern?
  - What shape (or object) goes in the empty space? Why?
  - Are some of these patterns alike? How?

Questions

1. What pattern block shape should come next?



2. One of the shapes is missing in this string of shapes. What shape and what colour would you insert? Why?



3. What animal should be in the next place? Why?



Can you see the cores in the repeating patterns below?

Pattern A:



Pattern B:



Pattern C:



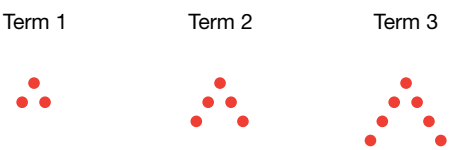
In pattern A, the core consists of 3 elements: 2 red squares followed by 1 blue rhombus. This is repeated over and over. In pattern B, the core consists of 5 elements: 1 pink circle, 1 green square, 1 pink circle, 2 green squares. Do you see that the core in pattern C is the same type as in pattern B?

Research from the mathematics assessment of the National Assessment of Educational Progress (NAEP) indicates that Year 4 students often have more difficulty identifying a missing element in a pattern (as in problem 2 from In the classroom 15.1) than identifying the element that continues the pattern (Kloosterman & Lester 2004). This difficulty may simply be the result of having been asked to continue patterns more often than to fill in missing parts. Be sure to ask children both types of question.

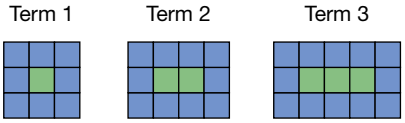
Growing patterns

The nonroutine problem using marbles involves a growing pattern. You can present such patterns verbally, through pictures only or with symbols. Both the marble problem and the garden problem, for example, could be presented just with pictures, and you could ask young children to continue the pattern:

Marble problem. What would be the next 3 pictures (terms 4, 5 and 6)?



Garden problem. The green squares represent the garden and the blue squares represent the weed mats. What would be the next picture (term 4)? Draw pictures to illustrate terms 5 and 6.



You can use pictorial representations like these as powerful links to analysing number patterns that are growing patterns. Mathematics curricula often include problems where children are asked to continue growing number patterns or sequences of numbers. For example, the beginning terms in the geese problem could be symbolised as 1, 3, 5, 7 . . . and the garden problem could be symbolised as 8, 10, 12 . . . Do you see how these growing patterns differ from repeating patterns? In growing patterns like these, each successive term changes by the same amount from the preceding term. (In both the marble problem and the garden problem, this change is 2 more.) Traditionally, children in primary school have been given only numerical patterns like these, where the difference between successive terms is some constant amount. However, numerical patterns need not be so simple. Growing patterns, like repeated patterns, can be thought of in different ways — we need to help children look at different ways in which patterns can grow. For example, consider the numerical patterns below:

Pattern A: 1, 2, 4 . . .
Pattern B: 1, 4, 9, 16, 25 . . .
Pattern C: 1, 1, 2, 3, 5 . . .

In a Year 4 classroom working with pattern A, Andy suggested that the next number in the sequence is 5. The teacher asked what the next number after 5 would be, and Andy answered 7 (1, 2, 4, 5, 7 . . .). Do you see what Andy may have had in mind? Some children said: 'Andy writes 2 numbers, skips 1 and writes the next 2', 'Andy skips all the threes — 3, 6, 9 and all those 3-times numbers' and 'Andy adds 1, then 2, then 1, then 2'. Sue suggested a different pattern from Andy: 'I think that the pattern goes 8, 16 and on. It just doubles.' Argon said, 'I think it goes 1, 2, 4, 6, 9, 12 and so on.' Can you figure out what rule Argon was using?

This episode illustrates several key ideas to consider as you work with children. First, do not expect children to see patterns the way you do. It is important to have them explain their thinking. It is also important to have them realise that there is often more than one way to look at a pattern and to explain it.

Pattern B is the sequence of perfect squares. Wells and Coffey (2005) gave this sequence to Year 2 students, who most likely were not familiar with perfect squares. One child did see that the differences between consecutive numbers were 3, 5, 7, 9 and used this to continue the pattern. The next difference would be 11, so the next number in the sequence would be 36. Another child just wrote the next differences as 11, 13 and 15. Another child used the 5 elements as the core of a repeating pattern and wrote 1, 4, 9 for the next 3 entries. This is a correct answer to another question. Think about how you could pose the problem so the child would not think a repeating pattern was a possibility. The fourth child wrote 34, 43 and 52 for the next 3 entries. What was this child thinking?

Pattern C is the Fibonacci sequence. The next numbers are 8 and 13 (1, 1, 2, 3, 5, 8, 13 . . .). Can you figure out the number after 13? Once you know the rule, it is easy to generate the sequence, but many people (adults as well as children) do not immediately see the rule. Children need practice in recognising and analysing patterns before we can start teaching them to represent patterns algebraically.

Relations

We will explore 2 types of relation that we can use to help children build algebraic concepts and thinking: *properties of numbers* and *functions*. Another relation — the *equality relation* — underpins both these types of relation as well as all of algebra.

Young children have a good sense of equality in terms of the same size or amount. Even young children know when they do not get a biscuit that is the same size as yours, when a block tower is not the same height as another or when they have 3 marbles just as you do. However, very few programs

build on the idea of presenting equality as being 'the same as', or equivalent to in size or amount and describing the relationship of 2 quantities as being the same (equal) or different (unequal). A balance provides a model of how to relate 2 quantities in this way (Mann 2004). In the classroom 15.2 shows some questions that children in primary school can investigate.

Properties of numbers

In an earlier chapter we reviewed some of the mathematical properties of numbers (commutative, associative, distributive and identity). In this chapter, we will discuss how to use number properties to promote algebraic thinking. Consider the following examples of relations based on the mathematical properties and meanings of operations that you could have students investigate. Which statement is not true? Why?

1. If you add a number to a given number and then subtract that same number from the sum, then you get the given number.
2. If you subtract 0 from a given number and add the given number to the difference, then the sum is twice the given number.
3. If you multiply any 2 numbers, the product is larger than each of the 2 numbers.
4. If you divide a positive whole number by a proper fraction (i.e. a fraction greater than 0 and less than 1), the quotient is larger than the positive whole number.

Functions

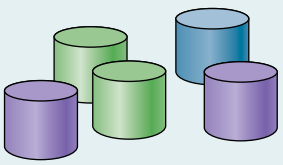
Suppose 2 sets of numbers are related in such a way that each number in the first set is related to 1 and only 1 number in the second set. A function is a way of expressing that relation. Young children can explore ways that functions work in a very informal way. For example, they can explore how the number of boys in a group is related to the number of hands. Begin by having students count the number of hands of 3, 4 and 10 boys; then write the numbers in a table, as shown below.

Number of boys	3	4	10	6	11	5	12
Number of hands	6	8	20				

IN THE CLASSROOM 15.2

Balance me

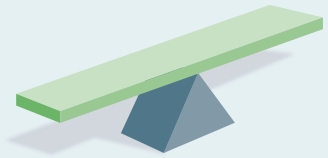
Objective: Explore what balances a balance scale.



Year level: 1–3

Materials: 5–10 one-gram masses or 1–10 washers or labelled containers weighing 1–10 units (2 of each)

(Note: You can fill containers with Unifix cubes to make masses.)



Balance scale

Questions

1. I am going to put some masses on each side of the balance. Which will balance?

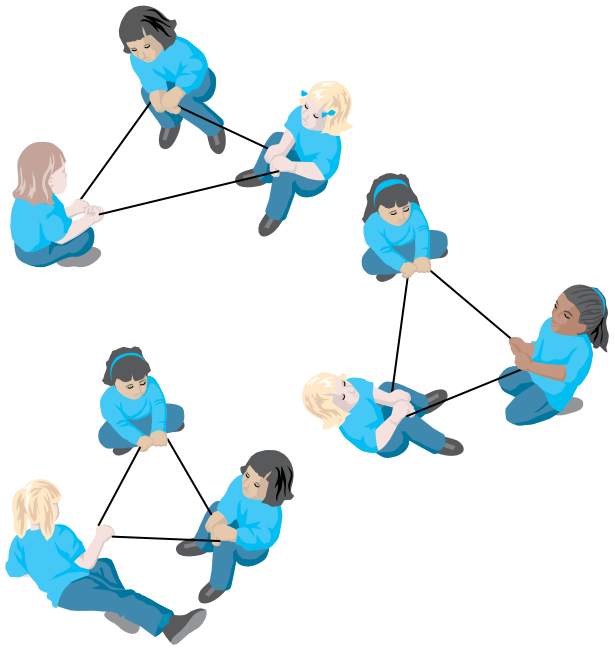
Left side	Right side
4	4
3	8
2 and 3	5
3 and 4	6
1, 2 and 3	6

2. If I put a 5 mass on the left side, what could I put on the right side to balance it?
3. If I put a 3 mass on 1 side and a 7 mass on the other side, what would I need to add to the side with the 3 mass to make the scale balance?
4. If I put an 8 mass on the left side and a 2 mass on the right side, can I balance the scale by adding a 6 mass to the left side?
5. I have the scale balanced with a 6 mass on each side. What could I add to each side to keep it balanced?
6. I only have 4 masses: 3, 4, 5 and 7. Could I put all or some of the masses on the scale to have it balance?
7. Could I balance the scale by using 1 of each of the 10 different masses?

Have students complete the table. You want the students to focus on the relationship between the number of boys and the number of hands — that is, to focus on the function. For this reason, it is best not to write the number of boys in order from 1 to 12 in the table, so students cannot easily see the pattern of the number of hands increasing by 2 with each additional boy in the group (see the table below).

Number of boys	1	2	3	4	5	6	7	8	9	10
Number of hands	2	4	6	8	10	12	14			

Another informal way for children to explore functions would be to consider the relationship of the number of girls to the number of triangles as the girls make triangles with string as shown below.



Unlike the 1-to-2 relationship of boys to hands, this is a 3-to-1 relationship (3 girls for each triangle). Some children may have difficulty describing this relationship in general terms because it calls for division rather than multiplication. Again, begin by writing the numbers in a table (in this case, it may be useful to write the number of girls in order, to help students see the relationship):

Number of girls	3	6	9	12	15	18	21	24
Number of triangles	1	2	3	4	5	6	7	8

At times, you may want to model relations using a 'machine' that outputs a number for each number input. The challenge for the students is to decide what the machine does and how to describe it. These machines can be simple, such as one that always gives an output of 1 more than the number input, twice as much as the number input or 2 less than the number input. A more complicated machine might relate input and output as shown in the function table below. The relationship involves 2 rules. Can you determine the function?

Input	3	6	7		20	53
Output	7	13	15	29	41	

You need to help children focus on the change from the input to the output. Some children will focus only on the change from one input value to the next, but this does not help them understand the input–output relationship. Two items with this same relationship (double the input and add 1) as in the function table above were given to Year 4 students on the 2005 and 2007 NAEP mathematics assessments (see sample items at www.nces.ed.gov/nationsreportcard). In one item, the children had to choose a rule that related the input and output, and in the other they had to produce an output for a given input. In both items, more students choose an answer that only considered the relation between the first input and output. For example, they would have chosen the option of 57 as the output for 53 since 7 is 4 more than 3. The other choices, including the correct answer, were selected at a chance level. Children at this year level often think additively (the first output is 4 more than the input). They need experience with multiplicative functions. They also tend to make judgements based on one instance. You need to ask them if their rule would work for the next input–output pair (and all the given pairs).

In the classroom 15.3 shows an activity with a 'function machine' that you can modify to suit the level of your students. Children love making a machine (a big box with a child in it or behind it). One student puts the number in the box; the child in the box uses the rule and writes the output. The other children guess the rule. Do not hesitate to give a rule involving multiplication or 2 simple operations.

IN THE CLASSROOM 15.3

In–out machine

Objective: Analyse the actions of a function machine.

Year level: 3–4

Dr de Zine invented an in–out machine. The machine adds, subtracts, multiplies or divides markings on animals as they go through it. He has kept a record of what happened each day as animals went in and came out of the machine. Your job is to fill in the blanks in his record and to figure out how the machine worked each day.

	IN	OUT
M	numbat with 3 stripes	numbat now has 12 stripes
O	leopard with 12 spots	leopard now has 48 spots
N	zebra with 5 stripes	zebra now has _____ stripes
T	dog with 5 spots	dog now has 15 spots
U	cat with 2 stars	cat now has 12 stars
E	rabbit with 3 marks	rabbit now has _____ marks
W	horse with 12 lines	horse now has 6 lines
E	pig with 8 spots	pig now has 4 spots
D	cow with 0 spots	cow now has _____ spots
T	elephant with 32 stars	elephant now has 16 stars
H	rhino with 24 diamonds	rhino now has 8 diamonds
U	hippo with 41 spots	hippo now has _____ spots
F	frog with 4 spots	frog now has 11 spots
R	toad with 3 spots	toad now has 9 spots
I	gecko with _____ stripes	gecko now has 15 stripes

15.2 Language and symbols of algebra

LEARNING OUTCOME 15.2 Identifying the language and symbols associated with algebra that are central to the algebra appropriate for primary school students.

Children can learn the language and symbols associated with algebra as they are learning about numbers. In this brief section, we look only at equality and inequality, variables, and expressions and equations, because these are the ideas that are central to the algebra that is appropriate for primary students.

Of course, there are many other algebraic terms and symbols; however, we will use them only when what is meant is clear from the context. For example, you probably understood what was meant by the 3 dots that we used in earlier sections of this chapter to represent the continuation of number sequences (e.g. 1, 4, 9, 16 . . .). When you are helping children learn to think algebraically, you should use algebraic terms and symbols as you would if you were teaching children a foreign language.

When children begin writing expressions for mathematical situations, they often use a mixture of symbols and words (Lannin et al. 2008). This is natural and many of these expressions communicate well. For example, a child may describe the rule in the previously discussed function table as 'double the input + 1.' Eventually, if the input was labeled x and the output was labeled y, as shown in the table below, you could write that the change was $2x + 1$ or that $y = 2x + 1$. This abstract version is the goal at this level, but not a place for children to begin.

x	y
3	7
5	11
6	13
	29
53	

Equality and inequality

As we have noted, young children understand sharing equally and balancing a scale. Nevertheless, they are baffled by the equal sign. It is well known that many children think that the equal sign means 'get an answer'. For example, many children in Years 1–6 respond that 12 or 17 is the number that should be placed in the box to make the following symbolic sentence true: $8 + 4 = \square + 5$. Research in the United States showed that only 5% of students in Years 1–2 and 9% of students in Years 3–4 responded with the correct answer, 7 (Carpenter & Levi 2000). Typically, children in these years have not learned what 'equals' means and have not learned how to interpret the structure of symbolic sentences. However, after teaching similar to that recommended in this chapter, children's results were greatly improved (66% correct in Years 1–2 and 72% correct in Years 3–4). For further information and research about developing meaning for the equal sign read the papers presented by Horne (2005) and Stephens (2005) at the Australian Association of Mathematics Teachers conference in 2005, which examine these understandings in an Australian and Japanese context. The different approaches discussed in these papers have important implications for the interpretation of the Australian Curriculum's Patterns and algebra sub-strand.

You can help children grasp the meaning of the equal sign (and the unequal sign) by introducing it in an activity like the one in In the classroom 15.2. You can tell children that 2 quantities are equal if the scale balances. Thus, the relationships in the first part of In the classroom 15.2 would be represented as shown in the table below.

You can help children use the inequality symbols ($<$ and $>$) to describe larger or smaller quantities. Children can describe the relationship using words and then use the symbols. For example, they can say '3 is less than 8' and then write $3 < 8$; similarly, they can say '12 is greater than 4' and then write $12 > 4$.

Left side	Right side	Representation
4	4	$4 = 4$
3	8	$3 \neq 8$
2 and 3	5	$2 + 3 = 5$
3 and 4	6	$3 + 4 \neq 6$
1, 2 and 3	6	$1 + 2 + 3 = 6$

In everyday life, the equal sign is used in different ways. For example, on a spreadsheet an equal sign signals the computer to do an operation. In the classroom, it is often carelessly used in carrying out a string of operations. Look at this statement:

$$5 + 3 = 8 - 6 = 2 \times 5 = 10 + 14 = 24$$

What does this mean? The equal sign was used to signal steps performing operations. It would be better to use another symbol such as an arrow. Certainly, $5 + 3$ does not equal 24.

Variables

You should be aware of the 3 different uses of *variable*. The most common use in primary school is as a *placeholder* — for example, in the sentence $3 + \square = 7$, the box is a placeholder for the number that makes the sentence true. This is no different from using a letter, as in $3 + a = 7$; children easily adjust to using both boxes and letters to represent variables. Variables are also used in *generalisations* — for example, the generalisation 'any number subtracted from itself is equal to 0' may be represented as $a - a = 0$. In this generalisation, the variable a represents any number, in contrast to the single number represented by a variable used as a placeholder.

Variables are also used in *formulas* and in *functions*. For example, the function relating the number of boys (B) to the number of hands (H) could be represented by $H = 2 \times B$, where H and B are the variables. If the value of 1 variable changes, then the value of the other variable also changes.

The following table summarises these three uses of variables:

Use of variable	Representation	Characteristics
Placeholder	$3 + a = 7$	Specific value for a
Generalisation	$a - a = 0$	All values of a make the sentence true.
Function	$H = 2 \times B$	Each value of B produces 1 and only 1 value of H.

Expressions and equations

Think of an expression as representing a phrase and an equation as representing a complete sentence. For example, the expression $n + 3$ represents the phrase '3 more than n' (or 'n plus 3' or any other equivalent phrase). The equation $n + 3 = 7$ represents the complete sentence '3 more than n is 7' (or 'n plus 3 is 7' or 'n plus 3 equals 7' or any other equivalent sentence). Knowing how to represent phrases will help students represent complete sentences. In many primary programs, an equation such as $n + 3 = 7$ or $\square + 3 = 7$ is called an *open sentence* because there is an unknown. In contrast, the equation $4 + 3 = 7$, which does not have an unknown, is referred to as a *closed sentence*.

Children often think there is only one number that will satisfy the open sentence. Consider this sentence, which was given to Year 4 students on the 2005 mathematics assessment of NAEP. (See sample items at www.nces.ed.gov/nationsreportcard/mathematics. Similar items in NAPLAN numeracy test papers for Year 5 can be accessed from www.naplan.edu.au/tests.) The students had numbered tiles (0–9) that fitted the open box, which was larger than shown here. Otherwise, they might have thought that the boxes must be replaced by the same number.

> Jan entered 4 numbers less than 10 on his calculator. He forgot what his second and fourth numbers were. This is what he remembered doing.
>
> $$8 + \square - 7 + \square = 10$$
>
> List a pair of numbers that could have been the second and fourth numbers. (You may use the number tiles to help you.)
>
> List a different pair that could have been the second and fourth numbers.

About one-third of the students gave 2 solutions; and another 22% gave one solution to the open sentence. A challenge for students would be to list all 10 solutions. Try it. What is the relationship between the 2 numbers of every pair? This task is a good example of how, with simple modifications, you can reach the needs of different children in your class.

Using only the digits 0–9 (you may want students to use the number cards in the Appendix C), what are the solutions to the following open sentences?

(a) $8 - \square = 2$
(b) $3 < \square$
(c) $0 > \square$
(d) $5 + \square = \square + 5$

Notice that there is only 1 number that is the solution to (a), there are several whole numbers (4–9) that are solutions to (b), there are no whole numbers from 1 to 9 that satisfies the inequality in (c), and all the numbers satisfy (d). Presenting children with a variety of types of open sentence such as (a)–(d) prevents them from thinking that every open sentence has 1 and only 1 number that satisfies it.

15.3 Modelling, generalising and justifying

LEARNING OUTCOME 15.3 Describing and using the interconnection between modelling, generalising and justifying to enhance students' algebraic thinking.

The Australian Curriculum: Mathematics calls for students to use mathematical models to represent and understand quantitative relationships, to represent and analyse mathematical situations and structures using algebraic symbols, and to analyse change in various contexts. In this section, we will examine how the processes of modelling, generalising and justifying relate to each of these topics in primary school mathematics.

Routine problems

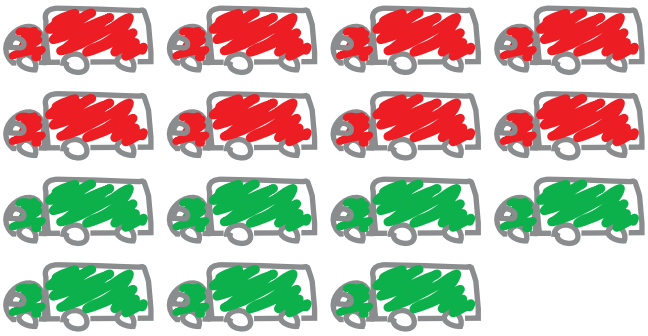
The following discussions show how you can use modelling and justifying to help children think algebraically about routine problems. Such problems, however, do not lend themselves to generalising, because the equations corresponding to those problems contain variables used only as placeholders, not to make general statements about numbers.

Modelling

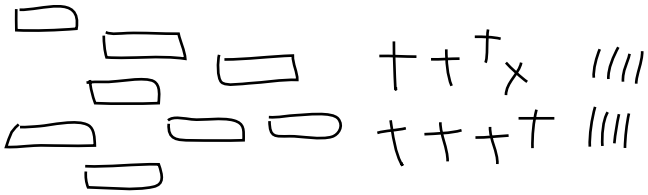
Many primary mathematics curricula include the use of equations to solve routine problems but, all too often, these curricula fail to emphasise helping children understand where these equations come from and how the equations make mathematical sense. For example, consider this routine problem:

> Before her birthday, Jane had 8 toy trucks. At her birthday party, she was given some more trucks. That night she counted all her trucks and found she had 15. How many trucks was she given at her party?

Young children can model this situation with physical materials (concrete models). For instance, they can put out 8 trucks and keep adding trucks until they reach 15 trucks. (A potential difficulty is that many young children will lose track of how many trucks they have added on and will have to repeat the process many times before they realise that it would be helpful to keep the original 8 trucks separated from the others. Therefore, you should begin modelling routine problems with simpler examples that involve just joining or separating existing groups of materials.) After modelling with physical materials, children can move to drawing pictures (semiabstract models). At first, the pictures will be realistic pictures. The child draws 8 red trucks and then draws green ones, 1 at a time, to determine that 7 more are needed to make 15.



With encouragement, children will move to more abstract drawing of the situations (showing only the original 8 trucks) such as those below:



A B C

The child who drew A needed to keep some semblance to a truck, while child B was happy with just using 't' for truck. Note that the child in C has simplified the drawing to a mark. All of these are more abstract than the truck drawings and certainly take much less time. If you model such behaviour, children will move to these stages.

Often, the purpose of modelling, whether with physical materials or with pictures, is to find an answer. However, we want to turn children's attention to the idea of representing a situation with an open sentence or with an equation (abstract model). Year 2 students who were already familiar with using open sentences to represent problems involving simple joining or separating explained their thinking about this more complicated problem as follows:

> Katie, who wrote $8 + \square = 15$, explained: 'Jane started with 8 trucks. She got some more; some more is a box because we don't know how many she got. She ended up with 15.'
>
> Chris, who wrote $15 - 8 = \square$, explained: 'Jane had 15 trucks after the party. If you took away the ones she had before, you could tell how many she got at the party, and that is what we want to know.'
>
> Sophie, who wrote $15 = 8 + \square$, explained: 'I thought about the balance. I put 15 here [*pointing to the left side of a make-believe balance*] and 8 on the other side. That is sort of like the before and after. I have to change the before side, the 8, to make it equal to the 15. Need to add on to the 8 to make it equal the 15.'
>
> Ronin asked Sophie: 'Why didn't you take away from the 15 to make it balance? Couldn't you write $15 - \square = 8$?'

What would you ask Ronin? The teacher, who had been working with her students on writing open sentences to model problems, asked Ronin what the open box stood for in his sentence. She was satisfied that his sentence made sense to him when he explained it was the number of trucks Jane was given. He said, 'Jane had 15, if she gave back those she got (the box), she would have 8. That is what my sentence says.'

This type of discussion does not happen the first time students write sentences. Look at the understanding of the problem and the open sentences that this group developed: They understood the structure of both addition and subtraction sentences, the meaning of the equals sign, and that many different sentences could model the same problem.

Now consider another routine problem:

> Bailee had 24 pencils to put in 4 boxes. She put the same number of pencils in each box. How many pencils did she put in each box?

How would you represent this with an open sentence or an equation? How would you explain each of the following possibilities, where p stands for the number of pencils in each box?

> Sentence A: $4 \times p = 24$
> Sentence B: $24 \div 4 = p$
> Sentence C: $p + p + p + p = 24$
> Sentence D: $24 - p - p - p - p = 0$

Sentence A focuses on the multiplicative nature of the problem. To understand this sentence, a child would have to understand that it is a sentence of the form $G \times N = T$, where G stands for the number of groups, N stands for the number of items in each group, and T stands for the total number of items. Sentence B also focuses on the multiplicative nature of the problem but puts it in terms of division. Sentences C and D focus on the additive/subtractive nature of the problem. Students writing these sentences were probably thinking: 'I have to put p pencils in 1 group and then in another group and so on until I have 4 groups, so $p + p + p + p$ must equal the 24 pencils.' At Years 3 and 4, you need to help children move from additive to multiplicative thinking. That is, they see the pencil problem as involving multiplication, not just addition.

These 2 problems and children's responses to them illustrate that different children often look at problems, even routine ones, in different ways. Therefore, you need to be careful about giving tests that

ask questions like 'What operation would you use to solve this problem?' As the pencil problem shows, there will often be more than one correct answer to such questions.

You should ask children not only to write open sentences that represent problems but also to write problems that correspond to given open sentences. For example, here are 4 different problems that correspond to the equation $28 - \square = 12$:

> Jennifer had 28 biscuits. She gave some to friends for lunch. After lunch, she had 12. How many biscuits did she give away at lunch? (separating)
>
> Jill has 28 books and Amanda has 12. How many more books does Jill have than Amanda? (comparing)
>
> There are 28 one-kilogram masses on the left side of a balance scale and 12 one-kilogram masses on the right side. How many one-kilogram masses need to be removed from the left side to balance the scale? (equalising)
>
> James has 28 stickers. 12 of them are blue and the rest are green. How many are green? (part–whole)

Note that these problems differ not only in the objects involved (biscuits, books, masses and stickers) but also in the type of activity involved (separating, comparing, equalising and part–whole). You should make sure that your students have experience with writing all 4 types of problem. Many mathematics programs do not require students to write problems of the equalising type, but we think they are important because of their close tie to the idea of balancing and the meaning of equality. They give children another way of understanding abstract representations and help them make sense of equations.

Justifying

Having children justify their answers or their approaches to problems can help them understand the mathematics and gain confidence in their knowledge and skills. You should apply this principle when children are working with open sentences and ask them to justify their solutions. For example, if a student says the solution to the open sentence $35 + a = 51$ is 16, ask 'How do you know?' Note that you should do this even when the solution is correct, as it is in this case. Often, because of time pressure, teachers ask students to justify only when their solutions are incorrect. Think how this could affect a child's confidence!

Figure 15.1 shows some open sentences and how students explain the thinking behind their solutions. Having students explain their thinking is a way to encourage justifying. The students' solutions depend on understanding equality and the abstract representation of the equation. See if you can follow each student's explanation.

Patterns

Modelling

Sometimes, you may ask students directly to describe a growing pattern; at other times, you may give students a problem whose solution requires them to use a growing pattern. In either case, modelling the pattern with a table showing every term will help students organise their thinking and describe the pattern.

The table below summarises the information that the students found from modelling the marble problem with concrete materials (actual marbles or counters) and then with pictures.

Marble V-term	1	2	3	4	5	6	...	V
Number of marbles (*n*)	3	5	7	9	11		...	?

The teacher's task would be to help students generalise the pattern so they can represent it symbolically. There are 2 ways to consider the generalisation: (1) as a *recursive expression* or (2) as an *explicit equation*.

A recursive expression is an expression that tells how to find the value of a term given the value of the previous term. In the marble problem, it is clear that if you know the value of a term, the value of the next term will be 2 more (i.e. 2 marbles are added to the pattern each time). Thus, if you know that there are 101 marbles in the pattern, you know that the next step in the pattern would have 103 marbles.

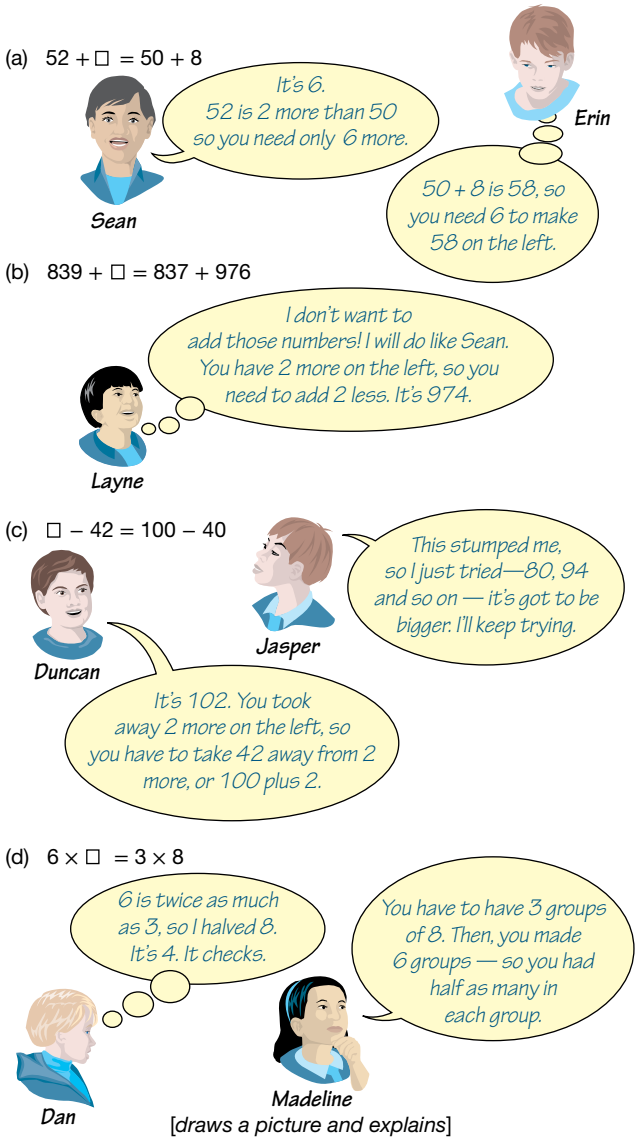


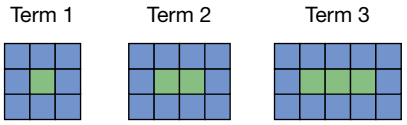
FIGURE 15.1 Explaining solutions to open sentences

Symbolically, we could express this as 'if n is the value of 1 term, then $n + 2$ is the value of the next term'. The recursive expression is easier for young children.

An explicit equation lets you calculate the value of 1 term given the number of the term. In the marble problem, an explicit equation would let you calculate the number of marbles given the number of the V-term. The students in the video expressed this as 'double the term number plus 1', which can be represented by the equation $n = (2 \times V) + 1$. An equation is a more powerful representation than a recursive

expression because you do not need any information about values of previous terms but can calculate the value of any arbitrary term.

Now let's consider the garden problem again. At the top of the next column is the picture showing the first 3 gardens along with a table of values that give the number of weed mats needed for the first 4 terms.



Term	1	2	3	4	5	6	...	*G*
Number of mats	8	10	12	14			...	*N*

Using the table, young children would find it relatively easy to tell how many mats would be needed for garden numbers 5 and 6. As in the marble problem, the recursive expression is $n + 2$ (2 more mats each time); thus, if garden 3 has 12 mats, garden 4 would have 14 mats, garden 5 would have 16 mats and garden 6 would have 18 mats. Young children with patience could extend this to show how many mats for garden 10 or even, say, garden 33! But what if they wanted to know how many mats for garden 200? An explicit equation would be more useful.

Look again at the picture of the growing garden above. There are several ways to look at the number of tiles without just counting them. Note that, no matter how long the garden is, there are 3 mats at each end. Also note that the numbers of mats above and below the garden are both the same as the term number (G) — that is, in term 2, there are 2 mats above and 2 mats below; in term 3, there are 3 mats above and 3 mats below; and so on. Thus, we can describe the sum (N) of all the mats as $N = 3 + 3 + G + G$ (i.e. 3 mats at one end plus 3 mats at the other end plus G mats above plus G mats below).

Justifying

If you have students working on these problems and trying to describe or model the pattern, it is also important to have them explain their thinking. At this level, you are having them justify their model. For instance, in the garden pattern, one might say $N = 6 + (2 \times G)$, another might say $N = (2 \times 3) + (2 \times G)$, and still another might say $N = 2 + 2(G + 2)$. Look at the picture and see if you can tell how these students might explain their thinking to justify their solutions. Of course, the power of algebra would let us show that all these expressions are equivalent, but children will find that out later in their study of algebra. At the primary levels, the important thing is that they should think about the models and patterns and explain their thinking.

When you show primary students number patterns that are not in a context such as a problem or a geometric figure, encourage them to explain the rule, either in terms of a recursive expression that relates 1 term to the next or, possibly, in terms of a closed sentence that expresses the value of 1 specific term. For example, students presented with the pattern 2, 4, 6, 8... could be encouraged to come up with a recursive expression such as 'each number is 2 more than the previous one' or to come up with a closed sentence such as 'the tenth number will be 20'.

You can also have students connect repeating patterns to algebraic expressions. Most primary curricula emphasise extending the pattern, filling in missing terms and doing other activities that help children analyse the overall pattern. Some curricula have children describe patterns in terms of the core. For instance, the simple alternating pattern ABABAB... is called an AB pattern because AB is the core. You can ask students questions such as 'What is the difference between an ABB and an ABC pattern?' and 'Can you give 2 examples of each?' to help them move to more abstract thinking, emphasising the justification of their answers.

Generalising

Finding a rule that generates the pattern is a type of generalisation. One of the most powerful ways to tie repeating patterns to algebra is to number the terms as you would do for growing patterns. This simple strategy can really help children begin to recognise and generalise the pattern. For example, consider the repeating pattern of geometric shapes in figure 15.2. Questioning guides student thinking about patterns in a more general way. Ask questions like the following.

- What shape is the ninth term? The twelfth term? [*They are both squares*.]
- What is the sixteenth term? (A hexagon) How do you know? [*Starting with term 1, every fourth term is a hexagon — terms 4, 8, 12, 16 and so on*.]
- What can you say about the terms of all the squares? [*They are all multiples of 3*.]
- How many of each shape are needed to extend the pattern to 20 terms? [*7 hexagons, 7 trapeziums and 6 squares*.]

Children in the upper primary years can begin to describe the terms of all the squares as multiples of 3. They may also see that the hexagons are always 2 places before the squares and the trapeziums are 1 place before the squares.



FIGURE 15.2 Repeating pattern with terms numbered

They can now begin to use symbols to describe their thoughts. The square's place can be described by $3 \times s$, where s represents the number of the squares thus far in the pattern; thus, the first square ($s = 1$) is in the third place. This means that the place of the hexagons can be described as $(3 \times s) - 2$ (i.e. hexagons are 2 places before squares). The first hexagon is in the first place and the second hexagon is in the fourth place. The place of the trapeziums (1 place before the squares) can be described by $(3 \times s) - 1$. These generalised rules written with symbols will come later, but describing what shape is in any place is well within the reach of students familiar with multiplication and division.

Nonroutine problems

We separated the discussion of modelling, generalising and justifying in the previous section; however, these processes are so intertwined that we will discuss each when appropriate for a particular problem, function or number property.

IN THE CLASSROOM 15.4

Painting cubes

Objective: Model and generate.

Year level: 5–8

Materials: Small cubes

Problem

If you painted the surface of a large cube made from smaller unit cubes, how many of the smaller cubes would be painted on no (0) face, 1 face, 2 faces, 3 faces, 4 faces, 5 faces and 6 faces?

Model each cube

Fill in the table below for the 1×1×1 cube. Make a 2×2×2 cube and refer to it to fill in the table below. Do the same with a 3×3×3 cube and a 4×4×4 cube. Some of the numbers have been filled in for you. Why is there no column for 4 faces or 5 faces?

Generalise

What pattern do you see? Can you predict the numbers for a $5 \times 5 \times 5$ cube? (You may have to build a model for the $5 \times 5 \times 5$ cube to verify your numbers.) How about for an $N \times N \times N$ cube? Justify your answers.

	Number of small cubes with painted faces					
Cube size	None	One	Two	Three	Six	Number of small cubes used
$1 \times 1 \times 1$	0				1	1
$2 \times 2 \times 2$	0			8	0	8
$3 \times 3 \times 3$	1		12			
$4 \times 4 \times 4$						
$5 \times 5 \times 5$					0	125
$N \times N \times N$					0	

In the classroom 15.4 shows an expanded version of the cube-painting problem, which we presented earlier in this chapter. This problem involves several different patterns that require analysis and description. The activity can be done with children in Years 5–8, but it is a good one for you and your peers to try as well. Go through the steps of modelling with actual cubes, find a systematic way of counting the painted faces, record your counts in the table, find a general rule for the counts in each column and justify your generalisations. To see the patterns, you may need to model and count up to a $5 \times 5 \times 5$ cube and visualise the counts for a $6 \times 6 \times 6$ cube. Why does it help to figure out the total number of small cubes in each large cube?

In this example, students who carefully count each of the cubes can often find the number in the next larger cube by extending the number pattern, but they often do not have the algebraic language to express the number of cubes that are painted on 1 or 2 faces since the general rule involves quadratic and cubic expressions. However, they all should be able to justify why each cube has exactly 8 smaller cubes that are painted on 3 sides and the number of cubes painted on only 1 face for any cube.

Figure 15.3 shows a number puzzle that also was presented earlier in the chapter. Number puzzles like this can help children think about generalising. First, have children do the puzzle with many different numbers chosen by the children. Then let them use materials to model the steps. As they use the materials, you should be able to see whether they have strongly developed concepts about the operations. If you discover, for example, that some children do not know how to model 'multiply by 2' or 'divide by 2', then you can reinforce the meanings.

You can move to a symbolic representation of the puzzle, as shown in the third column of figure 15.3. After the students model the actions with materials, you can ask them to describe the steps using symbols. Notice the concepts they can gain by doing this. For example, they can easily see that $2 \times (N + 4)$ is equal to $2 \times N + 8$ because both of these symbolic representations describe the same model.

Most children love number puzzles of all types, especially puzzles that let them predict a number, such as someone's favourite number. Find puzzles like these and first let children just enjoy them. Then challenge them to figure out how the puzzles work and then ask them to justify their solutions, by using either materials or symbolic representations.

You can also use calendars as a context for number patterns that will engage students in algebraic thinking. Figure 15.4 uses the calendar for a September as the context for the calendar number puzzle that we presented earlier. Have younger children look at how the numbers on the calendar change as you go across a row for a week. Also have them look at how the numbers change as you go down a column for a specific day of the week. The children may be able to recognise patterns such as that every other date is an even number or that the date of the next Friday after 12 September is 7 more, or 19 September.

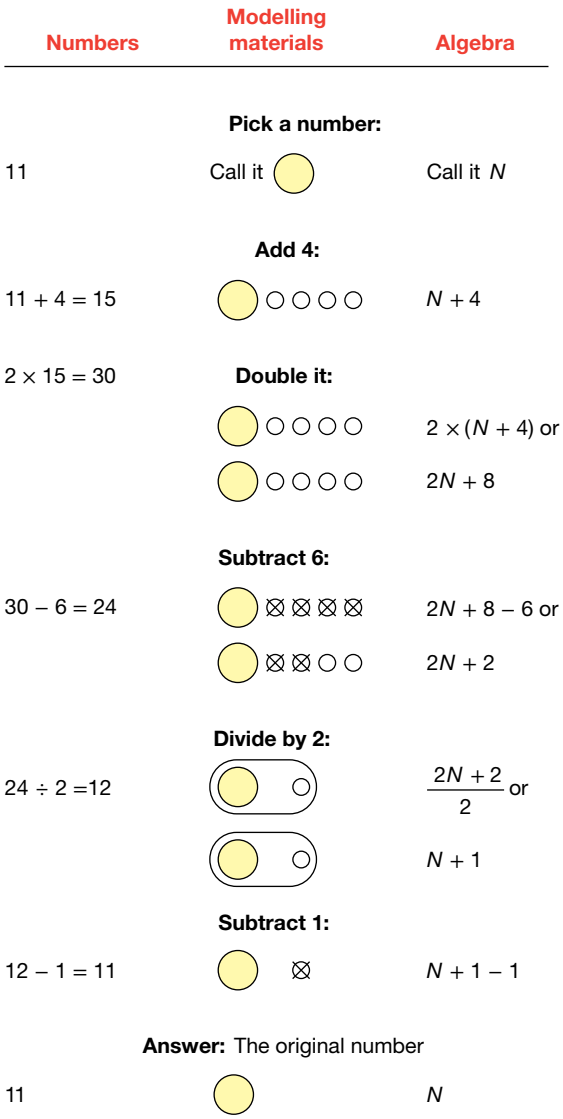


FIGURE 15.3 A number puzzle

Have students find the sum of all the numbers in a 3 × 3 square of dates (as shown in figure 15.4). After finding this sum in several different squares of the same size, students may realise that the sum is always 9 times the number in the centre. Older students should be able to see why this is the case. As shown in figure 15.4, using a letter (m) for the number in the centre and using expressions with that letter for the numbers in the other squares can help. Note that the sum of the expressions in the 9 squares is $9 \times m$. Children in the upper primary years or middle years should not find it difficult to make this generalisation.

Relations: functions

Functions may be treated as patterns, much like the growing patterns modelled in some of the previous tables in this chapter. By making tables and discussing the patterns that students see, you can help

students find the general rule (the function) that relates the 2 quantities. Children will find it easier to do this if the relationship involves only 1 operation. For example, the rules that describe how the In–out machine works in In the classroom 15.3 are easy to state for Monday through Thursday, as shown below (R is the result, or output, and I is the input):

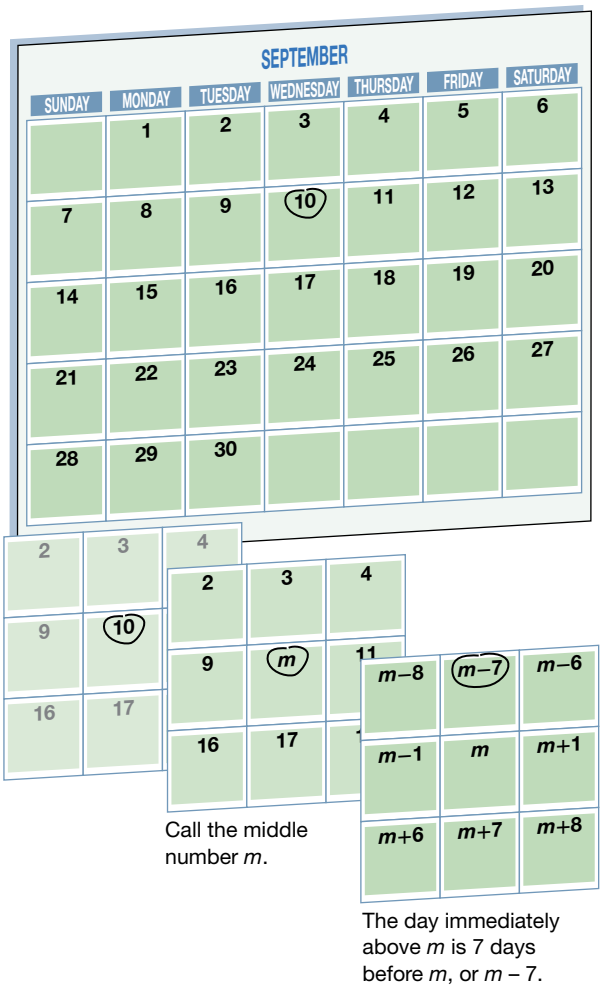


FIGURE 15.4 A pattern in calendars

Monday: $R = I \times 4$
Tuesday: $R = I + 10$
Wednesday: $R = I \div 2$
Thursday: $R = I - 16$

What about Friday? Remember, the machine does the same thing all day long. Student reactions tend to be like this:

> It looks like it added 7 spots to the frog. But that doesn't work for the toad. It's 3 times for the toad.

Here are some questions you can ask students to start them on the right track.

- Do the animals have more or fewer spots when they come out than when they went in? [*More*]
- Does adding the same number work? [*No*]

- Does multiplying by the same number work? [*No, it's 3 times for the toad, but it's not quite 3 times for the frog and the gecko.*]
- What if the machine used 2 operations? What if it multiplied and then added? [*Let the students try to work with just this hint.*]

Do not rush the students in their thinking. Later in their study of algebra, they will learn specific techniques for solving such problems, but at this point it is important for them to make conjectures and try to verify them.

In recent TIMSS and NAEP assessments, Year 4 students were asked questions about how pairs of quantities were related. Often, they found a rule that related 1 pair but not the others. They did not realise that every pair had the same relation and that they were supposed to determine it. Activities like In the classroom 15.3 are needed to help make children aware of this general idea of expressing patterns in terms of functions.

You should also ask children to 'undo' their thinking. For example, when doing In the classroom 15.3, you would tell children the output (how the animal came out) and ask them to tell how many marks it had going in. Educators call this *reversibility of thinking* (Driscoll 1999) and look at it as one of the crucial skills for children to develop as they learn to think algebraically.

Functions also are a good way to study change in subject areas other than mathematics. For example, in science lessons, students often collect data about how plants grow or how other natural phenomena change, and the patterns in which such changes occur can often be expressed as functions. Such changes may not always fit a simple rule. However, students can model the changes by graphing the data and then look at the graph to see if that helps them determine the rule. In the classroom 15.5 helps children see the relationship between rules (functions), numerical values of pairs of variables and graphs.

Connecting equations (the rules), the table of values and the graphs gives students ways to analyse situations. The questions asking the students to compare the graphs are a first step in having them look at how changing one representation (the rule) changes the other (the graph).

Relations: properties of numbers

Investigating properties such as distributivity can help students develop their ability to make and justify generalisations. Students may make conjectures entirely on their own, or you can initiate the process by asking thoughtful questions. Some teachers keep a written record available to the class showing students' conjectures along with the name of the child who stated each conjecture. Students can revisit these conjectures as they gather evidence as to whether or not they are true of all numbers. Remember that only 1 example is needed to prove that a conjecture is not true for all numbers. Consider, for example, the conjecture:

> When you multiply 2 numbers, the product is always larger than either number.

IN THE CLASSROOM 15.5

From a rule to a graph

Objective: Graph a function given a rule and analyse the graph.

Year level: 4–6 (for graphing), 5–6 (for analysing)

Materials: Graphing paper

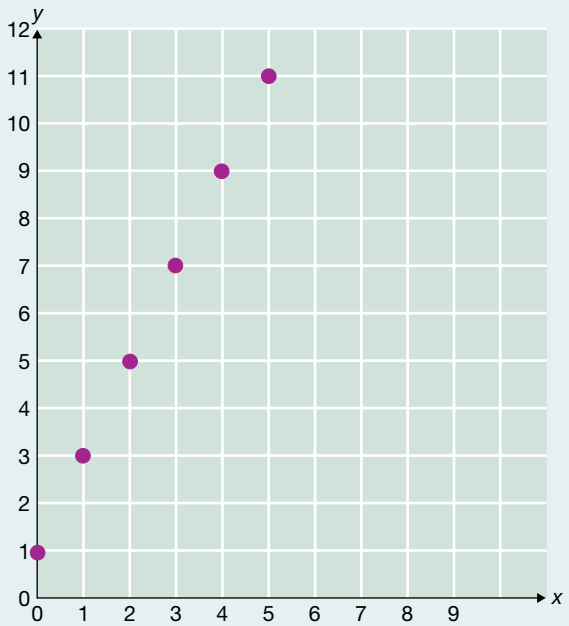
Make a table of values for each of these rules. Let x be 0, 1, 2, 3, 4 and 5.

Rule A

$y = 2x + 1$

Table	
x	*y*
0	1
1	3
2	5
3	7
4	9
5	11

Rule B: $y = x$
Rule C: $y = 2x$
Rule D: $y = 4$
Rule E: $y = 2x + 2$
Rule F: $y = 2x + 4$



Questions

1. Look at the graphs for rules B and C. How are they alike? How are they different?
2. Look at the graphs for rules B and D. How are they alike? How are they different?
3. Look at the graphs for rules C and E. How are they alike? How are they different?
4. How do you think the graph of the rule $y = 2x + 3$ would compare with rule E and rule F? Check your conjecture.

Although this is true for positive whole numbers, it is not true for all whole numbers. One example, $0 \times 6 = 0$, shows that the product (0) is smaller than 1 of the numbers (6). Thus, the conjecture is not true for all numbers and therefore the conjecture is false. Disproving conjectures is as important as proving them, and often it is simpler.

Young children use many of the mathematical properties intuitively because 'they work'. For example, most Year 1 students will find 2 + 9 by counting on 2 from 9. Actually, they are using the commutative property for addition, which can be stated like this: $a + b = b + a$, for any 2 numbers a and b. Of course, this language and symbolisation is too formal for very young children, but they can justify it by modelling it with concrete objects. Figure 15.5 shows how Years 3 and 4 students used linking cubes to model $8 \times 5 = 5 \times 8$, which is a specific example of $a \times t = t \times a$ (the commutative property for multiplication) (Carpenter & Levi 2000).

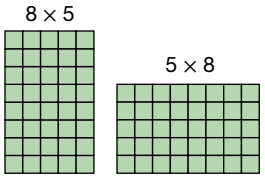


FIGURE 15.5 A model of $8 \times 5 = 5 \times 8$

This figure illustrates a particular case, but the students went on to argue that the model shows that all you have to do is turn 1 of the array of cubes on its side, so it would work for any array. Does this convince you that the commutative property holds for all whole numbers?

The distributive property states that $a \times (b + c) = (a \times b) + (a \times c)$ for any 3 numbers a, b and c. Again, students use this property naturally without realising it. For example, consider a problem that we presented earlier in the chapter:

> Pat took 4 buckets to the beach. She had 3 shovels and 2 rakes in each bucket. How many tools did she take to the beach?

Generally, children approach this problem in 1 of 2 ways.

1. They might add the number of shovels and number of rakes (3 + 2) first, to find out the number of tools in each bucket and then multiply the sum by 4. You could represent this by $(3 + 2) \times 4$ or use the commutative property to represent it as $4 \times (3 + 2)$.
2. They might find the total number of shovels (4×3) and the total number of rakes (4×2) and then add $(4 \times 3) + (4 \times 2)$.

The fact that both approaches give the same correct answer (20) demonstrates the distributive property — that is, $4 \times (3 + 2) = (4 \times 3) + (4 \times 2)$. How would you model the distributive property with physical materials? How would you justify the property more generally?

It is important to give children examples illustrating mathematical properties to see if they can generalise those properties. First show them closed sentences and have them verify that the sentences are true; then have them solve similar open sentences; finally, encourage them to make conjectures for generalising the properties illustrated by the example sentences. Do not expect children to immediately come up with good generalisations; it may help to give them some additional example sentences or to have the students themselves give more examples. The wording of the children's conjectures will probably need to be revised; a class discussion about how to word the conjectures so they make sense is important in helping children learn to write mathematically. Table 15.2 shows the kinds of closed and open sentences and corresponding generalisations that you can use with students.

TABLE 15.2 Sentences and generalisations

Closed sentences	Open sentences	Generalisations (mathematical properties)
79 + 0 = 79	37 + □ = 37	When you add 0 and any number, you get that number.
0 × 54 = 0	0 × □ = 0	When you multiply any number by 0, you get 0.
33 − 0 = 33	83 − □ = 83	When you subtract 0 from any number, you get that number.
67 − 67 = 0	3456 − □ = 0	When you subtract any number from itself, you get 0.
24 × 1 = 24	1002 × □ = 1002	When you multiply any number by 1, you get that number.

The wording of the generalisations in this table is informal but correct. Children will also use informal wording, and you should help them focus on whether or not their wording is clear and correct. For example, if a student says, 'when you subtract any number and 0, you get that number', encourage the class to discuss why this is not clear ('subtract any number *and* 0' doesn't make sense).

Often you can help older children explain their discoveries with algebra. Thus, you need to be familiar with using algebra. In the classroom 15.6 gives an example of how a teacher questioned Robin to help him make a more general rule for inserting fractions between 2 fractions and how she helped him prove it for the easier cases of 1 and 2 fractions being inserted.

Another look at modelling, generalising and justifying

In this section, we have shown examples of problems, patterns and relations that can be used to encourage modelling, generalising and justifying. These processes are often done together, but we often focus on only one of these at a time. For example, you can model without justifying the model or making a generalisation, or conjecture a generalisation without modelling or justifying it. However, the power of mathematics comes when we do include justification in whatever we do. This is a way to see if students are making sense of mathematics.

'The arguments that students typically use to justify generalisations fall into 3 basic categories: appeal to authority, justification by example, and deductive arguments' (Carpenter & Romberg 2004, p. 28). These educators then go on to point out that appealing to authority is really avoiding justification. For example, a student who says 'my teacher told us to divide fractions, you just flip the one after the division sign and multiply' is not providing a mathematical justification for doing so and is not showing an understanding of division of fractions. Children need to learn that, in mathematics as in most subject areas, they should not do something a certain way just because someone tells them to; rather, they need to understand why doing it that way makes sense (or doesn't make sense).

In this chapter, we have mainly focused on justification by examples and by informal explanations (i.e. explanations that aren't as tightly logical as formal deductive arguments). We have encouraged you to have children make models, move on to symbolic representations when they can and discuss how they arrived at their generalisations. These are important steps in understanding and fully justifying generalisations, and they may be as much as young children can handle.

IN THE CLASSROOM 15.6

Fractions discovery

Objective: Discover a rule for finding fractions between 2 fractions and prove the rule algebraically.

Year level: 6 and above

Robin found a rule for inserting 1 number of fractions between 2 special types of fraction. Answer these questions to see if you can discover the rule.

Questions

1. Robin's rule worked for these fraction pairs:

(a) $\frac{1}{3}$ and $\frac{2}{3}$ (b) $\frac{3}{5}$ and $\frac{4}{5}$ (c) $\frac{4}{11}$ and $\frac{5}{11}$

It would not work for $\frac{3}{7}$ and $\frac{4}{9}$.

How would you describe the pairs of fractions that worked?

2. Robin said:

If I want to put 1 number between the pair, I multiply the numerators and denominators by 2. For (b), I multiply like this:

$$\frac{3 \times 2}{5 \times 2} \quad \frac{4 \times 2}{5 \times 2}$$

That gives me $\frac{6}{10}$ and $\frac{8}{10}$. $\frac{7}{10}$ is between.

Try this rule with examples (a) and (b) above.

3. Robin's teacher asked:
What do you think you would do if you want to put 2 fractions in between these pairs?
Robin conjectured: Since for 1 between, I multiplied by 2, I think I will try 3.
Check his conjecture for all 3 pairs of fractions.
4. Write your own conjecture for putting 3 fractions between these pairs. Check it.
5. Try more examples and write a conjecture for putting n fractions between these pairs.

Extend

1. Prove that the rule for 1 and 2 fractions between works for this type of pair. *Hint:* If 1 fraction is $\frac{a}{b}$, how would you describe the second fraction?
2. Prove the rule for any number (n) of fractions for pairs like this.
3. What if the denominators are not the same? Can you discover a method?

We close this section with a summary of an interview with a Year 2 student, Susie. Her teacher was in a study group investigating how to encourage algebraic thinking in primary school. The class had been making and justifying conjectures such as the one Susie was asked to justify: $a + b - b = a$. The video of the interview (Carpenter & Romberg 2004) shows Susie thinking about the conjecture and the interviewer, Ms L, encouraging her to delve deeper.

> Susie initially proposes to justify the conjecture by trying a lot of different examples with different types of numbers. She recognizes that she cannot try all the numbers, and that she cannot be sure the generalisation is true for all numbers by trying a few examples.
>
> Ms. L realizes that Susie had moved from her initial strategy of operating on numbers to using properties appropriate for a more general proof. For example, she says 'I just know if you have a number and you minus the same number, it gets you to zero'. She then uses 2 conjectures that her class has discussed: $b - b = 0$ and $a + 0 = a$. She argues that if these conjectures are true, then her conjecture of $a + b - b = a$ must be true (Carpenter & Romberg 2004, pp. 30–1).

This is a powerful example of a young child beginning to make a convincing deductive argument. When asked if her way is better than trying numbers (the way she began), she answers: 'This way, because it would take your whole life, and you still wouldn't have tried all the numbers. And this way you only have to do one thing, and you proved it.' (Carpenter & Romberg 2004, p. 31).

SPOTLIGHT ON EARLY CHILDHOOD

Developing arithmetic and algebra

'The powers necessary for algebraic thinking are being used by children as soon as they leave the womb' (Mason, in Kaput, Carraher & Blanton 2008, p. 58). The babbling of babies in their cribs is not only the foundation for their language but can be interpreted in mathematical (algebraic) terms such as associativity through their pauses or ordering of sounds. The contention here is that arithmetic is not the necessary precursor to algebra, but that both may most effectively developed together. The development of cognition moving from physical experience to the remembered or imagined to the symbolised as words or arcane mathematical symbols is a natural, intuitive progression. Thus, the power of the very young as they engage in these natural activities can be harnessed by 'encouraging students to express perceived generalities, relationships, connections, properties, and so on' (p. 60).

MAKING CONNECTIONS

Development of classical algebra, which is often considered a generalisation of arithmetic, began in the fourth century. It was not until the nineteenth century, however, that algebra began to be studied in secondary schools around the world. During the twentieth century, algebra became a central subject for

anyone who expected to advance in their study of mathematics. Mason summarises 'significant moments in the history of the development of algebra' (Mason et al. 2005, pp. 311–12). Research suggests that students' understanding of algebra follows this historical trajectory. In the first half of the twentieth century, many of the topics included in algebra, such as positive and negative numbers and exponents, were made part of the junior high school curriculum.

Much of this material is now in many middle school curricula. Appropriate introduction to algebra in the primary school is being recommended and issues related to this are being researched.

The Trends in International Mathematics and Science Study (TIMSS) (Mullis et al. 2008) gathered information about which algebraic (pre-algebra in the primary years) topics were included in the Year 4 curricula of the countries that participated in the study. The study looked at 6 topics: number sentences, modelling unknown situations with number sentences, extending patterns, describing relationships between adjacent numbers in a sequence, generating pairs of numbers given a rule, and finding a rule for a relationship given some pairs of numbers (Mullis et al. 2008, p. 207). The most commonly included topic across the countries was number sentences, followed by extending patterns. More countries expected students to find pairs of numbers related by a given rule than to find the rule that related pairs of numbers.

A GLANCE AT WHERE WE'VE BEEN

The Australian Curriculum's Number and Algebra strand and the NCTM's Algebra Standard emphasise patterns and relationships, ways of representing and analysing mathematical situations using algebraic symbols, the use of models to represent and understand relationships, and the analysis of change in various contexts. These ideas have been blended into this chapter as we have considered problems, patterns and relations as topics to develop algebraic thinking. These topics occur in all strands of the mathematics curriculum as well as in other subjects. Helping children to think algebraically does not mean adding another topic; it does mean we probably need to teach those areas differently. We can help children develop algebraic concepts and habits of algebraic thinking through questioning; helping students model problems, patterns and relations; encouraging them to generalise; and expecting them to justify their thinking and statements. We have emphasised the need for teachers to ask questions and provide opportunities for children to discuss and defend their ideas with any of the mathematics they are learning.

THINGS TO DO: FROM WHAT YOU'VE READ

1. Why should algebra be studied in primary school? How would you describe that algebra?
2. What are topics in the primary curriculum that can be used to develop algebraic concepts and thinking?
3. What are 2 common types of pattern? How do they differ? How are they alike? In particular, how are they alike and different in developing the algebraic thinking discussed in this chapter?
4. What algebraic language and symbols are essential for primary school students?
5. What is the difference between an open sentence and a closed sentence?
6. How does representing routine problems promote algebraic thinking?
7. What 3 processes were used to describe algebraic thinking? Give an example of each.
8. What are the different meanings of *variable*?
9. Complete the cube-painting problem (In the classroom 15.4).
10. Justify the statements listed in the section Relations: properties of numbers (In the classroom 15.5).

THINGS TO DO: GOING BEYOND THIS TEXT

IN THE FIELD

1. *Observations.* Algebraic thinking is a part of developing number sense. Observe some mathematics lessons in a variety of classrooms. Are there elements of algebraic thinking being developed as part of these lessons? What are some of the key differences between the early and middle years? Think about the progressive development of algebraic thinking throughout the primary years and what the implications are for you as a teacher.
2. *Equality*. Ask students to explain to you their understanding of the equals sign. Are there misconceptions evident in their definitions? How might you address these misconceptions as a teacher? What can you learn about students' understanding of equality, mathematical sentences and justifying from this type of activity?
3. *In–out machine*. Use the activity sheet in In the classroom 15.3 or one that you design. Do a few other examples with children. Do you notice that children want to make a judgement of the rule after one in–out? Can they find the number but not describe the rule?
4. *Algebraic thinking*. Ask several students in Year 5 or 6 to examine the garden problem with you. Have students talk through their approach to solving this problem. Encourage the students to draw the next few gardens and weed mat paths (1×3 and 1×4) if they are having difficulty generalising to the 1×8 garden. Encourage them to describe what patterns they see between the garden and the number of mats. See if that helps them to generalise to larger gardens or if they still need to draw a picture. For students who have a good grasp of the problem, ask how many mats would be needed for a $1 \times x$ garden. What insights did you get into the development of algebraic thinking through this activity?

IN YOUR JOURNAL

5. Describe your understanding of what algebraic thinking means for primary students, giving examples in your explanation.
6. Describe your understanding of variables. How can you stop children from thinking of a variable as just a placeholder?
7. What was your experience in algebra classes? Do you think that if you had the background recommended in this chapter that it would have been different? Give ways and examples to explain your comments.
8. Describe why equality is an important idea in algebraic thinking.

WITH ADDITIONAL RESOURCES

9. Select an instructional idea in a current textbook series that is listed as an algebra lesson. Briefly summarise the lesson and explain how it supports the ideas or how it is contrary to the ideas in this chapter.

WITH TECHNOLOGY

10. Choose some of the growing and repeating pattern problems described in the chapter. How can you incorporate the use of calculators into the problem solving process? Do you consider using a calculator would help of hinder the development of algebraic thinking? Discuss this with your peers and justify your position.

CHILDREN'S LITERATURE CONNECTIONS

Ahlberg, J. & Ahlberg, A. (1986). *Each Peach Pear Plum*. Hamondsworth, UK: Penguin Books.
This is a rhyming book about nursery rhyme characters that can be used to introduce the concept of pattern to Foundation through Year 1 students.

Anno, M. & Anno, M. (1983). *Anno's Mysterious Multiplying Jar*. New York: Philomel.
This book is beautifully illustrated and explains the theory of factorials through a cumulative story featuring the improbable contents of a magic jar.

Barry, D. (1994). *The Rajah's Rice*. New York: W. H. Freeman.
A smart peasant girl outsmarts the rajah and ensures the village plenty of food forever after the rajah agrees to give the girl the rice that would cover his chessboard when he puts 1 grain on the first square, doubles it for the next square and then doubles the amount on that square and so on for all 64 squares.

Burns, M. (1997). *Spaghetti and Meatballs for All!* New York: Scholastic Press.
Problem-solving and algebra are brought together here in a story about arranging tables and seating for an ever-increasing number of people. This book is suitable for students in Year 3 and above.

Calmenson, S. (1991). *Dinner at the Panda Palace*. New York: HarperCollins.
This book for Years 1 to 3 counts dinner guests as they arrive at the Panda Palace. Each group is 1 guest larger than the group before. Children can be encouraged to predict how many guests would be in the tenth group or the fifteenth group. They might be challenged to make up a similar story with a different expanding pattern. The class can figure out how many groups it would take to fill the restaurant.

Emberley, B. & Emberley, E. (1967). *Drummer Hoff*. New York: Simon & Schuster.
This classic book is written in an increasing pattern style. As each new element is added, the list is repeated from the beginning, with the new element now at the end of the list. Students can model the patterns with manipulatives or label the elements and represent the patterns with symbols (e.g. *a*, *ab*, *abc* or *abcd*).

Geringer, L. (1987). *A Three Hat Day*. New York: Harper & Row.
A man who loves hats has a collection of 12 different hats. One day he decides to wear 3 hats and it brings him good luck. The students figure out how many different combinations of 3 he could make with the hats he has. The book is most appropriate for Year 3 and 4 students.

Martin, B. (1991). *Polar Bear, Polar Bear, What Do Your Hear?* New York: Henry Holt.
This predictable pattern book is a good illustration of patterns for kindergarteners and Year 1 students.

Murphy, S. J. (2000). *Beep, Beep, Vroom, Vroom!* New York: HarperCollins.
Kevin keeps his toy cars in a pattern on the shelf. Suddenly, the cars fall from the shelf and Kevin's sister, Molly, puts the cars back on the shelf in a different pattern. This happens again, but this time the pattern is the same as the original. There are activities and games for parents and teachers to use.

Swinburne, S. R. (1998). *Lots and Lots of Zebra Stripes*. Honesdale, PA: Boyds Mills Press.
This book explains with colourful illustrations how patterns encompass our world. The author explains that the patterns can have a purpose or be seasonal.

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CHAPTER 16

Geometry

LEARNING OUTCOMES

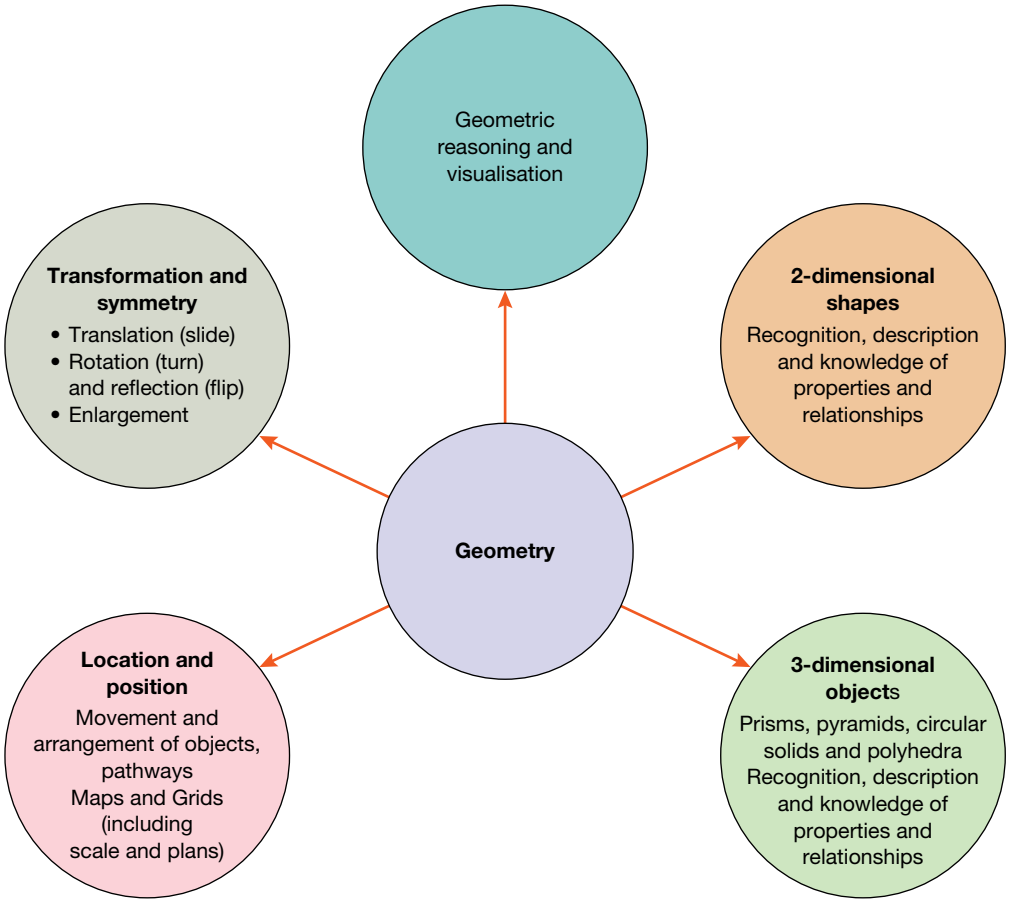
16.1 Introducing geometry and the attributes of 2-dimensional shapes and 3-dimensional objects.
16.2 Describing the roles that location and spatial relationships play in geometry.
16.3 Identifying applications of the 3 geometric transformations.
16.4 Understanding what types of exploration with geometry can help build children's spatial reasoning and visualisation skills.



'Geometry is the archetype of the beauty of the world.'

Johannes Kepler

Chapter 16 concept map



Introduction

Over time, perspectives about the importance of geometry have shifted away from:

> 'Oh, I could never do proofs.'
> 'The children don't understand geometry, so why do it?'
> 'We do geometry if we finish everything else first.'

There are many reasons for such views. Some are based on personal experiences such as unsatisfactory geometry classes in the secondary years. Some are based on inappropriate geometry curriculum materials for primary students that use, for example, an abstract, definitional approach. Some are based on a historical emphasis on computation — even though various professional groups have recommended geometry be included in the curriculum for over a hundred years.

As a pre-service teacher you might come across the following views:

> 'It amazes me who is good in geometry; it's not always my best number students.'
> 'What a joy it is to see a child's eyes light up as she discovers . . . '
> 'Some of my students could work on a geometry problem for hours.'
> 'Geometry gives me an opportunity to work on communication skills and to help children follow instructions.'
> 'The change in the spatial ability of children after they work with geometric shapes always surprises me.'
> 'I love to learn with my students; I never liked geometry before.'

Why should you include geometry in primary school? The previous responses certainly give several reasons, including geometry being a mathematics topic that engages children differently both in performance and persistence. It is a natural way to include other skills, such as following directions and reasoning about shapes and their properties. Children can make and verify conjectures about geometric figures. For example, by folding models of isosceles triangles on the line of symmetry, children can see that 2 sides are equal in length and 2 angles are congruent. It is also a topic that will help you teach many other mathematical topics. For example, many representations are geometric in nature — region models for fractions, area models for multiplication and spatial patterns that lead to algebraic expressions. Geometry can be applied to creative pursuits, as well. Artist Albrecht Durer understood that knowing geometry was essential to him as an artist, and ensured that all of his students learnt geometry.

Geometry, together with measurement is one of the 3 essential content strands of the Australian Curriculum. Table 16.1 includes the content description for the Shape, Location and transformation, and Geometric reasoning sub-strands for Foundation to Year 7.

This chapter is organised around the following content and processes:

1. geometry — 2-dimensional shapes and 3-dimensional objects
2. location — position and movement of objects, maps, pathways and plans
3. transformation — translation (slide), rotation (turn) and reflection (flip)
4. visualisation and spatial reasoning.

TABLE 16.1 **Geometry: Measurement and Geometry content descriptions in the Australian Curriculum**

Year level	Content description
Foundation	**Shape** • Sort, describe and name familiar two-dimensional shapes and three-dimensional objects in the environment. **Location and transformation** • Describe position and movement.
Year 1	**Shape** • Recognise and classify familiar two-dimensional shapes and three-dimensional objects using obvious features. **Location and transformation** • Give and follow directions to familiar locations.
Year 2	**Shape** • Describe and draw two-dimensional shapes, with and without digital technologies. • Describe the features of three-dimensional objects. **Location and transformation** • Interpret simple maps of familiar locations and identify the relative positions of key features. • Investigate the effect of one-step slides and flips with and without digital technologies. • Identify and describe half and quarter turns.
Year 3	**Shape** • Make models of three-dimensional objects and describe key features. **Location and transformation** • Create and interpret simple grid maps to show position and pathways. • Identify symmetry in the environment. **Geometric reasoning** • Identify angles as measures of turn and compare angle sizes in everyday situations.
Year 4	**Shape** • Compare the areas of regular and irregular shapes by informal means. • Compare and describe two-dimensional shapes that result from combining and splitting common shapes, with and without the use of digital technologies.

Year level	Content description
Year 4	**Location and transformation** • Use simple scales, legends and directions to interpret information contained in basic maps. • Create symmetrical patterns, pictures and shapes with and without digital technologies. **Geometric reasoning** • Compare angles and classify them as equal to, greater than or less than a right angle.
Year 5	**Shape** • Connect three-dimensional objects with their nets and other two-dimensional representations. **Location and transformation** • Use a grid reference system to describe locations. Describe routes using landmarks and directional language. • Describe translations, reflections and rotations of two-dimensional shapes. Identify line and rotational symmetries. • Apply the enlargement transformation to familiar two-dimensional shapes and explore the properties of the resulting image compared with the original. **Geometric reasoning** • Estimate, measure and compare angles using degrees. Construct angles using a protractor.
Year 6	**Shape** • Construct simple prisms and pyramids. **Location and transformation** • Investigate combinations of translations, reflections and rotations, with and without the use of digital technologies. • Introduce the Cartesian coordinate system using all four quadrants. **Geometric reasoning** • Investigate, with and without digital technologies, angles on a straight line, angles at a point and vertically opposite angles. Use results to find unknown angles.
Year 7	**Shape** • Draw different views of prisms and solids formed from combinations of prisms. **Location and transformation** • Describe translations, reflections in an axis, and rotations of multiples of 90° on the Cartesian plane using coordinates. Identify line and rotational symmetries. **Geometric reasoning** • Identify corresponding, alternate and co-interior angles when two parallel straight lines are crossed by a transversal. • Investigate conditions for two lines to be parallel and solve simple numerical problems using reasoning. • Classify triangles according to their side and angle properties and describe quadrilaterals • Demonstrate that the angle sum of a triangle is 180° and use this to find the angle sum of a quadrilateral.

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16.1 The geometry of 2-dimensional shapes and 3-dimensional objects

LEARNING OUTCOME 16.1 Introducing geometry and the attributes of 2-dimensional shapes and 3-dimensional objects.

Understanding the properties or attributes of objects and the relationships among different geometric objects is an important part of primary mathematics. Before discussing how to help children learn and use the ideas in the Australian Curriculum; it is worth introducing some of the research on children's learning of these ideas.

The research of Dina van Hiele Geldof and Pierre Marie van Hiele, a Dutch couple, has provided teachers with guidelines for the growth of geometric understanding and thought. They proposed and studied a model of levels of understanding. Clements and Battista (1992) have studied this model and offer a modification of the original van Hiele levels, as shown in figure 16.1. Although this model is not prescriptive, it will help pre-service teachers to analyse learning activities. Too often, we only have students recognise and name shapes. It is not difficult to see why our students do not do well in advanced geometry classes (Kloosterman et al. 2004), since many geometry experiences at the primary level only require students to be at the visual level (Level 1). We need to help them move to the next 2 levels in primary and secondary school.

Level	Description	Simple responses of children
Level 0 (Prerecognition)	Children only focus on some visual cues.	Shown a triangle, children may focus on straightness and say it is a square.
Level 1 (Visual)	Children view a geometric shape as a whole. They can describe properties based on visualisation but not based on analysis of the attributes.	Children may say it is a rectangle because it 'looks like a door'. It has 4 sides because they can count them.
Level 2 (Descriptive/Analytic)	Children focus on the relationships between parts of a shape and defining attributes.	Children may describe a rectangle as a 4-sided figure with opposite sides equal and parallel with 4 right angles.
Level 3 (Abstract/Relational)	'Students interrelate geometric properties, form abstract definitions, distinguish among necessary and sufficient sets of properties for a class of shapes…' (Battista 2007)	Students know that a sufficient definition of a rectangle is 'a quadrilateral with 2 pairs of parallel sides and a right angle'.
Level 4 (Formal Axiomatic)	Students use deduction to prove statements. This is the level needed to be successful in a formal, high school geometry class.	Students, given axioms, can write a deductive proof.

FIGURE 16.1 Levels of geometric thought (modified van Hiele levels by Clements and Battista 1992)

Very young children often attend to visual cues that are obvious to them. Later, children focus on visual cues from their environment saying it is a rectangle because it looks like a rectangle (Level 1). This is much like saying it is a dog because it looks like one. Just as very young children can identify many dogs when they see one, children can identify rectangles. Where can they see an object shaped like a rectangle? Our world provides plentiful examples: a door, a book and a table top for instance. From a young age, children can begin to look at shapes more critically, such as describing a rectangle as having 4 sides with square corners.

With many experiences that specifically require them to describe the properties of the figures (Level 1, Visual), children begin to recognise and more carefully describe properties of all rectangles and move to Level 2 (Descriptive/Analytic). At this level they are moving toward a more precise description of classes of shapes. For example, they might describe all rectangles as having 4 sides; 2 pair of parallel, equal or congruent sides; and 4 right angles. However, they do not realise that it is sufficient to define a rectangle as a quadrilateral that has 2 pairs of parallel sides and 1 right angle. This more abstract thought is at Level 3 (Abstract/Relational). At this level students establish relationships among properties and among figures.

A group of students doing the same activity might be observed working at different levels and, through questioning, you can help them develop a deeper understanding appropriate to their current thinking. We will return to these levels of geometric thought throughout this section.

3-dimensional objects

We live in a 3-dimensional world that can be represented and described geometrically. We begin with 3-dimensional objects because this is the world of young children — a world that they can explore in the geometric sense. It is important to begin their geometry explorations from their perspective with familiar 3-dimensional objects, such as balls and blocks. They may roll the ball and perhaps try to roll the block. Different experiences help them describe these solid, 3-dimensional shapes. A 3-dimensional object such as a ball is often referred to as a *solid* even though the object may be hollow; we will refer to these shapes as solids in this chapter.

So that you can see the development of ideas about 3-dimensional objects, we discuss them together in this section. This does not mean that we recommend doing everything discussed here before considering 2-dimensional shapes. Students need concepts from 2-dimensional shapes in order to more completely describe 3-dimensional objects.

Models play an important role in all of geometry, but especially in 3-dimensional geometry. If wooden or plastic solids are not available, you can make models, as suggested later in this section. You and your students should also collect real objects that have particular geometric shapes — including spheres (balls), cylinders (cans), prisms (boxes) and cones, as well as solids that may not have geometric names. The Australian Curriculum states that in the early years, children are able to sort, describe and name familiar 3-dimensional objects in the environment. Later they can recognise, visualise and classify them using obvious features. Over time they can represent 3-dimensional objects in drawings and models constructed from materials.

Studying geometric properties of 3-dimensional objects also provides an opportunity to reason and think geometrically. This section is built around the processes of:

1. describing, sorting and classifying
2. exploring, representing and constructing.

Describing, sorting and classifying

Children need to be able to describe properties of 3-dimensional objects to explain how 2 or more objects are alike or different geometrically. Describing and sorting are processes that are begun in early childhood and should be continued throughout primary school. New and more complex properties can be added to children's repertoire. Describing and sorting activities help children develop thinking at Level 1 and lead toward the Level 2 thinking described by the van Hiele model. They are essential to develop the abstract/relational (Level 3) thinking that is a precursor to high school geometry.

The activities that follow use vocabulary and properties appropriate for beginning, intermediate and more advanced students. Older children who have not been exposed to 3-dimensional activities will benefit from activities such as those described in the beginning activities. Their responses, of course, will be more complex.

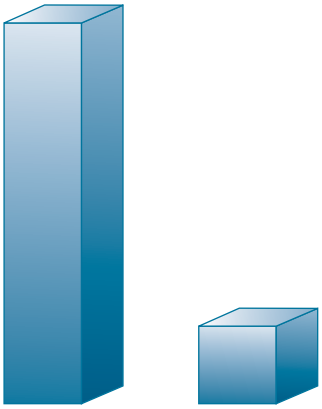
Beginning activities

Children are often taught the names of the geometric shapes; however, they do not develop the discriminating power they need to use the names with meaning. In these beginning activities you should build on the children's own vocabulary, adding new words as appropriate. Although the names of the solids can be used, they need not be formally introduced until children have done activities like these. Notice that they are not just asking questions like 'mark all the pictures of cones'; they require children to describe and to justify their answers. These activities are at Level 1, but pushing children toward Level 2 as they justify their answers.

1. *Who am I*? Put out 3 objects (such as a ball, a cone and a box). Describe one of them (it is round all over, it is flat on the bottom or its sides are all flat). Ask the children which object you are describing and have them explain why it is that object and not one of the others.
2. *Who stacks*? Provide a collection of solids for children in small groups to sort according to which will stack, which will roll and which will slide. A more complex sorting activity is one that requires 3 sets

of objects: (1) solids that can be stacked no matter what face is used, (2) solids that can be stacked if placed in some ways but not in other ways and (3) solids that cannot be stacked in any way.

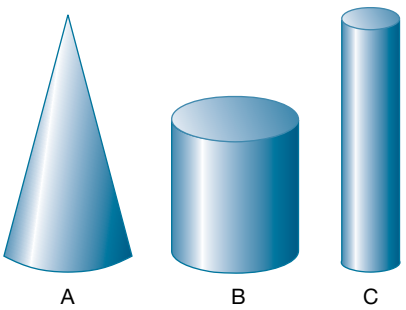
3. *How are we alike or different*? In a whole-class discussion, hold up 2 solids such as the following:



Ask children to explain how they are alike or different. For example, children may compare the 2 solids shown as follows:

'They are both flat all over.'
'One is tall.'
'One has bigger faces.'
'They both have some square faces.'
'They both have 6 faces.'

4. *Who doesn't belong*? For another whole-class activity, put out 3 solids such as these:



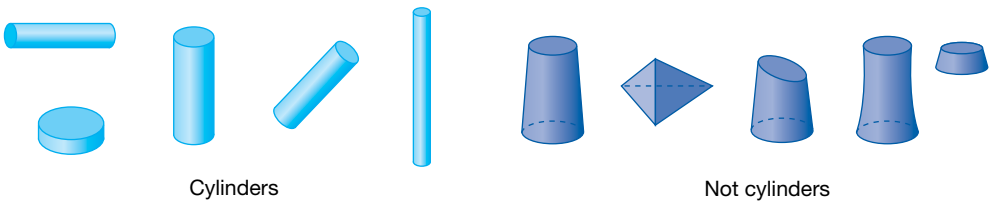
Ask the children which does not belong with the other 2 objects. Since there are many ways to solve this problem, be ready to encourage lively discussion. For example, some children may say A doesn't belong because it has a point. Others may say B doesn't belong because it's short or C doesn't belong because it is skinny or has a smaller base.

5. *How many faces do I have*? A *face* is a flat side of a solid object or a continuous curved surface. Have children in small groups count the number of faces on solids of various shapes. Then ask them to collect objects with 6 faces (boxes, books), with 3 faces (cans) and with 1 face (balls). You will be surprised at what they find! Some children have difficulty counting the number of faces on many-faced solids. You can help by using tape or stickers and numbering them as they count.

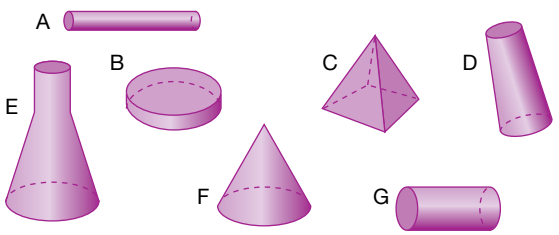
Intermediate activities

The following activities introduce the names of some solids and consider sizes (or measure properties) as well as edges, faces and vertices. Most of the thinking is at Level 2; the vocabulary and ideas are more mathematical than those in the beginning activities.

1. *Edges, vertices and faces: Who am I*? After introducing the concepts of *edge* (a line segment formed by the joining of 2 faces) and *vertex* (a point at which 3 or more edges come together), have children in small groups solve these riddles:
   I am a solid with:
   - 8 edges — who am I?
   - 6 edges and 4 faces — who am I?
   - 5 vertices — who am I?
   - the same number of vertices as faces — who am I?
   - 1 face and no corners — who am I?
2. *Classifying solids*. Introduce each type of solid — pryamids, prisms and curved objects such as cones, cylinders and spheres — by putting out solids that are examples and non-examples of each type of solid. As illustrated below, show 2 or 3 cylinders and 2 or 3 solids that are not cylinders. Then have the children select which are cylinders and which are not from the third set. Since it may be difficult to collect examples of such cylinders, you could use pictures. The important part of this activity is the discussion, having children describe why the object is or is not a cylinder. For example, many children think shape A below is too 'thin' to be a cylinder. They may be thinking at Level 0 — it does not look like one, but it is like a can. Whole-class discussions about what is common about all cylinders — they can slide and roll and have 2 identical faces that are flat circles joined by a curved surface — can help them move to a higher level of thought.



Cylinders Not cylinders



Which are cylinders?

3. *Searching for solids*. Make up a set of activity cards that children can use in small groups to search out solids according to the size and shape of the faces and the length of the edges. The clues you give will depend on the solids you use, but here are some samples to get you started:
   Search for a solid with:
   - exactly 2 faces that are the same size and shape (*congruent*)
   - exactly 3 faces that are the same size and shape
   - all edges the same length
   - edges of 3 different lengths.

Advanced activities

These activities focus on the properties of parallel, perpendicular and equal faces and edges as well as more careful definition and classification of the solids. These activities are at Level 2 and use more complex geometric ideas than in the previous activities. They also consider more than 1 property, leading students to see relationships between properties and helping them move toward Level 3.

1. *Parallel faces*. This activity can be done after parallel lines and parallel faces have been introduced. It consists of questions about real objects and why faces are parallel (equidistant). A few sample questions that you can use in a whole-class discussion are given here to start you thinking:

   Why are the top and bottom of soup cans parallel?
   Why are shelves of a bookcase parallel to the floor?
   Why are the roofs of houses in cold climates usually not parallel to the ground?
   Why is the front side of a milk carton parallel to the back side?

2. *Perpendicular edges*. In the classroom 16.1 provides clues about particular solids, focusing on perpendicular edges (at right angles) and asks students to construct the solids from sticks and connectors. (See the Edge models section that follows for a variety of materials that can be used to model the mystery objects.) Students enjoy making up their own mysteries for other students to solve. This experience provides an excellent opportunity to work on developing written descriptions that are clear, precise and not contradictory.

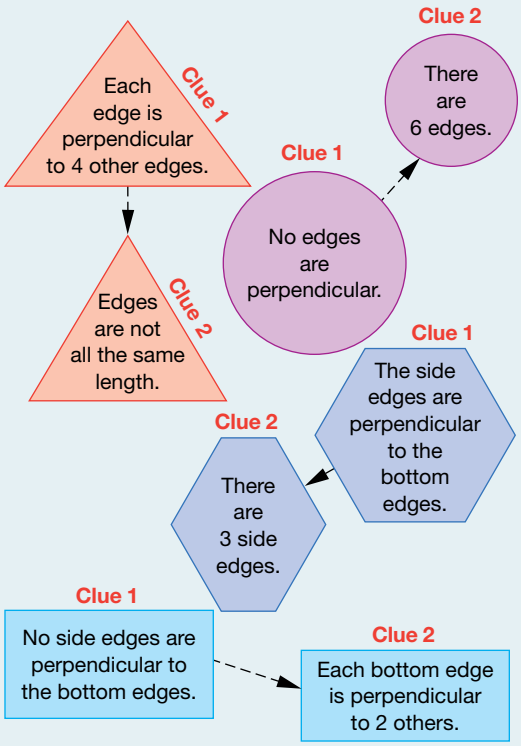
IN THE CLASSROOM 16.1

Solid mystery

Objective: Build 3-dimensional objects with specified conditions about the perpendicularity of the edges.

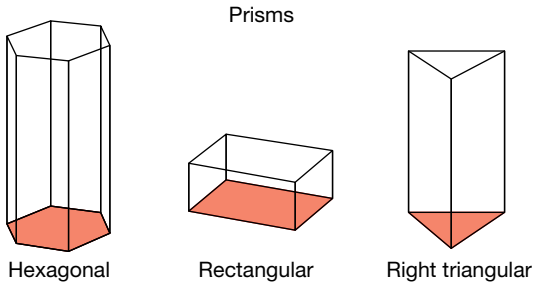
Year level: 4–7

Solve each of these mysteries by constructing a 'suspect' from sticks and connectors. If you think there is more than 1 suspect, look at clue 2.



3. *Right prisms*. This activity introduces the definition of right prisms and how to name prisms. Show examples and nonexamples of right prisms (as in the second of the intermediate activities), and ask students to describe the bases and faces, ultimately encouraging them to come up with the definition of *prism* — a solid that has 2 equal and parallel bases joined by rectangular faces. Then have students

in a whole-class activity discuss how prisms are named. See if they can determine a way to distinguish between prisms. For example, if the base is a triangle, it is a triangular prism.



4. *Platonic solids*. Regular polyhedra (3D objects that have all flat faces) with all faces of equal size and shape can be given special consideration. Give students equilateral triangles, squares and pentagon shapes (e.g. plastic polydrons) to discover the 5 Plato-inspired solids made from all the same faces.
5. *Semi-regular polyhedra*. The soccer ball (truncated icosahedron) is formed by 20 regular hexagons and 12 regular pentagons. Encourage students to make this and other solids from regular shapes.

Exploring, representing and constructing

One of the difficulties that children have with 3-dimensional geometry is visualising the solids. It is essential to have models of the solids; however, they can be expensive to purchase. There are many ways for children to make 3-dimensional models. Below are 2 suggestions for making 'face models' and 2 ways to make 'edge models'. Face models (construction paper tubes and polyhedra shapes) emphasise the faces, while edge models (toothpicks, magnetix and newspaper) focus on edges. As children are making models, they often discover many properties of solids. However, for some properties they may need structured investigations like those suggested here.

Face models

Some of the easiest and most versatile models can be made from heavy construction paper. Figure 16.2 shows how to make open-ended paper tube models of prisms. The top and bottom faces may be added to these models by tracing the top of the tube, cutting out the shape, and taping it to the tube. As this figure suggests, many variations of prisms can be made, and these can be cut (truncated) to create many strange shapes.

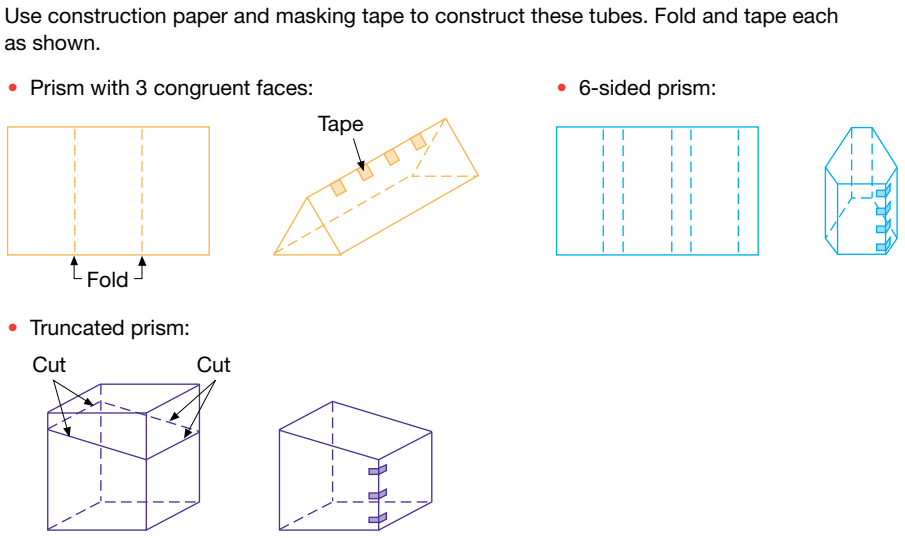


FIGURE 16.2 Constructing tubes to think about properties of prisms

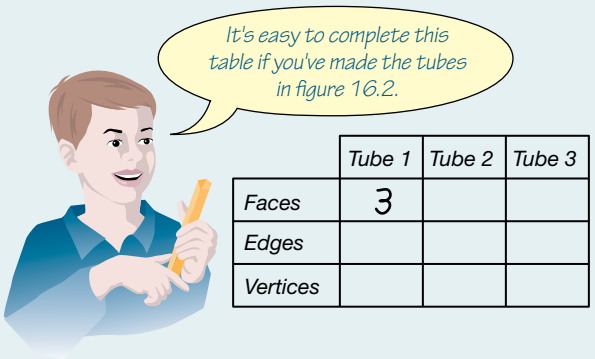
Any solid whose faces are all polygons is a polyhedron. Prisms and pyramids are polyhedra (plural of *polyhedron*), but spheres, cones and cylinders are not. The exploration in In the classroom 16.2 deals with a special case of Euler's formula. Euler's (pronounced 'oilers') formula relates the number of edges (E), faces (F) and vertices (V) of a polyhedron to each other: $V + F = E + 2$. In this activity, students investigate the relationship of the number of edges, faces and vertices in open-ended prisms (called tubes). The formula for these tubes is $V + F = E$. Have your students look for other relationships between 2 of these variables. For example, they may see that the number of edges is always 3 times the number of faces (not counting the top and bottom faces). This is another opportunity to use an algebraic description: $E = 3 \times F$.

IN THE CLASSROOM 16.2

What can you discover?

Objective: Explore the relationships among the number of vertices, edges and faces of prisms.

Year level: 4–5



	Tube 1	Tube 2	Tube 3
Faces	3		
Edges			
Vertices			

1. Count the number of faces of tube 1. (Remember there is no top or bottom, so don't count them.)
2. Count the number of edges of tube 1. (Don't forget the top and bottom edges.)
3. Repeat steps 1 and 2 for tube 2.
   - Do you see an easy way to tell how many edges if you know the number of faces?
     _____
   - Write your conjecture: _____
     _____
4. Count the vertices of tube 1 and tube 2.
   - Do you see an easy way to tell how many vertices if you know the number of faces and edges?
     _____
     _____
   - Write your conjecture: _____
     _____
5. Check your conjecture with tube 3.

Triangular or square grid paper (see appendix C) can be used to make polyhedra. A challenging activity is to find all the networks consisting of 6 squares and determine which will fold into cubes. Since there are 35 different networks, this is a good small-group project. If this is too advanced for your students, try using only 5 squares — known as pentominos — (12 different networks) and fold them into open cubes. Students often do not have the opportunity to work with the triangular grid paper. A larger triangle consisting of 4 smaller triangles will fold into a triangular pyramid. Can you arrange the 4 smaller triangles in any other way to make a pyramid? Experiment with other numbers of triangles.

Edge models

Models can be made from straws, pipe cleaners, toothpicks or other 'sticks' that can be connected with clay or tape. There are also reasonably priced, commercial materials designed for this purpose using

magnetic lengths and balls. Young children enjoy building stick models from toothpicks and soft jubes. In the classroom 16.3 uses this stick model to build specific objects given the number of vertices (jubes) and the number of edges (toothpicks). If your students need more of a challenge, have them make the shapes that are described in In the classroom 16.1.

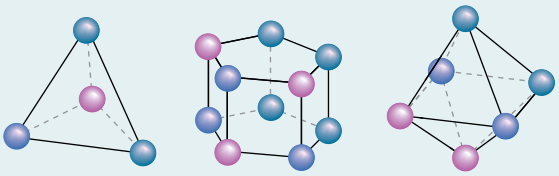
IN THE CLASSROOM 16.3

Soft jubes and toothpicks

Objective: Construct 3-dimensional figures with a given number of edges and vertices.
Year level: 4–6

Materials: Soft jubes and toothpicks (about 20 of each for each child). Clay can be substituted for the jubes.

Explore making 3-dimensional shapes with the soft jubes and toothpicks. Here are a few examples. How many jubes and toothpicks did each take?



Now, try some with the given number of jubes and toothpicks.

You will need to follow these 3 rules:
1. Two jubes can be used on each toothpick, one at each end.
2. Toothpicks should only be used to make the outline of a face (no extra ones stuck across the face).
3. The figure should be closed — that is, imagine it is a container with no openings.

Figure out which shapes (A–L) in the table below are possible and why. If you can make a shape, explain why it is a prism, a pyramid or another shape.

Shape	Jubes	Toothpicks	Possible? (yes or no)	Name of shape
A	6	9		
B	5	8		
C	8	5		
D	6	10		
E	7	12		
F	8	12		
G	10	15		
H	12	18		
I	8	14		
J	10	3		
K	6	12		
L	5	9		

Questions

1. What is your conjecture about the shapes that could not be made?
2. Which ones make pyramids? What is the relationship of the number? Using the models, explain why.
3. Which ones make prisms? What is the relationship of the numbers? Using the models, explain why.

In the classroom 16.4 illustrates how to make a 3-dimensional figure from newspaper sticks and tape. Rolling the paper tightly takes some practice for some children; however, as with many geometry activities, often you will be surprised at who can do this easily. When building 3-dimensional polygons, the children can investigate the rigidity of triangles, squares or other polygons. They should find that the pyramid they built is a rigid structure, though the cube needs bracing to be sturdy. This is why we often see angle braces added to furniture.

IN THE CLASSROOM 16.4

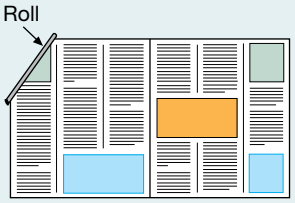
Build your own pyramid

Objective: Explore pyramids through construction.

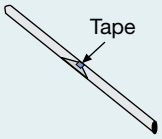
Year level: 5–6

Follow these easy steps to construct a 'stick' pyramid from newspaper. Use masking tape for the connectors.

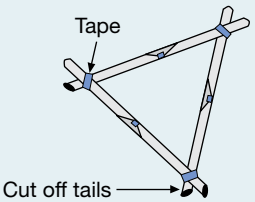
- *Step 1:* Take 3 sheets from a newspaper and roll tightly from corner to corner.



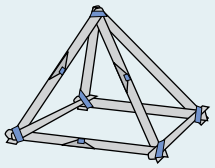
- *Step 2:* Use tape to hold the rod.



- *Step 3:* Make several rods and tape them together as shown.



- *Step 4:* Put the rods together to form a pyramid.



Now, use the same method to construct other 3-dimensional structures, such as a cube.

2-dimensional shapes

There are several general things that you should keep in mind as you work with children in extending their knowledge of 2-dimensional shapes. Children first recognise shapes in a holistic manner (Level 1) — that is, a triangle is a triangle because it looks like a shape they have seen in everyday life that someone has called a triangle. If an equilateral triangle with its base parallel to the bottom of the page is always used, then this shape will be the children's image of a triangle. Likewise, if children are only exposed to images of green triangles (for example, pattern blocks), you would not be surprised if they shaded all their triangles green. A variety of models and experiences are needed.

Children should begin to recognise types of shapes through examples and nonexamples, not through formal definitions. In the classroom 16.5 gives examples and nonexamples of triangles. Do you see why the different shapes were included? If a child says that C, B, F or G is a triangle, what property of triangles do you think is being ignored in each case? If a child fails to realise that D is a triangle, what do you think may be the reason?

Children need to be able to recognise geometric shapes as models for real objects. For example, you might have young children write a 'book' about circles. What is shaped like a circle? Let them find examples and draw pictures. Older children can be challenged to explain why certain everyday objects are shaped in a certain way:

Why are walls rectangular?
Why are support braces triangular?
Why are most buttons shaped like a circle?
Why is paper rectangular?

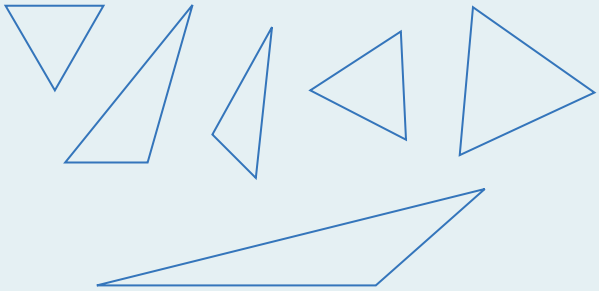
IN THE CLASSROOM 16.5

Find me

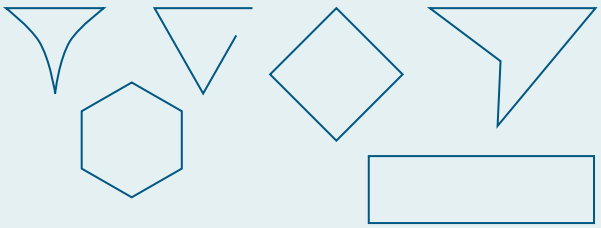
Objective: Identify examples and nonexamples of triangles.

Year level: 2–6

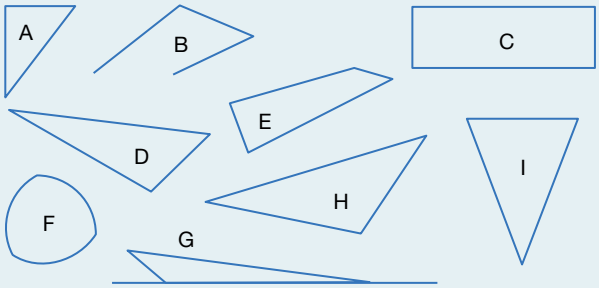
These are triangles.



These are not triangles.



Which are triangles? Explain why the others are not triangles.

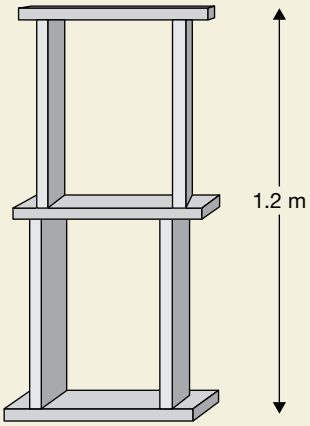


SPOTLIGHT ON EARLY CHILDHOOD

Construction of Benjamin's 3 windows

In the block corner at preschool, 4-year-old Benjamin has constructed a 2-storey rectangular structure that required him to use a number of problem-solving skills. First, he experiments with the large blocks in their upright position so that they balance correctly before they are bridged by another across the top. Although his first attempts were unstable, Benjamin discovers that by using a long block as a foundation, he is able to build successfully. The upright rectangle on the second level is even more difficult to build as he needs to gently balance each block precisely, at a suitable distance.

Benjamin then places the final block across the top. He can barely reach; the rectangles are the same height as he is. When the structure is complete, he exclaims with astonishment and goes to find an adult.



A staff member (Trina) enters and states that Benjamin's construction is fabulous. Benjamin describes in clear, precise language (with appropriate non-verbals) the process he used to build it.

Benjamin: Do you want to know how I did it?
I put 1 block down there and then put 2 more up.
I put 1 more block down there and put that 1 up there and these up there.

Trina: Very good Benjamin.

Melissa (Benjamin's preschool friend) comes around from the other side of the block area and stands behind Benjamin's construction. She puts her face through the top part of it.

Melissa: It's a window, ha ha ha!

Benjamin takes a good look at his construction and excitedly runs over to another place on the floor with some blocks in his arms. He then starts making another construction as Melissa watches.

Melissa: Are you going to make 2 windows?

Benjamin does not respond to her; instead, he speaks to himself, describing the building process again.

Benjamin: You put 1 block down on the floor.

Melissa interrupts.

Melissa: You're going to make 2 aren't you?

Benjamin: I'm going to make this.

Callum, who was building a nearby road out of blocks, has also been watching Benjamin. He now goes up to the first window and puts his face through.

Callum: Peek-a-boo, ha ha ha ha. I can watch through the window.

Benjamin completes the second window as Callum watches.

Benjamin: Now, we need 1 more, for the last 1.

Benjamin picks up a long block from the floor for the foundation.

Callum: Benjamin, I am making 1 for you.

Callum places the first 2 upright blocks. As he goes to the block trolley, Benjamin places the bridging block on top. It is a bit precarious, a little bit wobbly, but okay. Callum then places an upright block on the second level and Benjamin places the last upright block. Benjamin goes to place the top block but instead gives it to Callum.

Benjamin: This is the block for the top.

Callum carefully places the last block of the third window. He jumps around excitedly.

Callum: I hope it is not going to fall down.

Callum walks away from the window and goes back to Melissa. He picks up a couple of unit blocks, which he takes over to her. He is happy with his rectangle completed with Benjamin's assistance.

Callum: I made that one in the middle.

These young children working collaboratively were using the mathematical language of 2-dimensional shapes (including properties of sides and angles); spatial visualisation to make a stable construction; and problem solving, reasoning and visualisation skills. It was surprising how well they cooperated and shared knowledge and skills with their peers, how eagerly they experimented and practised new discoveries and techniques, and how readily they applied geometric knowledge later in their play. Note that, in this situation, the formal name 'rectangle' was not used; however, the children clearly visualised their construction as a 'window', a Level 1 van Hiele understanding.

Source: Rogers (1997).

Around Year 2 level, children should know the names of common shapes such as *triangle, square, rectangle, circle* and *parallelogram*. Over time they will become aware of other words that are used with shapes. Children should be able to identify the *centre, radius, diameter* and *circumference* of a circle or the *sides, vertices* and *angle*s of a polygon. Build vocabulary gradually, connecting the new vocabulary to other words. For example, how is a triangle like a tricycle and a tripod? The most important thing with all vocabulary is that, after it is introduced, it is used regularly.

Children are often taught the geometric names without being given much opportunity to explore the properties or to solve problems. A US assessment showed that students know the names of geometric shapes but have difficulty with complex geometric properties (Kloosterman et al. 2004). This section discusses each property separately, but the aim is to have children respond at more complex levels of understanding over time, as in figure 16.3. Notice the differences in the geometric vocabulary and attributes on which students focus. All the descriptions, however, could be based on visual clues that are only at the Level 1 or beginning Level 2. Asking children to explain or justify their answers would have them delve deeper into the properties.

Number of sides and vertices (or corners)

One of the first properties children focus on is the number of sides. They readily count the number of sides (line segments) of a shape, unless a shape has many sides — in which case, they may need to mark the place where they begin counting. However, sometimes children may have difficulty knowing what counts as a side. As you have children participate in activities such as those that follow, they will begin to make many conjectures about shapes and learn vocabulary.

1. *How many sides*? This simple activity uses pattern blocks (see Appendix C). Each child needs only 1 shape for this activity.
   - As a whole class activity the teacher can call a number and ask children who have a shape with that number of sides to stand. Call numbers such as 2, 5 and 7, for which no one will stand.

- Have the children search for all the different shapes in their environment that have 3 sides, 4 sides, 5 sides, 6 sides and 0 sides (the circle).
- Place a sample of each different shape somewhere within view of all children and call on them to explain how the 4-sided shapes differ (some are bigger than others, some are skinny and some are slanty — you can accept their everyday words at this point). It is important for them to realise that the number of sides does not determine the shape.

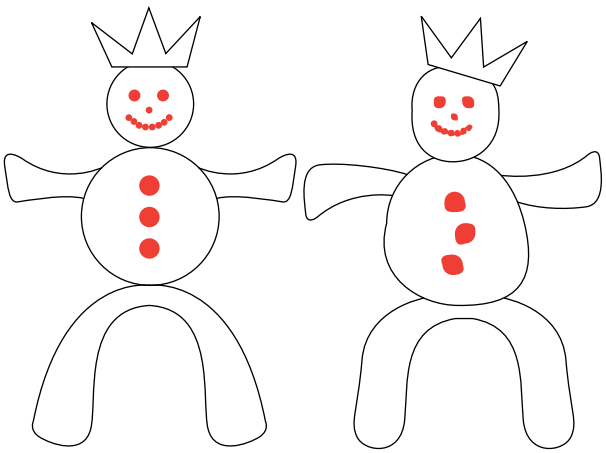
2. *Less is best*. In the classroom 16.6 provides a more advanced activity in which children put pattern blocks together to make new shapes with as few sides as possible. Two or more children choose 3 pattern blocks. Watch to see that they are counting the sides of the new shape, not the sides of the pieces. For example, if Marrietta said 10, it is likely that she counted the sides of the pieces. Composing and decomposing shapes is recommended in the content of Shape sub-strand at Year 4.
3. *Can you make*? In this activity, children make a figure of a given number of sides on a geoboard. (See appendix C for a geoboard model.) Give each child a geoboard and rubber bands. Begin by asking them to make simple shapes and then gradually add other conditions. For example:

   Can you make a 4-sided figure?
   Can you make a 4-sided figure that touches only 4 pegs?
   Can you make a 4-sided figure that touches 6 pegs?
   Can you make a 4-sided figure that has 2 pegs inside (not touching) it?
   How many different quadrilaterals can you make on a $3 \cdot 3$ geoboard?

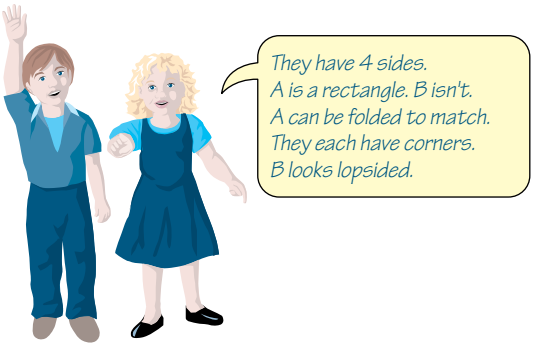
Closely related to counting the numbers of sides is counting the number of vertices or corners. Children will soon realise that any polygonal figure has the same number of sides as vertices or corners if they count both on each shape. The activities suggested for counting the number of sides can be modified for counting vertices or corners.

Symmetry

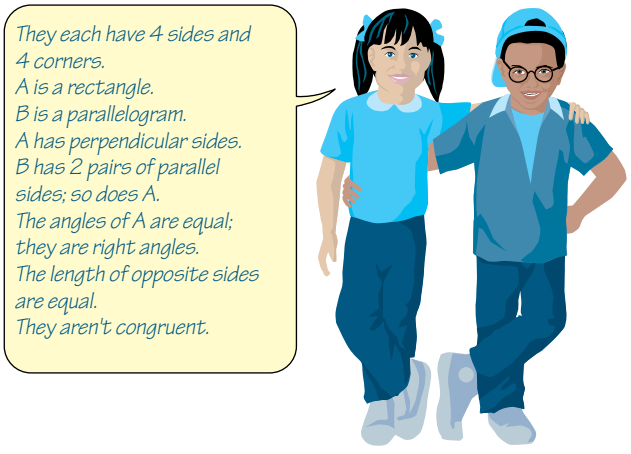
There are 2 types of symmetry — line or reflectional symmetry and point or rotational symmetry — used to describe geometric shapes as well as objects in the real world. To introduce line symmetry, ask children to compare an everyday example such as 2 gingerbread figures:



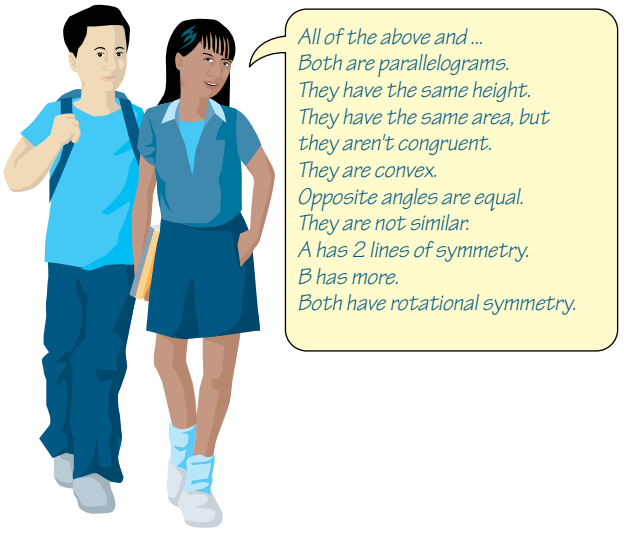
When they say one looks lopsided, show them how they can fold the drawings in the centre to see if the sides match. A child's first perception of symmetry is visual (Level 1). Use this visual perception to help build the idea of folding to match the sides or edges. You may also want the children to explore with mirrors or Miras (reflective geometry tools) to bring in the idea of reflection. A figure has line or reflectional symmetry if, when reflected over a line, the resulting image coincides with the original figure.



Year F–2 responses



Years 3–6 responses



Year 7–8 responses

FIGURE 16.3 Examples of children's responses that show their knowledge of geometric terms

IN THE CLASSROOM 16.6

Less is best

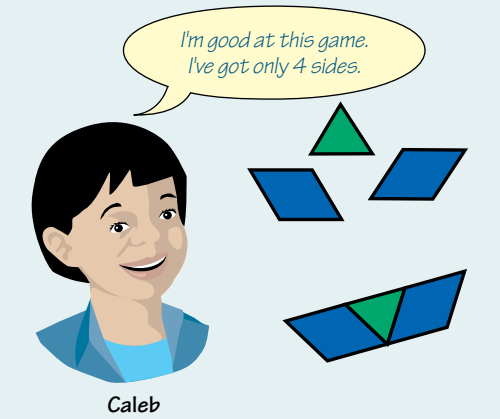
Objective: Make a shape with fewest sides.

Year level: 1–3

Choose a partner to play this game:

- Put an assortment of pattern blocks in a box.
- Without looking, each player chooses 3 blocks and puts them together to make a new shape.
- Count the number of sides of each of the new shapes. Whoever has the shape with the fewest sides wins — that is, 'less is best'.

Make a table to record your scores and play several rounds to determine the winner.





Caleb	4			
Marrietta	4			

Have students find the line(s) of symmetry of geometric shapes. Be sure to let them try folding a square (4 lines of symmetry), an equilateral triangle (3 lines of symmetry) and a circle (an infinite number) to find lines of symmetry before moving to activities such as those in In the classroom 16.7. Older children can often see the lines of symmetry without folding, but some shapes (such as parallelograms) are misleading. Many children say a parallelogram has 2 lines of symmetry. Try it yourself to see when this is true and when it is false. Children notice symmetry in things around them — both

pictorial and abstract. Looking for symmetry in the letters of the alphabet, photos of scenes or shapes in nature and fabric or tiling designs is appealing to students. Together, you might make a bulletin board of pictures of things that are symmetric.

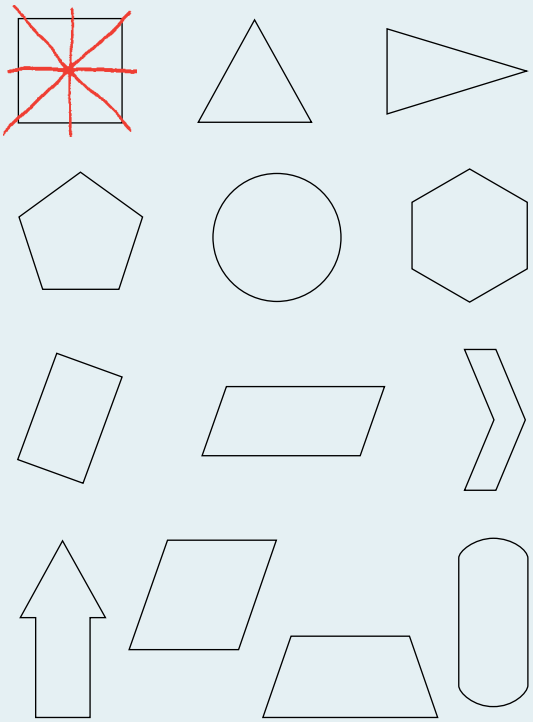
IN THE CLASSROOM 16.7

How many lines?

Objective: Identify lines of symmetry.

Year level: 4–5

Draw the lines of symmetry on each shape.



Children of many ages enjoy making symmetric shapes. One way to make a symmetric shape is to fold a piece of paper and cut the folded piece, leaving the fold intact. Can you figure out how to make a shape with 2, 4 or more lines of symmetry? There are many activities on the computer in which children can investigate symmetry. Battista (2003) describes children's activities and learning about line symmetry using the Geometer's Sketchpad. Many additional ideas may be found on the internet.

For older students (approximately Year 6), finding rotational symmetry in shapes will help them make connections with properties of regular and irregular shapes in a similar way to folding and reflective symmetry. In the classroom 16.7 could be redone but for rotational symmetry where students would look for angles of turn and a centrepoint.

Help students explore turning angles that give rotational symmetry from the centre of the shape — for example, 90° for the square, 120° for the equilateral triangle but 180° for the rectangle or isosceles triangle. Other regular shapes like the pentagon (72°) and hexagon (60°) will give rise to useful discussion and possibly some hypotheses about the angle of regular shapes ($360°/n$) and/or the connections between reflective and rotational symmetry.

Length of sides

Many definitions of geometric shapes, as well as classification schemes, use the property of lengths of sides. Help children focus on the length by having them find the shape with the longest side, find the shortest side of a given shape and measure lengths of sides. In the classroom 16.8 presents an example of a more advanced activity in which children make shapes on a geoboard according to certain specifications about the lengths of the sides. Try the items yourself and classify each as to whether it is easy, of medium difficulty or challenging.

Size of angles

There are many ways to examine the angles of geometric figures. A more complete introduction to angles can be found in a later chapter. However, there are some properties of shape specific to angle sum and measure the following.

- The sum of the angles of a triangle is 180 degrees.
- The sum of the angles of a quadrilateral is 360 degrees.
- The base angles of an isosceles triangle are equal.
- The angle opposite the longest side of a scalene triangle is the largest.
- Opposite angles of a parallelogram are equal.
- A polygon with more than 3 sides can have some equal sides without having equal angles.
- Regular polygons have equal angles.

A guided activity such as the one in In the classroom 16.9 will help students discover for themselves that the angles of a quadrilateral sum to 360 degrees.

IN THE CLASSROOM 16.8

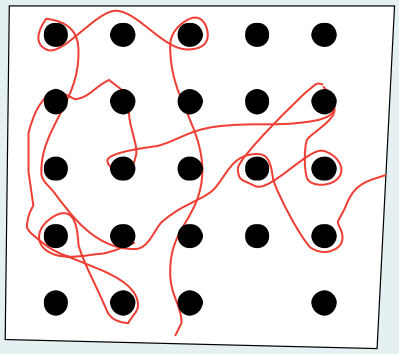
Show my sides

Objective: Make shapes with a specified number of congruent sides.

Year level: 5–8

Use a geoboard and ask students to show:

- a 4-sided figure with exactly 2 equal sides
- a 12-sided figure with all sides equal
- a 3-sided figure with 3 equal sides
- an 8-sided figure with 4 sides of one length and the other 4 of another length
- a 5-sided figure with exactly 4 equal sides
- a 4-sided figure with 2 pairs of equal sides that is not a parallelogram
- a 3-sided figure with 2 equal sides
- a 7-sided figure with no equal sides.



IN THE CLASSROOM 16.9

How many degrees in a quadrilateral?

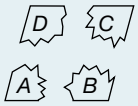
Objective: Explore the sum of the degrees of the angles in quadrilaterals and triangles.

Year level: 5–8

Try this method for finding the sum of the angles in a quadrilateral:



Trace and cut out.



Tear off the corners.

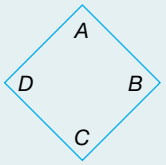


Put the angles around a point.

Name of shape parallelogram

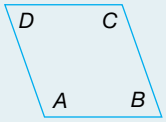
Number of degrees of $A + B + C + D$ 360°

Use the same method for these quadrilaterals. (Reminder: there are 360 degrees around a point.)



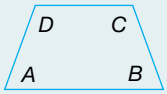
Name of shape _____

Number of degrees _____



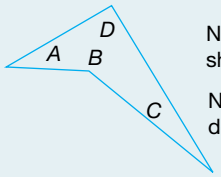
Name of shape _____

Number of degrees _____



Name of shape _____

Number of degrees _____



Name of shape _____

Number of degrees _____

Now try this with:

- more 4-sided figures
- a 3-sided figure
- a 5-sided figure.

What do you conclude?

Parallel and perpendicular sides

In addition to examining parallel and perpendicular sides in geometric shapes, children need to be able to identify parallel lines and perpendicular lines in a plane and, later, in space. Two lines in a plane are parallel if they never intersect. (Remember, a line can be extended indefinitely in either direction.) Another useful definition states that 2 lines are parallel if they are always the same distance (perpendicular distance) apart. Two lines are perpendicular if they intersect at right angles.

It is important that children recognise perpendicular and parallel lines in the world around them. Encourage them to search for perpendicular and parallel lines in the room. They might start a list on the board, adding to it as they find other examples. Here is a start:

- *parallel lines*
  - opposite sides of a book
  - the horizontal lines in **E**
  - the top and bottom of the chalkboard

- *perpendicular lines*
  - adjacent edges of a book
  - the vertical line and horizontal lines in **E**
  - the edge of the wall and the edge of the floor.

You may also have the children identify parallel and perpendicular sides on the pattern blocks. Ask them to find all the pieces that have 1 pair of perpendicular sides (none), the pieces with more than 1 pair of perpendicular sides (square) and the pieces with 1 pair of parallel sides (trapezium). Use the puzzles in In the classroom 16.10 to challenge children to arrange pattern blocks to make shapes with a specified number of parallel sides.

IN THE CLASSROOM 16.10

Parallel puzzles

Objective: Build shapes with parallel sides.

Year level: 5–6

Materials: Pattern blocks

Solve these puzzles using pattern blocks. Draw a sketch of the shape you made.

1. Use 2 different pieces and make a shape with:
- exactly 2 pairs of parallel sides
- exactly 1 pair of parallel sides
- no parallel sides.

2. Use 3 different pieces and make a shape with:
- exactly 3 pairs of parallel sides
- exactly 2 pairs of parallel sides
- exactly 1 pair of parallel sides
- no parallel sides.

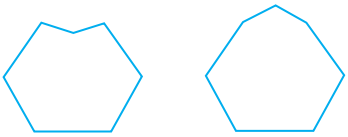
3. What is the largest number of pairs of parallel sides of a shape you can make from:
- 2 pieces
- 3 pieces
- 4 pieces.

4. Can you put all the pieces together to make a shape with no parallel sides?

Convex and concave shapes

Often children are exposed only to convex shapes (any polygon with all angles less than 180 degrees). Many of the activities suggested thus far have included both convex and concave shapes. When children are making shapes, concave examples will often give interesting variety.

Show children 2 shapes such as these and have them describe how they are alike and different:



They will probably express the idea that shape A 'comes back' on itself or 'caves in' (concave). Introduce the terms *concave* and *convex*. After children classify shapes as convex or concave, you might have them investigate questions such as the following:

Can you draw a 4-sided (5-sided, 6-sided, 7-sided) figure that is concave?

Can you draw a 5-sided (6-sided, 7-sided) figure that is concave in 2 places (or that has 2 angles greater than 180 degrees)?

Can you draw a 6-sided (7-sided, 8-sided) figure that is concave in 3 places?

It is challenging to try these exercises with Logo on the computer because the turtle turns the external angles of shapes. Have the children keep a record of the steps it takes to make each figure.

Altitude

The altitude (or height) of a geometric shape depends on what is specified as the base. Identifying and measuring the altitude is essential in finding the area of geometric figures. In the classroom 16.11 is designed to help children realise that a geometric object has different heights, or altitudes. After students do this with the triangular model, have them draw a triangle on paper and again measure each side and each altitude.

Classification schemes

What makes a parallelogram a parallelogram? When is a rhombus a square? What is a regular polygon? When students begin to understand the defining properties of 2-dimensional shapes, they are moving into Level 3. We will take a brief look at classifying triangles, quadrilaterals and polygons as well as some of the defining properties.

Triangles

Triangles are classified either by sides or by angles.

Triangle classification	
By sides	**By angles**
Equilateral: 3 congruent sides	**Acute:** all angles less than 90 degrees
Isosceles: at least 2 congruent sides	**Right:** 1 angle equal to 90 degrees
Scalene: no sides congruent	**Obtuse:** 1 angle greater than 90 degrees

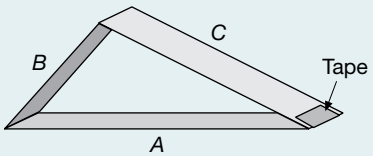
IN THE CLASSROOM 16.11

What's my altitude?

Objective: Identify and measure the altitude of a triangle.

Year level: 6–7

Make a triangle from a stiff piece of paper. Cut a strip 2 cm by 20 cm. Mark off segments of 9 cm, 4 cm and 7 cm and label them *A*, *B* and *C*, respectively. Fold and tape as shown.



For each of the following questions, draw a diagram to scale and label the altitude. Finally, measure the altitude, or perpendicular height, in centimetres

1. Set the triangle on side *A*. This is the base.
- How long is the base? _____
- What length is the altitude? _____

2. Set the triangle on side *B*. This is the base now.
- How long is the base? _____
- What length is the altitude? _____

3. Set the triangle on side *C*. This is the base now.
- How long is the base? _____
- What length is the altitude? _____

After children have learned to identify triangles by sides and by angles, the 2 classification schemes may be put together. For example, can you make an isosceles, right triangle? A scalene, obtuse triangle? An equilateral, right triangle?

What defines a triangle? Here are some definitions that students have provided:

'A triangle has 3 sides.'
'A triangle has 3 sides and 3 angles.'
'A triangle is a polygon with 3 sides.'

Although all 3 statements are true, only the last one defines the triangle. Can you think of a figure that has 3 sides and is not a triangle? Refer back to In the classroom 16.5.

Quadrilaterals

There are many special names for quadrilaterals. Common names are parallelograms, rectangles, squares, rhombuses, trapeziums and kites. These classes of quadrilaterals are not disjoint; one shape may fall into several categories depending on the definition. For example, a rectangle is also a parallelogram. This type of classifying process is more difficult for children than partitioning the whole set into disjoint classes, as is the case of classifying triangles by lengths of sides. It requires more than just recognising examples of figures — it requires understanding the defining properties (Level 3). For example, a parallelogram is a quadrilateral with 2 pairs of parallel sides and opposite angles equal. Assuming you know that a quadrilateral is a 4-sided, closed, simple figure, can you identify which of the following are parallelograms? What other names do they have?



You are correct: they all are parallelograms. Thus, a square, a rhombus and a rectangle are all special types of parallelogram.

Discuss these questions with a peer. A *rhombus* is a parallelogram with all sides congruent. Does that mean that a square is a rhombus? Why? A *rectangle* is a parallelogram with right angles. Does that mean a square is a rectangle? Why? Through the centuries, definitions of shapes have changed. At one period, we had oblongs and squares. Oblong was a rectangle that was not a square.

How do you begin to teach such relationships? Children must first begin to verbalise many properties of the figure. For example, they must be able to describe a square as:

- a closed, 4-sided figure (property 1)
- having opposite sides parallel (property 2)
- having all right angles (property 3)
- having all sides congruent (property 4).

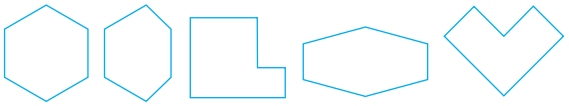
Properties 1 and 2 make it a parallelogram; properties 1, 2 and 3 make it a rectangle; properties 1, 2 and 4 make it a rhombus; properties 1, 2, 3 and 4 make it a square. A square also has the rare property that diagonals bisect each other at 90 degrees. In the classroom 16.12 helps students with this idea.

Polygons

Polygons are named according to the number of sides:

- 3 sides: triangles
- 4 sides: quadrilaterals
- 5 sides: pentagons
- 6 sides: hexagons
- 7 sides: heptagons
- 8 sides: octagons
- 9 sides: nonagons
- 10 sides: decagons.

This classification scheme is not difficult although often children are shown only regular polygons. Thus, among the shapes shown below, a child sees only the first as a hexagon instead of realising they are all hexagons.



Children should be encouraged to think of real objects that are shaped like these. A fun hunt for children at any age is to find different-shaped polygons in their neighbourhood. If the children live in a rural area, ask them to draw all the different-shaped pentagons they see (the sides of many roofed buildings). Buildings in cities often use wonderful geometric shapes.

IN THE CLASSROOM 16.12

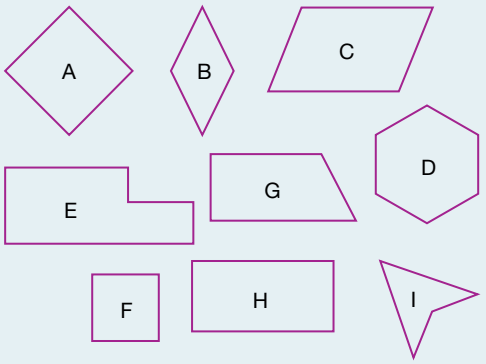
Classify me

Objective: Classify and name quadrilaterals.

Year level: 5–6

Mark each of the following figures with:

- 1 if it is a quadrilateral
- 2 if it has 2 pairs of parallel sides
- 3 if it has all right angles
- 4 if it has all congruent sides.

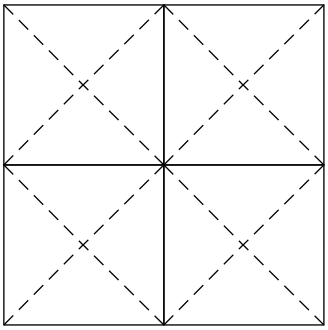


- Any figure marked 1 and 2 is a _____.
- Any figure marked 1, 2 and 3 is a rectangle as well as a _____.
- Any figure marked 1, 2 and 4 is a _____ as well as a _____.
- Any figure marked 1, 2, 3 and 4 is a _____ and a _____ as well as a square.

The names *heptagon, nonagon* and *decagon* are not widely used, so in doing activities you may have to remind children of these names. In the classroom 16.13 uses the names as well as other properties; students are asked to see shapes within other shapes, a task that is difficult for some. The geometric design paper in Appendix C may be used for similar searches. Children often have difficulty seeing figures within other figures. A national assessment held in the United States in 2007 included the following 2 items. Children were provided with 3 copies of the following grid. Students were asked to outline a square in each grid and were told that the squares must not be the same size. The second question was the same except the figures were to be triangles. The squares that are turned gave the students more difficulty than the 2 squares that are not turned. Less than half of the Year 4 students outlined 3 of the 4, and fewer students outlined the 3 triangles. Even more difficult was a third question that asked

students to draw a 4-sided shape that was not a rectangle nor a square. These results are a reminder of 3 things.

1. Students need to see many different examples and nonexamples of shapes in many sizes and orientations.
2. Students need to make shapes that illustrate given properties rather than always identifying properties in shapes.
3. Students need experiences with seeing and making shapes within shapes.



IN THE CLASSROOM 16.13

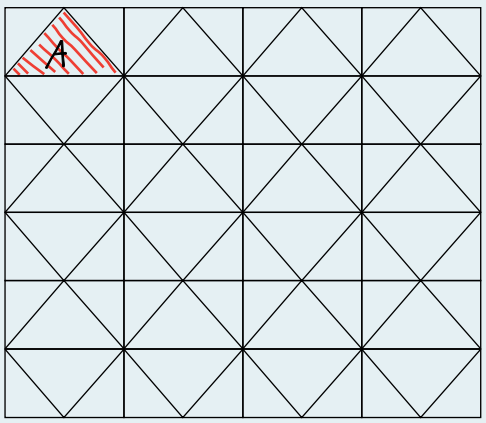
Can you find it?

Objective: Shade polygons with given properties.

Year level: 5–8

See if you can find each of the following shapes in the grid design. Fill in the shape and mark it with the matching letter.

A triangle — isosceles
B triangle — scalene
C quadrilateral — not symmetric
D quadrilateral — 4 lines of symmetry
E pentagon — concave
F pentagon — convex
G hexagon — exactly 2 pairs of parallel sides
H hexagon — symmetric
I heptagon (7 sides) — symmetric
J heptagon — not symmetric
K octagon



16.2 Locations and spatial relationships

LEARNING OUTCOME 16.2 Describing the roles that location and spatial relationships play in geometry.

Within the content strand of Measurement and Geometry in the Australian Curriculum, the topics of 2-dimensional shapes and 3-dimensional objects is one of 5 sub-strands. Another part of geometry deals with location, movement, maps and plans which describe direction, distance and position. Where are you? Are you above or below the floor? Are you in front of or behind your desk? Are you between the cabinet and the computer? Where would you be if you moved 5 steps forward? Starting at your desk, how far from your desk will you be if you move 5 steps forward and 3 steps backward?

Examining location and movement gives children a way to describe their world and give order to their surroundings. It also provides an opportunity to build mathematical concepts such as that of positive and negative numbers (forward and backward) and skills connected to other subjects, such as map skills useful in geography. Helping students explore movement by describing directions and distances travelled and by identifying positions of an object through several movements are useful activities. You might begin by taping to the floor of the classroom a map of your local area (see figure 16.4). Ask students to describe the directions and the distances between specific locations — such as between school and home — using informal or formal language, depending on their level of skill. Have the students describe a set of directions from home to school where they would use familiar landmarks and intersections (identified by traffic lights and stop signs). Have the students describe a set of directions from school to the playground and encourage a variety of solutions.

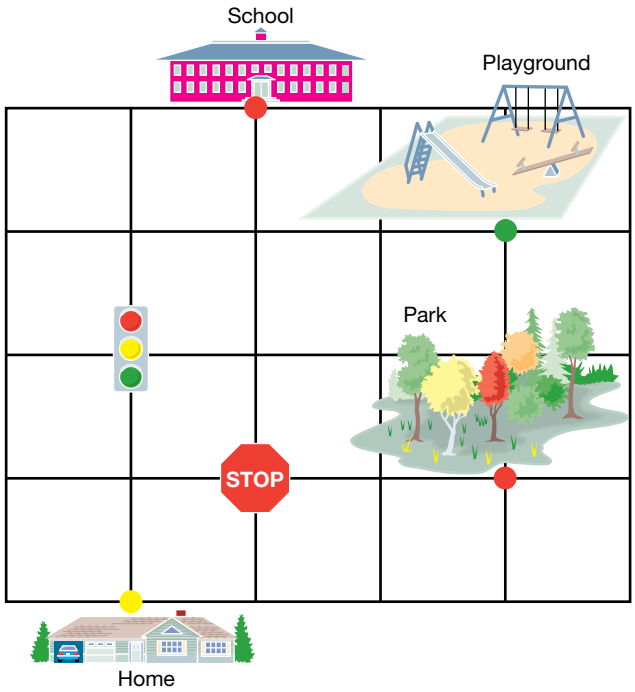


FIGURE 16.4 Map of nearby locations

In 2005, students were asked on a national mathematics assessment in the United States to give directions on a simple grid to go from a point marked ‘School’ to a point marked ‘Park’. They were asked to use the appropriate directions north, south, east and west which were illustrated in the picture. The first direction — go 2 blocks east — was given. Over two-thirds of the Year 4 students gave the remaining

2 directions (4 blocks north and then 3 blocks east). The others did not give the distance needed. According to the Australian Curriculum, Year 5 students should be able to achieve success on this type of question.

To formalise coordinate geometry, you can begin using the positive quadrant (similar to a map grid). There are many games to help students learn location in the spatial plane and the correct way to say coordinates (i.e. horizontal first, vertical second). Figure 16.5 shows an example of where a Year 5 student might use the positive quadrant: in a game of Battleships and cruisers.

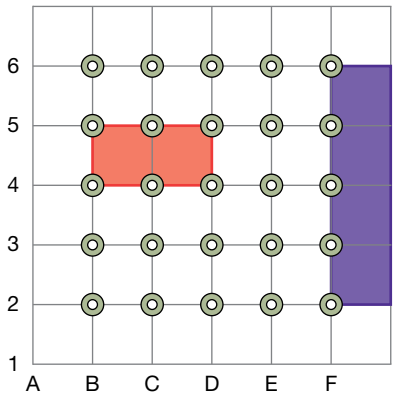


FIGURE 16.5 Sample ships on a game board

If the student says 'C3' it would be a *miss*; however, if the student says 'C4' it would be a *hit*. If the student had already hit 'B4' in a previous turn then the red boat here would be considered *sunk*.

By Year 7, according to the Australian Curriculum, students are able to plot points on the Cartesian plane using all 4 quadrants. The coordinate system illustrated in figure 16.6 using all 4 quadrants is not usually introduced until late in the primary years. However, this is a natural way for children to use negative numbers. When planning activities, you can tie movement and location together. For example, you could describe a movement and ask for a location:

> If I am at point (3, 1) and move 2 spaces, where could I be? [*Many answers*.]
>
> Describe 2 locations — for example, (3, 2) and (5, 6) — and ask what movements would take you from one point to the other. [*Many answers*.]

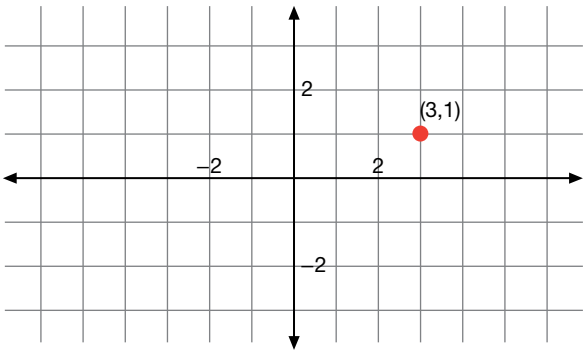
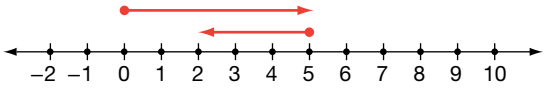


FIGURE 16.6 Coordinate system for describing spatial relationships

The number line is a linear representation that can help students understand movement on a line rather than in 2 dimensions. Movements like 'move forward 5 steps' and 'back 3 steps' can be represented

on the number line. Suppose you begin at 0, where would you end? The number line below shows this movement:



There are 3 parts to problems like this: the beginning point, the movement and the ending point. Knowing any 2, you can figure out the other. However, the movement problem could be open ended. For example, suppose you begin at 2 and end at 8, what are the ways you could have moved? The shortest movement is forward 6 — but consider forward 4, back 3; forward 5, back 4 and so forth. Will you get there? How many separate movements will it take? How far would you have moved in all?

The 'turtle graphics' feature of the programming language Logo, as well as other computer applications, offer a wide variety of activities for children to explore movement and location. In Logo, an angle is considered as a turn and a figure is a path created as the 'turtle' cursor travels with its pen.

With only a few simple commands (FD for ForwarD, BK for BacK, RT for Right Turn, LT for Left Turn), children can begin drawing pictures and hypothesising about geometric figures. An important feature of Logo is the angle that is created when the turtle is instructed to turn, which is an exterior movement to the shape. Figure 16.7 shows some turtle commands and the resulting figure that is drawn. To complete an equilateral triangle, FD100 is needed after the first 4 commands. A simplified way of creating the same triangle would be to use the command REPEAT 3 [FD 100 RT 120]. Note that the measure of the angle the turtle needs to turn in order to create an equilateral triangle where each of the internal angles is 60 degrees, is 120 degrees.

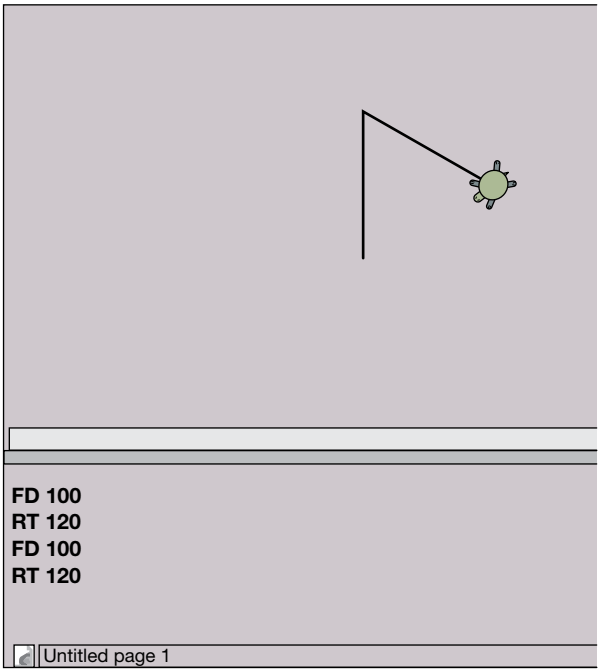


FIGURE 16.7 Logo commands: instructing the turtle to begin creating an equilateral triangle

Have children in the early year levels begin exploring Logo through the Logo simulation, Turtle Pond, at the NCTM's Illuminations website, http://illuminations.nctm.org. In this activity students are to move the turtle to the pond. It is helpful to first have students act out the path described in the instructions to

gain a sense that the turtle is moving forward (walking) and turning through the exterior angles of the object being drawn. The mathematical toy, *Bee Bots*, is also a great tool.

Children love to plot points on a coordinate system and connect the dots to make pictures. This certainly is one way to have them practise plotting points, but it is a rather low-level activity that requires no spatial reasoning. It is more challenging if they draw a picture and create their own directions. There are many problem-solving activities that involve coordinates and geometric shapes. In the classroom 16.14 has challenging problems involving shapes and changing shapes. If you are teaching younger children, then consider asking them to draw shapes and give descriptions in the first quadrant (the one with all positive coordinates). For example, you may ask them to draw a rectangle with 1 vertex at (3, 2). As many answers are possible, ask them to give the length of each side of the shape — for example, the length of the longer sides is 4 units and shorter sides is 3 units.

IN THE CLASSROOM 16.14

Shapes on a grid

Objective: Explore various shapes on a coordinate grid.

Year level: 5–7

Materials: Grid paper (see Appendix C).

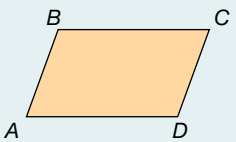
Use these as a problem of the day (or week). Have students share the different solutions. There is only 1 rule: the vertices must be on the grid.

Questions

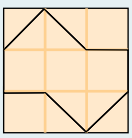
1. Two vertices of a square are (2, 0) and (4, 2). What are the other 2 vertices? [3 solutions]

2. Two vertices of a right angle triangle are (0, 0) and (3, 0). What could be the third vertex? How many possibilities are there? Give at least 5 examples.

3. Parallelogram *ABCD* has vertices at (0, 0), (0, 4) and (1, 3). What are the coordinates of the fourth vertex shown in the picture? Find other parallelograms with these 3 vertices. Give the fourth vertex.



4. The octagon below is enclosed in a 3-by-3 grid. What other octagons can you draw in a grid this size? Remember that the vertices must be on the grid.
Which of your octagons are:
(a) symmetrical
(b) convex?
Which has:
(a) the largest area
(b) the largest perimeter?



16.3 Transformations

LEARNING OUTCOME 16.3 Identifying applications of the 3 geometric transformations.

Geometry is often studied in terms of patterns. Think of wallpaper patterns that you have seen or look at the 3 border patterns in figure 16.8. What basic design is repeated? How is the basic design moved? Describe the moves in your own words. Did you use words such as *turn, slide* or *flip*?

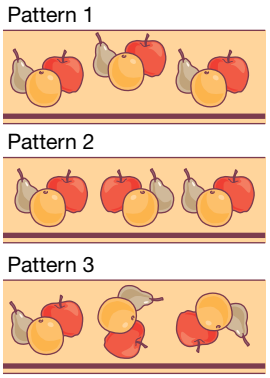


FIGURE 16.8 Wallpaper borders

The Australian Curriculum expects students between Years 2 and 7 to become familiar with the 3 geometric transformations: slide (translation), turn (rotation) and flip (reflection), as illustrated in figure 16.9.

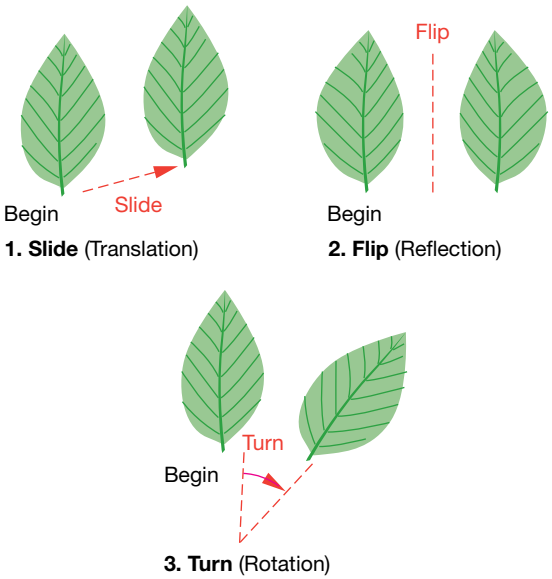
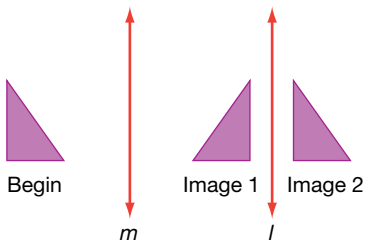


FIGURE 16.9 Motions for determining congruence

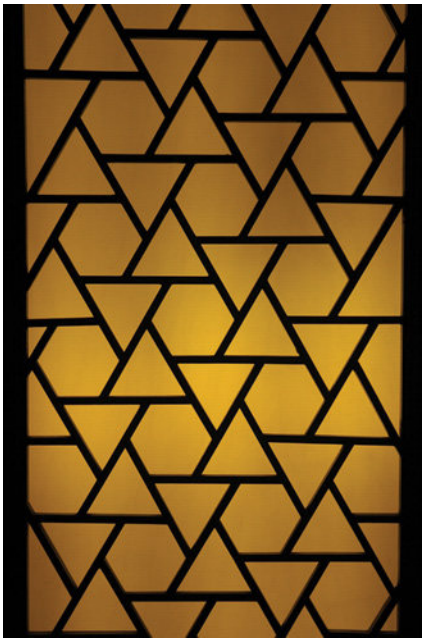
Research (Jacobson & Lehrer 2000) points to the crucial role of the teacher when children are investigating transformations. If students are asked to make conjectures and justify their thinking, they develop a deeper understanding of the transformations. For example, older children might be asked to investigate what 1 motion would be the same as 2 flips over parallel lines, first over line *m* and then over line *l*:



What if the lines are not parallel? What if the lines are perpendicular? What if you first flipped it over line l and then over line m?

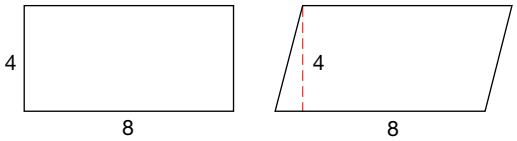
Some questions about transformations can be answered just by looking, but for others children will need to develop mathematical techniques using distance and angles. For example, children can fold on a line of reflection as they have done with lines of symmetry to see if the image is a reflection over that line. They can trace and turn or slide shapes to see the rotation or the slide. Computer programs such as the Geometer's Sketchpad provide a way for primary school children to investigate many of the characteristics of transformations.

One of the most interesting applications of transformations for students involves tessellations. A simple transformation of a shape can produce an interesting design. The window below uses many geometric shapes (triangles and hexagons) in regular 2-dimensional patterns known as tessellations. You can find many ideas on the internet and a simple example in the article 'Tessellating T-shirts' (Shockey & Snyder 2007).



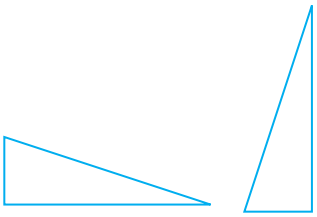
Congruence

Two shapes are said to be *congruent* if they have the same size and the same shape. In everyday language, *equal* is used. Young children grasp this idea when they see that one shape can be made to fit exactly on the other. If the 2 shapes are line segments, they are congruent if they have the same length. If the 2 shapes are 2-dimensional and they are congruent, then they have the same area. The converse is not true. Two shapes with the same area may not be congruent. Children have difficulty with this concept, and many middle school students would respond that the parallelogram shown here is congruent to the rectangle:



Young children have little difficulty identifying figures with the same shape and size through observation such as with these 2 right angle triangles. They can cut the shapes out and show that one fits exactly on top of the other to demonstrate the idea. Thus, the task becomes one of asking children to

match figures to see if they are the same size and shape and gradually introducing and using the word *congruence*.



Congruence is often investigated through transformations. If 2 shapes are congruent, they can be made to fit by 1 or more of the 3 motions (flip, slide or turn). What motion or motions could you use to move 1 of the triangles above onto the other? In terms of transformations, why are the rectangle and parallelogram shown above not congruent?

Similarity

Another type of transformation is that of enlargement (or reduction). You have probably used these terms when making a photocopy. What happens to a triangle on a sheet of paper when you reduce or enlarge it by a certain percentage? We say the 2 triangles are similar.

Similar is a word used in everyday language to mean alike in some way. In mathematics *similar* has a very specific meaning: 2 figures are *similar* if corresponding angles are equal and corresponding sides are in the same ratio. In contrast, the idea of *same shape* might be used in the early years to describe shapes of the same type — such as rectangles. However, not all rectangles are similar. So, part of your task in teaching similarity will be to refine intuitive notions about similarity to fit the mathematical definition stated above.

This definition of similar is too formal to be used initially. Instead, you can begin by using models like a geoboard and geopaper. Children can make a design on the geoboard and transfer it to smaller geopaper, or they can copy designs from one size of graph paper to another. Older students can explore doubling, tripling and so on the lengths of the sides of geometric shapes to see what effect this has on the perimeter and area on the electronic geoboard. After students have been introduced to ratio, they can investigate similarity in a more rigorous way. (Similar triangles are discussed in an earlier chapter as an example of proportions.)

16.4 Visualisation and spatial reasoning

LEARNING OUTCOME 16.4 Understanding what types of exploration with geometry can help build children's spatial reasoning and visualisation skills.

Visualisation, spatial reasoning and problem solving are important content and proficiency strands in the Australian Curriculum. Furthermore, if you have been doing the activities in this chapter thus far, you have probably been using all 3 of these processes. Geometric modelling using number lines, coordinate grids and rectangular arrays for multiplication have been used throughout the text. In this section we consider several aspects of visualisation and spatial reasoning that we have not emphasised in the previous discussions.

Battista (2007 pp. 483) defines spatial reasoning as the:

> ability to 'see,' inspect, and reflect on spatial objects, images, relationships, and transformations. Spatial reasoning includes generalising images, inspecting images to answer questions about them, transforming and operating on images, and maintaining images in the service of other mental operations.

Using geometric physical and pictorial materials

One way to develop visualisation and spatial reasoning is to use physical and pictorial materials that can be moved, subdivided, put together and otherwise changed. Research shows that the use of physical

materials — including paper models, plastic or wooden geometric shapes, geoboards and solids — can be helpful in developing geometric representations, but they must be used wisely (Clements 2004). Although we have illustrated much of this chapter with the use of physical materials, we have only done a little with the idea of putting together (combining) and taking apart (subdividing) shapes. The ability to see shapes rearranged to make other shapes or seeing shapes within shapes is important to develop. The more that students work with materials to do this, the more ways they have to visualise geometric shapes and properties.

You can use simple paper folding to make shapes from a square. The tangram puzzle (see Appendix C) provides pieces for endless explorations of ways to put the shapes together to make animals, figures and other shapes. Children love to make designs and fill in pictures with the pattern blocks. All of these types of activity are a beginning; however, students need to be encouraged to take these activities further — to make predictions, to investigate various ways to make the same shape, to describe the pictures they make with geometric terms and to make shapes with given properties. The Australian Curriculum places emphasis on this in Year 4 leading students to make conjectures and reason about putting together and taking apart shapes — particularly triangles and squares. In the classroom 16.15 uses the pattern blocks (Appendix C) to present some problems of combining shapes to make other shapes.

IN THE CLASSROOM 16.15

Shape maker

Objective: Make shapes with given characteristics.

Year level: 3–5

Materials: Wooden or plastic pattern blocks



Use as many of the blocks as you need. Record your answers with drawings showing the blocks used.

Questions

1. How many different ways can you make the hexagon?
2. Can you make a triangle with 2 pieces? With 3 pieces? With 4 pieces? With 5 pieces? With 6 pieces?
3. Can you find 2 different ways to make a triangle with 4 pieces?
4. Make a rhombus using 1 hexagon and other pieces.
5. Make a convex hexagon using 1 hexagon and other pieces. Find another way.
6. Make a concave hexagon using 1 hexagon and other pieces. Find another way to do this.
7. Make a nonregular hexagon (all sides are not equal) with 3 pieces, with 4 pieces, with 5 pieces and with 6 pieces.
8. Make a large triangle using only the green triangles. How many did it take? Make another larger triangle and another. What pattern have you found?
9. Now make a larger rhombus using only the blue rhombi. How many did they take? Is the pattern the same as triangles? Why?
10. Make a larger trapezium using only the red trapeziums? How many did it take?
11. You cannot make a larger hexagon using only yellow hexagons. Can you explain why?

Pictures or diagrams are essential tools. Can you imagine learning what a rectangle is without seeing examples? You should think about this if you have students who have visual difficulties. What can you do to help those students with pictures and diagrams? Think about using models with raised interior lines (a piece of string glued) that they can feel. It is not only these students who have difficulty; many students see diagrams and pictures differently than intended. It is important to have them create their

own pictures and diagrams from work with the physical materials. For example, In the classroom 16.15 question 8 might give rise to the triangle as shown in figure 16.10. The question 'how many triangles can you see?' will lead to good quality problem solving.

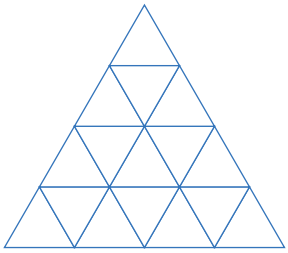


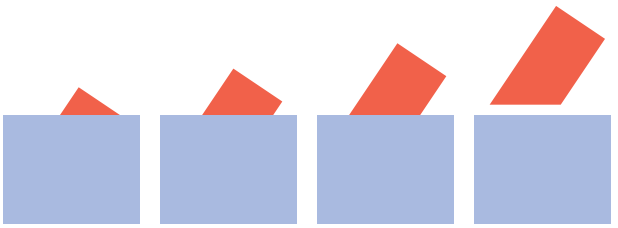
FIGURE 16.10 How many triangles?

Computers open up many possibilities for many students. The electronic environment is motivating and allows students to work at their pace. For example, students who have trouble moving the pattern blocks into place and recording their work can often use a shape-making computer program. Computers allow students to create many more examples of shapes they are trying to explore. For example, the exploration of altitude and bases (In the classroom 16.11) is done with only 1 triangle. On a computer, students could extend this to many examples and realise that to find the area of a triangle, it does not matter which side is the base. Many of the teacher resource books you might use refer you to special computer programs; see what is available in your school.

Using mental images

Although you can help children learn geometric ideas by using physical and pictorial materials, you also need to help them develop visual images of geometric shapes. As your students mature, you can help them manipulate these images in their minds. For example, visualise a rectangle that is longer than it is wide. Now turn it (in your mind) to the right 90 degrees. What does it look like now? Draw a line from the top right corner to the middle of the bottom side. What 2 figures did you create?

One of the classic tasks to encourage developing mental images is to cover up most of the shape and reveal it slowly, asking students to predict what shape is covered. See the following example. There are questions to get you started, but let children come up with what they think it is before you ask them these specific questions.



What could the shape be? Could it be a triangle? Square? Rectangle? Trapezium? Other?

Why can't it be a triangle? Could it still be a square?

Why can't it be a square? Could it still be a rectangle?

What is it? How could you change the bottom so it would not have been a trapezium?

Another version of this type of activity can be found in the overlap problem (Tayeh 2006). Student solutions are given to the problem of overlapping 2 transparent squares and asking what shape the overlap is. Before reading the article, try this yourself. Are you as flexible as the students?

Visualising 2-dimensional patterns that can be folded to make 3-dimensional patterns is an important aspect of geometric thinking. In the process, you might visualise the folding or you might use spatial reasoning to help you decide. For example, imagine trying to fold an open cube (no top face) from 5 connected squares, known as pentominoes. Which of the configurations in figure 16.11 could be folded into an open cube? Which would be the bottom square? Being able to make these visual manipulations mentally is enhanced by experience in doing such manipulations physically.

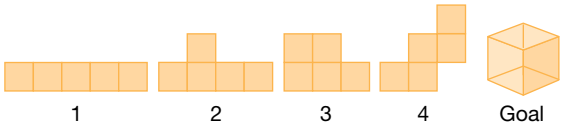
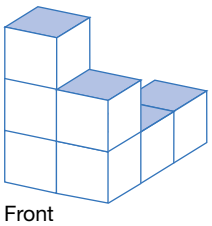


FIGURE 16.11 Which figure can be folded into the shape of the open box?

Visualising 3-dimensional objects from different perspectives is another aspect of geometry. What happens if you view a solid from different perspectives? Pretend you are a fish, swimming below a pyramid that is sitting on top of the water. You can see the entire bottom of the pyramid (base view). What do you see? Begin with putting the children in easy locations where they see only 1 face and then move to ones that show more than 1 face. Pretend you are a bird and you are looking down on the point of a (square) pyramid (top view). Can you describe what it would look like? Then have children put out solids and ask them to draw what they see from different views (top, base, directly in front, right or left, front when the solid is turned to see 2 faces).

Students need to work with models to help them make these mental images. Young children can begin with the basic solids and describing the faces. Make imprints in play dough of the faces of wooden geometric solids. Ask children to match each solid with a face. The faces can be traced instead of making imprints, although younger children can work easily with imprints.

One way to help students see shapes from different perspectives is to look at structures built with cubes. Ask students to build a structure with cubes and then draw the different faces on square grid paper. There are several variations of this task. Give the children the drawings of the faces and see if they can construct the building. Alternatively, have the children draw the structure using the isometric paper as shown below. (A master sheet of isometric paper can be found in Appendix C.)



Another visualisation task involves cross-sections of solids. Real-life examples such as a loaf of sliced bread (square or oval) and toblerone chocolate (triangle) can help students connect 3-dimensional solids to their cross-section. Cutting paper tubes (see figure 16.2) helps children to see the cross-sections. Using other objects that can be cut (e.g. oranges, carrots shaped in cones or zucchini shaped in cylinders, for example, can help the children see the cross-sections. The next time you eat cheese, cut some small cubes of cheese and then try slicing each cube so the slice shows different geometric shapes. Can you slice the cube to see a triangle? A different triangle? How can you slice the cube to make a rectangle that is not a square? Try a pentagon and a hexagon.

MAKING CONNECTIONS

Geometry opens the doors to many cultures. Literally, an internet search for doors or windows from other countries can be an exciting adventure. We found African doors from the thirteenth century, including a beautifully carved wooden door with an intricate geometric pattern around the edge and a picture of wheat in the centre. Mexican doors often have many rectangular panels, some plain and some decorated with geometric designs; Chinese doors have elaborate screen designs. Of course, the doors of every country differ by the period when they were made, the availability of materials and the socioeconomic status of the owners. Have you ever thought about what cultures have doors that are not rectangular? Why are most doors rectangular? What geometry do you see in the doors in your neighbourhood?

Cultures have their own designs for their arts and crafts — quilts, rugs, pottery, jewellery — the list is endless. Many of these designs are geometric at some period in any culture's history. Homes and other structures also have distinctive geometric properties. In many countries homes are some variation of rectangular prisms with roofs that are a mixture of triangular prisms. Look around your neighbourhood or in house magazines for homes that are made up of rectangular prisms. What other shapes do you see? In Australian there is an innovative housing company who builds yurts which are hexagonal or round in design. In the nineteenth century, Orson Squire Fowler claimed that octagonal houses were the most economical (less siding material for the same area of floor space) and the healthiest (more windows and ventilation). Although he promoted his idea through several books and his own designs, there are only a handful of octagonal buildings. What other shapes are used for the structure of buildings or roofs? A classic article (Zaslavsky 1989) looks at round houses.

Geometry as a content area is treated differently in many countries. The French curriculum concentrates more on geometry in middle year levels than we do, but it does much less before those year levels.

While a variety of geometry expectations are given in the Australian Curriculum, we need to be mindful not to spend the majority of time teaching geometric vocabulary. The curriculum in Japan expects students to learn the vocabulary naturally as they use it in solving geometric problems. The geometric vocabulary in many countries is aligned with other common words. For example, *tri* (from *triangle*) is a variation of *tres* (Spanish) or *trois* (French). Also, the names of many geometric shapes are often descriptive. In English, this would be like using 'six sides' for hexagons.

In the diverse classrooms of today, you will find students with many different backgrounds. The visual nature of geometry helps you accommodate these differences in many geometric activities. Building on the cultural knowledge students bring to school can also help celebrate their diversity.

A GLANCE AT WHERE WE'VE BEEN

Geometry is a topic that is often neglected in primary school, yet it has many benefits for children if it is presented in an intuitive, informal manner. This chapter presented a variety of activities that provide this type of informal experience. The discussion of 3-dimensional shapes included ways to have children describe and sort solids as well as ways to make 3-dimensional models. The discussion of the 2-dimensional shapes focused on properties of shapes. It emphasised moving from simple recognition of geometric shapes (modified van Hiele Level 0) to the higher levels.

Geometry helps us represent the space in which we live and to describe location, movement and the relationships between objects in space. Intuitive ideas as well as more formal ideas using coordinate geometry were given. Transformation and symmetry can be used to describe our world and to solve problems. Building children's visual skills and reasoning is an important aspect of geometry. The ultimate aim is to be able to use geometry to solve problems and to appreciate the geometry in the world around you.

This chapter included a few suggestions for helping children build concepts and skills in geometry, solve visual problems and apply geometry to everyday life. There are many other fascinating topics and

activities that you can source and implement in your teaching. Use these ideas wisely by helping students deepen their understanding of geometry, progressing from simple recognition to analysing and justifying geometric statements and to solving problems involving geometry.

THINGS TO DO: FROM WHAT YOU'VE READ

1. Name 3 properties of solids that children should learn at each of these levels: early years, primary and middle years.
2. Explain how the children's answers in figure 16.3 differ from Years 3–6 to Years 7–8. What more is shown by the middle years students than the primary and early years students? On what level of the modified van Hiele scheme do you think they are operating?
3. Describe different geometry ideas that you find in quilting references.
4. Give the reason for including each example and non-example of a triangle in In the classroom 16.5.
5. Design a sorting activity for primary students based on one of the ideas in the beginning activities with solids.
6. Explain how quadrilaterals can be classified. Why are squares a type of rectangle? How could rectangles be defined so that squares would not be rectangles?
7. Complete the activity in In the classroom 16.9. Build a list of at least 10 objects in the world around you that have lines of symmetry.
8. What are the 3 geometric transformations that leave shapes unchanged except for position and location?
9. Do one of the challenging activities from In the classroom 16.14 or 16.15.

THINGS TO DO: GOING BEYOND THIS TEXT

IN THE FIELD

1. *Solid activities*. With children, try 1 of the 3-dimensional activities at the early primary or middle years levels in the section on describing and sorting objects (section 16.1).
2. *Find me or classify me*. In the classroom 16.5 provides examples and nonexamples of triangles. Ask young children to explain which of the mixed examples are not triangles and to justify their selections. Or use In the classroom 16.12 with older children to see what they understand about the classification of quadrilaterals.
3. *Compare shapes*. Interview a range of F–7 students, asking them to compare the 2 shapes in figure 16.3. Compare their responses with those in the text.
4. *Jubes and toothpicks* In the classroom 16.3 is one of the activities with building 3-dimensional shapes. As you try this with children, observe which shapes children make and whether they can tell the number of edges and vertices. Can they make shapes with a given number of edges and vertices?
5. *Less is best or show my sides*. In the classroom 16.6 is a game for younger children, and In the classroom 16.8 has challenging puzzles. Try 1 of these with students and observe how they react to games or puzzles.

IN YOUR JOURNAL

6. Give 4 reasons why geometry should be included in a primary mathematics program. Explain, in your own words, what 1 of these reasons means to you.
7. Defend spending time in a classroom constructing models of solids.
8. A fellow teacher says that they cannot start to teach any geometry until the students know all the terms and definitions and that the Year 5 students just cannot learn them. What misconceptions about teaching geometry does this teacher hold?

WITH ADDITIONAL RESOURCES

9 Investigate the use of geometry in everyday life in another country or culture. Describe how it is different and how you could use this information in your teaching.

10 Look at the geometry in a textbook at a given year level. Make a list of the activities from this chapter that would complement the text. Have any important ideas been omitted from the textbook you investigated?

11 Read one of the children's books listed in the Children's literature connections or another book of your choice. Describe how you would use the book in teaching geometry.

WITH TECHNOLOGY

12 Write a lesson plan for children to use the Turtle Pond activity from the iMath activities of NCTM's Illuminations website.

13 An excellent activity for Year 3 students and beyond is lesson 134 from maths300, Pentagon Triangles. Explore triangles of different sizes using 2 types of isosceles triangle formed from the diagonal cuts within a regular pentagon. Can you make triangles using 2, 3, 4, 5, 6, 7 or 8 pieces? Was there a pattern in your findings? You can download the coloured pieces from the website, www.maths300.esa.edu.au, print them onto light card, cut and package them into individual student sets.

14 Explore drawing geometric shapes with Geometer's Sketchpad. What happens as you drag the shape around? How could this software be used with primary students?

CHILDREN'S LITERATURE CONNECTIONS

Ayture-Scheele, Z. (1990). *Beautiful Origami*. New York: Sterling.
There are lots of beautiful animals in this book, as the title implies. The photographs help you figure out the steps. There are books on origami appropriate for any age.

Burns, M. (1994). *The Greedy Triangle.* New York: Scholastic.
This story is about a shape that wants to be another shape. A shapeshifter turns the triangle into another shape; however, the triangle still is not happy. This continues until the triangle realises that he really wants to be a triangle. There are activities for children, parents and teachers.

Flournoy, V. (1991). *The Patchwork Quilt.* New York: Four Winds Press.
Tanya loves listening to her grandmother talk about the quilt she is making from pieces of colourful fabric from the family clothes. When Grandma becomes ill, Tanya decides to finish Grandma's masterpiece with the help of her family. This book can lead to discussions about different shapes used in the quilt pieces.

Friendman, A. (1994). *A Cloak for the Dreamer.* New York: Scholastic.
A tailor has been asked by the archduke to make new clothes for a very important journey that he will take. The tailor asks his sons — Ivan, Alex and Misha — to help in making the clothes. The tailor is concerned when Misha, the dreamer, uses geometric shapes in his tailoring designs. There are extended activities to use in the classroom.

Gowler Greene, R. (1997). *When a Line Bends…A Shape Begins*. New York: Houghton Mifflin.
From a line to squares, circles, triangles and more, children can look for shapes in each page while appreciating the rhymes about familiar things.

Grifalconi, A. (1986). *The Village of Round and Square Houses.* New York: Little, Brown and Company.
This is based on a true story of how people in Central Africa live in a village called Tos in Cameroon. The women live in round houses and the men live in square houses. A young girl listens to her grandmother as she tells the story of a volcano erupting and burning everything except 1 round house and 1 square house.

Hoban, T. (1992). *Spirals, Curves, Fanshapes and Lines.* New York: Greenwillow Books.
Intriguing shapes — spirals curves, fanshapes and lines — are all around. This book helps to heighten children's attention to the geometry in their own world.

Pilegard, V. (2000). *The Warlord's Puzzle*. Gretna, LA: Pelican.
When the artist presents a special tile to the warload, it is broken into 7 special pieces. It takes a young boy to piece it back together as he thinks about his fisherman father.

Tompert, A. (1990). *Grandfather Tang's Story.* New York: Crown.
A tangram adventure in which the animals change into different animals. Just rearrange the 7 tangram pieces to form a rabbit, a dog, a squirrel, a hawk and a crocodile. Using the tangram pattern (see appendix C), children can explore different shapes.

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ACKNOWLEDGEMENTS

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CHAPTER 17

Measurement

LEARNING OUTCOMES

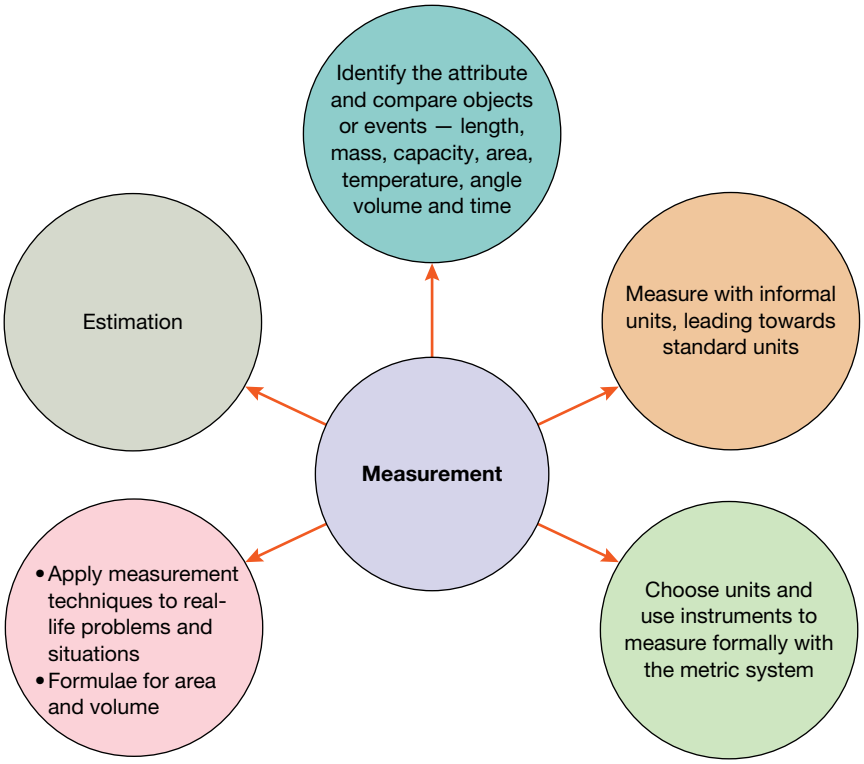
17.1 Explaining the measurement developmental sequence and how it can be applied to all attributes.
17.2 Understanding the nature of each attribute and developing appropriate language for comparison of objects and events.
17.3 Identifying the 10 measurement concepts that apply to all units.
17.4 Exploring the measurement of length, area, volume and capacity, mass, time, temperature, and angle with informal units.
17.5 Exploring the measurement of length, area, volume and capacity, time, temperature and angle with formal units, and using instruments and tools for standard measurement.
17.6 Developing formulae for perimeter, area and volume measurement.
17.7 Discussing the importance of equivalence relationships when converting units.
17.8 Articulating strategies for the skill of estimation.
17.9 Formulating connections between attributes.



‘There are two possible outcomes: if the result confirms the hypothesis, then you’ve made a measurement. If the result is contrary to the hypothesis, then you’ve made a discovery.’

Enrico Fermi, Italian Physicist (1901–1954)

Chapter 17 concept map



Introduction

This chapter discusses measurement with examples for all measurable attributes. The teacher's role is to be aware of, and build upon, prior experiences of length, area, volume and capacity, mass, time, temperature and angle, which children bring to their learning. What concepts, skills and processes are important for children to develop? This and other questions will be examined in this chapter, but first let's consider why measurement should be included in school mathematics.

Stop and think about how you have used mathematics in the past few days. Did you tell someone how long it took you to drive to school/university, how far it is to the nearest shop, how many cups of water or coffee you drank, what the bathroom scales are saying when you stand on them, how much money you spent on lunch? All of these are measurements.

Measurement is the topic from the primary mathematics curriculum that is used the most directly in students' daily lives. In fact measurement could be seen as the intuitive and binding notion for all the concepts: number, pattern, geometry, and data. Together with geometry, measurement is 1 of the 3 content strands in the Australian Curriculum (ACARA v8.1 2015). Table 17.1 lists the proficiencies and content descriptions for *using units of measurement* sub-strand within the Measurement and Geometry strand. Measurement is also one of 5 key content strands in the *Standards and Expectations for School Mathematics* (NCTM 2000), partly because of its power to help students see the usefulness of mathematics in everyday life.

Another reason is that measurement uses many other topics in mathematics. For example, children may count the number of grams it takes to balance an object on a scale, multiply to find a volume, divide to change minutes to hours, subtract to see how close an estimate was to the actual amount or add to find the perimeter of a triangle. To report the number of units, children might use whole numbers, common fractions, decimals or negative numbers.

TABLE 17.1 **Measurement: Proficiencies and content descriptions in the Australian Curriculum**

Year level	Content description
Foundation **Fluency** includes comparing the lengths of objects. **Reasoning** includes explaining comparisons of quantities and explaining processes for indirect comparison of length.	• Use direct and indirect comparisons to decide which is longer, heavier or holds more, and explain reasoning in everyday language. • Compare and order the duration of events using the everyday language of time. • Connect days of the week to familiar events and actions.
Year 1 **Fluency** includes naming the days of the week. **Reasoning** includes explaining direct and indirect comparisons of length using uniform informal units.	• Measure and compare the lengths and capacities of pairs of objects using uniform informal units. • Tell time to the half-hour. • Describe duration using months, weeks, days and hours.
Year 2 **Fluency** includes using informal units iteratively to compare measurements and describing and comparing time durations.	• Compare and order several shapes and objects based on length, area, volume and capacity using appropriate uniform informal units. • Compare masses of objects using balance scales. • Tell time to the quarter-hour, using the language of 'past' and 'to'. • Name and order months and seasons. • Use a calendar to identify the date and determine the number of days in each month. • Identify and describe half and quarter turns.
Year 3 **Understanding** includes using appropriate language to communicate times. **Fluency** includes using familiar metric units to order and compare objects. **Reasoning** includes comparing angles.	• Measure, order and compare objects using familiar metric units of length, mass and capacity. • Tell time to the minute and investigate the relationship between units of time. • Identify angles as measures of turn and compare angle sizes in everyday situations.
Year 4 **Understanding** includes using appropriate language to communicate times. **Fluency** includes using instruments to measure accurately. **Problem solving** includes comparing time durations. **Reasoning** includes comparing angles.	• Use scaled instruments to measure and compare lengths, masses, capacities and temperatures. • Compare objects using familiar metric units of area and volume. • Convert between units of time. • Use am and pm notation and solve simple time problems. • Compare the areas of regular and irregular shapes by informal means. • Compare angles and classify them as equal to, greater than, or less than, a right angle.
Year 5 **Fluency** includes choosing appropriate units of measurement for calculation of perimeter and area and using instruments to measure angles. **Problem solving** includes formulating and solving authentic problems using whole numbers and measurements.	• Choose appropriate units of measurement for length, area, volume, capacity and mass. • Calculate the perimeter and area of rectangles using familiar metric units. • Compare 12- and 24-hour time systems and convert between them. • Estimate, measure and compare angles using degrees. Construct angles using a protractor.

(continued)

TABLE 17.1 *(continued)*

Year level	Content description
Year 6 **Understanding** includes making reasonable estimations. **Fluency** includes measuring using metric units and interpreting timetables. **Problem solving** includes formulating and solving authentic problems using . . . measurements and finding the size of unknown angles.	• Connect decimal representations to the metric system. • Convert between common metric units of length, mass and capacity. • Solve problems involving the comparison of lengths and areas using appropriate units. • Connect volume and capacity and their units of measurement. • Interpret and use timetables. • Investigate, with and without digital technologies, angles on a straight line, angles at a point and vertically opposite angles. Use results to find unknown angles.
Year 7 **Fluency** includes calculating areas of shapes and volumes of prisms. **Problem solving** includes formulating and solving authentic problems using numbers and measurements.	• Establish the formulas for areas of rectangles, triangles and parallelograms and use these in problem solving. • Calculate volumes of rectangular prisms.

Source: © Australian Curriculum, Assessment and Reporting Authority 2015.

Although not immediately apparent, measurement can help children learn about other topics in mathematics. Many of the numeration models are based on measurement. For example, the number line is based on length. One model for multiplication is the array or area of a rectangle. Fractions, angles and the analogue clock have much in common. There are many concepts and processes that underlie both measurement and number ideas. As shown in figure 17.1, measuring to the nearest unit is similar to rounding to a given unit. Thus measurement ideas may be used to complement numerical ideas.

Not only is measurement useful in everyday life, it is also useful in other areas of the curriculum. If you are trying to connect mathematics with other subjects, consider the ways that measurement is used in art, music, science, history, geography and language arts. Much additional practice with measurement skills could be accomplished as children learn other subjects.

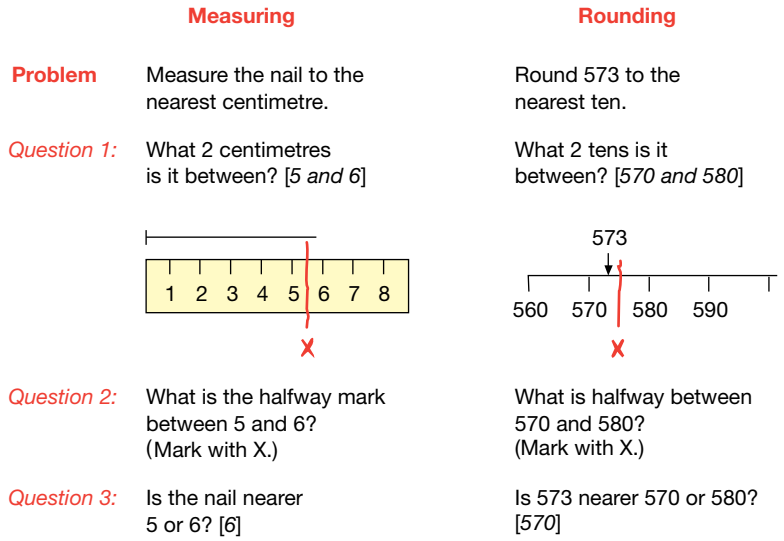


FIGURE 17.1 The similarity between measuring and rounding numbers

Another reason measurement is an important part of the mathematics curriculum is not so much mathematical as pedagogical. Measurement is an effective way to engage diverse student needs and interests, some of who may be less motivated to learn in other maths topics. Some students are more motivated to learn about measurement than other mathematics topics. They often see the usefulness of learning when it relates to them personally. Think about your school experiences. In primary school were you fortunate enough to experience active mathematics classes when you were in primary school?

Research from international studies has shown that measurement is a content domain in which Australian Year 4 students were stronger than in number (Thomson et al. 2009, p. iv). However there is considerable evidence that many secondary students do not thoroughly understand length, area and volume measurement (O'Keefe & Bobis 2008). In particular, 'students do not understand the attributes being measured or the units that are used for measurement' (O'Keefe & Bobis 2008, p. 391).

In summary, measurement should be an integral part of the mathematics curriculum and occupy a considerable slice of learning time each and every year. There are several important reasons why measurement should be studied in depth.

- Measurement provides many applications to everyday life.
- Estimation skills are enhanced through measurement activities.
- Measurement can be used to help learn other mathematics.
- Measurement can be used across the curriculum, in other areas of learning.
- Students find learning in measurement to be engaging and interesting.

The NCTM Principles and Standards (2000) call for measurement to be a continuing part of the mathematics program, rather than being presented in a few isolated lessons (see Appendix A). The first measurement statement concentrates on the conceptual understanding of the attributes, units and processes of measurement; the second statement focuses on techniques and skills. This chapter integrates both of these as it considers the process of measurement, which depends both on concepts and skills.

17.1 The measurement process

LEARNING OUTCOME 17.1 Explaining the measurement developmental sequence and how it can be applied to all attributes.

Much of the research about how children measure and think about measurement has focused on children's conceptual development. However, there have also been classroom studies (Lehrer 2003) that shed light on what students can do when given the opportunity. Wilson and Osborne (1988) give the following recommendations that are as appropriate for today as they were then.

- Children need to measure frequently and often, preferably using real problems rather than textbook exercises.
- Children need to develop estimation skills with measurement in order to develop common referents and as an early application of number sense.
- Children should encounter activity oriented measurement situations by doing and experimenting rather than by passively observing.
- The activities should encourage discussion to stimulate the refinement and testing of ideas and concepts.
- Planning for teaching should emphasise the important ideas of measurement that transfer or work across measurement systems.

In the chapter that follows, look for these 5 characteristics in the suggestions and activities. More importantly, keep these recommendations in mind when planning teaching and learning experiences.

Measurement is a process by which a number is assigned to an attribute of an object or event. Length, capacity, weight/mass, area, volume, time, temperature and angle are the measurable attributes considered in most primary mathematics programs. Although each of these attributes is different, the process of measuring is common to all of these attributes. The following outline, based on the measuring process, can be used to plan learning experiences.

1. *Identify the attribute being measured.* Physical objects need to be handled to raise awareness of attributes and develop language.
2. *Compare objects and events.* Perceptual, direct or indirect comparison will lead to ordering of 2 or more objects.
3. *Measure with informal units.* Everyday non-standard units will lead to measurement processes that assign a quantity and unit.
4. *Measure with standard units including the concept of repeated units.* In Australia and New Zealand the metric system is standard. Familiarity with quantities, naming, reading, writing and symbol conventions is needed together with estimations.
5. *Apply measurement to real-life contexts.* This stage could involve use of formulae and processes used extensively in real life.

When a new attribute is introduced, students need to cycle through the measurement process. Focus first on understanding the attribute and building language to describe objects and events, this is followed closely by comparison. Next students will begin measuring using informal (nonstandard units) and assign a number of units to the attribute. This process will lead to an awareness that units can vary and in order to accurately describe the object or event then a standard unit needs to be used. Estimation is an important skill for students to be comfortable with at this stage and later with using standard units of measurement, tools and instruments. When a thorough understanding is gained, and students are at an appropriate developmental level, formulae can be introduced.

This cycling may take place over several years for the first four attributes studied — length, mass, volume/capacity and time. The process should not be rushed, however after several attributes have been introduced and developed, the amount of time spent on the measurement cycle can be shortened as students make important connections with other attributes. For example, in the Australian Curriculum, Foundation year children directly and indirectly compare length, mass and capacity. Children are able to use informal units for length and capacity in Year 1 and begin to use metric units for these attributes in Year 2. A balance scale for comparing mass can be used in Year 2, as can comparison of objects by area and volume. Finally, from Year 3 onwards, students can use real-world benchmarks to estimate length, mass and capacity and measure with familiar metric units such as the centimetre, metre, kilogram and litre.

Benchmarks based on personal situations are more meaningful to children — for example:

> My little sister, who has just started school, is about 1 metre tall and she can reach up to Mum's waist.
> Measuring across my little finger with a centicube, I found it to be the same: 1 centimetre.
> When my water bottle is full, it holds a litre and can balance just over a kilogram mass on the scales.

Over time it is important to develop skills and processes associated with estimation, use of appropriate language, and choosing and using measuring units and instruments. Estimation is an important process in itself but can help to establish the idea that measuring can yield different results depending on perception and precision. If students practise the skill of making an informed prediction before measuring, then they will be able to construct a deeper understanding of the number and the units. As students describe their problem-solving processes, their strategies and their results when dealing with measurement tasks they will be learning to use appropriate and accurate language. Further encouragement to record (using words, diagrams, units and symbols) helps children to remember and reflect upon the processes used. Initially children will work with informal units for finding length, capacity and mass, then area, volume, time, angle and temperature before using standard units. Nonstandard units are easy to work with initially and can be readily available, for example pop-sticks, cubes and body measures. The hand can be closed or stretched to give a measure across (hand span or hand) for length; it can be traced around (handprint) for area; it can be cupped or scooped for volume (filled with cubes or beads). A variety of experiences will help children learn that the size of the unit chosen and the number of units used are related (inversely). For example, big cubes will fill a lunchbox much quicker and with less gaps of space (if stacked neatly), compared to beads or centicubes. As their developmental level progresses, students will see the need for, and comfortably use standard units.

If a student has some gaps or weaknesses in their measurement understanding and abilities, it might be that they have not attained the fundamental concept of conservation. That is, quantities are unchanged if rearranged. For example, a strip of frieze tape is the same length if it is curled or straightened out; a kilogram of apples is the same as a kilogram of blueberries even though the second fruit has many more; a cup of juice in a tall, thin glass is the same amount as in a short, wide glass; a minute spent colouring a mandala design is the same as a minute spent doing star jumps. Conservation understanding can be developed alongside the use of formal and informal units and is not a pre-requisite, however the teacher needs to be attuned to students who might be using units inconsistently. In other words, the teacher needs to regularly observe students and ask open questions leading to students' describing, reasoning and justifying their results.

17.2 Identifying attributes and comparing

LEARNING OUTCOME 17.2 Understanding the nature of each attribute and developing appropriate language for comparison of objects and events.

To measure with understanding, children need to know what attribute they are measuring. For example, young children will find it difficult to measure the area of an object because they might not yet understand the concept of area or surface. This is an attribute that would normally be met after length, when children are a little older, however they will gain intuitive understanding of area from their real life experiences, like throwing a cloth over the mini dinner table during dramatic play or putting some blue coloured paper in their block construction to signify a swimming pool. Another example pertinent to older children would be if they have difficulty measuring an angle due to not understanding the concept of angle as dynamic, or an amount of turn. Therefore, one of the first tasks is to build an understanding of each measurable attribute.

There are 3 types of comparison that can help build understanding of attributes: comparing 2 objects perceptually (they either look the same or they look different), comparing 2 objects directly (they are placed next to each other), and comparing 2 objects indirectly (a third object is used to compare objects). Comparison is also relevant to attributes that cannot be easily seen, such as the passing of time, duration of an event or temperature. As children make these types of comparison, not only are they gaining an understanding of the particular attribute and the associated vocabulary, but they are also learning procedures that will help them in assigning an amount to a measurement.

Awareness of the attribute and comparison will be considered for each attribute found in primary mathematics programs because of the importance of gaining a sound understanding and participation in comparison experiences for every attribute. The 3 types of comparison described will help children develop an understanding of that attribute.

Length

Length is one of the most easily perceived attributes of objects. It has many forms: the 1-dimensional line joining 2 points, the property of a shape or an object, a height, a distance, a depth, a measure around or through an object, plus more. Very young children who have experienced a variety of play situations often have good language skills and can estimate and compare objects by length effectively. However, conservation of length takes time to develop and many children may still have misconceptions about length. For example, they may say that a belt is shorter when it is curled up than when it is straight. These misconceptions disappear as children develop cognitively and are involved in constructive experiences. In the classroom 17.1, 17.2 and 17.3 illustrate comparison activities that are appropriate for early childhood from preschool through to Foundation and Years 1 and 2. The Focal Points (NCTM 2006) recommend beginning in preschool with length by making pairwise (2 objects) comparisons and moving to ordering (more than 2 objects) in their Foundation year.

IN THE CLASSROOM 17.1

Perceptual comparison of lengths

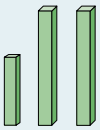
Objective: Compare lengths of physical objects perceptually.

Year level: F–1

Materials: Collections of long, thin objects such as rods, candles, chopsticks, straws, spaghetti, pencils and crayons.

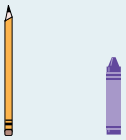
Teacher holds up 2 long objects and 1 short object.

- Ask child to state which 1 is different.
- Use vocabulary of *shorter, smaller, higher, taller* and *longer*.
- Repeat this activity with other objects that differ only with respect to length and are obviously different in length.



Hold up a long pencil and a short crayon.

- Ask how they are different. Expect answers based on other characteristics such as colour, type of object, paper wrapper, eraser.
- If there is no response about the crayon being shorter or the pencil being longer, ask 'Which is longer?'
- If the child mentions that 1 of them is thick, thin, wide or narrow then agree with their response, as length has more than 1 dimension here.



IN THE CLASSROOM 17.2

Direct comparison of length

Objective: Compare lengths of physical objects directly.

Year level: F–1

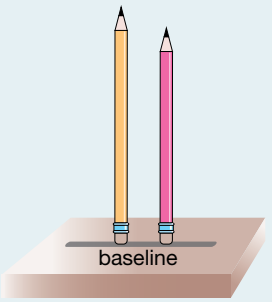
Materials: Collection of long, thin objects and 3 sheets of construction paper (labelled 'Shorter', 'Same' and 'Longer').

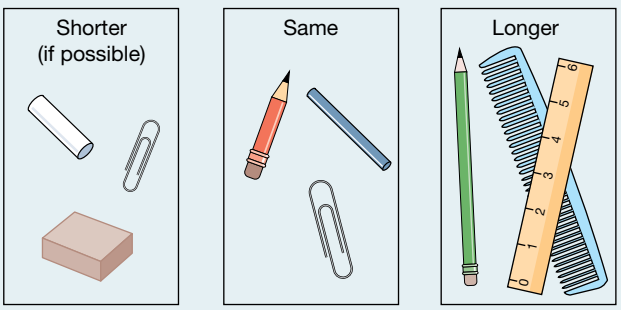
Present the children with a problem by holding up 2 pencils that look similar in length. Ask the group which is longer.

There might be some disagreement. Ask a child to compare the lengths by putting the pencils upright on a table. Repeat.

After the children have had an opportunity to compare 2 objects directly (the ones available in the container, their height with a partner, the length of 2 shoes or length of classroom objects) using a baseline, set up the following centre at the small group table.

Place the box of objects in the centre and choose 1 object to be the reference. Place it on the sheet labelled 'Same'. Ask the students to sort the other objects as shown below. Observe if the 3D objects have been compared by length, height or width.





Students can then change the reference object and re-sort.

Gearing down
Work with individual students to help them with the words and with the idea of a baseline. Ask them to justify their sorting by showing you why they placed an object in that category. Begin by asking about an object they placed correctly.

Gearing up
Challenge the students to pick a reference object so that 1 category has no objects. See if they can generalise that if they choose the shortest (longest) object, there will be no objects in the shorter (longer) category.

IN THE CLASSROOM 17.3

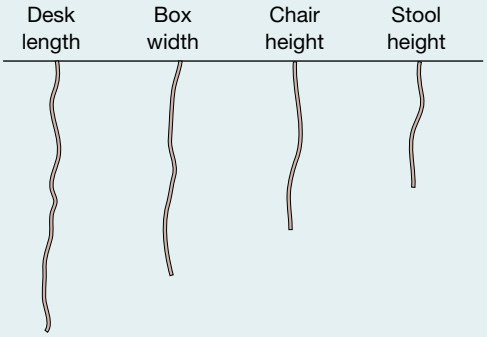
Indirect comparison of length

Objective: Indirectly compare 2 objects by length.

Year level: F–2

Materials: String or connecting links, index cards, tape and objects in the room selected for children to represent.

- Choose 2 lengths that are about the same from objects that cannot be moved to compare directly. For example, the height of the teacher's desk with the width of the door. Ask the children to choose which is longer and let them explain how they will find out.
- Discuss the ways they propose and then if no one has come up with the idea, show them how you would use the links (or string) to represent each of the lengths and then compare.
- After this discussion and demonstration, give each pair of children a string and have them represent 1 of the objects that were prepared in advance with a label.
- Compare in size by recording with a 'graph' of all the lengths found.



- Allow students to discuss any problems that arose during their measurement activity.

In the classroom 17.1 involves comparisons made perceptually and is designed to be done with very young children. Many children in preschool have no difficulty with this task, yet some will have difficulty when the objects are not so perceptually different. In the beginning of this activity, all irrelevant perceptual attributes have been masked (i.e. the objects are the same except for length). This allows the child to build the concept of length as an attribute of long, thin things. Note that the teacher doesn't need to ask which is longer: it is best to encourage the child to observe and describe their findings using their own language. In the second activity, objects that differ with respect to many attributes (e.g. colour and function) are used, but the focus remains on length. The teacher can evaluate how well the students are answering these questions. Do they focus on length? Do they know the vocabulary associated with length? Observing individual students in each small group (over time) it may be necessary to gear down the activity for some students who have not had the opportunity to learn the vocabulary (especially

shorter and longer) by asking more questions and letting the students learn from each other. It can be possible to gear up by asking questions about a set of 3 to 5 objects:

Which object is the longest?
Which object is the shortest?
Which objects are shorter than the longest object?

In the classroom 17.2 is designed for use at learning centres after introducing the procedure of placing 2 objects side by side on a common baseline (comparing directly). The teacher presents the problem of comparing 2 objects that are not perceptually different in length. Asking the group which object is longer and how they can find out might lead to the suggestion of putting the 2 objects side by side on a baseline. This activity may be extended to organising the objects by length from shortest to tallest. For the young child, this task is more difficult because multiple comparisons must be made. Later when children are using rulers, they are directly comparing 2 lengths — the object to the ruler.

In the classroom 17.3 presents the problem of indirectly comparing 2 objects when they cannot be placed side by side. Children need to represent the lengths with another object to help them make a comparison. Notice that in launching this activity, the students are presented with a problem to solve. Which is longer — the height of the teacher's desk or the width of the door? This would be a very real question for the furniture removal company. Children can represent both the height of the desk and the width of the door with frieze tape, string or connecting link chains. As the width of the door is an 'invisible' measure except at the base or top, the children will need to decide how to do this accurately. They can then directly compare the string or link chain to determine which object is longer. Later, a ruler or tape measure will be used to represent length and the objects will be compared indirectly using numbers and units.

Through open-ended activities, children initially develop an understanding of length as an attribute of long, tall, high, deep, wide or thick objects; however, length is used in other ways. For example, length is the *distance around* your wrist, your waist or the girth of a tree. Young children can estimate first, then use string to compare their wrists with their ankles. In the classroom 17.4 is an activity appropriate for older children in which they estimate the distance around a can and the height of a can, and then check their conjecture by comparing the lengths with a representation (a piece of string).

The *distance between* 2 points is also measured by length. Distance is often more difficult to perceive than the length of an object, because you have to imagine the straight path between the 2 endpoints. People often confuse distance and speed when travelling, which is quite common in Australia. People speak about travel time more often than kilometres, which can be confusing to children learning about the distance between 2 places. Asking a teenager recently what was the distance from Willunga to Adelaide, the immediate response was 'an hour'. Asking further about why time was given as the measurement instead of distance, the response was 'that people care more about how long it takes than how far it is'. Curious about this, the author did a Google search with the key words *distance Adelaide to Willunga* and found that the answer was given first in time — 53 min, second in distance (48.2 km), and then information about the route, via M2 and A13 with a visual map to show that it was a direct path. Very interesting! Finally the thoughts came that people in our local community do an annual walk to Adelaide city (Hutt Street) to raise money for charity, cyclists regularly ride this distance for the breakout *Tour Down Under* or for leisure, would they measure their travel by time or distance?

Perimeter, or the distance around a region, is a special type of length. Children should be given the opportunity to measure the distance around a region with string or a measuring tape or trundle wheel (if they are familiar with these tools). Later, they can add the lengths of the various sides of the region to find the perimeter. There are many everyday examples of problems that require finding the perimeter, such as finding the distance around a bulletin board to determine how much border is needed or how far it is around a city block or around a track on a sports oval. *Circumference* is a special word for the distance around a circle and closely linked to this measure are radius and diameter, 2 lengths within the circle.

IN THE CLASSROOM 17.4

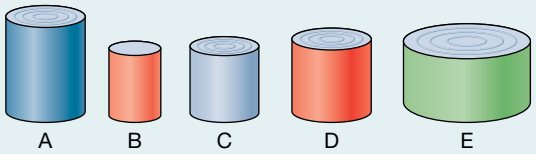
Comparing height and circumference

Objective: Compare the distance around an object with its height.

Year level: 2–3

Materials: Cans or jars of various sizes (at least 1 for each pair of students) and string.

Label the cans and make a recording sheet as suggested below. If cans are unavailable, roll construction paper to represent cans of various sizes.



Using 1 example, ask students to make a conjecture about which they think is longer: the distance around the can or the height of the can. Students are often surprised about the comparison. Help students measure each length with string and encourage them to record their conjecture and the result after measurement for the first example. Discuss as a group before breaking away into pairs.

Working together in pairs for all examples, students should first decide which is the longest attribute the height or circumference for each can or jar by looking at it. Then they can verify, or disprove their estimate by measuring each with a piece of string and recording the results in a chart.

Questions

1. Which cans did you find gave you a correct estimate and measure? Why?
2. Which cans fooled your eyes? Why do you think this was so?
3. What have you learned about the properties of 3-dimensional objects, particularly cylinders?

Can	My eyes said	The string showed
A	height	distance around
B		
C		
D		
E		

Capacity

Capacity is considered the attribute that describes ‘how much a 3-dimensional container can hold’. Language used to describe capacity can include pour, fill, full and empty. Although perceptual comparisons can be made between 2 containers, young children often make the comparisons based on length (the height of fluid or packing material) rather than on capacity. When asked which holds more, a tall, thin container or a small, wide container, most children will choose the taller container even if the shorter one may actually hold more. Play experiences with construction blocks, packing boxes, pouring water and sand into containers, submerging objects and measuring ingredients are essential for learning about this attribute. In the early stages it is not necessary to discriminate between volume and capacity, that conceptual difference can come later. When students are ready to use direct comparison, fluid type materials, such as beans, rice, seeds, beads or blocks, are needed. Water and sand are very natural and easy for young children to use. Given a variety of containers, they can fill one and pour the contents into the other to see which holds more. After children have experimented with direct comparison through filling

and pouring activities they will be more confident with perceptual comparison. For example, children greatly enjoy activities in pairs or small groups, where they are able to estimate which container holds more, and then check the results. Figure 17.2 displays a possible comparison.

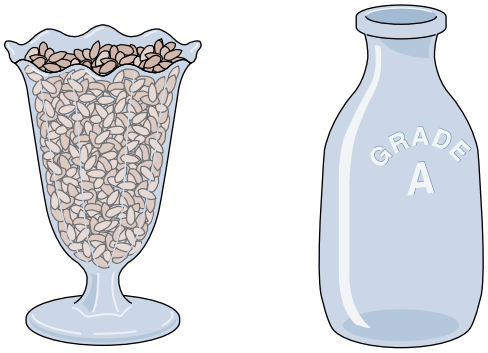


FIGURE 17.2 Which container holds more seeds? Why do you think the one you chose holds more?

Indirect comparisons are used when 2 containers are difficult to compare perceptually or directly. For example, suppose you have 2 containers with small openings that make it difficult to pour from one into another. By pouring the filler in each into a pair of identical large-mouth containers, the capacities can be compared. Note that this activity is similar to what might be done when graduated cylinders are used in a science lab to identify amounts of liquid.

Mass (weight)

These 2 attributes are different and the right terminology should be used, which in most cases will be mass. The mass of an object is the amount of substance, or matter it contains, whereas weight is the force of gravity acting upon that substance. The attribute of weight is scientifically determined by the formula for force and is measured in Newtons. For example if a person has a mass of 60 kilograms, their weight is approximately 600 Newtons on Earth. If that same person (astronaut) were on the moon, which is a much smaller object, it is easy to imagine the person weighing less and taking very light, bouncy steps. The gravitational pull on the moon is approximately $\frac{1}{6}$th of Earth and so the person would have a weight of about 100 Newtons. Figure 17.3 helps to illustrate this concept. It is important for teachers to model the correct use of the language for 'mass' and 'weight' as these terms are often used incorrectly in everyday life. The force of gravity might be involved in comparing and 'weighing' objects but the result is best described as the 'mass of' the object. The same descriptive and comparative words are used by children for both situations and include light, heavy, hard to push, and easy to lift.

The mass of 2 objects is compared perceptually by hefting or lifting the 2 objects. Children should be given a variety of pairs of objects, one that is much heavier than the other, and asked to hold one in each hand. They will feel that the heavier object 'weighs down' on their hand more than the other; in effect their body is perceptually the balance scale. Comparing mass is done by feel rather than sight. Experience with holding and sorting objects into 2 categories: heavy and light, is important to include initially. Children often hold a misconception that a larger object has a greater mass than a smaller one. In this case they are not yet conserving mass and will need opportunities to compare small and heavy objects (like a fishing lure) with similar but light objects (like a water balloon). Various real world objects such as small toys, balls, flowers, leaves, feathers, shells, stones and books will be useful. A sorting tray containing at least 3 sections heavy, light about the same (middle) is a good resource for comparison.

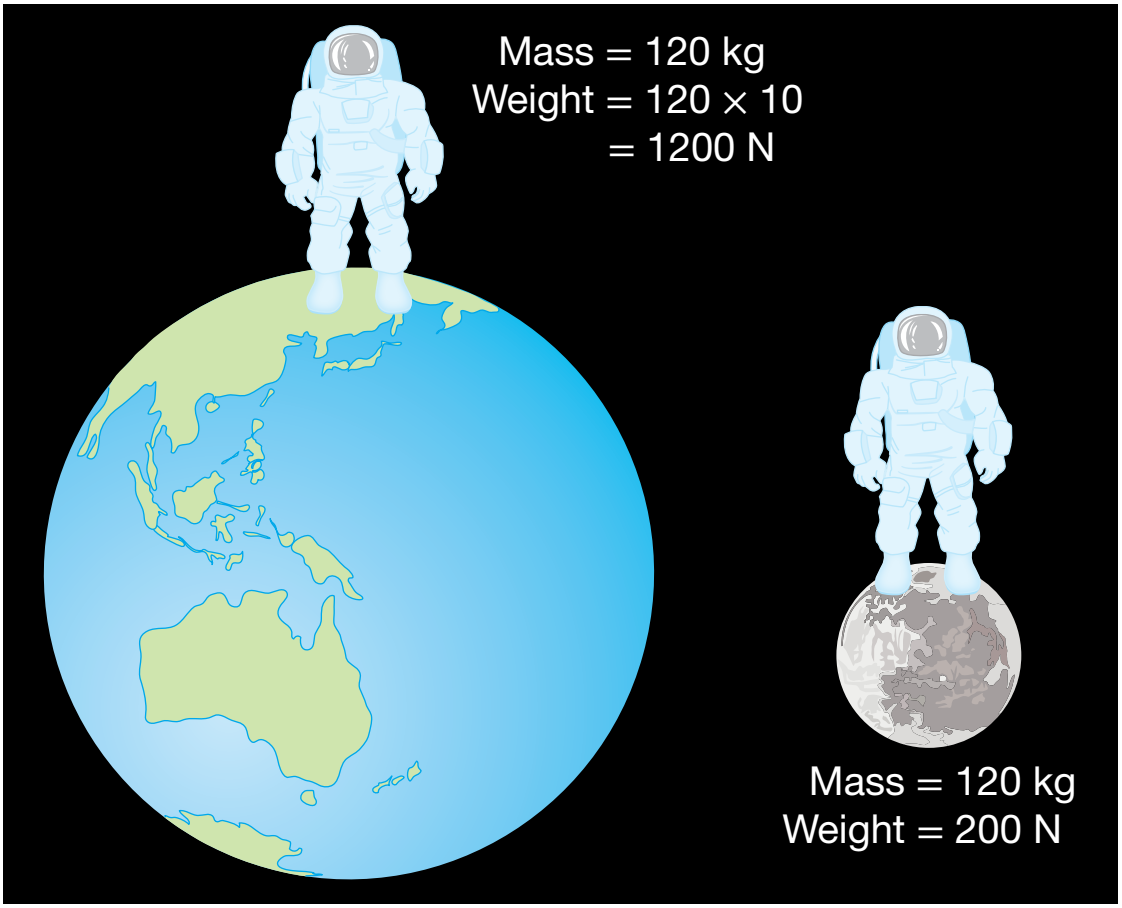


FIGURE 17.3 Comparison of mass and weight with correct units

To find which objects are heavier children will need to do more than look at the object. If 2 objects are very similar in mass, the teacher might need to provide a balance scale to assist in the comparison. To introduce the balance, choose 2 objects that differ greatly in mass so children can see that the heavier object 'goes down' on the balance. Figure 17.4 illustrates that even though the white rock is larger, it is not the heavier of the 2 rocks. Another method to provide this experience is by having children compare 2 identical containers with lids (mystery boxes) that are filled with different types or amounts of objects. For example, there could be 5 metal keys in 1 container and 5 plastic pegs in the other. Alternatively, cloth bags can be used. For example, there could be a kilogram of rice in 1 bag and a similar amount (size) of lightweight wooden cubes in the other. Children might be able to hear, by shaking the boxes, or see by looking at the bags, that there are similar quantities but recognise that 1 container or bag holds objects, which are much heavier.

Many activities can be set up for children, working in small groups: doing, talking, recording and reporting back to the class will help children to compare the mass of objects. Vocabulary such as: *heavier, lighter* and *about the same as* will naturally arise as they are hefting these objects. One child might explain to their peers, 'my pencil case has about the same mass as a banana and an apple'. To verify this statement the teacher could encourage the child to demonstrate with an equal arm balance and the children would then find out that the mass of the pencil case is about the same as the fruit. They will physically see that the downward pressure of gravity on both groups of objects is equal, so the balance is about level.

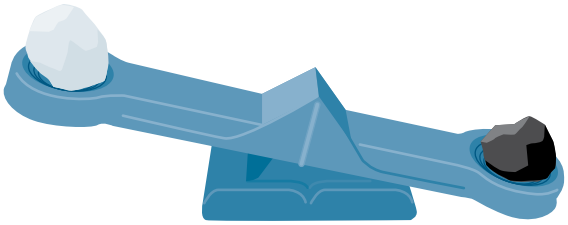


FIGURE 17.4 Using a balance to show which rock has the greatest mass

To extend children's thinking, they could compare 5 identical opaque containers, with lids secured and filled with different amounts of objects, and put them in order from lightest to heaviest. If the masses are not significantly different, then this task might require multiple comparisons using the equal arm balance. Finally, indirect comparison is not needed until units of mass are introduced. Whenever the mass of 2 objects is compared to a third on the balance scale, it would be much simpler to compare the 2 objects directly.

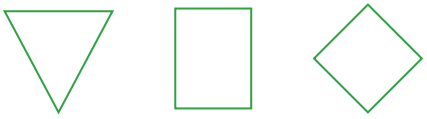
Area

Area is the amount of surface within a 2-dimensional shape or the region inside a boundary. Children can practise making closed shapes with boundaries of chord or ribbon and discuss why their shape has an enclosed surface compared to a line or open shape that does not enclose a surface. They might also observe that not all the shapes have the same amount of surface inside even when everyone had the same length of chord or ribbon in the beginning. Two regions can be compared by sight (perceptually) if the differences are large enough and the shapes are similar enough. For example, the area on top of the teacher's desk can be compared to the same surface of the student's desk. Similarly, the area of the whiteboard can be compared to the area of the door where children enter the room. Horizontal surfaces and vertical surfaces are more easily compared than hidden, curved or unusually shaped surfaces. For direct comparison of area, children can work in pairs, draw a shape and colour it, then try to cover 1 shape over the other to compare which is the biggest. There might be some disagreement and the need to justify their ideas about size. Everyday objects like stickers, stamps, envelopes and cut out shapes can work well. Using 3 sheets of construction paper or fabric, 2 of which are the same size but smaller than the third sheet the children can be asked which sheet is different and to describe their reasons for this. They might say that 1 is 'bigger' but their vocabulary may need refining to say that this sheet is 'larger in area'.



FIGURE 17.5 Two student-designed shapes of similar area

Begin to challenge students to compare regions like the regular shapes A, B and C. If the regions can be cut out, then students might start by comparing regions B and C, which are similar shapes, by placing 1 on top of the other. It is more difficult to compare either B or C with A because of the triangle shape. When children have some idea of *conservation of area* — that a region can be cut and rearranged without changing the area — this experience will become more meaningful. Nonetheless, it is surprising what solutions young children will propose; they are often more perceptive than older students, who often rely on measuring length and formulae.



To compare area when objects cannot be moved to place them 1 on top of the other, children can trace the objects and use these representations to make an indirect comparison. For example, choose 2 objects in the room that are about the same area and cannot be moved, such as the power switch cover and the label on the fire extinguisher. Students can re-create these surfaces on tracing paper and then compare them directly.

To help children understand that regions can be rearranged without affecting the area, many experiences with geometry activities are helpful. Personal areas such as handprints and footprints are meaningful to work with. Tangram activities are one of the best resources for this type of learning but if unavailable in plastic or wood, paper or card cut outs of the tangram pieces (Appendix C) can be used. In the classroom 17.5 will work effectively with just 4 of the small right triangles. If children are encouraged to discuss the area of the shapes A, B and C then this will help them focus on the objective of comparing shapes indirectly. The right triangles (the smallest of the 3 sizes in the tangram set) represent a unit of area that will help children make comparisons between the middle-sized right triangle, the square and the parallelogram. Although this activity was designed for younger children, it is appropriate for children in middle and upper primary years if they haven't had previous experiences with tangram pieces.

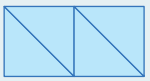
IN THE CLASSROOM 17.5

Are these shapes the same size?

Objective: Indirectly compare areas.

Year level: 2–3

Materials: Cut-outs of 4 right, isosceles triangles (as shown below) or 2 sets of tangrams.



Questions

1. Estimate if the area of A, B or C are the same, larger or smaller than each other.
2. Experiment with making shapes A, B and C using the small isosceles triangles.
3. Draw a line to show how you placed 2 triangles to make each of the shapes.



4. Of the shapes A, B or C which looked the smallest to you in the beginning? Explain to your classmates why each shape actually is the same size in terms of area.
5. What is actually different about these shapes and why?
6. Experiment with making shapes by joining 3 of the isosceles triangles. Draw a sketch of each.
7. Explain what the area is of the new shapes you made using the small triangles as the unit.

Extension: What would be the area if the unit were changed to be the square B shown above?

The Australian Curriculum (ACARA v8 2015) in the Measurement strand recommends that students compare and order shapes based on area using appropriate informal units in Year 2 and that in Year 4 they compare objects using familiar metric units of area (and volume). However in the Geometry strand

it is recommended that students compare the areas of regular and irregular shapes by informal means. These Year 4 statements appear to be incongruous but imply that there needs to be a strong foundational understanding of the attribute of area and comparison of areas in the early primary years before moving to formal measurement. The United States Focal Points (NCTM 2006) includes the majority of the developmental sequence for area in Year 4. Building on an understanding of the attribute, students 'learn that they can quantify area by finding the total number of same-sized units of area that cover the shape without gaps or overlaps. They understand that a square that is 1 unit on a side is the standard unit for measuring area' (p. 16). Furthermore students estimate and solve problems involving area, make connections to the area model of multiplication and work with the area formula for rectangles.

SPOTLIGHT ON EARLY CHILDHOOD

Discovering key measurement concepts

Children bring abundant experiences of measurement with them when they come to school. In their daily life and in preschool play activities, children explore the size of objects in many ways. They also understand about the passing of time, know routines and anticipate special events in their lives.

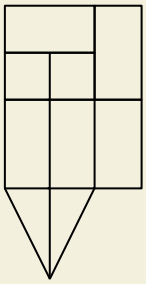
Over a 6-week research project, preschool children were seen to demonstrate intuitive understanding of measurement as they played in the block area, outside with water and sand, in the thematic corner, with puzzles, and during craft and cooking activities (Rogers 1997).

The attributes of length, area, mass, volume, capacity and time were evident in the descriptive language used and in children's reasoning about measurement concepts and issues. With respect to length during block play, children compared the size of blocks to determine whether they were 'equal' or 'not equal', 'twice as big', 'very big' and 'half as small'. They used direct comparison by matching blocks 1 to 1 along their length, width or depth. Additionally, children made indirect comparisons of the length of blocks by using other objects and informal units based on parts of the body. Children used the concept of units of measurement when they readily substituted a block of a certain size with a number of equal blocks of a smaller size, or vice versa.

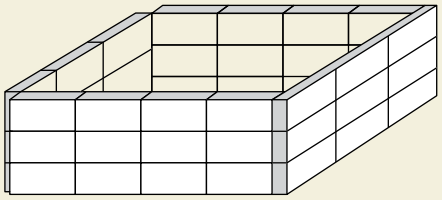
It was clear that play with a variety of materials helped children discover key concepts about measurement. The following table summarises the measurement language used by 3 to 5 year-old children who were observed and recorded (in observation notes, on video and in transcripts) during a comparatively short time in a NSW regional preschool.

Concept	Language and sample phrases used
Length, distance and estimation	• long; small; thin; thick; little; big; really big; wide; narrow; high; tall; bigger; smaller; shorter; higher; longer; equal; not equal; twice/half as big; too long • 'a huge house'; 'now we need the long blocks'; 'this is as big as me'; 'the longest road ever'; 'as tall as your building'; 'we had enough blocks to make it around the whole school'
Mass, balance and pressure	• heavy; light; heavier than; lighter than; strong; stronger; strongest; crooked; wobbly • 'it doesn't stand up very well'; 'not balanced in the middle'; 'wow'; 'I will press on this building to see if it will collapse'
Area, covering surface and tessellation	• cover • 'room for everyone in the house'; 'do you want a window?', '1, 2 . . . 15, now the floor is covered'; 'this is like our own bricks, you walk all over that'
Capacity and volume	• more; less; most; least; empty; full; nearly full; the same amount; big/small (with regard to jug, cup, bucket, scoop or building) • 'Stop! It is enough'
Time	• before; after; in a while; minute; morning; afternoon; night time; weekend; tomorrow; yesterday; my birthday; hours; next week; days (e.g. Monday) and months (e.g. July) • '5 hours to drive to Lismore'
Speed	• 'we need these speed bumps to slow the vehicles'; 'the ramp makes them go faster'

The diagram on the left illustrates the way children grouped unit and half-unit blocks together to cover an area leading from a building to the 'beach'. The diagram on the right shows how blocks were used to make a regular building with a floor area of 4 × 3 units and a height of 3 blocks.



Coverage using unit and half unit blocks



Block construction of 4 × 3 × 3 building

Volume

If volume is considered as 'how much space a 3-dimensional object takes up', then initial understanding will be created by stacking and packing activities. Perceptual comparison will be the main focus before units are introduced. The language to be developed is larger, smaller, biggest, smallest and the same as. Different arrangements of a certain number of cubes can help children understand that volume of an object can be the same but look different. A building made of 24 cubes in a row or column will be very long or tall but a building made with 24 cubes can also be wide and made with layers. For example, it might be 4 cubes long by 3 cubes wide in the first layer but there will be 3 layers (or it is 3 cubes high). Another building, which is 6 cubes long by 2 cubes wide will have 2 layers (2 cubes high) and has the same volume of 24 cubes but will look different again.

Children could explore the concept of 'how big is my hand?' by grasping cubes from a pile or container of cubes with their left hand and comparing the result with their peers by counting the amount. They can repeat the experiment a few more times to find their best attempt. Children can repeat the process again (3 times) with their right hand and compare the difference.

Normally, volume will receive little formal attention until Year 4 and 5 when students begin quantifying 3-dimensional objects. Because there is a close connection between volume and capacity however, background experiences can be provided for young children by filling objects and containers with solids such as blocks, balls or other small objects. Initially there might be gaps but eventually the objects found to fit the best will be cubes, which have been matched carefully.

Figure 17.6 uses blocks to fill a box, allowing for the units to be counted in 3 dimensions. This type of activity leads to volume comparisons with other boxes. Incomplete diagrams of containers similar to figure 17.4 will encourage students to figure out how many cubes it would take without filling the entire box. Real world pictures of packing and stacking will help children to understand the concept of volume and compare the size of containers. Experiences like this are essential to help children build spatial awareness and make sense of volume formulae. The Focal Points (NCTM 2006) and the Australian Curriculum (ACARA v8 2015) both recommend that volume be a focus in Year 5 mathematics.

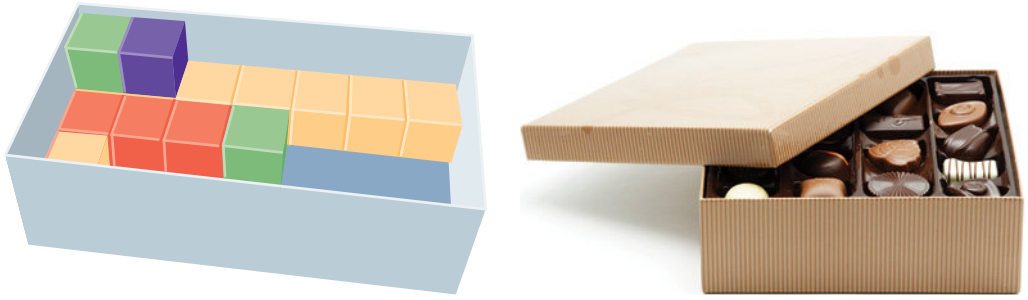


FIGURE 17.6 How many blocks will fit? How many chocolates in the box?

Angle

The most common definition for angles is the amount of turn. Unfortunately students are rarely given the time to understand the meaning of both dynamic and static angles and the opportunity to compare angles before they are taught to classify them, name them and measure them formally in degrees with a tool such as a protractor. If an angle is considered a turning (such as a clock's hands) or a door opening then young children can imagine the creation of an angle by considering 1 arm, as moving through a distance. The angle is anchored at a centre point, for example the centre of a circle (for a clock in 2D) or a doorframe with hinges (in 3D). Young children can also compare angles directly by comparing the amount of space the turn would make.

The concept of a right angle is helpful for children to use as a starting place for comparison. A circle cutout can be folded in half and half again to illustrate a right angle. The ideas of: smaller than a right angle, equal to a right angle and larger than a right angle are relatively easy to understand. Children are familiar with right angles because they see them in their everyday environment and in common shapes and often call them 'corners'. They can create their own diagrams using string or pop sticks and glue and label them as small, medium and large angles. Each diagram should have the amount of turn marked in near the vertex as shown in figure 17.7.

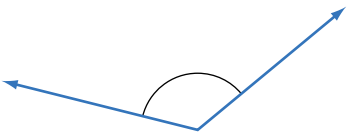
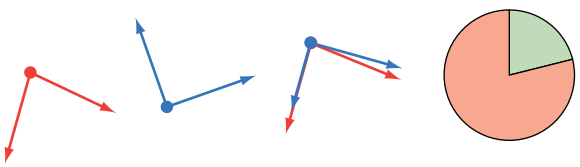


FIGURE 17.7 A medium sized angle

Before moving on to measuring angles, students can compare angles indirectly by tracing 1 angle then comparing the tracing with the other angle. Both the red angle and blue angle appeared very similar until 'mapped' onto each other. An excellent tool for angle comparison and informal measurement is the 'rotogram'. It consists of 2 pieces of plastic, 1 piece is fixed and has a reference line marked on it (black), the other is a moving disc with an arm (blue) on it. Angles like the ones below can be indirectly compared using a rotogram.



Time

Time is an abstract attribute because it is not an attribute of objects that can be seen or felt. There are 2 qualities of events that can be measured: *time of occurrence* and *length of duration*. Children can describe both the time of occurrence of an event and sequence them in their day, their week or their year using simple language related to their environment and their actions. Many and varied children's books are useful to help children gain the language of time and concepts of time passing and events occurring. For example, having breakfast, walking to school, eating lunch, bath time, time for bed and reading a story are all events that might occur in the daily life of a young child. These events can be described in sequence with language of before, after, now, soon, in a short while, and much later. Children are often exposed early to standard units for duration and for moments in time that they will not yet understand, such as: wait a minute, grandma will be here in an hour, when your brother comes home from school, at the weekend we will go to the beach, in winter time it rains a lot, and when it is your birthday. They will hear formal language for events such as talk of the clock (8:30 am we need to be there) and the calendar (25th December is Christmas day) but will be unable to process the meaning of a particular time or day until much later in their experiences. However, young children become familiar with routines and moments in the day and week early on. They might hear events being labelled such as morning, afternoon or night and discussion of regularly events like 'swimming class was yesterday', 'today we are catching the train' and 'tomorrow is Friday' are all examples of time of occurrence.

Another aspect of learning time is becoming aware of time passing. Events can happen in a slow of fast manner. 'Are we there yet?' is a common question when children are travelling somewhere, because the journey always seems to take a long time. Growing up might seem to take a long time, even when it is '10 sleeps to your birthday', the days seem to pass slowly. In countries where the climate changes significantly over a year, photographs of the seasons can help children learn about time, seeing images over years of human development like a baby, toddler, child, teenager, adult and grandparent might help children understand the passing of years. Something that might happen quickly in a child's view, is the time it takes someone to read a story book, to watch their favourite television show, or eat an ice-cream.

From preschool through to Year 2, children need to develop the vocabulary of days of the week, months and seasons of the year. They also need to hear occurrences described in hours and minutes: school begins at 9 o'clock, you eat lunch at 12 o'clock and you have 30 minutes to play. Most of this learning takes place naturally if children are made aware of the time of familiar events. Regular activities involving children's literature and classroom routines will help develop time vocabulary. For example, the book *Diary of a Baby Wombat* by Jackie French (2009) is a beautiful story that includes the following vocabulary and concepts of time: day, early morning, late morning, afternoon, night, days of the week, sequence of events and routines. The famous book *The Very Hungry Caterpillar* by Eric Carle (1969) includes these concepts of time: morning, night, next day, 2 weeks, days of the week, Sunday again, and the beautiful description of night as 'in the light of the moon'. In the classroom 17.6 describes a lesson plan for sequencing 5 pictures from *Alexander and the terrible, horrible, no good, very bad day* by Judith Viorst (1986). Please note there is a very good reading by Moriah Vermette (October 9, 2012) of this book at the following weblink: www.youtube.com/watch?v=GOGv3Cptb7I.

IN THE CLASSROOM 17.6

Things that we do in our day

Objective: Compare and order events in a day, based on a children's book.

Year level: Foundation to Year 1

Materials: Cut-outs of pictures from the book, or sketches of similar events, 1 envelope per table group of 2 to 4 children.

Activity: Children choose at least 5 pictures including getting up and bedtime and place them in order of events that happened in the book. They discuss the order of similar events in their 'normal' day and justify how they order the events in a similar or different way to Alexander.

Altogether they can choose from: Getting up, eating breakfast, car ride to school, at school in art, singing, maths, during play time, lunch time, dentist after school, falling in the mud and fighting with siblings, shoe store, dad's office, dinner, bath time and going to bed.

Extension: Ask children to create their own sequence of 5 events over a day, week or year.

Children can compare which of 2 events takes longer if their length of duration is greatly different. Does it take longer to brush your teeth or read a story? If the events are similar in duration, children can tell which activity lasts longer by beginning at the same time. Children can act out certain events like walk to the school front office and back or skip 20 times on a rope. They can come up with their own ideas using materials available in the classroom.

Temperature

Temperature is a measure of the heat or cold of an object or event. Perceptually it is possible to compare temperature and be aware of large differences in temperature. For example, this room is cool because the fan is working effectively, or the shady tree provides a cool place to rest a moment but the outside temperature is quite hot today. On another day this room is warm because the wood fire is burning but outside it is cold and windy weather. Before introducing children to reading a thermometer with either formal or informal units, the teacher can ask children to compare which of 2 objects is colder (or warmer). Beakers filled with cold, cool, warm and hot water can be ordered by temperature. Pictures of clothing worn in different seasons or scenery during different seasons can be compared. Later comparisons can be made with an instrument (thermometer).

Other attributes

There are many other measurable attributes appropriate for upper primary students; for example, speed and density. For the purpose of helping children learn mathematics connections can be made to science and children can become aware of these attributes through experiences across the curriculum.

17.3 Measurement concepts for all units

LEARNING OUTCOME 17.3 Identifying the 10 measurement concepts that apply to all units.

After children have begun to develop a firm concept of an attribute through awareness and comparison activities, it is important to help them move through the rest of the measurement process. To answer the question 'How long is the pencil?' you can say 'It's longer than my thumb' or 'It is shorter than my arm'. These are relative statements that give a range of possibilities for length but do not do an accurate job of describing the pencil. To be precise, the pencil needs to be compared to a unit of measurement. An informal or nonstandard unit such as a paperclip can be used to measure the pencil end to end, and the result of 7 paperclips long can be stated. Alternatively, a standard unit that is known and easily recognised can be used to measure the pencil with a result of 16 centimetres being found. In each case the amount of accuracy will depend on the situation and the choice of unit. The following considerations are important for all attributes.

Conceptual understanding develops over time. A variety of activities for each concept will be needed; 1 learning experience alone will be insufficient. The role of the teacher will be to provide many opportunities within any measurement activity — whether it is with nonstandard, historical or metric units — to

help children develop these 10 concepts. Children will learn these concepts as they proceed through the measurement process.

1. *A unit must remain constant*. Young children (Clements & Sarama 2007) might not understand that units need to be of equal size. They may measure a length using pencils of different sizes and report the result as 6 pencils or they might measure a pathway with cubes but not line them up without gaps. If they use different sized shapes to cover an area it would be incorrect to report that the area is 8 squares. Children might compare measurements made with different units such as different sized teddies (from the family of bears) when finding the mass of an apple and see no conflict. They could be viewing measurement as a process of counting rather than the process of assigning a value to an object or event.
2. *A measurement must include both a number and the unit*. Teachers frequently remind students to write the unit. The child engaged in an activity of measuring in metres might not see why it is necessary to record the unit. When children measure the same object with many different nonstandard units, they are more likely to see the need to report the unit. Giving children the task of measuring the length of a book with post it notes or coloured paper strips, erasers and cubes, or finding the mass of an object with iron washers, coins or counters will encourage them to write (or draw) the unit of measurement. In the classroom 17.7 provides an activity that encourages children to focus on the importance of reporting the unit in the result.

IN THE CLASSROOM 17.7

Measuring length with informal units

Objective: Measure lengths with different informal units and compare those measurements.

Year level: 1

Materials: Easily available classroom and real world objects such as pop-sticks, chopsticks, straws, candles, pens, books, boxes and other objects that can be used as informal units of measurement

Give each pair of children a different unit and the item or action to measure. For example their stride (from 1 heel to another when walking, their fathom (from fingertip to elbow) or their reach (from shoulder to fingertip).

Ask the children to record their measurements at a place everyone can see. Observe whether they record the number and the unit (if they do not, it will lead to the discussion below about the necessity of recording units).

Ask children to give a brief report about their findings. If there is an omission, ask them why stating their unit is necessary when measuring. If they haven't consistently used the same unit, ask them what happened with different units.

3. *Estimation is an important process to encourage children to use whenever measuring*. By estimating first, the student becomes more fully involved in the activity and they build a deeper understanding of common benchmarks as well as formal units. In making a best guess they will take real ownership in finding how close their real answer will be to their estimation. No right answer is needed, although a number and a reasonable unit is required. For example, if asking older students to estimate the size of the longest side of a business card, the answers of 'approximately 8 centimetres' and the 'width of my hand' would both be accepted as good estimates. All children need to be encouraged to build their estimation skills even if they have difficulty giving an estimate because of concern about an incorrect answer.
4. *Two measurements can be easily compared if the same unit is used*. Comparing the results of measurement with different units might clarify if there are misconceptions. Young children might be thinking about the numbers rather than the units to make comparisons. For example, if a pencil has a length of 2 blue strips and another pencil is 6 red strips long, some children will say that the pencil that measured as 6 strips is longer. They may not yet have reached the stage where they can coordinate the

number with a unit. Students with special needs will need more experiences of measuring the same object with different units to realise that different numbers may occur if different units are used.



When students realise that it is easy to compare sizes if the same unit is used, they will understand the need to convert to the same unit when comparing measurements such as 3.4 metres and 1459 millimetres or when comparing $\frac{4}{5}$ of a kilogram with $\frac{2}{3}$.

5. *One unit may be more appropriate than another to measure an object*. The size of the unit chosen depends on the size of the object and on the degree of accuracy desired. If children are encouraged to choose the unit of measurement rather than being told what unit to use, then the best unit to measure an object develops naturally. To encourage this concept, the teacher can ask questions to help children think about the task of measuring with an appropriate unit. For example, 'would you use a tablespoon, a jug or a bucket to measure how much water the fish tank will hold?' Similarly, 'would you cover your desk in playing cards or an eraser to find the area?' is a suitable question to consider.
6. *There is an inverse relationship between the number of units used and the size of the unit*. When measuring the same object with 3 different units, children realise that the larger the unit, the fewer are required. For example, you could ask each child to find the mass of an object with coins, washers and cubes. If each child makes a graph of the results, as depicted in figure 17.8, a pattern becomes apparent when all the graphs are compared. The cubes are lighter than coins and washers and it takes more cubes than washers to weigh each object. When rulers are introduced, children can use rulers made of different units (Clarkson et al. 2007) to reinforce this and other ideas.

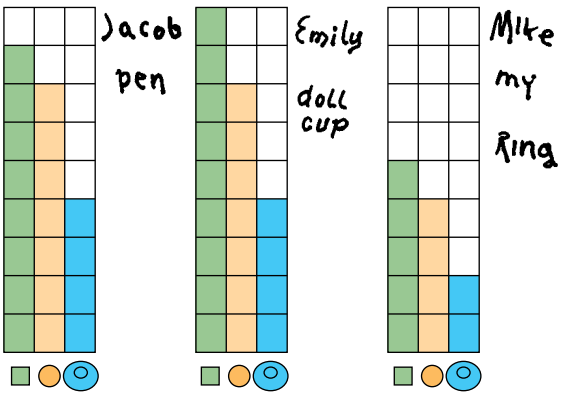


FIGURE 17.8 Children's graphs of cubes, coins and washers used to measure the mass of 3 objects

7. *Standard units are needed to communicate effectively*. Many concepts about units can be developed with nonstandard units. At the same time, children will be learning procedures for measuring with units (for example, how to line up units, use a balance, cover a region, accurately fill a cup or clap hands to count time). There will come a time when children are ready to learn about standard units. In Australia, and most places in the world, the metric system of standard units is used. The basic units are metre, litre and kilogram. In some everyday situations, imperial measures continue to be commonplace — for example, the mass of a newborn baby (in pounds), the height of an adult (feet and inches) and the measure of large areas of rural land (acres). Some adults working in certain occupations and people communicating their knowledge about these measures have not yet made conversions to metric.

Stories and activities that demonstrate the difficulty in communicating size when there is no standard of measurement are 1 way to present the necessity of a standard unit. The classic story retold in *How Big Is a Foot*? (see Myller, 1991 in the Children's literature connections) is an interesting and amusing resource to use in helping children see the necessity for a standard unit. Another enjoyable activity for children is making a recipe of powdered drink (like milo), using a very large cup for the water or milk and a very small spoon for the powder. Be sure to have extra powder available once children have tasted the drink!

8. *A smaller unit gives a more exact measurement*. The majority of measures, except those done on scientific equipment are approximate. When children measure real objects, they usually record approximate measurements without being fully aware of it. A practice of saying 'about 6 metres', 'more than 6 metres' or 'between 6 and 7 metres' helps develop this idea.

   To establish the need for a more precise measurement, 1 child can be given a strip of paper that is 27 cm in length and another child, sitting a distance away, is given a strip that is 29 centimetres long. By visual perception both strips look about the same. Giving each child a decimetre (the 10 stick in MAB base which has a value of 10 cm) they measure the paper. Both strips will be less than 3 decimetres. After each child reports the measurement, other children can be asked whether they thought 1 strip was longer (without comparing the strips directly). The difference is clear but the unit of decimetres won't give an accurate result. Each child can repeat the measuring process using centimetre cubes. Discuss with the class why a smaller unit was needed.
9. *Units may be combined or subdivided to make other units*. When students are given the experience of making their own measuring tools and instruments, they often combine units to make a larger unit. For example, children might say 'hands and fingers' to find the width of the book or 'big beads and small ones' to find the volume of a jug. It is worthwhile for them to experience subdividing or partitioning units to make smaller units. For the attribute of length, folding a strip of frieze tape to make fractional parts of a length unit, using fractional parts of square units when covering areas, or using measuring cups of different (but related) size to fill another container with sand or rice are all helpful tasks.
10. *Units must match the attribute that is being measured*. Length is measured with a length unit, area with a shape that covers the region without gap or overlap, and volume with a cubic unit that 'fills' the same space. Similarly a unit of mass needs to determine how heavy the object is, a unit of angle needs to measure amount of turn from a fixed point and a unit of time needs to measure duration of an event or record a specific moment. That is, we compare the object to be measured to a unit of the same type. Children who say the area of an object is so many centimetres have misunderstood the concept of surface. They have only learned to use 1 attribute of the shape or surface, which is the length of it. Possibly they have missed the concept of square units in the rush to describe the object or use formulae.

17.4 Measuring with informal units

LEARNING OUTCOME 17.4 Exploring the measurement of length, area, volume and capacity, mass, time, temperature, and angle with informal units.

The first 2 sections of the measurement process focused on identifying attributes and comparing 2 objects or events with respect to a specified attribute. Measuring with units is very similar to this. Instead of comparing 2 objects or events with each other, the child compares and measures the object or event with a unit to find out how many. In all attributes, measurement involves iteration of the unit to make the comparison. For visual attributes such as length, area, capacity, volume and angle, copies of the physical unit are repeated to 'fill the space'. When measuring area, for example, the size of the region is compared to a square unit to find how many square units cover the region. For physical attributes such as mass, temperature and time, the process can be done with informal units but might also require a measuring tool.

Children need to gain experience with nonstandard and informal units before progressing to standard units (the metric system). These units naturally arise from everyday experiences of measuring, including the use of parts of the body. Measuring with informal units is relatively easy but the results will vary, as we are not all the same size. The complications that arise will help children with the processes of problem solving and reasoning and build a deeper understanding of the language of measurement. Researching and using units of historical and cultural importance helps to enrich this content strand for all students.

The following informal units and others can be used to further explore each of the attributes:

- *length* — digits, inches, hands, hand spans, cubits, feet, paces, fathoms, paperclips, links, pencils, pop-sticks, straws, Cuisenaire rods, connecting blocks like Unifix, card strips
- *area* — handprint, footprint, stamps, stickers, envelopes, tiles, bread tags, playing cards, tangrams, triangles and quadrilaterals (all types), circles, coins, regular hexagons, grid paper, mosaic paper
- *capacity* — spoonfuls, cups, scoops, jugs, jars, bottles, jelly moulds, buckets, cans
- *volume* — cubes, matchboxes, wooden blocks, lunch boxes, cereal boxes, pyramids, marbles, large beads, table tennis balls, baskets
- *mass* — rocks, shells, buttons, fruit, metal washers, coins, play dough, marbles, golf balls, cubes, teddies and tools such as seesaws and pan balances
- *time* — years (as the passing of 4 seasons or the intervals between birthdays), days (as the passing of day and night or the interval between getting up/going to bed), months (as cyclical movements through moon phases) and minutes and seconds (perceived through imagination and counting tasks).

Length

Length is the first attribute that most children measure. The beginning activities should be with multiple copies of a nonstandard unit, which are placed end to end along the object being measured. Children need to gain understanding through experience by using equal units in a straight line with no gaps and no overlaps. From previous learning about direct comparisons of 2 objects, children should know when the lengths are the same or almost equal. They will also need to record the number and type of units and the approximation at the end (is it a half or a small/large part of the unit). Extension activities using curved or zigzag lines are important but more challenging. The teacher shouldn't expect too much precision with young children, but keep encouraging the development of the skill. Giving students problems to solve like 'How long is your desk in pop-sticks?' 'How far is the doorway in steps from your chair?' 'Who is the tallest in feet?' are all useful activities. Children particularly enjoy working in pairs to measure their body as described in the activity below:

My height: Cut a piece of ribbon the same length as your height. You might like to lie down, just the same as the Queen in the book *How Big is a Foot?* by Rolf Müller (1962). Alternatively, stand against a straight section of wall and ask a friend to mark your height. Cut a star, heart or other shape from cardboard and attach your name to the ribbon. Find the height of each member of your group using their hand spans (outstretched hand from thumb tip to fingertip) or cut out their foot as a template (like the King or apprentice). Who is the tallest by this informal measure? Who is the tallest for real? Are the answers the same? Why/why not?

All students, particularly those with special needs, enjoy measuring with units that connect, such as connecting cubes or links. These are useful informal units because they are easy to handle and line up without gaps or overlaps. Later, children can take 1 unit and move it along a line or object (an *iteration*), marking the beginning and each endpoint as they go. This more advanced skill is needed for proper use of a ruler when measuring objects longer than the ruler. It is a skill that should not be pushed too early. If children are unable to move the unit and mark the end, then postponing this skill until later is recommended.

Area

Area activities will involve children covering a surface or shape with other equal shapes. Initially some of the units will make neat tiling patterns (tessellations) with no gaps and overlaps but some units will

be more problematic. From previous learning about direct comparisons of 2 regions, children should know when the surface is almost covered. They will also need to record the number and type(s) of units used and make approximations where the shapes do not fit exactly on an edge or boundary. Extension activities can involve irregular shapes like hands, footprints, body outlines, puddles or ponds. Giving students problems to solve like 'How many envelopes covered your desk?', 'How many playing cards are equal to the surface of your laptop/book?', 'How many coins cover this dinner plate?' and 'How many coloured paper squares cover your shadow (body outline)?' are all useful activities and will give rise to discussion about units that do not fit together well. Children particularly enjoy working in pairs to measure their body as described in the activity below:

My handprint and my shadow: Trace around your hand (with fingers closed, up to your wrist) using a piece of grid paper with large squares (2 cm × 2 cm). Decide who has the biggest handprint. How many squares does your hand cover? What will you decide about part squares? Did the tallest/smallest person have the biggest/smallest handprint? With the help of your buddy, trace around yourself, that is, find your shadow. Estimate first, then using copies of your handprint find out approximately how many will fit into your shadow. Do you know that 100 of your handprints is equal to the amount of skin on your body? A person working in what kind of job might know this information?

Children can use tiles of different shapes which connect like Polydrons® to cover a region such as a large square, rectangle, triangle, pentagon or hexagon. They can investigate tiling patterns in their home, school or community space. Children can explore different arrangements of squares and rectangles with a particular number of square tiles like 12, 16, 24 and explore triangle tiles like 1, 4, 9, 16 and reason about their findings. Arrangements of areas are an important skill for understanding arrays and multiplication and will lead to developing an understanding for area formulae in later years.

Volume and capacity

Volume activities will involve children stacking and packing with objects such as cubes, beads, marbles and balls. Capacity activities will involve children pouring from containers and measures of fluidity like cups of water, scoops of sand and ladles of rice. Sometimes the container will not be able to be filled completely due to pockets of air. Learning to pack units such as cubes systematically will take time as children need to learn about layers and depth and further complications arise when containers do not work simply. From previous learning about direct comparisons of 2 objects or containers, children should know when the space is almost filled. They will also need to record the number and type of units used and any approximations. Extension activities can involve creating their own cylinders from paper or card and estimating then measuring volume. Children can use their own drink bottles and lunch boxes to measure volume and capacity with informal units. Giving students problems to solve like 'Who has the biggest rock by water displacement?, 'How many eggcups of seeds will fill this science beaker?' and 'How many bottles of soft drink would we need for a class party if everyone can have 2 glasses each?' are all useful activities which should promote real world thinking and applications. Children particularly enjoy working in pairs to measure themselves or their belongings:

How much can my hand hold? How big is my shoe? Create a pile of cubes and/or use a small container of beads to 'grab' a handful. How many cubes can you hold in 1 hand? How many beads can you grab with 1 hand? Make 3 attempts for the members of your group and discuss the similarities and/or differences. Did your attempts improve with practice? Is your right hand different to your left? What if you held your hand upright and placed cubes into it — would there be more? Repeat your experiment by filling your shoe with cubes or cups of beads. How will you know when it is filled enough?

Children can use Multilink® cubes that connect together, to make rectangular prisms of different dimensions. Children can explore for example what kind of buildings they can make using 24 cubes. Three-dimensional views also lead to an understanding of multiplication and the formulae for volume of a rectangular prism based on length, breadth and height.

Mass

Measuring mass with informal units will usually involve the use of an equal arm balance. Children need to gain understanding by adding units to the pan or bucket until the given object is equally balanced. The units should be of an equal mass, but if not, the activity will lead children to an interesting discovery. For example, if a child were counting the number of strawberries that balance the mass of an orange, the answer would vary a little compared to using a consistent unit such as centicubes. Children will need to record the number and type of units used to find the mass of an object. Extension activities using mystery objects are likely to be more challenging but worth doing. Giving students problems to solve based on real classroom objects like 'How can I find out how heavy my pencil case/ lunch box/ mobile phone is?' will be meaningful to them.

Over time children will be likely to make the connection that the volume of an object and mass of that object are proportional if it is made of the same substance. This understanding leads to the concept of density.

Time

Measuring time with informal units can be very enjoyable for young children. It is difficult to make the units of equal size but well worth having an attempt. For example, children can count numbers, count beads from 1 container to another, clap hands, do a regular exercise, write a name or sentence over and over, sing a song, recite a poem, read a story, listen to their heartbeat or other everyday tasks. Recording would include the number and type of units used for the duration of an event. A common example is when people count seconds informally, during a storm. The duration of time between a flash of lightning and the sound of thunder can tell us approximately the distance away the storm might be to our current position. In school we were often encouraged to use the word Mississippi between counts. Questions could be asked such as 'How many swings could you do while I ran around the oval?' 'Time how long it takes you to walk/ride home from school compared to the number of vehicles driving past'.

Temperature

It is extremely difficult to measure temperature with informal units. There appears to be very few teacher resources or recommendations for this stage between comparison and measuring with standard units. Bobis et al. (2012) record a classroom activity where children created their own thermometers with markings to describe differences in the temperature of water.

Angle

Angle activities often jump from comparing and classifying, to naming and drawing angles. Units can be developed for measuring an angle informally without a protractor. For example, an informal unit can be made by folding a circle into equal parts (fractions), which are smaller and smaller until an eighth or sixteenth is developed. Millsaps (2012) reports on a lesson with US Grade 5 students where they are exploring wedges as a unit of measure for angles. In the classroom 17.8 is adapted from a lesson developed for teacher education students connecting possible units for angles with time and sectors of circles. Recording could include such as 'The big angle is equal to 5 sectors, the medium angle is the same as 3 sectors and the small angle is 1 sector'. Given time for reflection and follow up a whole 'protractor' can be created based on sectors or wedges.

IN THE CLASSROOM 17.8

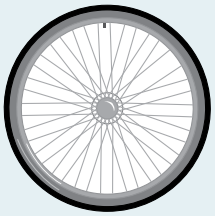
Measuring angle with informal units

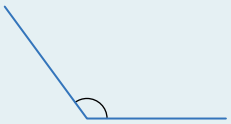
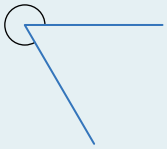
Objective: Measure angles with informal units based on wheels, clocks and circles.

Year level: 4

Materials: Circular objects, cut-out circle paper, 5 different colours for each student

Keep 1 circle to represent a full turn, like the wheel below or the full rotation of the minute hand on a clock, e.g. from 5 o'clock until the next hour which is 6 o'clock; fold 1 circle in half to represent a half turn, like half past on a clock, or North to South on a compass; fold the next circle twice to represent a quarter turn, just like on a clock showing the 3 and 9 positions (past and to the hour) or an apple cut into quarters.



Ask the children to experiment with the 2 remaining circles to make wedges of smaller size. Discuss in pairs and small groups, encouraging children to justify their ideas.

17.5 Measuring with formal units

LEARNING OUTCOME 17.5 Exploring the measurement of length, area, volume and capacity, time, temperature and angle with formal units, and using instruments and tools for standard measurement.

If children have been measuring with nonstandard units, then using standard units will naturally follow. The process of measuring builds on children's understanding of each attribute, comparing and measuring with informal units; therefore the purpose for the next stage will be to gain a feel for standard units.

Table 17.2 illustrates the metric units for each attribute. The Australian Curriculum (v 8.1 2016) can be used as a guide to the appropriate year level to introduce these units. Quality teacher resources and inquiry-based activities where students are actively involved will help develop measurement understanding and proficiency. The main prefixes kilo, meaning one thousand; centi, meaning one hundredth part; and milli, meaning one thousandth part are important for metres, litres and grams and children need to know them early when working with the metric system. Units of time can help children remember 2 of these prefixes. For example, 1 year is one hundredth part of a century and 1 year is a thousandth part of a millennium, Additionally, a decade (for the prefix deci) is made up of 10 years.

TABLE 17.2 **Standard units in the metric system**

Attribute	Standard unit	Additional relationships	Common everyday objects and benchmarks
Length	metre (m)	millimetre (1 m = 1000 mm) centimetre (1 m = 100 cm) kilometre (1000 m = 1 km) centimetre (1 cm = 10 mm)	1 cm is the width of the nail on the small finger. 10 cm is the width across the hand. 20 cm is approx. a handspan. 1 m is the distance from shoulder to opposite finger of outstretched arm. 2 m is the height of a doorway. 100 m is a short running track.
Area	square metre (m^2)	hectare (ha) A square 100 m × 100 m = 1 ha (10 000 m^2 = 1 ha) or equivalent shape	6 cm^2 area of a postage stamp 40 cm^2 area of a business card 100 cm^2 area of a mobile phone 1 m^2 area of a shower base 1000 m^2 a good sized house block 1 hectare approx. 10 suburban house blocks

(continued)

TABLE 17.2 *(continued)*

Attribute	Standard unit	Additional relationships	Common everyday objects and benchmarks
Volume	cubic metre (m^3)	100 cm x 100 cm x 100 cm = 1000 000 cm^3 = 1 m^3	1 cubic metre of water has a mass of 1 tonne and fills a space 1 metre by 1 metre by 1 metre or equivalent.
Capacity	litre (L)	millilitre (1 L = 1000 mL) kilolitre (1000 mL = 1 kL) mega litre (1 ML = 1000 kL)	1 metric cup holds 250 mL. 1 level tablespoon holds 20 mL. 1 level teaspoon holds 5 mL. A wine bottle holds 750 mL normally. A large bucket holds 10 L.
Mass	kilogram (kg)	gram (1 kg = 1000 g) milligram (1 g = 1000 mg) tonne (1000 kg = 1 t)	5 g = 10c coin 60 g = large egg 500 g = 4 medium apples 1 kg = 1 litre of water
Time	second (s)	minute (60 seconds = 1 minute) hour (60 minutes = 1 hour)	Counting at a steady rate to 60 Time to eat an enjoyable lunch From dawn to dusk is approximately 12 hours. Seven days (7 sunrises) is a week.
Temperature	kelvin (K)	degrees Celsius, °C freezing point = 0 °C and 273.15 K boiling point = 100 °C and 373.15 K	25° = temperature of an air conditioned building or spring day 37° = normal body temperature 70° = cup of tea with milk 180° = a moderate oven
Angle	radian (rad)	degrees (°) (1 rad = 180°/π) 360° = 1 revolution 180° = straight line 90° = right angle	Angles in a triangle add to 180°. Angles in a quadrilateral add to 360°. Corner of a picture frame is made of two 45° angles (mitre).

Length

In the beginning, activities are based on a decimetre (dm) because it is similar in length to a base-ten 10 stick (or long) or a pop-stick and can be easily seen and handled by young children as a transition unit. Initially a metre is too long and a centimetre is too small. The language 'ten centimetre' can be modelled and recording in symbols as 10 cm initially if the word decimetre seems too complex. The examples that follow are a good foundation for children in Year 1 or 2. Give each child a paper strip that is 10 centimetres long by approximately 2 centimetres wide. Strips can be easily cut from card, frieze tape or otherwise wooden 10-rods from Cuisenaire or MAB_{10} materials can be used. Here are some simple ways to have children compare, estimate and construct a mental image of a decimetre.

- *Decimetre list-up*. (Comparing 10 cm) Pose the problem for children to find something that is smaller than, the same length as, and bigger than their 10 cm strip or stick. Encourage children to record their findings in a table and share their results on the whiteboard. Ask each child to find a decimetre length on his or her hand. For example, 'from the base of my thumb across my palm diagonally to the base of my pinky finger is exactly 10 centimetres', which is a handy measuring tool.
- *Decimetre hold-up*. (Estimating 10 cm) Pair the children and ask each child to hold up their pointer finger vertically (to the sky, with thumbs across and overlapping) so that there is a distance of approximately 10 cm between them. Ask their buddy to measure with the 10 cm strip or stick. Repeat for 2 more positions, horizontally and obliquely. Their buddy checks each time; then the roles are reversed.

- *Decimetre stack-up*. (Constructing 10 cm) Set up learning centres with coins, counters, tiles, cubes, beans and more. At each centre, children stack the counters to make a tower 10 cm high or a line of counters 10 cm long.

After children are somewhat familiar with the length of a decimetre, they can begin to measure with decimetres. A good way to begin is by asking them to estimate the lengths of objects and then measure. As children are measuring with decimetre strips or rods, they should notice that putting down strip after strip is not the easiest way to measure. Encouraging them to create a new longer strip, by taping 3 strips together, end to end will provide them with a measuring tool equal to 3 decimetres or 30 centimetres. If they would like further precision, then they can make a 'centimetre ruler,' from sticky notes of different colours, 1 centimetre wide, placed side by side and taped on. When they become a little tired of counting them, the suggestion might arise about numbering the strips. The product of their effort might be similar to figure 17.9. This activity helps children understand how rulers are made and that they are counting repeated units. Children will often place the number in the middle of the unit but encouraging them to record the number at the end of each strip, to signify the distance of each centimetre, the same as on a ruler or number line which is more mathematically correct.



FIGURE 17.9 A homemade ruler of coloured strips, numbers not yet placed correctly

Once children are familiar with the decimetre measurement of objects, the next stage is measuring something very long. Each child can place together 10 decimetre (10 cm) strips end to end along a piece of dowel (pre-cut) to create a new tool. The teacher can question the children to find out if anyone knows about this unit, or if they have used something this long before when measuring. If no-one is forthcoming with ideas, explain that it is called a *metre* and is used to measure longer distances. Activities for comparing, estimating and constructing a metre will help children become familiar with the unit. Estimating and drawing a straight line, a squiggly line, a zigzag line of one metre will help children gain a visual image. When ready children will delight in using a trundle wheel to measure longer distances, particularly outside. Combining or subdividing units that children have already used will help them understand each new unit and assist them in making conversions from one unit to another. It is important that not too many standard units are introduced at one time, that the unit is not too small or too large for a child of that age to handle, and that the numbers generated are not too large.

Finding length with a ruler involves recognising the number of units (decimetres, centimetres or millimetres) represented. Children need to be aware of what unit they are using and line up the ruler properly with the object. Children will focus best on the unit being used if the scale on the ruler has only that unit. For example, if the unit is centimetres, choose a ruler marked only in centimetres, not in centimetres and millimetres. If the ruler is a metre in length, then it should only show centimetres.

It is important that children measure real objects with the ruler. When planning activities, the teacher needs to include activities in which children measure objects longer than one ruler. Can children move the ruler (iterate) and do they have the addition skills to add the units? For example, suppose the children have a 25 or 30 cm ruler, how are they going to measure something that is 43 cm long? Encourage them to place a marker and then add on from 25 or 30 centimetres.

Number concepts are closely linked to measurement. Allowing students to use techniques that rely on their counting, place value and addition background (e.g. 25, 35, 36, 37… 43 or 25 + 10 + 8 more is 43) will help develop both concepts. Although they may not yet be proficient at addition, encouraging them to solve authentic problems might lead to surprising results. The role of the teacher in measurement activities is to ask questions and scaffold their learning.

Fraction concepts will naturally occur, as measuring will give rise to parts of a whole unit. Building confidence and understanding (making friends with fractions) can be developed by using simple

benchmarks like a half or quarter. The ability to measure to the nearest unit can be supported by estimation and rounding up or down to the nearest whole or half unit in most cases.

A common misconception when using rulers is the starting point, which is often a short distance in from the left of the ruler rather than at the very edge. Helping children recognise that the start of the measurement aligns with zero and then counting or reading off the numbers from that point is worthwhile. Children learn to count numbers beginning from 1, so sometimes they begin measuring at 1 rather than 0. Activities like making a ruler and seeing that the 1 is placed at the end of the first unit are helpful. If children are having difficulty, return to directly comparing the object with the ruler using a baseline as described previously in the section on direct comparison.

It is true to say that children enjoy learning to use a ruler and learning to create their own line segments. Planning for learning should include a variety of activities, both measuring objects and constructing line segments, shapes and objects of a given length. When children have difficulty drawing a line segment of a given length then it may stem from inexperience with creating lines. This is particularly important for later work in area and volume and for geometry activities with 2-dimensional shapes and 3-dimensional objects.

Finally, students need to gain experience with measuring objects that are not straight, as well as measuring around the perimeter of shapes. In the first instance, a piece of string can be accurately cut to measure a squiggly line or circular object, so it can then be measured on a ruler. Separate sections of a shape can be measured and added together to gain experience with perimeter.

The teacher can encourage children to measure real objects in a variety of interesting ways. Indoor and outdoor measurement activities like those that follow, will help create authentic situations of value to students. Children might also bring to class 'measurement problems' they have encountered outside of school. For example older students might be measuring the length of furniture in their bedroom to see what arrangement is best for their bed, desk, chest of drawers or other large objects so there is also access to their window, door or built in.

Outdoors: Choose 5 different leaves. Estimate and find their length, width, and perimeter.

Is there a relationship between any of these measures? What can you say about your leaves?

Indoors: Estimate then find the length (in centimetres and metres) of:

- a packet of spaghetti, every piece
- a packet of large paper clips, end to end
- a roll of toilet paper.

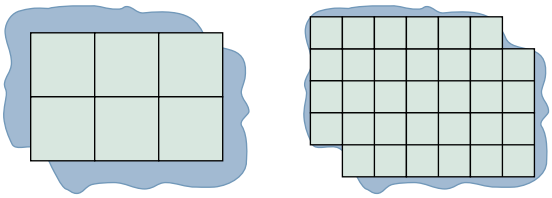
What other information can you find about these items? Why is the total length a useful measure in each case?

Area

The starting place for activities involving standard units of area is with the use of cm^2 and m^2. Finding area in Years 3 and 4 is based on a square 10 cm × 10 cm made of paper, card or wood (base-ten 10 flat) and smaller square tiles 2 cm × 2 cm which can be sourced easily from kits such as pattern blocks, flip tiles or square counters. Students are given regular and irregular regions to cover with their large squares (100 cm^2) and their small squares (4 cm^2). For a regular shape such as a square or rectangular object e.g. book, laptop computer, table, mat or other everyday item, students cover it and find how many 'squares' there are. They will need to describe the process and the result in quantity and units. For example, 'my book was covered by 6 large squares (600 cm^2) but it was a bit more than 2 large squares across and not quite 3 large squares down'. Later, 'I used the small square tiles, and there were about 10 in each row and 14 rows so there were 140 small squares, which was a lot to work out!'

To find the area of an irregular region, students can count the number of whole units. Covering many different shapes helps students see the need to approximate and to use smaller units. For example, the following region was first covered by 1 size of squares and then by smaller squares in order to more

adequately report its area. Here, a decision was made not to use part squares. If the large squares are the same as before, the first diagram represents an area measurement of approximately 600 cm^2 (6 10 cm × 10 cm squares were used). The second diagram shows a better approximation of 825 cm^2 (if 5 cm × 5 cm squares are used). There is much less wasted space in the second example.



Once students are familiar with using concrete materials i.e. squares of different sizes (large 100 cm^2, medium 25 cm^2 and small 4 cm^2) they can consider covering shapes with plastic overlay of large and small squares of standard size. Children will also be ready to work with larger areas in their classroom and use/create square metres of dowel with joiners or newspaper.

Outdoors: Choose a solid section of brick wall. Estimate the number of bricks per square metre, for the whole wall.

What size is the wall? Why is this useful information?

Indoor: Measure the area of your table using MAB_{10} flats. Is it more or less than you expected?

Estimate first the number of square metres you think would measure the classroom floor space. Now using construction paper, measure the area of the classroom floor using as many square metre papers, as you need. Why is this a difficult task?

Discuss how you could overcome the problems. Which method is the quickest? Which method is the most accurate/why?

Volume and capacity

Using a litre jug as the standard unit, children can measure cups of water to find how many fill 1 litre (a cup should be $\frac{1}{4}$ litre), compare a litre jug to their water bottles, compare a litre to larger drink bottles and find how many litres are in a bucket, fish tank and sink. Please do not waste water in these investigations but recycle the water into the garden. Smaller objects like 100 millilitre tubs (from yoghurt or snacks) can be used to help students understand $\frac{1}{10}$th of a litre. Larger standard measurements like the kilolitre would be difficult to recreate but students could discuss the amount of water their rainwater tank stores. To model the size of a kilolitre, a cubic metre, made from 12 sticks a metre each with 3 part joiners will help give a visual image. This model also applies to volume and students can do meaningful estimations and investigations based on 'How many children fit in a cubic metre?' and 'How many cubic metres fill our classroom?'

IN THE CLASSROOM 17.9

Make your own measuring bucket

Objective: Make a measuring container based on litres.

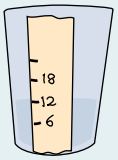
Year level: 3–4

Materials: Litre jug, big bucket slightly transparent, masking tape, water and felt-tip pen

Directions

1. Put a piece of tape on the side of the bucket.
2. Fill the litre jug with water and pour it into the bucket. Mark the level of the water and then write 1 litre on the tape.

3. Fill the litre jug again. Empty it into the bucket and mark with the number 2.
4. Continue until the markings are nearly at the top.



Now that you have your own measure, use it to see how much other containers hold.

Question
What kind of scale works best? Why?

To measure volume with standard units in Years 3 and 4 students can use 10 cm × 10 cm × 10 cm cubes (base ten block) worth 1000 cm^3 and individual centicubes. They can fill regular and irregular objects like shoeboxes, shopping bags, backpacks, small garbage bins, laundry baskets and cartons to find their volume. They will need to describe the process and the result in quantity and units. They can use a fixed number of decimetre cubes to create rectangular prisms (buildings) of different types. For example a 2 × 3 × 4 arrangement will yield 24 dm^3 but so will a 6 × 4 × 1 arrangement, an 8 × 3 × 1, a 12 × 2 × 1, and a 2 × 2 × 6 rectangular prism. If each object were an empty container then they would all be filled with 24 litres of water. Create real world measuring problems for the attributes of volume and capacity for children to solve. Plan in advance the materials and methods to be used in the investigation.

Outdoors: Visit the creek and determine the volume of water, in litres, that flows past a given point in one minute.

Option 2: If you were Noah, how would your ark (300 cubits long, 50 cubits wide, 30 cubits high) compare with the size of this classroom, the biggest building in the school or the Adelaide Oval complex [insert local context] or something else? Make sure you convert the units into metres but you might need to find the average size of a cubit for your group before measuring using metres.

Indoor: How many cups of centicubes would fill your shoe? Consider the different shoe styles people are wearing and decide on something similar. Record your results for each group member. Is this an indication of who has the biggest feet? Why/why not?

Mass: Using the kilogram mass as the standard unit children need experience feeling the heaviness of objects equal to a kilogram, or part thereof. Items will vary a great deal in size. Students can hold/heft bags containing stones, sand, flour, packing foam or plastic, nails, books and other items. Cooking activities where children measure standard amounts like $\frac{1}{4}$ kilogram, $\frac{1}{2}$ kilogram, 100 grams are very real and important. Older students can do comparative shopping activities to calculate the cost of items on a standard 100 grams. Going beyond the equal arm balance students can use spring balances, kitchen scales, and bathroom scales to help them estimate and measure objects and understand the force of gravity working on objects in different ways. Encourage children to estimate, find out and compare the mass of their pets, their siblings and their computer (but not themselves due to confidentiality). When students are confident with the kilogram they can research objects that have a mass of 1 gram (0.001 kg) and 1 tonne (1000 kg).

Time

Learning to measure with standard units of time is a complex process that will be met by children from Year 1 to Year 5. Initially students will become familiar with duration of events and the minute, second and hour. Later they learn to read moments in time from an analogue and a digital clock. The passing of minutes on either clock is the first time that students will meet base 60. That is 60 seconds equal 1 minute and 60 minutes equal 1 hour. Working with a minute as the standard unit is a good place to start. Activities such as a quiet minute, a busy minute, a counting minute and an active minute are well

worth doing with children so that they can gain the sense of duration of an event lasting a minute. They can also watch a minute using a sand timer.

Indoor/outdoor: Estimate a quiet minute by closing your eyes and thinking of nothing or quietly colouring a picture. Look up when you think that 1 minute is over. Ask your buddy (in pairs) or someone in your small group to time you. Alternatively the teacher can have a timer counting on the IWB for all to see but don't make a distraction when you think the time is up!

- Estimate a busy minute by writing something mundane over and over. Again look up and record the time it was.
- Estimate a counting minute by putting beads or counters from 1 container to another. Stop when you are done and record.
- Estimate an active minute. Erin chose star jumps, but you can do anything the class agrees upon. Again record the time.

In your small group or as a whole class record the minimum, maximum and average time for each activity. What did you find out? During which activity were you closest to one minute? Why do you think this was so? What do you think a child understands when an adult, friend, teacher, parent says to them 'Just a minute!'. Is there any truth in the saying that time flies when you're having fun?

Everyday events can be timed and shown on the classroom clock(s) to help build familiarity with minutes. The teacher can reinforce the time between events by making connections with the movement of the clock. Language such as 'Now it is time to have our 10-minute break, looking at the clock, it is 10:20 am and you will need to come back by 10:30 am (half past ten)'.

The passing of seconds can be understood by using counting rhythms, observing the movement of the second hand on a watch and by looking at a digital display as it changes. In relation to a minute, seconds pass very quickly. The passing of an hour is more difficult but can be learned through classroom routines such as using an alarm to 'bing' at every hour, making special mention of the passing of an hour on the clock, or time taken when doing an activity 'art is our next lesson, you will have an hour to do a painting'. In relation to a minute, hours pass very slowly. The recognition of an hour or half hour on a clock is 1 of the first aspects of time that children will learn, before or during Year 1. Regular access to clocks with movable hands and digital time devices are beneficial. Encouraging parents and/or caregivers to help with the teaching of time is useful.

Reading time on an analogue clock is a complex process and children need a great deal of practice to develop the skills required. The developmental sequence should not be rushed! Some adults are unable to tell the time on an analogue clock, which can be both embarrassing and inconvenient for them. At some stage in their learning, the different aspects of telling the time using an analogue clock either became too confusing to them or there were misconceptions with the clock that were not resolved between Year 1 and 5 at school or at home. Using digital representation of time without making connections to the traditional clock might have been easier.

There are many ways to read the scale on a clock, which is very different to a linear measurement like a ruler. The hour, minute and second hands all move in a circular and repeated fashion. The clock uses fraction parts (half, quarter and twelfths) and these are done in 2 ways. Children have different readiness when it comes to learning how to tell time. The teacher will often notice a wide range of ability within a class due to different experiences outside of school. The following list of skills associated with telling the time are developed by children in a variety of ways, that is, order will vary but they are all important for gaining understanding.

- Identify the hour hand as the small hand and the direction in which it moves (clockwise).
- Identify the minute hand as the big, long hand and the direction it moves.
- Orally tell time by the hour by noticing that the minute hand is on the 12 (pointing directly up) and the hour hand is on a number 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 or 12. It is helpful to demonstrate the movement of the minute hand of an analogue clock through 1 revolution to show the way the clock moves through time for 1 hour.

- Orally tell time by the half hour by noticing that the minute hand is on the 6 (pointing directly down) and the hour hand is halfway between 2 numbers. It is helpful to demonstrate half a revolution of the minute hand.
- Orally tell time by the quarter hour by noticing that the minute hand is on the 3 (pointing directly right) for quarter past the hour and the minute hand is on the 9 (pointing directly left) for $\frac{3}{4}$ past the hour. This is commonly said as quarter to the next hour. It is helpful to demonstrate a quarter turn and a three quarter turn with the minute hand.
- Connect analogue clock and digital time for each number by skip counting in 5 around the clock. Each number on the clock also represents an interval of 5 minutes in the hour. Counting 5, 10, 15, 20, 25, 30, 40, 45, 50, 55, 60.
- Identify that a time 'after' is shown in the first half of the clock, using the language 'quarter past' and 'half past' the hour for times of 15 and 30 minutes respectively (for example 4:15 or quarter past 4 o'clock).
- Identify that a time 'before' is shown in the second half of the clock, using the language 'quarter to' the hour for 45 minutes or 15 minutes before the hour (e.g. 9:45 or quarter to 10 o'clock).
- Count by 5s up to 30 for numbers 1, 2, 3, 4, 5 and 6 to tell time after the hour (e.g. 20 past 8 o'clock or 8:20).
- Count by 5s from 25 backwards to 5 for numbers 7, 8, 9, 10 and 11 to tell time before the hour (e.g. 25 to 6).
- Count on by 1s adding on from multiples of 5 to tell time to the nearest minute (e.g. 25, 26, 27 minutes past 3).
- Recognise that time cycles through the clock twice in a day and night. The starting places are 12 o'clock (midnight) which is called a.m. or occurring before noon and 12 o'clock (midday) which is p.m. or after noon.
- Match the time on a digital clock to an analogue clock.
- Write the time in digital notation for 12 and 24 hour time (for example 4:20 am and 16:20 pm).

Before using the clock to tell the time, children need abundant and varied experience with sequencing events that show a clock face and different times in the day and night. The beautifully illustrated book *The Bad Tempered Ladybird*, by Eric Carle (1977) includes times from 5 o'clock in the morning until 6 o'clock in the evening plus language of time such as night, day, moon, sun came up, morning, breakfast, noon, dinner and specific times of 5:15, 5:30 and $\frac{1}{4}$ to 6 pm included. Clock faces are included on most pages. The book also includes concepts of size.

By matching pictures in a story to sequential events in a day, children can understand the passing of hours and match what happens in the book to time on a clock. In Year 2 or 3 it is developmentally appropriate for children to make their own clocks.

The teacher can begin problem solving with young children if they have clocks or watches. Activities relevant to physical education can be included to help children measure time. For example: time yourself doing various activities: walking, jumping, running, hopping, skipping and sitting. Take your resting pulse and exercise pulse and compare them for each member of the group. There are a large number of 'worksheets' available in print and on the internet that require children to answer questions about time however caution needs to be taken with how many of these (if any) are given to children. It is better for the teacher to come up with their own recording sheets based on interesting and relevant questions for their own group of children. Questions such as 'What time will it be in 2 hours?' can be asked. Children can submit a relevant question to the teacher for their peers to answer. As children become more familiar with the clock, they can be give more challenging and more abstract questions such as the ones in figure 17.10.

Although reading digital clocks is considered easier than reading analogue clocks, solving the problems in figure 17.10 are more difficult with a digital clock. In learning to read a regular analogue clock, a child learns the relationship of the minutes and hours and has a model to use in solving such problems. If all clocks were digital, teachers would focus less on how to read time but more on how our 24-hour time

system works. As students move through school they need help to learn to read timetables, schedules, timelines and understand time zones around the world. They will also need to learn about converting units of time for seconds, minutes, hours, days, weeks, months, and years. All of these tasks will require quality number skills including algorithms for addition, subtraction, multiplication and division.



FIGURE 17.10 Time problems to practise reading clocks

To integrate mathematics and history, an interdisciplinary unit on the history of time, clocks and calendars could prove to be interesting to older students. Students might enjoy seeing how the measurement of time has evolved through the years. A good resource (Branley 1993) for these activities is described in the Children's literature connections, and includes making sand clocks, sundials or other historic instruments. Further investigations can include world time zones and Greenwich Mean Time. What does *daylight saving* mean?

Temperature

The standard unit for measuring temperature is degrees Celsius (°C) but scientific measurements are based on Kelvin (K). Everyday experiences to do with daily temperatures: minimum and maximum for different seasons of the year and different countries of the world will help to build an understanding of the standard units for temperature. Similarly, experiences and discussion involving real world activities such as cooking (oven temperatures), refrigeration and heating (food temperatures), measuring cool water, tepid water, hot water, and ice with thermometers are all worthwhile. Probably boiling water is best omitted due to health and safety issues. Helping children to understand room temperature, normal body temperature and fever measurements through the sharing of experiences will make mathematics interesting and meaningful to them. For example, raising consciousness about why a baby, child or small pet should never be left in a car during hot weather is a valuable discussion.

Thermometers can be both analogue and digital in nature. The first type has a graduated scale that needs to be read and understood; the second type gives a number value.

Angle

Degrees are used to measure 2-dimensional shapes and 3-dimensional objects. Degrees are also used to measure longitude and latitude which describe positions on Earth. Children need to gain experience with the size of different angles for 0° through to 360° which is a full rotation. It is interesting to note that

time and angle, which both use base 60, are historical measures which remain with us today. The degree is a very small unit so it is easiest for children to begin work with angles using common benchmarks as discussed previously.

The *protractor/geoliner* is often a difficult instrument for students to learn in that it commonly has 2 scales on it. The example shown in figure 17.11 is a half circle protractor which has the angle measures from 0° to 180° marked on it with the right angle 90° shown as the centre of both scales. Children need to understand angles as an amount of turn and they need to decide which direction they are measuring from to use the tool correctly. The fixed arm of the angle is aligned with zero and the central point of the angle is placed at the centre.

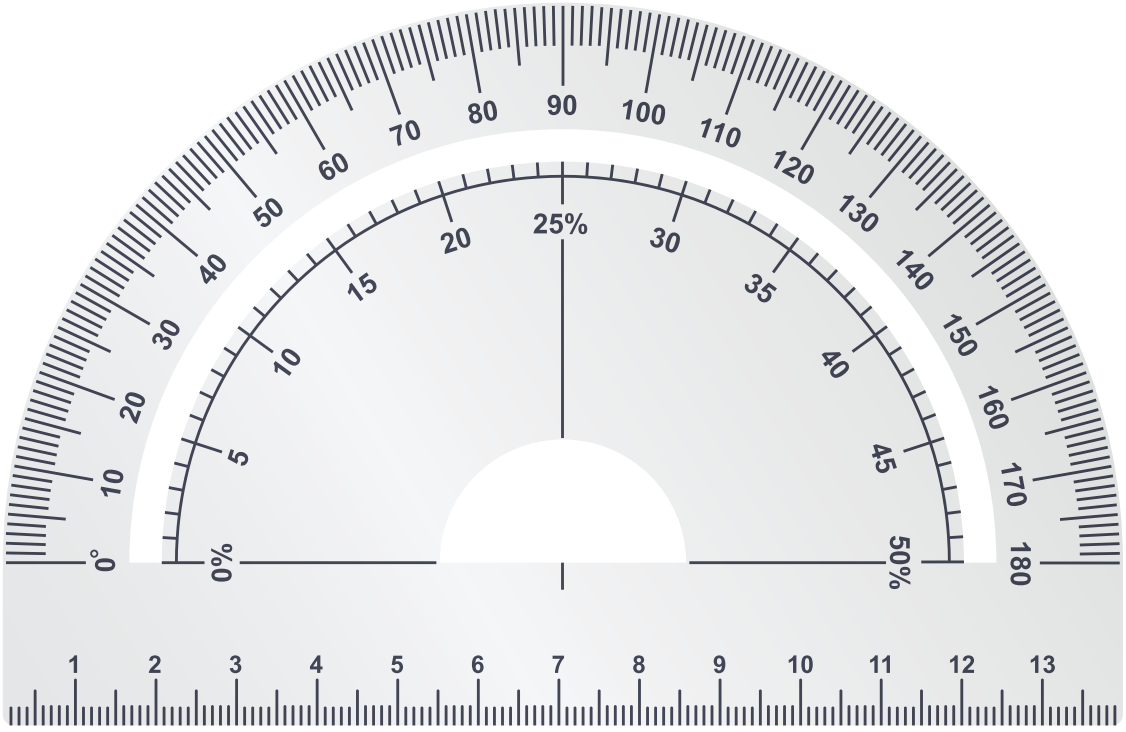


FIGURE 17.11 The angle measuring tool, protractor

Young children who have been asked to compare angles, especially to the benchmark of a right angle, have less trouble with the protractor because they already know whether the angle is greater or less than 90°. Estimating sizes of angles, for example it is about half of a right angle, or 45°; it is about a 5-minute turn, or 30°, will assist students when making sense of the protractor. The geoliner is similar but the tool is created within a right-angled triangle and 1 vertex is 90° and 2 vertices are 45° each.

Scaled instruments

The most common instruments children meet in primary school: rulers, scales, graduated containers, thermometers, protractors and clocks, have been met in the previous section. Instruments such as bathroom scales, kitchen scales and thermometers are more easily read if they are digital or if each unit is clearly marked. For cylinders and cups or jugs, students have been known to have difficulty with markings representing intervals of unusual size. As number lines become more prevalent in primary schools, it is helpful to work with containers that are marked in units such as 1, 2, 5, 10, 20, 50 and 100. One way

to help children become more aware of the markings on a scale is to have them make their own instruments. Some attributes (such as area and volume) are assigned a measurement by the use of a formula after an instrument has been used to measure essential dimensions.

Much of the emphasis in the upper primary curriculum (Years 6 and 7) is on instruments and formulae, and some children encounter difficulty with both. The most feasible source of difficulty in the upper primary years is where the student does not understand what they are measuring and what it means to measure with formal units. Hence, the measuring process discussed throughout this chapter is developmentally important. The activities and suggestions already presented in this chapter build children's understanding of all the key attributes: length, area, volume and capacity, mass, time, temperature and angle. Later, other attributes are derived from more than one attribute (for example, speed is derived from distance and time).

17.6 Applications including formulae

LEARNING OUTCOME 17.6 Developing formulae for perimeter, area and volume measurement.

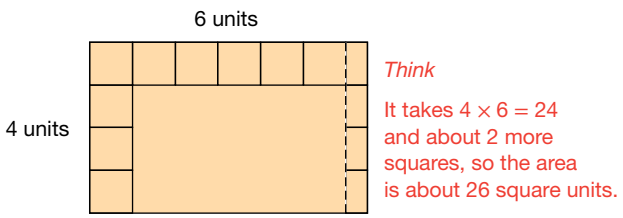
Formulae for perimeter, area, volume and surface area are discovered and introduced in the upper primary years. Although formulae are necessary in many measurement situations, they should not take the place of careful development of measurement attributes and the measuring process. The skill of using formulae should be developed once students are confident about measuring with standard units. Formulae should not be introduced too early, at the expense of helping students build meaning for the formulae themselves.

Rectangles

The formula for the perimeter of a rectangle is derived by the sum of the lengths of each side. Because 2 of the sides are equal in length, the formulae $P = 2(l + w)$ where l = length and w = width is commonly learned. The order of doing the perimeter will not affect the answer, that is, 2 different sides can be added and doubled or 2 lengths can be added to 2 widths. Squares which are a special kind of rectangle with all 4 equal sides have the formula $P = 4 \times length$. Using grid paper with tiles (or cardboard cutouts) of squares and rectangles is an effective way to develop the formulae for area.

The area of a rectangle is often the first 2-dimensional formula children encounter. Use of the rectangle can be developed easily, building on models that children may have used for multiplication. Figure 17.12 shows a sequence that can be used to develop learning activities that lead to the formula for the area of a rectangle, $A = b \times a$ (where A is area, b is base and a is altitude). This form of the formula generalises to other polygons better than the one using length and width $A = l \times w$.

When children measure with square units, the base and altitude may not be an exact number of units. They can begin by estimating how many squares it would take to cover the shape. Later, the idea of using smaller units or fractional/decimal parts of the unit can be developed.



The teacher needs to be aware of the steps as listed in figure 17.12 but should not teach them. It is given as an outline to help sequence experiences and assess students' understandings. Although some of the steps could be combined into a single activity, step 3 should not be done until children have the experiences listed in step 1 or 2. Physically covering a shape with smaller units is an important stage that cannot be rushed. Problems may arise when children are doing step 7 if they have not had prior

experience in covering real objects. The time children spend thinking mathematically through each step of the process will lead to a deeper understanding and a better retention of the formula for area.

Developing the formula for the area of a rectangle

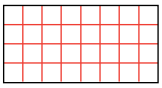
Prerequisites:
- Identifies rectangle
- Compares areas directly
- Assigns a measurement by covering with units
- Models multiplication as an array

Step 1: Covering with nonstandard units



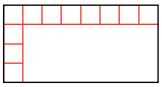
- Reviews covering of a rectangle with units
- Develops finding the area by multiplying the number of rows by the number in each row

Step 2: Covering with standard units



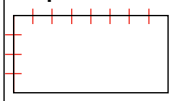
- Uses standard square units such as square centimetres
- Continues to find area by multiplying the number of rows by the number in each row

Step 3: Shortcut to covering



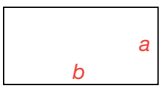
- Develops a shortcut to covering the entire rectangle by showing that it is only necessary to see how many rows and how many in each row

Step 4: Shorter shortcut



- Marks how many squares could fit across and down
- Continues to multiply to find the area

Step 5: Identifying base and altitude



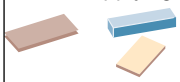
- Identifies base and altitude of rectangles (begins with cut-outs of rectangles and measures their bases and altitudes)

Step 6: Formula for area



- Measures base and altitude
- Notes the number of squares across the base and the number of squares for the height (or altitude)
- Multiplies the base by the altitude (number of rows and the number in each row) to give the number of square units
- Uses the formula $A = a \times b$

Step 7: Applying formula to real objects



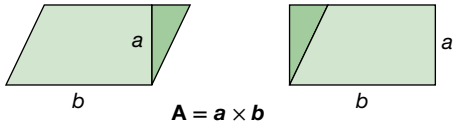
- Practises finding the areas of regions and of real objects

FIGURE 17.12 One sequence for developing the area formula for a rectangle

Misconceptions can occur with the area of a rectangle if children have learned, by rote, that the area is the length times the width (or base times altitude) but not seen or experienced where these ideas have come from, i.e. the underlying concepts. Therefore, when they are asked to find the area of a square, which is actually simpler than a rectangle, they run into difficulty. Rote learning might lead to the misconception that there is no side longer than the others. This section has concentrated on the area formula for rectangles because it can be used to develop the formulae for other common shapes. Visually there is a very real connection between the shapes.

Parallelogram

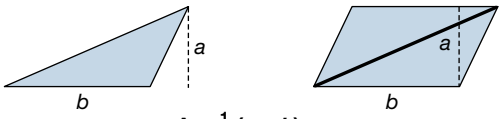
Every parallelogram can be transformed into a rectangle, by creating right angles as shown below. Students often see the rectangle and the right angle but they might use the side of the parallelogram instead of the altitude in their measurement. Children need experiences in measuring and recording the altitude of parallelograms. They might be convinced by cutting out copies of parallelograms and standing them up to see the altitude and measuring from the base to the top.



$\mathbf{A = a \times b}$

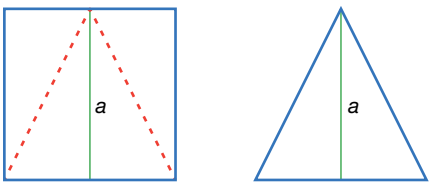
Triangle

Every triangle is half of a square, rectangle or parallelogram depending on their angle sizes. Before students are introduced to the formula for the area of a triangle, they should experiment with cutting a variety of quadrilaterals in half and duplicating a variety of triangles to create quadrilaterals, Ask each student to cut out 2 identical (congruent) triangles and make a 4-sided figure. After sharing their results, they are likely to hypothesise their own formula, using diagrams like these, which follow. The area of a triangle is half the area of the square, rectangle or parallelogram will be remembered if derived from scratch.

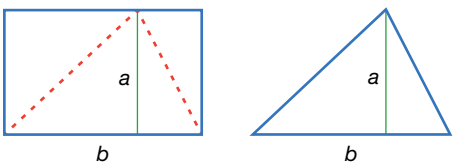


$\mathbf{A = \frac{1}{2}\,(a \times b)}$

An obtuse-angled triangle when doubled leads to a parallelogram.



$\mathbf{Area = \frac{1}{2}\,(a \times b)}$



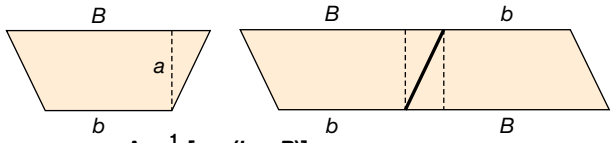
$\mathbf{Area = \frac{1}{2}\,(a \times b)}$

An acute equilateral triangle or an acute scalene triangle when copied and cut into 2 parts leads to a square or a rectangle.

Trapezium

Two congruent trapeziums can be put together to make a parallelogram. As with the triangles, have students experiment with a variety of trapeziums. See if they can generalise to give the formula.

In other words, the base is an average of the 2 lengths, so area (A) = $\frac{1}{2}$ (B + b) multiplied by altitude (a).



$$A = \frac{1}{2}[a \times (b + B)]$$

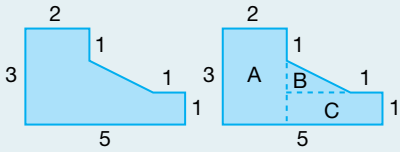
After learning how to use these formulae, children need experiences like those in In the classroom 17.10, in which they cut the shapes into different smaller shapes to find area formulae that they know. There are many different ways to subdivide the shapes. When working with students on these types of problems, ask them to show each other and the teacher the different ways they subdivided the shapes.

IN THE CLASSROOM 17.10

Breaking up the shape to solve it

Objective: Find the area of irregular shapes using familiar shapes.

Year level: 7–8

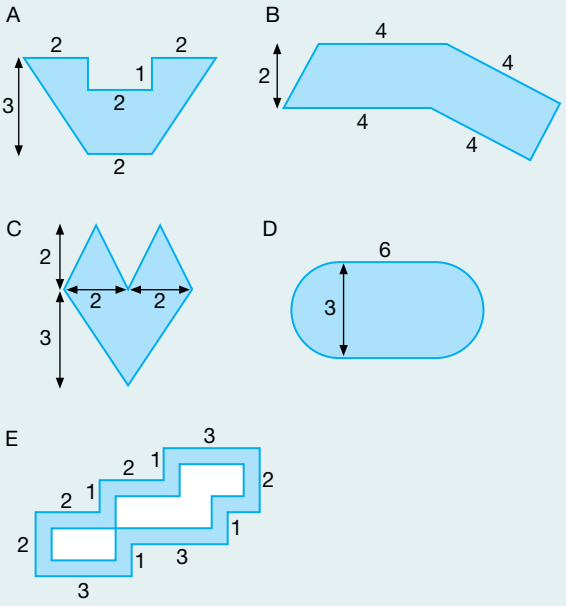


A: 6
B: 1
C: 3
10 square units

Justification: Rectangle A is 2 x 3 so its area is 6 square units. Rectangle C has a base of 3 and an altitude of 1, so its area is 3 square units. The base of triangle B is 2 units and its altitude is 1, so the area is 1 square unit. The total area is 10 square units.

Tasks

1. Find the area of each of these shapes. The white shading indicates that part is missing.



2. Design your own strange shape and show how to find its area.

Circumference and area of a circle

While this is an extension idea in the primary years, students might come across real world examples where these calculations are of value. To help children discover the formula for a circle they can roll round objects and mark the beginning and end point or use string to go around the object. Measuring across the diameter or to the centre (radius) and recording the results in a table will help lead to a discovery based on the relationship of approximately 3 times the diameter or 6 times the radius. The teacher can introduce the constant pi (π) at an appropriate time. The formula $A = \pi r^2$ is met in secondary school.

Volume and capacity

The formula for the volume of a rectangular prism is derived from the product of all the dimensions for length. Once students are confident with measuring containers and objects with standard units then they will be ready for exploration of a formula and they will begin to see the connection between volume and capacity. Rectangular prisms constructed from a number of blocks in length *(l)* a number of blocks in width *(w)* create the bottom layer or base of the prism. Further layers to a particular height *(h)* will replicate the bottom layer each time. $V = l \times w \times h$ or $V =$ area $\times$ height.

For a cube where each side has equal length, the volume is length^3. Any prism of the bases described above in area has a volume that is multiplied by 1 more dimension. Triangular prisms have a volume of $\frac{1}{2} \times$ base $\times$ altitude $\times$ height; parallelograms have a volume of base $\times$ altitude $\times$ height and trapeziums are $V = \frac{1}{2}(a \times [b + B]) \times$ height. In every case for a prism (with equal cross-section) the volume is found by multiplying the area $\times$ height. This is an important relationship for finding the volume or capacity of a container. For example, a student can trace their drink bottle base onto cm^2 grid paper and then find the volume in cm^3 and capacity in litres by multiplying by height.

The best relationship of all for volume and capacity is the fundamental metric equivalence that 1 litre of pure water has a mass of 1 kilogram and occupies the space of 10 centimetres $\times$ 10 centimetres $\times$ 10 centimetres — that is, 1000 cm^3 (cubic centimetres) or 1 dm^3 (cubic decimetre) as shown in figure 17.13. Visually, this can be demonstrated by pouring a litre of water onto the pan of a balance scale and by pouring it into an open plastic cube of size equal to one MAB$_{10}$ 1000 block.

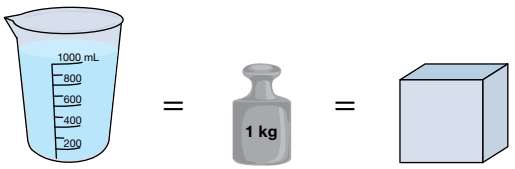


FIGURE 17.13 A single litre of water has a mass of 1 kilogram and fills a space of 1000 cubic centimetres

Time

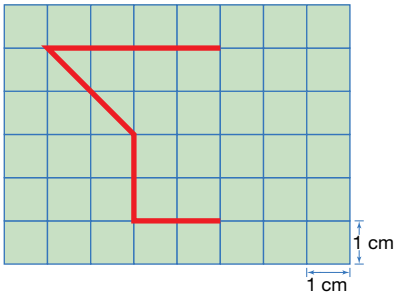
Formal units and formulae in time are complex as discussed previously. They do not fit our base-ten number system — 60 seconds in each minute, 60 minutes in each hour, 24 hours in a day and 365.25 days in a year are learned through experience with daily events, diaries, calendars and timelines.

Problem solving with measurement

The focus of this chapter has been measurement of an object or event and working through the measurement process. Equally important is creating an object of a given measurement. For example, children might measure a line segment that is 8 centimetres long but have trouble drawing a line that is 8 centimetres long. More problematic is creating an area given the measurement in square units. This is partly because of different shapes which can be used and other restraints. For example, 'create as many different

shapes as you can with an area of 12 square centimetres'. There are many rectangles that would have an area of 12 cm^2 if sides were not restricted to whole numbers. Even with this restriction, there are 3 possible and different rectangles (12×1; 2×6; 3×4).

The following problem was given on the 2003 Trends in International Mathematics and Science Study (TIMSS) assessment:



The squares in the grid above have areas of 1 square centimetre. Draw lines to complete the figure so that it has an area of 13 square centimetres.

The international average for the correct answer for students at Year 4 was 29%. In some countries, over half of the students completed the figure to show 13 cm^2. It is a good item for a teacher to give to upper primary students, and extending it to encourage problem solving for other variations: draw any figure that has an area of 13 cm^2, draw a rectangle that has an area of 13 cm^2, complete a figure that shows 6 squares and 1 half square with a mirror line (draw the other half, using symmetry). Which of these is an easier task? What successes did students have?

This example was shown to emphasise the difficulty that students have in problem solving. In this case, it is not a simple reversal. Drawing any figure with an area of 13 cm^2 is not straightforward. This kind of problem involves much more depth of knowledge because the understand, strategise and check approach requires the learner to bring other things to mind and/or ignore certain conditions. In the TIMSS problem, it is tempting for children to draw a symmetric figure but they have to ignore symmetry. This problem reminds the teacher that children who can measure do not automatically create objects of given measurements. Regular exposure to problems where the answer is not immediately obvious is important. Children love these challenges when given time and encouragement. This allows for creative mathematical thinking.

17.7 Comparing and converting measurements

LEARNING OUTCOME 17.7 Discussing the importance of equivalence relationships when converting units.

Solving problems involving measurement often involves a comparison or an arithmetic operation. In so doing, it may be necessary to change from 1 unit to another (*conversion*), which relies on the *equivalence* relationship between the 2 units. As each new standard unit is introduced, the teacher can help children look for relationships with other units already learned and familiar. For example, when introducing the millimetre, children are building on their understanding of a centimetre. A millimetre, millilitre and milligram are all one-tenth smaller than a centimetre, centilitre and centigram (the latter is rarely if ever used). Similarly, a kilometre, kilolitre and kilogram are all one thousand times larger than a metre, litre and gram. Measures in time are frequently used and the conversions need to be repeated often for recall: 7 days = 1 week, 24 hours = 1 day, 60 minutes = 1 hour and 60 seconds = 1 minute. Given regular use, metric equivalences can become second nature to children, rather than a requirement or constant reference.

It is a reasonable expectation that children will know by heart the most commonly used prefixes for decimal multiples and submultiples: *milli*, *centi* and *kilo*. The metric system expresses these standard

prefixes as base-ten relationships. For example, the metre (m) is the base unit for length, so a kilometre (km) is 1000 times longer than a metre whereas a centimetre is 100 times shorter than a metre. The litre (L) is the base unit for length, so a kilolitre (kL) is 1000 times bigger than a litre whereas a centilitre (cc) is 100 times smaller than a litre. The base unit for mass is the kilogram (kg) so the tonne (t) is 1000 times heavier than a kilogram and a gram (g) is 1000 times lighter than a kilogram.

Area and volume equivalences can be difficult for children to understand because they are multiplied in 2 and 3-dimensions from the linear equivalences. For example, knowing that $1\,m = 100\,cm$ allows you to derive $1\,m^2 = 100 \times 100\,cm^2$ and $1\,m^3 = (100 \times 100 \times 100\,cm^3)$. In the classroom 17.11 illustrates the relationship for area that $1\,cm = 100\,mm^2$ derived from $10\,mm \times 10\,mm$.

Conversions

To change from 1 unit to another, children must know the equivalence or relationship between the 2 units. By itself, however, this information is not sufficient to make conversions. Now look at the following example of a class discussion.

Mr Bane: It seems that several of you are stumped on this assignment. Dylan, please read the first exercise and let's look at it together.

Dylan: How many metres equals 5 kilometres?

Mr Bane writes on the board: _____ m = 5 km

Mr Bane: Who can tell me what we are looking for?

George: How many metres there are in 5 kilometres.

Mr Bane: What do you know about metres and kilometres?

Alana: A metre is about this big and a kilometre is about the distance to the park.

Mr Bane: Which unit is larger?

Alana: Kilometre.

Mr Bane: Right, so will it take more or less than 5 metres to make 5 kilometres?

Karina: It'll take more. It takes 1000 metres to make 1 kilometre.

Mr Bane: If 1 kilometre is 1000 metres, then what would 5 kilometres be?

Dylan: I see, it is 5 groups of the 1000 metres or 5000 metres.

Mr Bane: Good. Let's try how many metres in 184 centimetres?

Leon: Centimetres are smaller, so it takes more of them to measure something.

Mr Bane: Right.

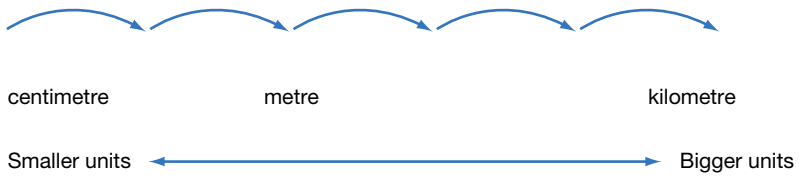
Dave: If we need to know how many hundreds in 184, then we divide.

Jim: There is more than 1 metre but less than 2 metres.

Mr Bane: What about the 84 left over?

Paula: Those are centimetres. We have 1 full metre and 84 centimetres.

Mr Bane: Let's write that down:
184 cm = 1 m 84 cm or 1.84 m (and he draws the diagram)



This discussion points out many good techniques to use in developing conversions. First, Mr Bane had the children decide whether their answer would be larger or smaller than the number given. This relies on children knowing the relative sizes of the units and understanding that the smaller the unit, the

more it takes to represent the attribute. Second, Mr Bane tried to have the children visualise the relationship between the units. Third, he related the operation to be used to their understanding of what multiplication and division mean.

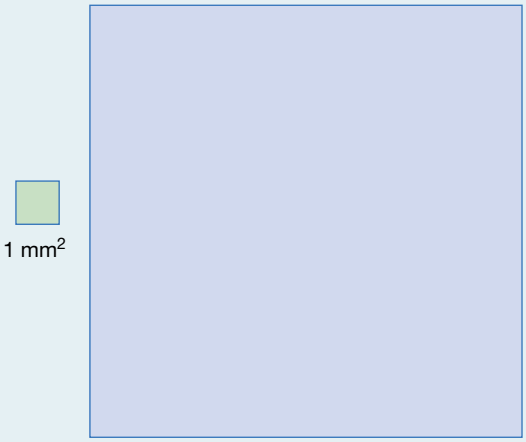
IN THE CLASSROOM 17.11

Solving conversion problems with square units

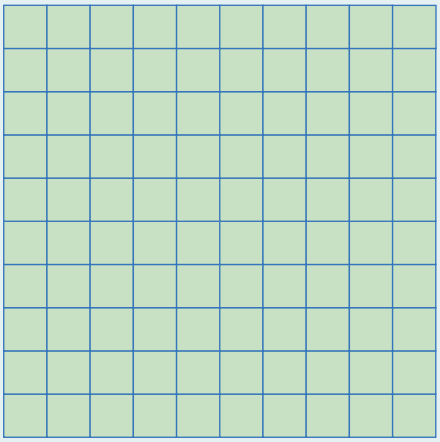
Objective: Determine the equivalents between cm^2 and mm^2.

Year level: 7–8

Review 1 cm = 10 mm. Ask students how many square millimetres (mm^2) are in 1 square centimetre (cm^2). An enlarged representation of the relationship is shown below.



Show that each row would have 10 squares and there would be 10 rows. So 1 cm^2 = 100 mm^2.



Use sketches and the model above to answer the following question:

- Connor has a sticky note that is 20 cm^2 (i.e. 5 cm × 4 cm). How many mm^2 is that? Challenge the students to work out:
- Maria has a blanket that is 2 m × 1 m. How many cm^2 is it?
  Note: 100 cm = 1 m

17.8 Estimating measurements

LEARNING OUTCOME 17.8 Articulating strategies for the skill of estimation.

Estimating is the mental process of arriving at a measurement without the aid of measuring instruments. There are many reasons to include estimation in the development of measurement understandings. Firstly, it helps reinforce the size of units and the relationships among units. Secondly, it is a practical application, think of all the times you want to know approximately how long, how heavy or how much something holds. Year 5 students were encouraged to use measurement and to make estimations of inaccessible objects during an excursion to a science museum (Sedzielarz & Robinson 2007). The Adelaide Museum contains a model of a giant squid that can be viewed through more than 3 levels of an elevator shaft, which children always find fascinating.

There are 2 main types of estimation. In the most common type, the attribute and object are named and the measurement is unknown for example, 'about how long is your arm?' In the other type of estimation, the measurement is known and the object is to be chosen. For example, 'what piece of furniture in this room is about 1 metre long?' By keeping the 2 classes of estimation in mind, children are able to expand their experience of estimation. Identify these 2 types of estimation in the activities in In the classroom 17.12.

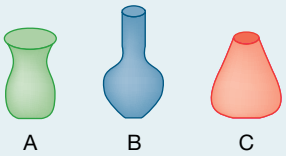
IN THE CLASSROOM 17.12

Objective: Estimate measurements of objects and compare measurements with those of other objects.

Year level: 3–5

About how many centimetres long?

	Estimate	*Measure*
Little finger	_____	_____
Nose	_____	_____
Foot	_____	_____



Which holds about 2 cups?

Make a decimetre

Draw a snake that you think is a decimetre long.

Draw a tree that you think is a decimetre high.

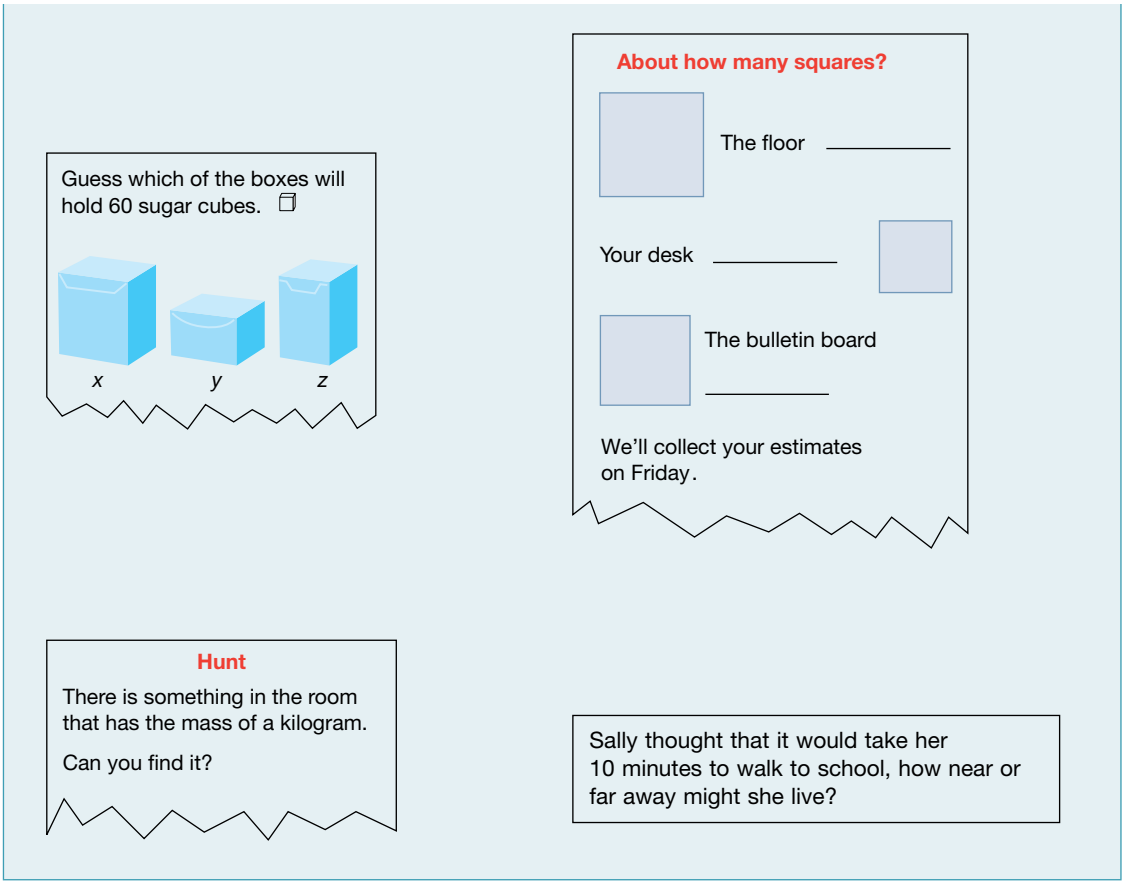
Contest for Wednesday

About how much does the wonderful watermelon weigh?

Name	Guess
_____	_____
_____	_____

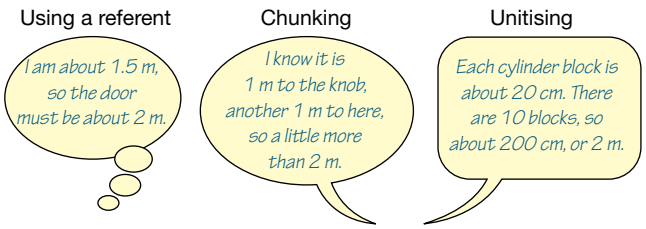


Winner gets the largest piece at Joanne's picnic.



Several common strategies can be used with either type of estimation. Helping children develop these strategies can be done through small group discussion of different estimation methods used.

What is the height of the door?



1. One strategy is to *compare to a referent*. If a child knows they are 1.2 metres in height, then when they see someone of a similar height standing by the front of the school they can estimate the height of the school building. Alternatively, if you know there are 4 oranges in 1 kilogram, then the mass of a bucket of oranges is fairly easy to estimate. The NCTM Standards call for developing common referents or benchmarks to help with estimating measurements. As suggested previously, it is helpful to have a referent for each of the standard metric units: metre, litre, kilogram, square metres, cubic metres and minutes.

2. Another strategy is that of *chunking*. In this process, the object or event is broken into subparts and an estimate is made for each part. For example Jacqui estimates that she needs 35 minutes to get ready for school in the morning. If she has a shower (5 minutes) and gets dressed in uniform (5 minutes), prepares and eats her breakfast (10 minutes), brushes her teeth (3 minutes) and packs her bag for school (3 minutes) she should be able to walk to the bus stop in time.
3. A strategy related to chunking is *unitising*. In this case, you estimate 1 part and see how many parts are in the whole. For example, a recipe calls for 600 grams of flour but only a standard measuring cup is available. From experience it is known that 150 grams is a good estimate for the mass of 1 cup of flour, so 4 cups would be about the right amount. Similarly, a child's water bottle contains approximately 750 mL of water. They know they need to drink 2 litres on a hot day, so just under 3 water bottles of water should be consumed.

Estimation should be included as a regular part of the teaching and learning program, so that children are able to build confidence when measuring.

- Encourage children to see if they can estimate before measuring: How long is that rope? How far away is that tree? How heavy is that book? How much juice is left in the jug? How much time do you spend in the playground?
- Look for ways to include estimation in other subject areas, such as how far did you jump? What size art paper will you need? How long do you think it will take to read this book? How toy cars can fit in the lunchbox before it sinks?
- Plan estimating activities for their own sake or use brief estimations as daily openers for several weeks throughout the year. In the classroom 17.12 presents some ideas to get started. Once the teacher and children begin thinking about estimating things in their room, many variations will come to mind. Talk to colleagues in the same year level about estimation and share ideas!

Finally, there is one important thing to remember: Do not state that an estimate is right or wrong. Helping children develop better ways to make estimates, can take time. Do not discourage them. Let them check their estimates by measuring. They will know whether they were close or not. You may be surprised to find out how all children can become great estimators.

17.9 Connecting attributes

LEARNING OUTCOME 17.9 Describing connections between attributes.

Activities involving 2 attributes can help children see how attributes are related together or that 1 attribute does not depend on the other. For example, by doubling the dimensions of a rectangle, children can see how the area is changed. By examining figures with the same area but different shape, children can see that the distance, or perimeter, around the shape varies. The following sample activities are suggestions for variations or extensions.

Area and shape

In the classroom 17.13 encourages children in Years 2 through 5 to investigate the different shapes they can make using 2 to 4 squares. Younger children enjoy this activity using square tiles. For older children, extending this activity to more squares enables quality problem solving to occur. The restrictions that squares need to have sides touching (not corners) and that 2 arrangements of shapes are the same if 1 is a reflection or rotation of the other, should be clarified before the investigation. A challenging variation is to use triangular grid paper (Appendix C) rather than the square grid paper. Allow for problem solving by giving a total (e.g. 24 squares or 24 triangles) and asking for all the different closed shapes that can be found. Make certain with all of these explorations that students come to a generalisation: shapes made from the same number of tiles can be arranged in different ways but have the same area. In the upper year levels, children should explore similar shapes with different areas.

IN THE CLASSROOM 17.13

With my area, I can change my shape!

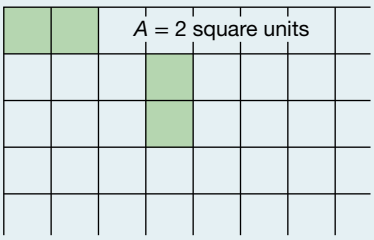
Objective: Find connected shapes with equal areas.

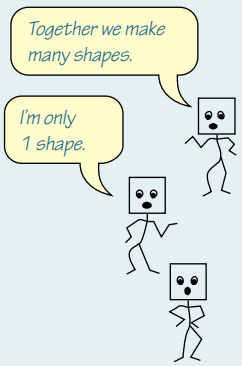
Year level: 2–5

Materials: Square grid paper (Appendix C), square tiles and coloured markers.

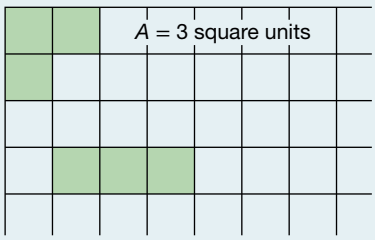
Show 2 arrangements for dominoes that can be made from squares that touch. Colour the shape on the square paper.

Can you see they are the same but have a different orientation?

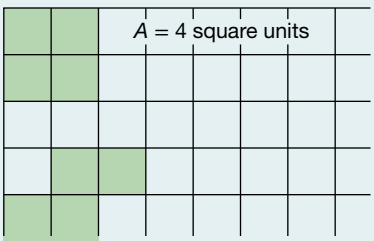


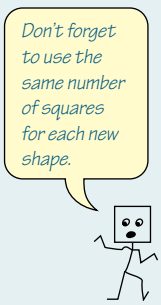


See how many different shapes you can make from 3 squares. Are they the same or different? Why?



Now, try 4 squares but look out for repeats (flips, slides or turns) of each shape.





Finally, attempt the arrangements that can be made with 5 squares (pentominoes). Remember they must share a full side, not just a corner. Aim for 12 of these shapes.

Volume and shape

An activity such as In the classroom 17.14 can be done with cubes. Not only does this activity look at the generalisation that solids with the same volume may be made in different arrangements, it also ties the investigation to number theory (primes and composites).

IN THE CLASSROOM 17.14

Same volume, different shape

Objective: Explore the different shapes of solids that can be made with a given volume.

Year level: 5–8

Materials: Cubes



I'm 8 cubes: 8-by-1-by-1.

My volume is 8 cubic units.



I'm 8 cubes: 2-by-2-by-2.

My volume is 8 cubic units, too!

Tasks

1. See how many different rectangular solids you can make with 12 cubes. Record the dimensions and volume of each.
2. Now use 2, 3, 4 . . . 17, 18 cubes.
3. How many different solids can you make if the number of cubes is:
   - prime? _____
   - a product of 2 primes? _____
   - a perfect square? _____
4. How many solids can you make with 24 cubes? _____

Perimeter and area

Children are often confused about the relationship between perimeter and area. This confusion may be caused partly by a lack of understanding of area and partly by premature introduction of the formulae for calculating area given the dimensions of a shape. For example, undertaking the investigation 'Country Maps' maths300.com lesson 50, an adult student suggested that you could find the area of land in South Australia by using a string to measure the perimeter and compare it to the string used to measure the perimeter of Victoria. Discussing this misconception another student explained that the string could be stretched out to a very long thin rectangle, which would have almost zero area, or it could become the circumference of a circle and have a very large area. This explanation helped her classmate to understand but it wasn't until the student covered both states with square tiles that an 'ahah' moment dawned.

Many and varied activities can help children see that a figure with a given perimeter may have different areas. In the classroom 17.15 is a sample of this type of activity. Children need to realise that figures with the same area can have different perimeters. In the classroom 17.13, met previously, can be modified by asking for the perimeter of each of the shapes made by 3 or 4 squares. Furthermore, children can be challenged to take 2 5-square pentominoes and combine them in different ways to see what shapes they can make that have the largest perimeter or the smallest perimeter.

Geoboards are an excellent resource for challenging children to make shapes of certain size perimeter or area. An electronic geoboard will allow students to make a shape and then check the area and the perimeter. For example, asking students to explore and record (see geoboard paper in Appendix C) the areas and

perimeters of various shapes is an excellent problem-solving activity. A challenge can be given like finding all 16 non-congruent quadrilaterals that can fit in a space of size 3 nails by 3 nails. Alternatively students can choose an area on the geoboard (say, 6 squares) and see how many shapes with different perimeters they can make. Which shape has the largest perimeter? What shape has the smallest perimeter?

IN THE CLASSROOM 17.15

Do you know how to connect perimeter and area?

Objective: Explore different areas that can be made from a given perimeter.

Year level: 5–8

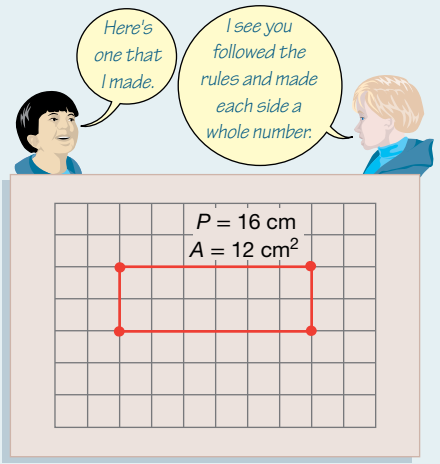
Materials: Centimetre grid (see Appendix C) and pencil.

How many different rectangles can you make with a perimeter of 16 centimetres? Find the area of each.

With 3 other students, try each of the following perimeters: 8 centimetres, 9 centimetres, 10 centimetres, 11 centimetres, 12 centimetres and 13 centimetres. Make a table of the number of different rectangles you can make from each perimeter.

Do you see a rule that will help you work out the solutions?

How many different rectangles do you think you could make if the perimeter were 52 centimetres? What is your conjecture?



Volume and surface area

Just as the area of a figure does not depend on the perimeter, the volume does not depend on the surface area. The experiment illustrated in In the classroom 17.16 looks at the relationship of lateral surface area to volume. The teacher can vary this activity by asking children to fold the papers into thirds or sixths to make prisms or make cylinders (a long, thin one and a short, fat one). During the middle school years, after developing the formula for a rectangular solid, students could calculate the volume of each of the tubes created in In the classroom 17.16.

IN THE CLASSROOM 17.16

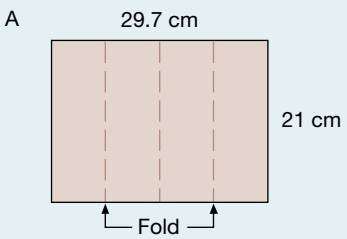
What is the connection between volume and area?

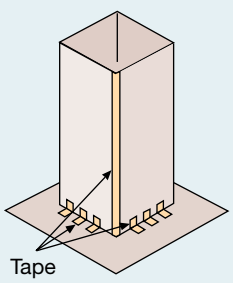
Objective: Explore the volume of prisms and cylinders with the same lateral area.

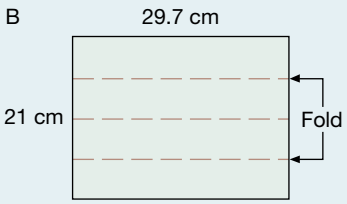
Year level: 6–8

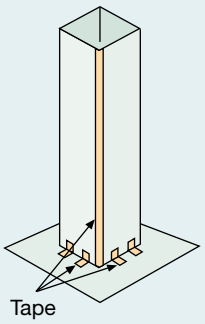
Materials: An A4 sheet of construction paper (21 cm by 29.7 cm), tape and dry filler such as rice.

Use construction paper to make 2 tubes:









Guess which tube (A or B) holds more, or do they hold the same?
Fill the tubes to test whether you were right.
Change the number of folds to 2 and repeat the experiment by making triangular prisms.
Further extensions are hexagonal prisms, octagonal prisms and cylinders.
Do you have a conjecture about which shape — the shorter or the taller of each pair — will hold the most?

MAKING CONNECTIONS

Most of the world uses the metric system however there is some diversity. In any classroom there could be students from other countries, who might need extra support to understand our measurement system. The TIMSS survey by Mullis et al. (2004) notes that 'the fourth-grade mathematics curriculum included a strong emphasis on measurement topics in many countries' (p. 217). However, that does not mean that all students have had the opportunity to learn many of the measurement concepts and skills, and to apply these to realistic situations.

Children may be interested in learning about measurement in the past. Do you know that mouthfuls, jacks and jills were once used as measures of capacity? A mouthful was used to sell honey or oil. So, how many mouthfuls are in a pint if a jigger is 2 mouthfuls, a jack is 2 jiggers, a jill is 2 jacks, a cup is 2 jills and a pint is 2 cups? This historical detail puts a new spin on base-2 number concepts as well as the old nursery rhyme about Jack and Jill fetching a pail of water. Measurement has always been crucial to trade, so people need proficiency in this area in order to buy and sell effectively. Today, many of our purchases come packaged and premeasured but there continues to be a need for using units of measurement both formal and informal.

Investigating historical resources can serve as an interesting introduction to our metric system. Similarly, the variety of nonmetric units that are still in use by some cultures and in different countries can be quite fascinating. Further investigation of other cultures will help to broaden the knowledge base of pre-service and in-service teachers and enrich their teaching skills. In some countries, many things are not sold by number but by mass — for example, you may buy 600 g of eggs rather than a dozen.

In addition to historical and cultural connections it would be remiss not to include concepts of sustainability in the measurement strand of the curriculum. Drought and water restrictions are a regular part of life in Australia and children's awareness of water conservation and other issues are important. Mathematics plays a crucial part of this raised awareness. The teacher reference *Counting on a Small Planet: Activities for Environmental Mathematics* (Baker & Baker 1991) contains 2 measurement-focused chapters that include activities worth undertaking with primary level students. Further information can be found on the internet and in books such as *365 Water Saving Tips* (Coulthard 2006). Finally, the reference *Education for Sustainability in the Primary Curriculum: A Guide for Teachers* (Littledyke & Taylor 2009) includes a focus chapter on mathematics that is also worth investigating.

A GLANCE AT WHERE WE'VE BEEN

Measuring is a process that may be used when determining the size of attributes belonging to objects and events. The measurement process consists of identifying the attribute and comparison, measuring with informal and then standard units, and applying measurement to real world situations. To help children understand an attribute, it is suggested that they compare 2 objects using that attribute: perceptually, directly and indirectly. Finding the number of units that would quantify that attribute for an object should arise from natural measures before using metric units. Children may first assign a number by counting and later by using instruments or formulae. Other suggested ways to help children learn about measuring are estimating the size of objects or the duration of events, finding equivalent measurements and relating 2 attributes. In the Australian Curriculum (ACARA 2016), the attributes of measurement to be studied in the primary years are length, area, volume and capacity, mass, time, temperature and angle. The language of measurement is crucial to develop at all stages, as is the process of estimation.

By including measuring in the primary classroom, teachers have the opportunity to show how mathematics is practical and enjoyable, to develop other mathematical ideas, to relate mathematics to other topics, and to make mathematics meaningful for many children.

THINGS TO DO: FROM WHAT YOU'VE READ

1. What are the stages in the measurement process?
2. What attributes, units and instruments are included in most primary mathematics programs?
3. Explain how you will include estimation throughout your measurement activities and why this will be helpful for children.
4. Why should you include measurement in your mathematics program? Choose 1 reason and give examples of how measurement achieves that reason.
5. What 10 concepts related to units need to be developed as children have experiences with measurements?
6. Describe 3 difficulties or misconceptions children might have with measuring instruments.
7. Why do students need to be able to convert from 1 unit to another?
8. Show how you would find the areas of each of the figures in In the classroom 17.10.
9. Give 3 examples of connections that can be made between different attributes.

THINGS TO DO: GOING BEYOND THIS TEXT

IN THE FIELD

1. *What's an angle?* Develop and teach a lesson for Years 3–5 based on the video created by ABC Splash, all about angles http://splash.abc.net.au/home#!/media/1244894/what-s-an-angle-. This video can be used to introduce the concept of angles and comparison of angles and would be useful to activate prior knowledge. It only goes for 2:57 and is a great visual tool.
2. *Angles are nice but are they cute?* Following on from the previous video from ABC Splash, this brief video introduces the terms acute, obtuse and right angle. http://splash.abc.net.au/home#!/media/1244916/angles-are-nice-but-are-they-cute-. Use the ideas presented in the video to encourage children to go on an angle hunt and classification around their school environment.
3. *Monumental measurement mess ups!* This animation from ABC Splash could be used as a discussion starter about the connection between metric and imperial measures. http://splash.abc.net.au/home#!/media/1566108/monumental-measurement-mess-ups Children could think about and record instances where people they know use imperial measures and why it could be problematic.

IN YOUR JOURNAL

4 Reflect on your use of measurement in the last 24 hours. Classify according to each attribute — length, area, volume/capacity, mass, time, temperature; and angle — and whether you used formal or informal units.

5 You have a child in your class who has difficulty physically handling a ruler. What would you do in this situation? Give at least 2 solutions.

6 Describe a set of activities that you could use to introduce a centimetre, a kilogram, a litre or a square centimetre based on those for introducing a decimetre.

7 Design an activity card that uses group work to investigate a fixed area and varying perimeters (e.g. a fixed area of 4 square units made with 4 square tiles and find the different perimeters). Include the answer to the investigation.

8 For a year level of your choice, design 5 estimation activities that include both types of estimation (estimating measurement for a known object and choosing an object that fits a known measurement).

9 Outline a series of steps for developing the volume formula for rectangular solids, similar to those for developing the area formula for a rectangle.

10 Keep a record of how many times you used measurement in a week. Classify them as to whether you made an estimate, used an instrument or calculated the measurement with a formula.

WITH ADDITIONAL RESOURCES

11 Check the scope-and-sequence chart of a teacher resource to see how the introduction of units corresponds to table 17.2.

12 Read an article about the adaptation of the Russian curriculum in Hawaii (Dougherty & Venenciano 2007) in *Teaching Children Mathematics*. How do the children solve the problem about comparing line segments without moving them? What do you think about the use of letters to describe their comparisons?

13 Read the article by J. Dwyer, 'Ordering rectangles: Which is bigger?' (Dwyer 2003). How would you have ordered the rectangles? What did you learn about children's understanding of 'bigger'?

WITH TECHNOLOGY

14 Watch the video *Table for 22* http://lunaproductions.com/a-table-for-22/ which makes the connection between perimeter and area. What strategies did the students use to solve the challenge of the biggest table to fit the most food and the smallest table to fit the least food. Discuss the teaching strategies used by the teacher, Suney Park and why the lesson is effective.

15 Visit the NCTM's Illuminations website and investigate the Cubes volume tool, http://illuminations.nctm.org/ActivityDetail.aspx?id=6. How would you use this in your teaching?

CHILDREN'S LITERATURE CONNECTIONS

The best children's books is a comprehensive website to support the use of children's books with mathematics. http://www.the-best-childrens-books.org/measurement-lesson-plans.html

A number of children's books have been given throughout this chapter. Each of those chosen has a video link available of a teacher, parent, student or author reading the book. They are well worth a look.

Branley, F. M. (1993). *Keeping Time: From the Beginning and into the 21st Century*. Boston: Houghton Mifflin.

Early timekeepers were the sun, moon and stars. Directions for making ancient timekeepers such as the sundial, candle clock, water clock and sand clock are given. Mechanical clocks are introduced and the need for more accuracy is discussed. An index and bibliography of other books on time for young and more advanced readers are also included.

Hightower, S. (1997). *Twelve Snails to One Lizard: A Tale of Mischief and Measurement*. New York: Simon & Schuster Books for Young Readers.
Animals introduce measurement facts in a delightful book for young children. A break in the dam sets the need for nonstandard and standard measurements. This book succeeds in introducing customary length measures and the relationships among such measures in a humorous way.

Lionni, L. (1995). *Inch by Inch*. New York: Morrow/Avon.
This classic book is about an inchworm who convinces a hungry robin that he is useful for measuring the robin's tail. The robin spreads the word about being measured by the inchworm to other birds. Unless the inchworm measures the nightingale's song, the nightingale will eat the inchworm. The inchworm measures far, far away as the nightingale sings.

Murphy, S. J. (1999). *Room for Ripley*. New York: HarperCollins.
Carlos visited a pet shop to buy his favourite type of fish: a guppy. His sister Ana shows Carlos how to set up a home for his guppy using different types of liquid measurement. Carlos decides to buy another guppy, which prompts him to add more water to the fish bowl using different liquid measures. There are activities and games for teachers and parents to use.

Myller, R. (1991). *How Big Is a Foot*? New York: Dell Publishing.
This is a humorous tale about nonstandard measures in which a king decides to have a bed made for the queen as a surprise for her birthday. The large king marks off the dimensions for the proposed bed with his feet. Unfortunately, when the bed is delivered, it is the wrong size. The apprentice who made the bed solves the problem from his gaol cell.

Reisberg, J. (2006). *Zackary Zormer: Shape Transformer*. Watertown, MA: Charlesbridge.
This is the story of Zack Zormer's favourite day: Measurement Day.

Zack has forgotten to bring something to show at school. All he has is a piece of paper and an imagination. Zack cleverly shows length, width, perimeter and area with his paper.

Schwartz, D. M. (2003). *Millions to Measure*. New York: HarperCollins.
Marvelosissimo the Magician explains the difference between nonstandard and standard units of measure and how different cultures of the past used to measure using nonstandard units. In the back of the book, the author talks about measuring and the metric system.

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ACKNOWLEDGEMENTS

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CHAPTER 18

Data analysis, statistics and probability

LEARNING OUTCOMES

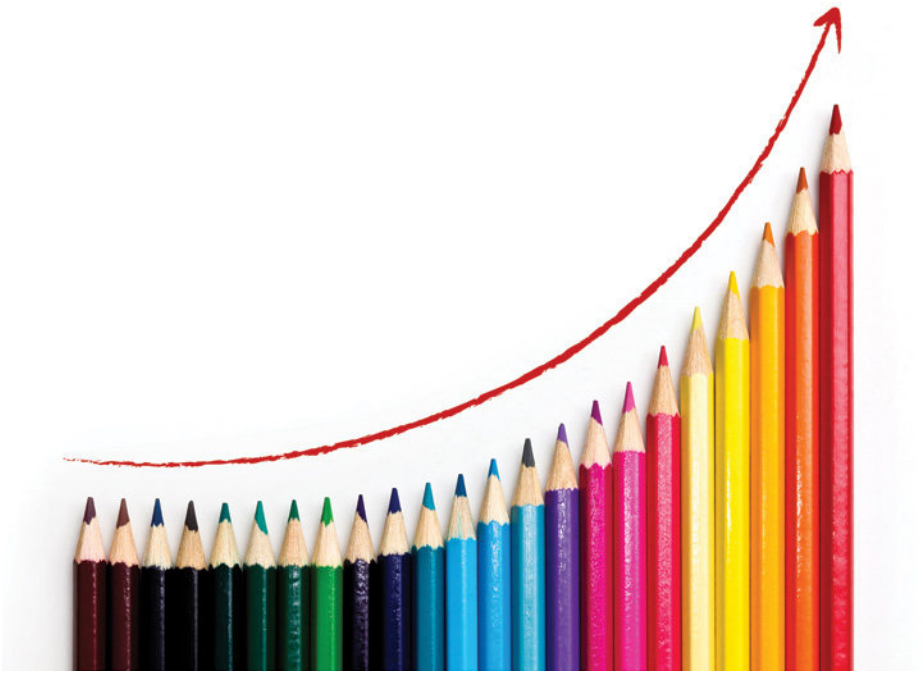
18.1 Understanding how to formulate a question of interest and the skills and methods required to effectively collect and record data.

18.2 Describing when to use the different methods of organising and presenting data.

18.3 Recognising what descriptive statistics are appropriate to introduce in the primary year levels.

18.4 Understanding how to develop your students' data sense and ability to communicate results.

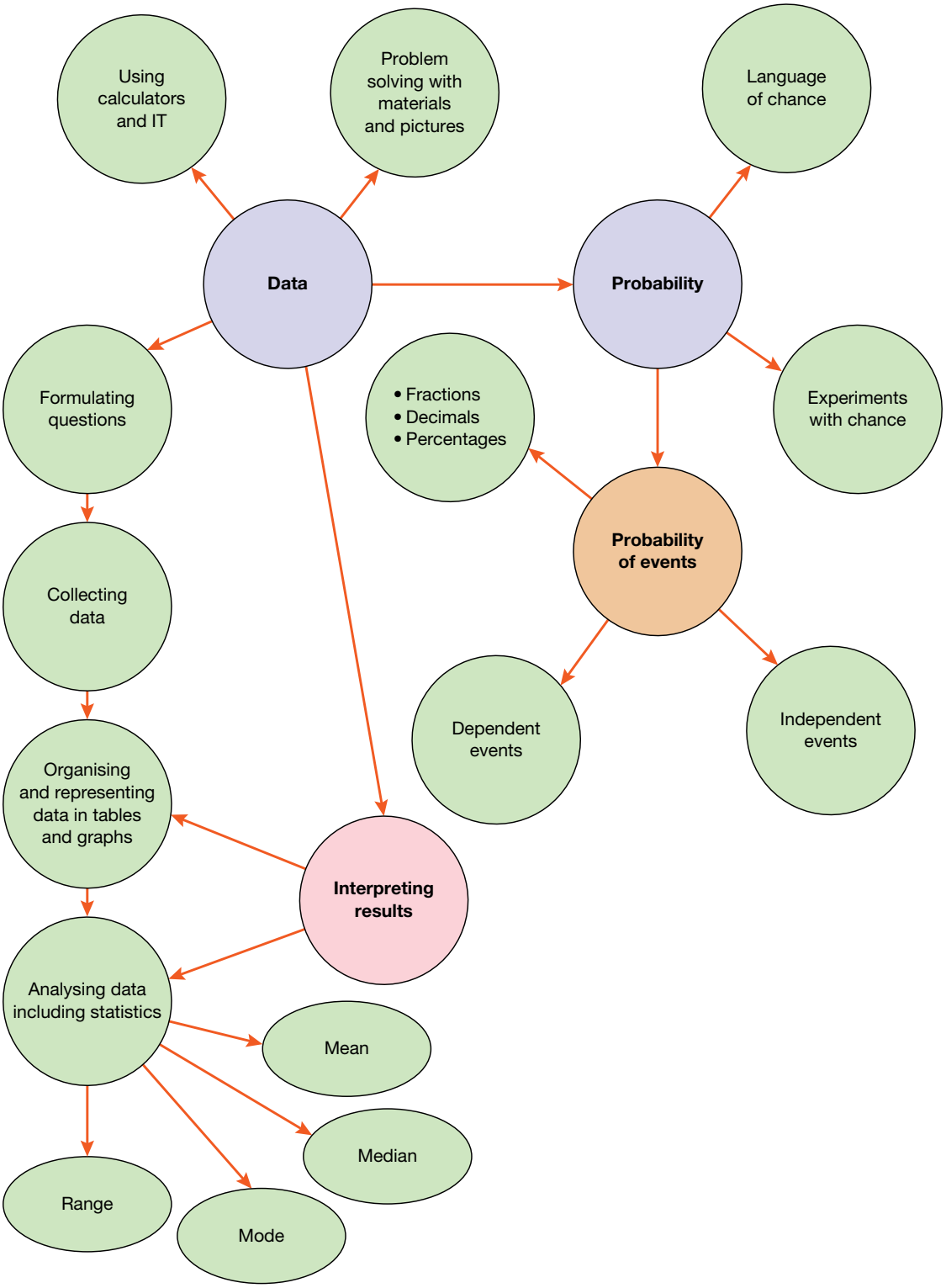
18.5 Recognising the linkages between the studies of probability, data representation and data interpretation.



'Statistical thinking will one day be as necessary for efficient citizenship as the ability to read and write.'

H. G. Wells

Chapter 18 concept map



Introduction

Data analysis, statistics and probability provide a meaningful context for promoting problem solving and critical thinking, enhancing communication, developing number sense and applying computation. The study of these topics supports a problem-solving or investigative approach to learning and doing mathematics. As a result, statistics and probability are now highly visible topics in primary school mathematics programs. The shift of attention to these areas has been dramatic and reflects the growing importance of analysing data in our daily lives.

Increased attention to data analysis, statistics and probability in primary mathematics programs has increased student performance in these areas. In 2003 and 2007, Trends in International Mathematics and Science Study (TIMSS) indicated that Australian Year 4 and Year 8 students were performing higher than the international average in the Data content area. In 2006 and 2009, the Programme for International Student Assessment (PISA) found that 15-year-old students in Australia were producing good results in the Uncertainty content area. These favourable international findings for Australian student performance are likely to be due to these topics receiving relatively more attention through the Chance and Data strands of various Australian curricula documents over time (Stacey & Stephens 2008; Thomson et al. 2007).

Let's look at some reasons for including the study of data analysis, statistics and probability in primary school.

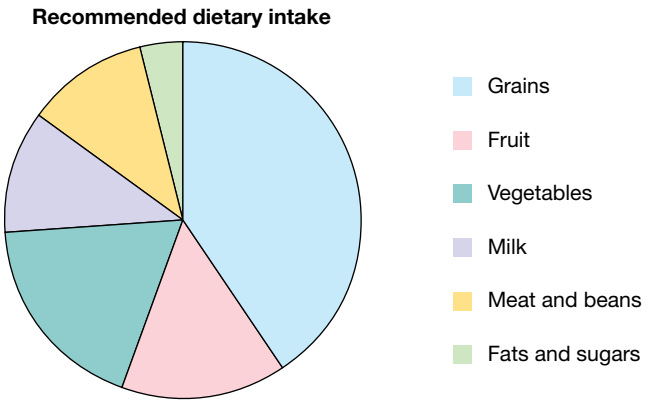
Children encounter ideas of statistics and probability outside of school every day

Radio, television, newspapers and the internet bombard us with information. For example, news reports present national economic and social statistics; opinion polls; weather reports; and medical, business and financial data.

The current demand for information-processing skills is much greater than it was 25 years ago and technological advances will place a far greater premium on such skills in the years ahead. Many consumer and business decisions are based on market research and sales projections; however, statistics about issues or the quality of commercial products can be misleading to the public. If data are to be understood and used widely, every educated person must be able to process such information effectively and efficiently. An intelligent consumer, who is able to make informed decisions, must be able to understand and use statistics and probability.

Data students encounter outside of school are often presented in graphical, statistical or probabilistic form.

- *Graphical*. Which food nutrient is the largest portion of a healthy eating plan?



- *Statistical.* The mean salary of an Australian Football League player in 2012 was $173 000 a year.
- *Probabilistic.* The probability of rain today is 0.35.

Each of these statements needs to be understood if meaningful interpretations are to be made. The context and format in which information is presented vary greatly, but correct interpretation of the information often requires the application of mathematics. Consider, for example, the mathematical concepts involved in weather reports (decimals, percentages and probability); public opinion polls (sampling techniques and errors of measurement); advertising claims (hypothesis testing); and monthly government reports involving unemployment, inflation and energy supplies (percentages, prediction and extrapolation).

Data analysis, statistics and probability provide connections to other mathematics topics or school subjects

The study of data provides an excellent opportunity for curriculum integration. For example, each day kindergarten children might create a picture graph with their photos to show who is present and absent. In doing so, they practise basic counting skills. Year 1 students may estimate the number of treats in single-serving packages and then sort and graph the treats in concrete form. After counting the actual number of treats, a second graph may be created, displaying the actual number of treats in each package in picture form. The children are able to compare their estimates with the actual results. Year 4 students measure the height of each student in the class and then present the data in a table. Year 7 buddies might find the mean, median, mode and range so they can better understand and summarise the data. Year 6 students conduct a probability simulation with dice, then use fractions, decimals and percentages to report their results.

Statistics and probability are also easily integrated into other school subjects. For example, Year 2 students might use graphs in literacy to keep track of numbers and types of books they have read. Year 6 or 7 students conducting a science experiment on rolling a car down ramps of various heights may calculate the mean distance for the number of trials at each height. Graphs and charts are frequently used to display information about populations or geographic areas. In physical education class, students may graph pre-test data from physical fitness tests and compare it with post-test results at the end of the year. Year 1 students reading folktales in language and literacy may develop a chart listing which events from the stories are possible and which are impossible. Effective teaching involves finding ways to make data analysis an integral part of the primary and middle school curricula.

Data analysis, statistics and probability provide opportunities for computational activity in a meaningful context

Data are not merely numbers but numbers with a context. The number 11 in the absence of a context carries no information, but saying that there are 11 possible sums for 2 dice gives a clear probability statement. Data provide many opportunities to think, use, understand and interpret numbers, rather than simply carrying out arithmetic operations. Using data helps further develop number sense. Working with real data requires judgement in choosing methods and interpreting results. Thus, statistics and probability are not taught in primary school for their own sake but because they provide an effective way to develop quantitative understanding and mathematical thinking.

Data analysis, statistics and probability provide opportunities for developing critical-thinking skills

When students learn how to design and carry out experiments that utilise data analysis and probability, they develop skills to help them answer questions that often involve uncertainty and they draw conclusions based on their interpretations of the data. As students learn how to approach situations statistically, they can face up to prejudices, think more consistently about arguments and justify their thinking with numerical information (Burns 2000). This approach has applications in many areas of our lives — social and political.

Data analysis, statistics and probability should not be viewed or treated in isolation. Their study provides numerous opportunities to review and apply mathematics in a variety of real-world situations. For example, whole numbers, fractions, decimals, percentages, ratios and proportion are essential ideas for understanding a wide variety of situations. Many computational skills are reviewed and polished as they are applied in graphing or doing statistics and probability.

The Australian Curriculum (ACARA 2015) devotes a complete strand to statistics and probability with sub-strands Data representation and interpretation and Chance from Year 1 to Year 7 and beyond (see table 18.1).

TABLE 18.1 Statistics and Probability content descriptions in the Australian Curriculum

Year level	Content description
Foundation year	**Data representation and interpretation** • Answer yes/no questions to collect information and make simple inferences.
Year 1	**Chance** • Identify outcomes of familiar events involving chance and describe them using everyday language such as 'will happen', 'won't happen' or 'might happen'. **Data representation and interpretation** • Choose simple questions and gather responses and make simple inferences. • Represent data with objects and drawings where one object or drawing represents one data value. Describe the displays.
Year 2	**Chance** • Identify practical activities and everyday events that involve chance. Describe outcomes as 'likely' or 'unlikely' and identify some events as 'certain' or 'impossible'. **Data representation and interpretation** • Identify a question of interest based on one categorical variable. Gather data relevant to the question. • Collect, check and classify data. • Create displays of data using lists, table and picture graphs and interpret them.
Year 3	**Chance** • Conduct chance experiments, identify and describe possible outcomes and recognise variation in results. **Data representation and interpretation** • Identify questions or issues for categorical variables. Identify data sources and plan methods of data collection and recording. • Collect data, organise into categories and create displays using lists, tables, picture graphs and simple column graphs, with and without the use of digital technologies. • Interpret and compare data displays.
Year 4	**Chance** • Describe possible everyday events and order their chances of occurring. • Identify everyday events where one cannot happen if the other happens. • Identify events where the chance of one will not be affected by the occurrence of the other. **Data representation and interpretation** • Select and trial methods for data collection, including survey questions and recording sheets. • Construct suitable data displays, with and without the use of digital technologies, from given or collected data. Include tables, column graphs and picture graphs where one picture can represent many data values. • Evaluate the effectiveness of different displays in illustrating data features including variability.

Year 5	**Chance** • List outcomes of chance experiments involving equally likely outcomes and represent probabilities of those outcomes using fractions. • Recognise that probabilities range from 0 to 1. **Data representation and interpretation** • Pose questions and collect categorical or numerical data by observation or survey. • Construct displays, including column graphs, dot plots and tables, appropriate for data type, with and without the use of digital technologies. • Describe and interpret different data sets in context.
Year 6	**Chance** • Describe probabilities using fractions, decimals and percentages. • Conduct chance experiments with both small and large numbers of trials using appropriate digital technologies. • Compare observed frequencies across experiments with expected frequencies. **Data representation and interpretation** • Interpret and compare a range of data displays, including side-by-side column graphs for two categorical variables. • Interpret secondary data presented in digital media and elsewhere.
Year 7	**Chance** • Construct sample spaces for single-step experiments with equally likely outcomes. • Assign probabilities to the outcomes of events and determine probabilities for events. **Data representation and interpretation** • Identify and investigate issues involving numerical data collected from primary and secondary sources. • Construct and compare a range of data displays including stem-and-leaf plots and dot plots. • Calculate mean, median, mode and range for sets of data. Interpret these statistics in the context of data. • Describe and interpret data displays using median, mean and range.

Source: © Australian Curriculum, Assessment and Reporting Authority 2015.

The Data representation and interpretation sub-strand is about investigating questions of interest and typically progresses in several steps including:

1. formulating questions
2. collecting data
3. organising and representing data
4. analysing data
5. interpreting the results.

18.1 Formulating questions and collecting data

LEARNING OUTCOME 18.1 Understanding how to formulate a question of interest and the skills and methods required to effectively collect and record data.

Begin with a good question or problem that interests students, one for which the answer is not immediately obvious, and one that also clearly gives students a reason for collecting and analysing data and then interpreting the results. There is a real benefit to having students identify their own questions or problems, for they will take ownership of the investigation and be motivated to undertake the investigation more fully (Bohan, Irby & Vogel 1995). At the primary level, the teacher may have to help in formulating the questions to be sure students are focused on questions for which they can collect data available to them. At the middle-school level, students should be encouraged to ask questions that will require the

collection of data from a larger sample. Questions may come from a variety of sources on many different topics. In the classroom 18.1 gives a few examples. These ideas may be shared with children to help them brainstorm the kind of questions they might ask.

Once a suitable question has been identified, students will need to plan how to collect the data needed to answer it. Communication skills are very important during this stage. To be successful, students must be able to develop clear survey questions and logical steps for their experiments or simulations. They must communicate with others to negotiate the details of the investigation. They must find a clear and efficient method of recording their data. Students may collect data from surveys, experiments and simulations, all of which typically involve counting or measuring. Data may be recorded in a variety of ways, such as using tally marks or placing information in a table. Computer spreadsheets may also be used for recording data.

Surveys

Survey data result from collecting information. Data may range from taking a national public opinion poll or observing vehicles parked at school, to simply tallying the ages of students in a class. A wide array of data are available from the Australian Bureau of Statistics. The actual data used depends on student interest and maturity, but survey data collected by students provides a freshness that increases student interest and sustains persistence in related problem-solving activities. Computer software is also available that allows students to design a survey and have the persons surveyed enter their responses directly into the computer. Once the data have been collected, the program can display results in a table or in several types of graph.

IN THE CLASSROOM 18.1

Let's find out

Objective: Plan and conduct a survey.

Year level: 3–5

Steps

1. Think of a question you would like to answer. Here are some ideas to get you started.
- **Questions about ourselves:** Who can whistle a tune? How far can we throw a softball? What is our class's typical height, eye colour, shoe size, number of family members, amount of pocket money or type of pet?
- **Questions about opinions or feelings:** How do you feel about fractions? Does life exist on other planets? What should be done about pollution? What country do you want to study in geography class? What is your favourite television show, song, book, sport, colour or food?
- **Questions about the world:** Which month has the most birthdays? What is the most popular colour of car in the school parking lot? Which brand of biscuit has the most chocolate chips? How many paper towels do we use in a day? What is the effect of fertiliser on bean plant growth? What type of paper plane will fly the farthest?

My question: _____
_____

2. Plan the survey by answering these questions.
(a) Where or from whom will I collect the data?
(b) How will I collect the data?
(c) How much data will I collect?
(d) When will I collect the data?
(e) How will I record the data as I collect them?
(f) What else do I need to do before I start collecting data?

3. Collect your data!

4. What is the best way to display my set of data?

5. What purpose can my data serve when it is collected?

Each question from In the classroom 18.1 gives students an opportunity to collect data themselves. In order to sharpen data-collecting techniques, students may consider the following questions.

- What questions will this survey answer?
- Where should I conduct my survey?
- When should I conduct the survey?

Why is it important for students to think about these questions before conducting their survey? In planning a survey, students are required to refine and polish their questions to get whatever information they are seeking, which in itself is an important and valuable experience. A host of other idea starters are available (Bright, Frierson, Tarr & Thomas 2003; Lindquist et al. 1992).

Experiments

Experiments may be somewhat more advanced than surveys. When students conduct experiments, in addition to using observation and recording skills, they often incorporate the use of the scientific method. For example, students may design an experiment to measure the effect of applying different types of fertiliser on plant growth or compare flight times of different paper planes. They may try to determine which brand of tissue is the most absorbent. Students may play a 'spinner' game to determine if the outcome is fair to all players.

Simulations

Although a simulation is similar to an experiment, random number tables or devices such as coins, dice, spinners or computer programs can be used to model real-world occurrences. Students may start with a probability question such as 'If I flip a coin 20 times, how many times will it land on heads?' Then, they carry out the simulation and record data as they are generated. Simulations help students gain insight and understanding of empirical or experimental probability.

Sampling is another method of data collection that students can simulate. In statistics, the whole group you are studying is called the *population*. In real-life data collection, there are times when it is impossible or impractical to collect data from a complete population. A *sample* is a subset of a population. Samples are often collected to learn more about public health issues or people's buying habits, or to predict election results. It is important for students to realise that the use of a particular sample may make a survey biased and to discuss ways to reduce bias. For example, consider a student survey of the students in a classroom in order to determine which games are to be purchased for use during recess. The population to be surveyed is a Year 5 class; however, if the students interview only the girls in the class (a sample), their results will probably not reflect the wishes of all students. Alternatively, students' names could be randomly selected by drawing them out of a hat, which would more likely reflect the total population.

Researchers have developed methods of collecting samples so they may learn more about a population that would be impossible to survey completely. For example, when wildlife biologists want to count the number and types of fish in a lake, they use a mark–recapture technique. The biologist captures a number of fish, counts them, marks them and then releases them. More samples are collected and the number of marked fish in the sample are compared in a ratio to the unmarked fish in the sample. The biologist may then use the data from the samples collected to estimate the total population of fish and the ratio of marked and unmarked fish. These techniques and how they can be simulated with primary and middle school students using beans, biscuits or games are explained further in Morita (1999) and Quinn and Wiest (1999).

18.2 Organising and representing data

LEARNING OUTCOME 18.2 Describing when to use the different methods of organising and presenting data.

After data have been collected, the first step of analysis is to organise the information so that the results may be interpreted. A *graph* is a type of diagram that may be used to visually present or organise data. At the primary and middle-school levels, students frequently encounter and create tables, tallies and

concrete picture, bar and circle graphs. As they move through school, the types of graph they read and create become more complex. Knowledge related to constructing and interpreting various graphs is an important part of mathematics learning and should begin in the primary year levels. Children gain competence with age and experience.

For young children, initial work with data means they begin by working with concrete objects in their environment. This will involve collecting and counting objects, sorting them into categories and then displaying them in an organised way. For example, children might each bring a favourite book and then collaborate to organise their collection. A *real graph*, or concrete graph, is developed as the actual books are arranged in rows. As students work with their 'data', they also practise counting skills. Older students might work with larger collections of objects and, by placing the objects into groups of 10, place value is reinforced. Helping children move from the concrete, real-graph representations to more pictorial and then symbolic representations is essential.

An example of an early experience is the task of sorting preferences towards real fruit. First, ask each child to choose one piece of his or her favourite fruit from a basket and position this piece of fruit on a table, as shown in figure 18.1(a). The resulting rows of fruit represent the children's preferences in a concrete way as a real graph. Next, ask each child to draw the fruit on index cards or post-it notes, or use pre-drawn pictures of fruit. Have the children use the cards to build a picture bar graph, as shown in figure 18.1(b). Although this graph is a less concrete means of showing the information, most children still find it a meaningful way to represent their preferences. Finally, this same information can be expressed more symbolically in the form of a bar graph — see figure 18.1(c). Children can also be given blank coloured paper and encouraged to organise and report their data on the page in a way that makes sense to them. In this way, children learn to communicate their information in ways that are meaningful both to themselves and others (Folkson 1996).

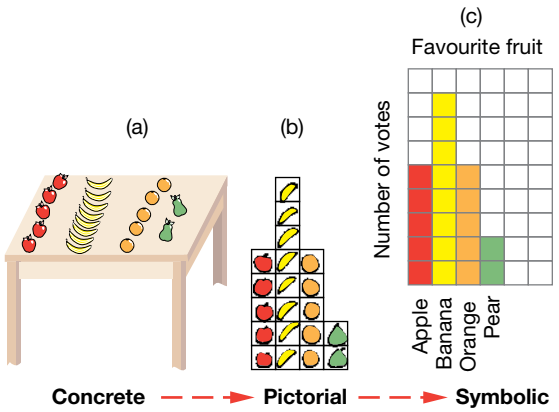


FIGURE 18.1 One method for introducing graphs

Regardless of how the data are presented, suitable questions can be asked to encourage thoughtful interpretation of the graphs. Open-ended questions might also be included, such as:

What do you notice about the graph?
How many children prefer apples?
What is the favourite fruit?
How many different types of fruit are shown?
How many children contributed to the graph?
If you were conducting this survey what fruit would you include?

These types of question result naturally from the data and provide valuable opportunities for students to ask as well as answer questions.

Quick and easy graphing methods

For primary school children, the process of constructing a graph helps them learn about its critical features and is a valuable activity; however, some graphs are more difficult to create than others, and construction may develop into a lengthy process. The examples in the 'Things to do with technology' section at the end of this chapter provide ways of collecting data and show different ways to represent results graphically.

If the main purpose of visually displaying the data is simply to learn more about the data and easily examine results, several alternatives exist to having students create complete publication-quality graphs by hand. By using some of the materials illustrated in figure 18.2, data may be displayed immediately as they are being collected. These graphs, called 'sketch graphs' or 'working graphs,' can be created quickly and provide a visual representation of the shape of the data. They should be clear; however, they do not have to be neat. Sketch graphs do not require labels or titles and do not require time-consuming attention to construction. Sketch graphs may be made with concrete materials or with paper and pencil.

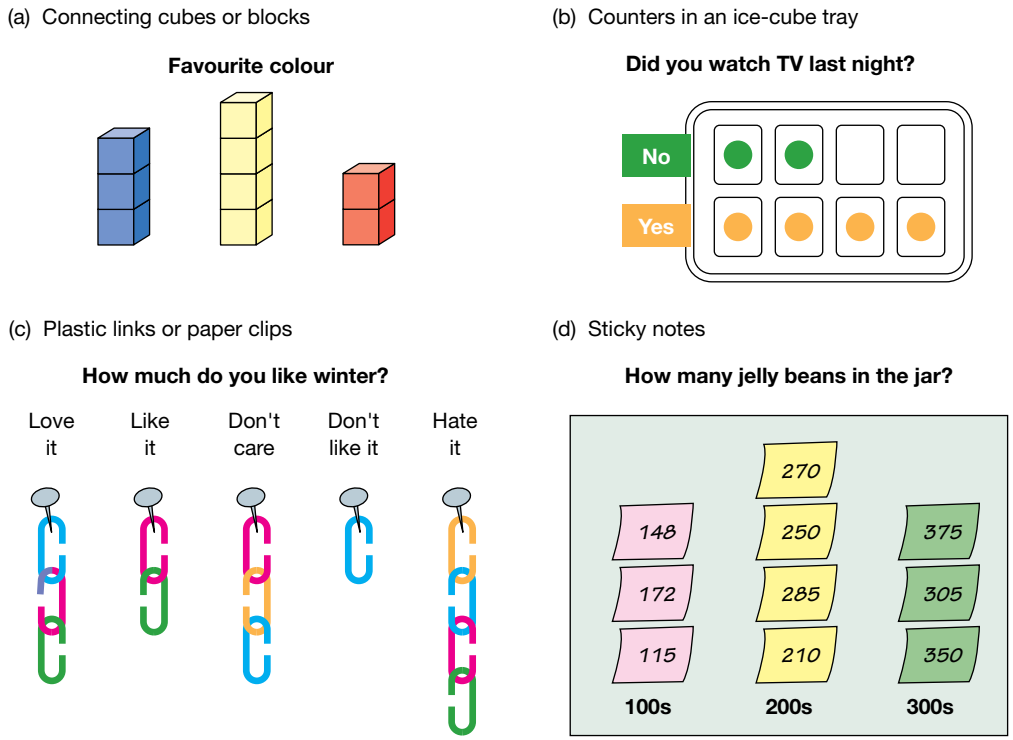


FIGURE 18.2 Quick and easy graphing materials

Spreadsheets or statistical programs may be used to quickly graph data. As an added benefit, once the data have been entered, students can quickly and easily see their data displayed and printed in more than one type of graph or with a different scale. The graphing calculator is another tool that may be used for making graphs. Graphing calculators designed for middle year levels allow students to create picture graphs, bar graphs and pie graphs.

Plots

A *plot* is another type of graph used to visually display data. In recent years, plots have been used frequently in magazines and newspapers because they provide efficient ways of showing information as well as comparing different sets of data. Some plots, such as line plots and stem-and-leaf plots, are quick and easy to make and can be used as sketch graphs to get an initial look at the shape of the data.

Line plots

A *line plot* may be used to quickly display numerical data with a small range. The range and distribution of the data may be clearly seen in the display. Line plots may be successfully used at all levels. Young children can more easily create a line plot than a bar graph, and older students will enjoy the quick feedback they receive when sketching a line plot. When creating line plots, the data should be plotted with the same scale. This allows for a quick visual interpretation of the data when attempting to identify trends in the data.

Suppose that at a class party, some Year 3 students are given mixed lollies in single-serving packages. The children open the packs and someone suggests they count to see how many lollies they each received. The class begins counting, and soon each child begins calling out his or her total. You write the totals on the whiteboard as they are randomly called out:

17, 19, 21, 20, 15, 18, 22, 17, 20, 18, 17, 18, 22, 17, 20

Discussion continues as students try to draw conclusions from the data. You suggest putting the data into a line graph so it will be easier to see and understand. The children notice that the smallest value is 15 and the largest is 22, so they direct you to draw a horizontal number line beginning with 15 and ending with 22. Next, an *X* is placed above each number to mark its frequency in the set of data. Figure 18.3 shows the completed line plot. The students are easily able to see that 17 occurred most often, followed by 18 and 20. The student who had only 15 in the package felt they missed out and those who had 22 felt lucky. The class talked about why there might not always be the same amount in each package.

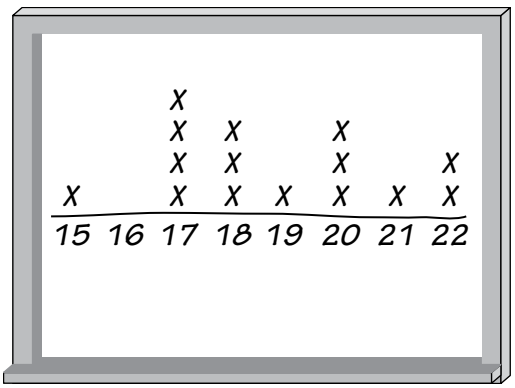


FIGURE 18.3 Line plot of number of mixed lollies per package

Line plots are also useful for teachers to quickly display the results of class assessment tasks. The highest, lowest and most frequent scores as well as the spread of scores can be easily seen in a line plot.

Stem-and-leaf plots

A *stem-and-leaf plot* is another quick way to display data and provides a quite different representation than when the data are arranged in a line plot or bar graph. It works best with data that span several decades, since the plot is usually organised by tens. This plot is a little more abstract than the line plot, but it may be used successfully with students in the middle years.

It is often useful to display data in more than 1 way. Consider the mixed lollies data just displayed in a line plot and rearranged in a stem-and-leaf plot (see figure 18.4).

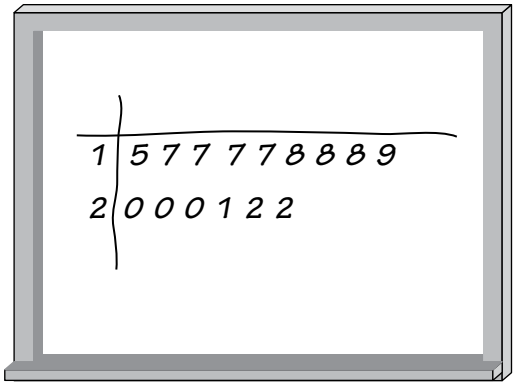


FIGURE 18.4 Stem-and-leaf plot of number of mixed lollies per package

To begin, divide each value into tens and ones. The tens become the 'stem,' and the ones will be the 'leaves'. Notice that the data fall into 2 decades: the tens and twenties. A vertical line is drawn with the tens values (1 and 2) on the left of the line; this is the stem. The ones values are placed on the right side, evenly spaced and in numerical order from lowest to highest for each 'leaf'. By examining the plot, it is easy to see that 17 occurred most often in this set of data. From this arrangement you can also see that more packages contained lolly amounts in the tens than in the twenties. Now, all those students who had 20 or more feel lucky.

Stem-and-leaf plots also allow for easy calculation of the mean (or arithmetic average) of the data.

Stem-and-leaf plots are very effective when used to compare 2 sets of data in the same plot. In the middle year levels students can successfully compare more than 1 data set. Suppose you wanted to explore some questions related to the height of students in a Year 4 class. Comparing the heights of the boys and girls will generate some interesting discussion. Questions such as 'Which group is tallest?' and 'Which group has the most variability?' will naturally arise. After some conjectures have been made, it is time to have students measure their heights and begin to analyse the data. The following shows the construction of a stem-and-leaf plot for student height. The heights of 27 Year 4 students (15 girls and 12 boys) are reported in centimetres in the following table.

Boys				Girls			
118	132	135	137	122	155	114	125
120	125	147	129	155	137	136	137
133	148	153	125	134	130	133	145
				148	148	147	

Rather than use a traditional frequency distribution, the values are organised in a stem-and-leaf plot, shown in figure 18.5. The stem represents the hundreds and tens places of the data on student height and the leaves represent the ones place.

Thus, in the bottom row of figure 18.5, 8 | 11 | 4 means 1 girl has a height of 114 cm and 1 boy is 118 cm tall. The stem-and-leaf plot preserves the individual measures while revealing the general shape of the organised data. The plot presents all of the information (in this case for both groups) and provides a clear visual picture of it.

In addition to the quick graphs and plots mentioned previously, students should be able to represent information with graphs that require more construction time and effort both on paper and electronically.

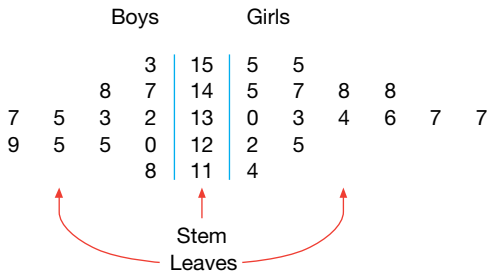


FIGURE 18.5 Stem-and-leaf plot of heights of boys and girls

Box plots

A *box plot* (also called a box-and-whisker plot) summarises data and provides a visual means of showing variability — the spread of the data. The box plot shown in figure 18.6 compares the heights of girls to boys. The median is a key reference point; we will talk more about the median later in the chapter. The lower hinge, or lower quartile, is the median of the lower half of the data; the upper hinge, or upper quartile, is the median of the upper half of the data. These are found by computing the medians of the data in the lower and upper halves, respectively. The interquartile range (IQR) is a measure of variability and is the difference between the upper and lower quartiles. In figure 18.6, the IQR for the boys is 17 (142–125) and that for the girls is 18 (148–130). The smallest and largest heights represent the lower extreme and the upper extreme. The lines (also called 'whiskers') extending from the top of the upper quartile to the maximum value and then from the bottom of the lower quartile to the minimum value provide another visual indication of variability.

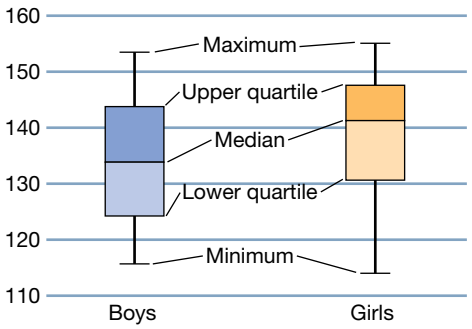


FIGURE 18.6 Box plots for heights of boys and girls

The box plot shows many things. For example, it shows that the median height for the girls is greater than that for the boys. Although the groups have about the same IQR, the boys are a bit more evenly distributed throughout the box than are the girls. (Why? Because the median of the boys is closer to the middle of the box than is the median of the girls.) The box plot is derived naturally from a stem-and-leaf plot or line plot. The box plot shows many important characteristics of a group visually and, when 2 or more groups are shown on the same graph, it allows comparisons to be made easily. Since each section represents 25% of the data, multiplicative comparisons can be made. For example, about 75% of the girls are taller than about 50% of the boys (the median for the boys and the lower quartile for the girls are about the same height). The box plot is an appropriate display for middle-school students to construct. It provides a useful connection to descriptive statistics such as median and range. In addition, many standardised test results are reported as box plots, so this is an

important kind of graph for teachers to understand even if they don't teach it. Finally, box plots allow 2 groups with unequal amounts of data to be compared to each other since both sets are converted to fractional representations.

Picture graphs

In *picture graphs*, data are represented by pictures. For example, children may graph pictures of their favourite food or the pets they own. A picture can represent 1 object (figure 18.7) or several (figure 18.8). To properly interpret picture graphs, children must know how much each object represents. Research shows that students often ignore such coding information when interpreting graphs (Bright & Hoeffner 1993; Friel, Curcio & Bright 2001).

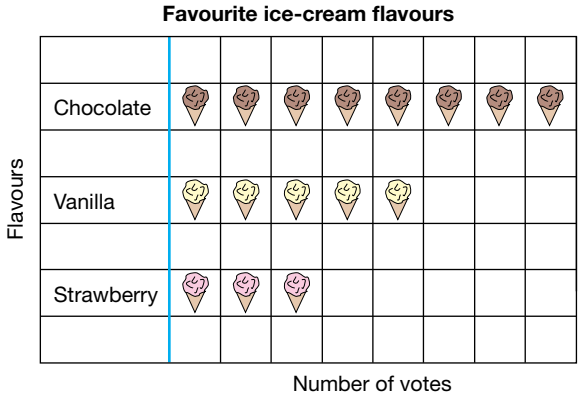


FIGURE 18.7 A picture graph in which each picture represents 1 object

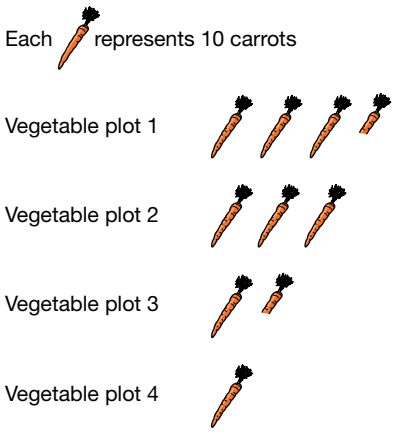


FIGURE 18.8 Number of carrots Alex pulled from each vegetable plot

Bar graphs and histograms

Bar graphs are used mostly for discrete, or separate and distinct, data; the bars represent these data. For example, they might graph the number of children's birthdays in each month or the number of students who travel to school by bus, by car or on foot. Figure 18.1 shows that values can be read from the axis. Figure 18.9 shows that values can be reported directly on the graph. Bar graphs are often used for quick visual comparisons of categories of data and are appropriate for all ages.

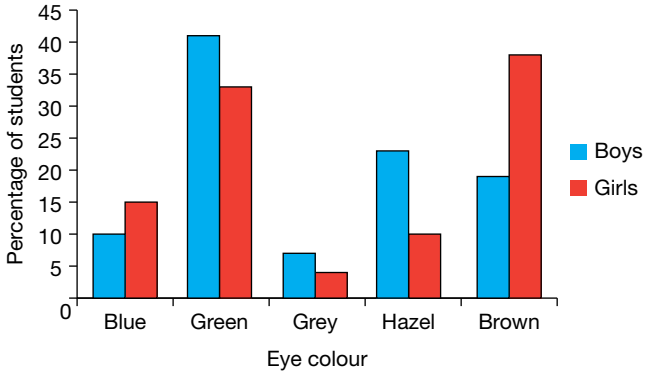


FIGURE 18.9 Bar graph showing the percentages of eye colours of a Year 5 class

Although a *histogram* looks like a bar graph (figure 18.10), there are some key differences that discriminate them from each other. A histogram is used with continuous data, not discrete data. Therefore the data are represented with connected bars, each representing an interval. Any data that falls within that interval appear in the bar. If a student uses the internet for 22 hours, his or her data point is located between 20 and 29.99 hours. The interval includes all of the numbers up to, but not including, the maximum value of the interval. Therefore, individual data points may not be distinguishable from reading a histogram as they would be in a bar graph. Histograms are more appropriate for middle year level students, although many younger children draw them mistakenly when attempting to create bar graphs.

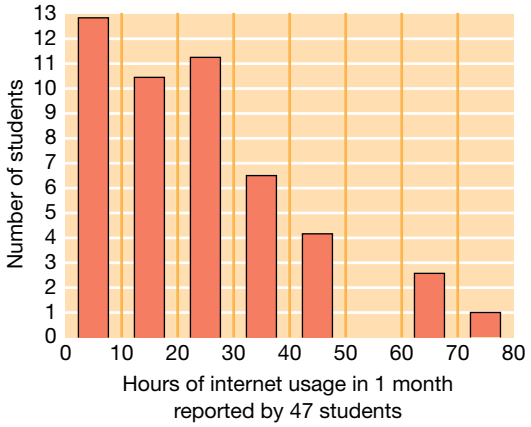


FIGURE 18.10 A histogram showing students' hours of internet usage in 1 month

Pie graphs

A *pie graph* is a circle representing the whole, with wedges reporting percentages of the whole, as illustrated in figure 18.11. The pie graph is popular because it is easy to interpret; however, it has major limitations in that it represents only a fixed moment in time, and it cannot exceed 100%. For example, students may graph how they spend their weekly allowance or the favourite colours of the class. While students in the senior primary year levels can read and interpret pie graphs, construction manually (not electronically) requires the use of fractions, percentages, proportions and measurement of angles. Pie graph construction is more appropriate for middle-school students; however, there is an effective, concrete method of creating pie graphs successfully with primary students.

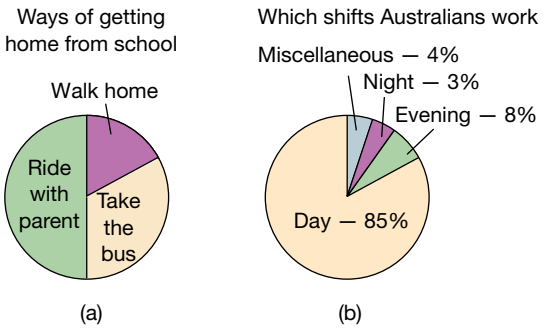
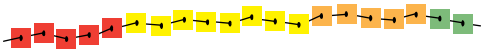
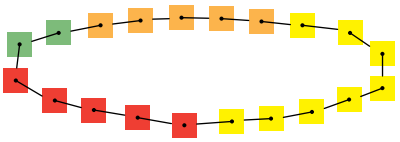


FIGURE 18.11 Pie graphs

As illustrated with the line and stem-and-leaf plots, showing the same data in different displays can be both useful and effective. For example, the data shown in the bar graph in figure 18.1 can be easily shown in a pie graph. Cubes of different colours to match the fruit could be strung together as shown here an equidistance apart:



The string can then be tied in a circle:



This experience helps make a connection between bar and pie graphs. It also provides a natural context for fractions and percentages. For example, the pie graph suggests that one-quarter of the fruit are oranges.

This model can be extended by placing a 'metre' tape or a 100-centimetre strip of paper marked with similar units, around the circle of blocks to form concentric circles, as shown in figure 18.12(a). Comparing the sections suggested by the different groups of coloured blocks with the marks on the strip or metre tape will identify percentages that can be easily read.

Note that the first fruit, apples, represented by red cubes (or beads) is placed at 0 and the other fruit are grouped together until the last fruit, pears, represented by green cubes (or beads) joins up to the beginning. The most common representation in pie graphs commences at the 12 o'clock position and continues in a clockwise direction back to 12 o'clock.

As a more concrete visualisation, 20 children can be arranged in a circle and the 'wedges' of a circle graph duplicated with string, as illustrated in figure 18.12(b). Both of the models in figure 18.12 make it easy to estimate or read the percentages and conclude, for example, that more than 50% of the children chose apples or bananas.

Similar observations might have been made directly from the bar graph in figure 18.1, but conclusions involving fractions and percentages are much more obvious from the pie graph. Technology applications, such as spreadsheets, are also useful tools that allow students to create bar graphs and pie graphs of the data. The process of moving from a bar graph to a pie graph provides different perspectives for the same set of data, and research suggests that developing such multiple perspectives helps promote greater understanding (Shaughnessy 2006).

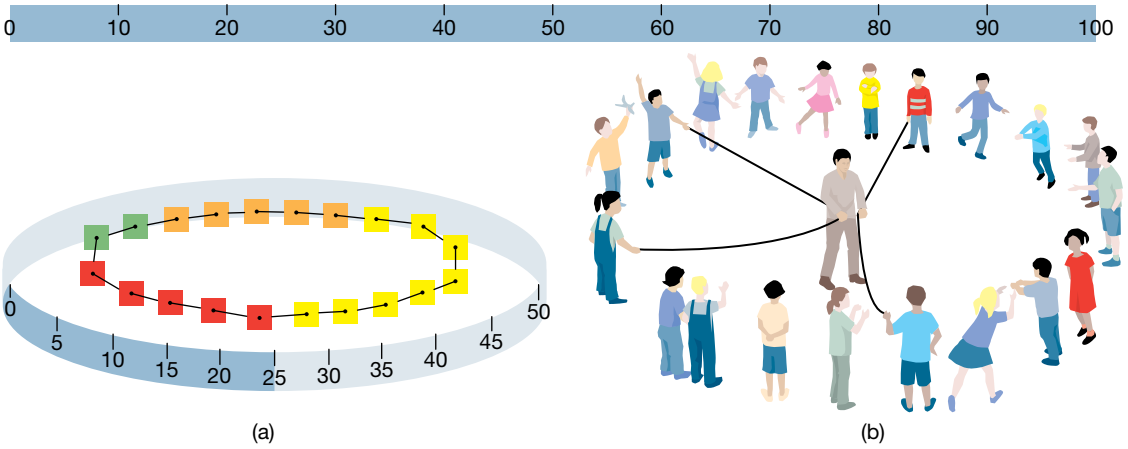


FIGURE 18.12 Models for interpreting circle graph data

Line graphs

Line graphs are effective for showing trends over time. In line graphs, points on a grid are used to represent continuous or uninterrupted data. Each axis is clearly labelled so the data shown can be interpreted properly. A wide variety of line graphs exist and are used, but 3 basic assumptions are inherent.

1. The data are continuous rather than discrete. This means that the data are grouped along a continuous scale and cannot be 'counted'.
2. Data can occur between points with continuous data. Because of this, the line can also be used to interpret values between plotted data.
3. Change is accurately represented with linear functions (i.e. by lines) rather than some other curve.

As figure 18.13 shows, line graphs are particularly good for showing variations or changes over time, such as hours of daylight, temperatures, rainfall and so on. For example, students could graph plant growth in a science experiment. Line graphs are also an effective visual means of comparing several sets of data. Constructing or interpreting line graphs requires students to examine both horizontal and vertical axes, which is good preparation for coordinates. Line graphs are more frequently used at the middle-school level.

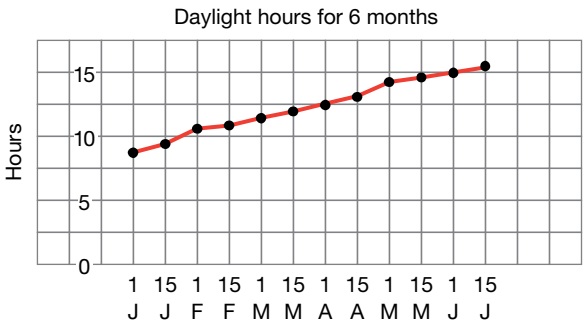


FIGURE 18.13 A line graph of a single set of varying data

Graphical roundup

Each of these types of graph deserves classroom time as students examine ways to display their data. Children need experience constructing them and interpreting information that is represented. In the children's book *The Best Vacation Ever* (Murphy 1997), a little girl collects and displays data for her family so they can plan a holiday. In the classroom 18.2 gives students ideas to think about when planning to display a collection of data.

IN THE CLASSROOM 18.2

Sharing data

Objective: Develop techniques for handling data.

Year level: 3–5

Collect some data but before sharing it, seek agreement on the following issues for graphing and reporting the results.

Before graphing

- Shall we start with a sketch graph?
- What kind of graph will best display our data?
- Can we show our data in more than 1 way?
- How shall we label our graph?

Before reporting results

- What questions can we answer with our data?
- Who might be interested in our results?
- How will we report our findings?

Questions

1. Why do these questions need to be answered before graphing and planning our report?
2. Name 2 other important questions that need to be answered before beginning.

The availability of graphing calculators and graphing programs allows for easy construction of a variety of graphs. This availability of different graphs via technology places a greater premium on interpreting and understanding the graphs that are so easily produced. As children become familiar with different graphs, they should recognise some characteristics associated with them. Figure 18.14 highlights specific characteristics of graphs that are encountered in primary and middle school. The focus here is not on memorising characteristics of these graphs, but rather on becoming aware that each type has strengths. The selection of a graph should capitalise on these strengths while recognising any of its limitations. Figure 18.15 shows graphical complexity and a suggested progression for introducing different types of graph (Friel, Curcio & Bright 2001, p. 147). Again, memorising this information is not the goal. Instead, be aware of how some graphs — due to their complexity — are better understood at different levels, which includes students' ability to read and interpret a graph versus progress for introduction of types of graph (includes creating the graph, and both reading and constructing graphs).

Type of graph	Characteristics
Real	• Used by young children • Actual objects are placed on a graph
Line plot	• Provides a quick way to examine the shape and variation or spread of data • Gives a bar-graph-like representation

FIGURE 18.14 Characteristics of graphs

(continued)

FIGURE 18.14 *(continued)*

Type of graph	Characteristics
Stem-and-leaf plot	• Efficient way to show detailed data • Provides similar visual patterns as a bar graph but has more detailed information • Uses stem-and-leaf coding that needs to be understood • Technology has facilitated its use
Picture	• Frequently encountered in newspapers and reports • Generally easy to use and interpret but visuals may be misleading • Codes/keys which accompany graph need to be understood
Bar	• Mostly used for discrete data • Frequently encountered in newspapers and reports • Easy to interpret • Uses scales/codes that need to be understood
Histogram	• Used for continuous data, data grouped in intervals • Similar to bar graph but individual data are not visible • Effective to display data with a large range but limited values
Pie	• Frequently encountered in newspapers and reports • Shows fractional parts, which are based on a whole or 100% • Easy to use and interpret • Difficult to construct by hand but easy to do so with technology
Line	• Frequently encountered in newspapers and reports • Used for continuous data • Effective to show patterns, trends, comparisons and change over time • Uses vertical and horizontal scales that need to be understood • Provides good readiness for coordinate graphs
Box plot	• Provides useful information about the variability of data • Requires knowledge of range, median and quartiles to interpret • Can be used to compare 2 data sets with unequal amounts of data • Technology has facilitated its use

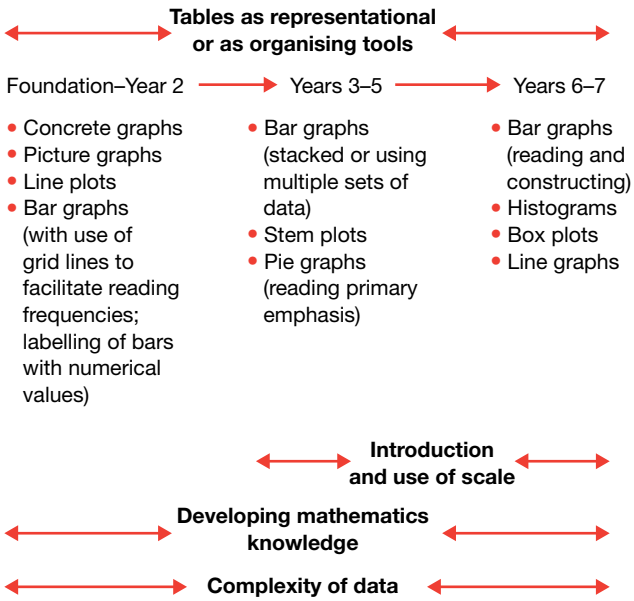


FIGURE 18.15 Graph/display complexity: suggested progress for introduction of types of graph (includes both reading and constructing graphs)

18.3 Analysing data: descriptive statistics

LEARNING OUTCOME 18.3 Recognising what descriptive statistics are appropriate to introduce in the primary year levels.

Another way to analyse data is to use descriptive statistics. So much information exists today that it must often be simplified or reduced in ways other than by graphs. The organisation and summarisation of data is called *descriptive* statistics. Descriptive statistics are in common use. They are introduced in the primary years through data collection and graphs, and then extended with further exploration and practice activities in the middle years. Here are some familiar examples:

'Most children in Year 5 are 10 years old.'
'The median family income in our region is $45 250.'
'The average temperature today was 29 °C.'

Each of these statements uses a number to summarise what is typical for a current situation or condition. Two of the most common types of descriptive statistics include measures of variation and measures of central tendency. For each of the descriptive statistics that follow, we provide a concrete example using cubes and an example using numbers. For some children, manipulating the physical model not only helps them understand the formula but also promotes retention.

Measures of central tendency or averages

The word *average* is a popular statistical term that many children have heard. It is used to report such things as average temperature, average family income, cricket batting averages and average life expectancy.

Any number that is used to describe the centre or middle of a set of values is called an *average* of those values. Many different averages exist, but 3 — mean, median and mode — are commonly encountered in primary and middle school. Simply being able to state the algorithm for finding these statistics is not enough. To support the development of data sense, each of these 'averages' should be developed meaningfully through concrete activities before introducing computation. Such experiences provide greater conceptual knowledge or understanding of the concept of average. Spreadsheets and software programs provide explorations that allow middle school students to investigate what happens to various descriptive statistics when different values are used (Wilson & Kraft 1995).

Mean

The *mean* is called the arithmetic average because it is determined by adding all the values involved and dividing by the number of addends. The mean is the most difficult to compute, although it can be understood by children beginning in the upper primary years. It is used with numeric data only. When people talk about finding the average of a set of data, they are often referring to calculating the mean. It is important that children realise that the mean is not the only type of average.

Simply being able to state or use the algorithm does not indicate understanding of the mean. Difficulties in interpreting the mean were shown on a national assessment conducted in the United States. Figure 18.16 shows a question that required Year 8 students to determine what data would be reasonable for a given mean. Less than 40% answered all 4 choices correctly. These assessment results illustrate that many middle year level students are able to calculate averages when asked to do so, but the depth of their understanding of the concept of average is shallow.

Providing conceptual interpretations for the mean helps develop understanding of average. Say that 4 students each have 9, 5, 3 and 7 trading cards respectively. The cubes in figure 18.17(a) represent the number of cards each child has. The first interpretation involves the concept of equal distribution or sharing, an idea very familiar to students. In this interpretation, the mean is identified as the number that describes the data if each piece of data was 'evened out' or the same as all others. People often think

of this as what is typical for the data. If children are asked to even out or share the cards so that each student has a fair or equal share of cards, this process produces a mean of 6 cards per student — see figure 18.17(b).

Akira read from a book on Monday, Tuesday and Wednesday. He read an average of 10 pages per day. Circle whether each of the following is possible or not possible.

Possible	Not possible		Pages read		
			Monday	Tuesday	Wednesday
A	(A)	**(a)**	4 pages	4 pages	2 pages
(B)	B	**(b)**	9 pages	10 pages	11 pages
(C)	C	**(c)**	5 pages	10 pages	15 pages
D	(D)	**(d)**	10 pages	15 pages	20 pages

FIGURE 18.16 US-based Year 8 question on interpreting an average (correct responses are circled)

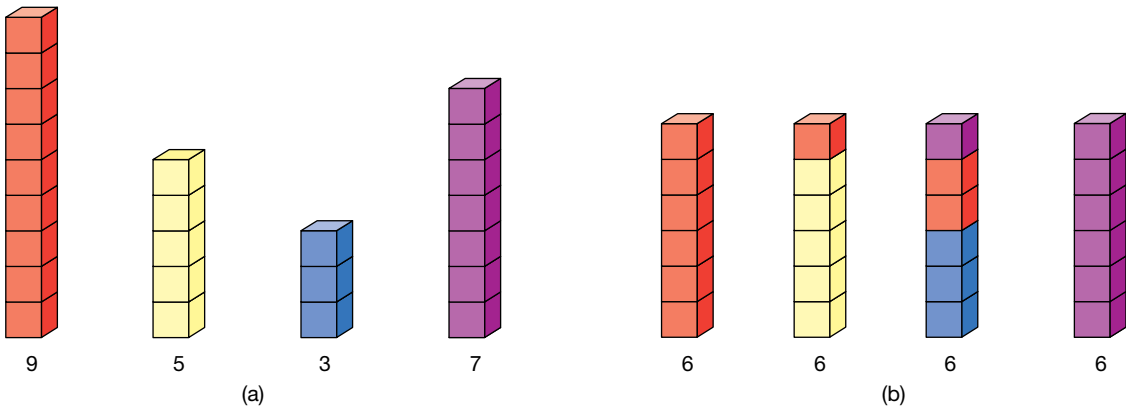


FIGURE 18.17 Equal distribution or sharing model for the mean

As students gain experience and become comfortable with the equal distribution interpretation, they also begin to realise that with large data sets, such as whole-class data, trying to share equally in a concrete way becomes cumbersome. A more numeric approach is needed. Students often discover the add-then-divide algorithm on their own, or you can introduce it with concrete materials. The mean also could have been determined by computing:

$$\text{Mean} = \frac{9 + 5 + 3 + 7}{4} = \frac{24}{4} = 6$$

Figure 18.18 shows a pictorial model for the add-then-divide algorithm for the mean. Test scores are returned to children on pieces of adding machine tape, and the length of each strip is determined by the score (e.g. a score of 88 is 88 centimetres long and a score of 64 is 64 centimetres long). Scores can be physically compared using the tapes (e.g. it is clear that the score on test 2 was higher). To show the mean score, simply tape the 2 strips of paper together (add) and then fold the resulting strip in half (divide by 2). It also illustrates the effect of a further test score of 0 when the fold number on the tape is increased by 1. Using this example, the 3 test scores would be 88, 64 and 0. The length of the tape is still 152 centimetres. However, if we fold the tape into thirds (as there are now 3 scores) rather than in half, the length of the strip is a little over 50 centimetres. Therefore, the mean test score dropped over 25 points! Students find this technique is appealing and enlightening.

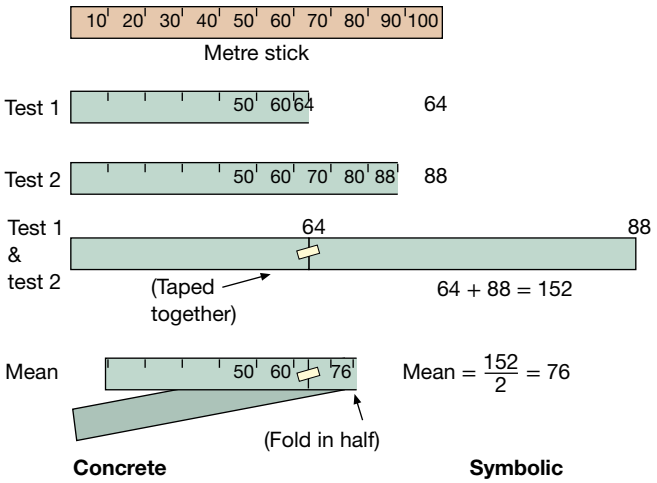


FIGURE 18.18 Model for finding the mean using lengths of adding machine tape to indicate test scores

Not all data sets divide evenly. Once students are comfortable with the concept of evening out, they can discuss what to do with remainders, which makes a natural connection to decimals. They can also discuss how some data cannot be divided and some averages that are calculated are not realistic. For example, in one data set the mean indicated that the average number of children per family was 2.5. Children will be quick to point out that you can't have five-tenths or one-half of a child.

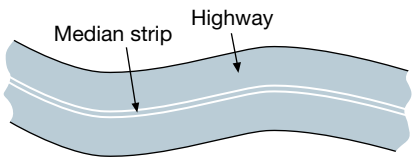
These types of experience help students understand some fundamental notions related to the mean; namely, that the mean must be somewhere between the values averaged and that it is the typical value or balancing point for a set of data.

Median

The *median* is another type of average that can easily be identified without the need for computation. It is used with numeric data only and is the middle value in a data set of ordered data. The median divides the data into 2 equal subsets. Thus the same number of values are above as below the median. The median of the children's ages listed below is easy to find (and also in figure 18.20). Before looking for the median, the data must be ordered usually smallest to largest — that is,

6, 6, 7, 7, 7, 8, 8, 8, 8, 8, 8, 8,
9, 9, 9, 9, 9, 9, 9, 9, 9, 10, 10.
A total of 23 children.

Once ordered, the outside numbers can be removed, one at a time, until the middle value remains. In this case, 11 numbers are removed and the middle age or median is 8 years. Reference to a highway median strip will remind students that a median strip in statistics is a middle position. Notice that the median is not the middle of the range of data because data aren't always spread symmetrically over the range. One way of visually representing this is to provide children with a strip of grid paper that has exactly as many boxes as data values. Have them put each ordered data point into a box and fold the strip in half. The median is the fold. In the children's age example, the median simply tells us that there are as many children in the group from ages 6 to 8 as there are in the group containing ages 8 to 10.



The median, too, can be modelled with numbers. For example, consider the 5 test scores shown on cards in figure 18.19(a). They have been placed in order from lowest to highest. To find the middle score, or median, simply remove the highest and lowest cards simultaneously. Continue this process until the middle card remains, as shown in figure 18.19(b). This score, 88, is the median.

A test is then given to 6 students with the results shown in figure 18.19(c). Again, the highest and lowest cards are removed simultaneously until 2 cards remain. In this case, as shown in figure 18.19(d), the median is the middle point between these 2 scores — that is, 86. If you add together the 2 remaining cards and divide them by 2, you will obtain the median as well $(84 + 88)/2 = 86$. It is recommended that younger children first be introduced to the median using data sets with an odd number of entries so that the median is one of the entries in the data set.

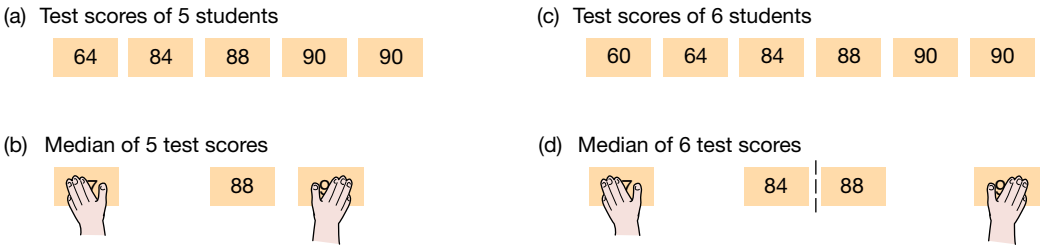


FIGURE 18.19 Finding the medians of test scores

Mode

The *mode* is the value that occurs most frequently in a collection of data. Using concrete materials such as cubes, the mode is easily identified as the tower height that is greatest. In figure 18.20, each cube represents 1 child. It is easy to see that 9 is the mode because it has the highest tower, while the other values are less. In graphical terms, this is also the largest portion — the tallest column in a bar graph, for example. In figure 18.21, the most frequently occurring test score is 90 (it occurred 3 times), so the mode is 90. The mode is a versatile average in that it may be used with both numeric and nonnumeric data (also called categorical data). The mode is easy to find and is affected very little by extreme scores. Young children are often interested in which item on the graph received the most votes or is the most popular. Therefore, their initial experience with average begins with mode. The children are comfortable using the mathematical term, *mode*, if the teacher introduces it to them.

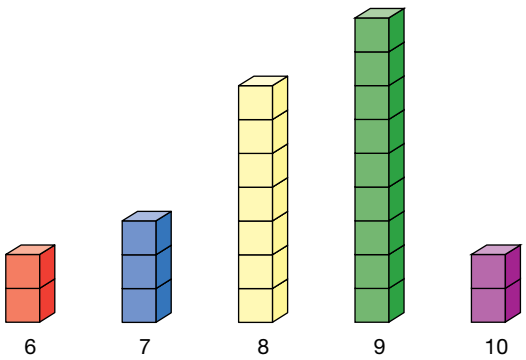


FIGURE 18.20 Ages of students at a swim centre

Students' ages within a class provide an excellent application of mode because within a given class a large number of children are the 'same' age. Businesses also frequently rely on the mode to select merchandise. Suppose, for example, that you own a shoe store. The mean and/or median size of shoe you sell has no practical value for restocking; however, the modal shoe size holds clear implications because you want to stock

the sizes most people wear. In some cases, a data set will have multiple values that occur most frequently. For example, if a class of students is asked how old they are, and there are 9 students who are 10 and 9 students who are 11, and no age occurs more than 9 times, then the data set has 2 modes. In the case of 2, the set is considered to be bimodal; 3 modes is called trimodal, and so on.

Choosing averages

As students use averages, they must be aware of how averages are influenced by data. For example, in figure 18.21 the mean for 19 tests is 87 (rounded up from 86.89) and the median is 88. If an extremely low score of 17 is included in the data, the mean drops to 83 (rounded down from 83.4) but the median changes only slightly to 87.5. One negative characteristic of the mean is that its value is affected by extreme scores.

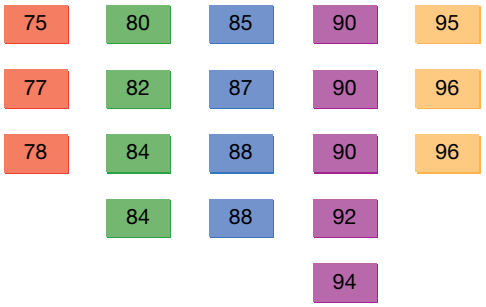


FIGURE 18.21 Model for finding the mode and median of 19 test scores

Grasp of average is a powerful tool in estimation and problem solving. Problems such as the one in figure 18.22 provide opportunities to apply averages and estimation in everyday situations.

Finding the mean, median and mode for the same data can generate discussion about when certain averages give a better result. You can use In the classroom 18.3 to allow children to calculate averages with concrete materials or numbers and to discuss how one might select a particular average to describe the data. For another example, look at basketball team salaries in In the classroom 18.4. Calculating the mean, median and mode provides practice in computational skills. More important, however, is deciding which of these averages to report. The median salary of $500 000 or the modal salary of $480 000 seems more representative than the mean salary of $605 000. If salary negotiations were taking place, the players might cite one average and the owners a very different average. Discussing which average is appropriate for what purpose helps students better understand why different ones exist and are used. Once children have collected their own data, they should determine which averages to calculate and report.

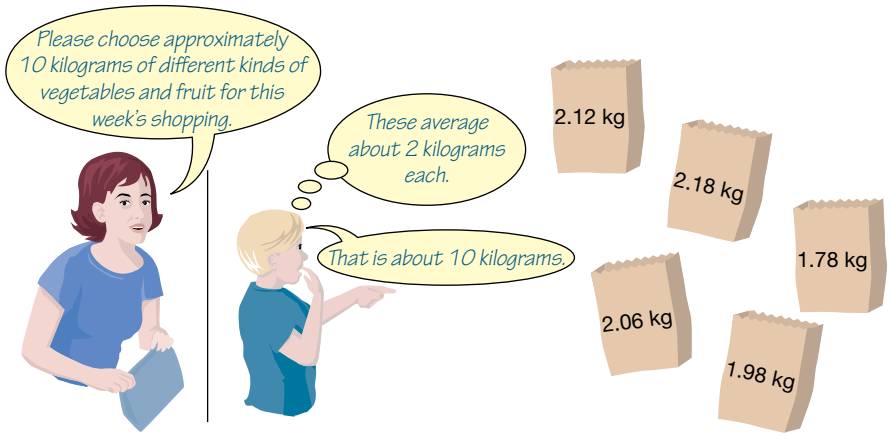


FIGURE 18.22 Example problem for developing averaging and estimating skills

IN THE CLASSROOM 18.3

Peanuts

Objective: Choose the best average.

Year level: 5–8

Suppose you have opened some Nutty Bars to check the company's claim of an 'average' of 8 peanuts per bar. Here is what you found after opening 10 bars.

Bar	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
Number of peanuts	5	8	8	8	11	7	8	6	6	6

Create some different sketch graphs so you can examine the shape of the data. Which graph would the company probably use to promote their product? Why?

Calculate the averages. You may use counters such as beans and grid paper to represent the peanuts if you wish.

- What is the mean number of nuts?
- What is the median number of nuts?
- What is the modal number of nuts?
- Which average did the company probably use? Why?

Write at least 3 questions that can be answered by your graphs and statistics.

1. _____

2. _____

3. _____

IN THE CLASSROOM 18.4

What's the average?

Objective: Calculate mean, median and mode, and decide which average is most appropriate to use for a particular purpose.

Year level: 5–8

The following shows the yearly salaries of one professional basketball team.

Salary ($)	
380 000	520 000
420 000	600 000
480 000	840 000
480 000	900 000
480 000	950 000

Questions

1. What is the mean salary for the team?
2. What is the median salary?
3. What is the mode?
4. What is the best average to report for the team? Why is your choice better than other averages?
5. Access other examples of salaries online and report the averages.

Each of the averages can be modelled and developed in ways that are appealing, interesting and meaningful. No new mathematics is required, yet learning about averages provides a vehicle for applying many mathematical concepts and skills that students are developing.

Care must be taken to ensure that statistics is viewed as more than a series of skills or techniques. For example, finding an average is an important skill that should be developed; however, the teaching of statistics must not stop with the 'how to'. Rather, it must raise questions such as 'When is an average useful?' or 'Can average be misleading?'

For example, the ice-cream survey in figure 18.7 reports that 8 students like chocolate, 5 like vanilla and 3 like strawberry. This picture graph clearly and accurately shows student ice-cream preferences. These preferences are an example of nonnumeric data, or categorical data. A mean or median of these data could be computed but would be inappropriate. In fact, a mean or median is meaningless for these data! Before any statistics are computed, challenge students to decide what questions are to be answered and discuss what statistics, if any, are needed to answer them. Nonnumeric data such as ice-cream flavours are better analysed with frequency tables or graphs. If an average is desired in this case, the idea of favourite or mode would be the appropriate average to use.

Additional questions might include:

'Why should the average be reported?'
'What average is most appropriate? Why?'
'What degree of precision is needed?'

These questions are essential and must be asked regularly. This sort of discussion will support students as they develop their data sense. The teaching of statistics in primary school must aim higher than skill development. Students should know how to get a statistic; however, they must also know what the statistics tell them.

Measures of variation

As students investigate measures of central tendency, they should be presented with multiple data sets that have the same median and mode. Figure 18.23 shows the number of points scored by 2 basketball players for their last 10 games. When comparing the data, it should be noted that the mean, median and mode do not help in distinguishing between the 2 players. In this case, a measure of variability could help distinguish their performance by illustrating a difference in the consistency of the 2 players. Measures of variability are used to describe how much the data are spread out. One measure of variability is the *range*. The range is a simple measure that tells the difference between the maximum and minimum values in the data set. Students may find the range by comparing the maximum value to see how much greater it is than the minimum value. The range for player 1's scores may be found by subtracting 16 from 20, resulting in a range of 4; for player 2, you would subtract 0 from 40, resulting in a range of 40. The difference in the ranges shows that player 2 has a great deal more variability in his or her scores.

Points scored in last 10 games		Mean	Median	Mode
Player 1	16, 20, 20, 18, 22, 24, 20, 20, 20, 20	20	20	20
Player 2	10, 2, 20, 36, 4, 20, 38, 0, 30, 40	20	20	20

FIGURE 18.23 Comparison of scores of basketball players for the last 10 games

We have already mentioned another measure of variability, the interquartile range (IQR), which is found when subtracting the first from the third quartile. (See the earlier discussion of box plots.) One

more measure of variability that may be explored by middle school students is the *mean absolute deviation*. The mean absolute deviation has to do with the distance (always positive) of each data entry from the mean. The distances may be found by taking the absolute value of the difference beween the mean and the values. These distances are then averaged to obtain the mean absolute deviation of the data from the mean. Figure 18.24 shows the calculation of the mean absolute deviation of scores for the basketball player data seen in figure 18.23. The mean absolute deviation shows that player 1 was much more consistent than player 2.

The introduction of range, IQR and mean absolute variation to measure variability are important for primary and middle years students to prepare them for the study of variance and standard deviation in secondary year levels.

Player 1		Player 2	
16	$\lvert 16 - 20 \rvert = 4$	10	$\lvert 10 - 20 \rvert = 10$
20	$\lvert 20 - 20 \rvert = 0$	2	$\lvert 2 - 20 \rvert = 18$
20	$\lvert 20 - 20 \rvert = 0$	20	$\lvert 20 - 20 \rvert = 0$
18	$\lvert 18 - 20 \rvert = 2$	36	$\lvert 36 - 20 \rvert = 16$
22	$\lvert 22 - 20 \rvert = 2$	4	$\lvert 4 - 20 \rvert = 16$
24	$\lvert 24 - 20 \rvert = 4$	20	$\lvert 20 - 20 \rvert = 0$
20	$\lvert 20 - 20 \rvert = 0$	38	$\lvert 38 - 20 \rvert = 18$
20	$\lvert 20 - 20 \rvert = 0$	0	$\lvert 0 - 20 \rvert = 20$
20	$\lvert 20 - 20 \rvert = 0$	30	$\lvert 30 - 20 \rvert = 10$
20	$\lvert 20 - 20 \rvert = 0$	40	$\lvert 40 - 20 \rvert = 20$
Sum	12	**Sum**	128
Average (mean absolute deviation)	$12 \div 10 =$ **1.2**	**Average (mean absolute deviation)**	$128 \div 10 =$ **12.8**

FIGURE 18.24 Calculation of mean absolute deviation for basketball players' scores for the last 10 games

18.4 Interpreting results

LEARNING OUTCOME 18.4 Understanding how to develop your students' data sense and ability to communicate results.

Data sense

Once data have been collected and displayed, they should be analysed and interpreted. Just as children develop knowledge about numbers or number sense, they can also develop knowledge about statistics or *data sense*. Data sense is gradually developed as students formulate questions, collect data, construct graphs, find descriptive statistics and interpret them in a variety of contexts. Students with data sense are able to determine how data should be interpreted. They are able to read and evaluate statistical information being presented, such as material presented by the media. Students also display data sense when they are able to use statistical language when reasoning about data. The aim for this strand is for students to develop both procedural knowledge (how to construct a graph or calculate a statistic) and conceptual knowledge (understanding what a graph or statistic is communicating) (Friel 1998).

One way to begin interpreting data is through the use of questions. Students should be encouraged to examine their results and discuss questions that may be answered by the data. There are 3 levels of graph

comprehension, which progress from lower-level to higher-level questions. Your aim should be to teach students to move beyond lower-level thinking and to ask and answer higher-level questions.

- *Reading the data*. The student is able to answer specific questions for which the answer is prominently displayed. For example, 'Which player averaged the most points?'
- *Reading between the data*. The student is able to find relationships in the data, such as comparison, and is able to operate on the data. For example, 'Which player had a median less than their mean?'
- *Reading beyond the data.* The student is able to predict or make inferences. For example, 'Which player had the greatest range? The smallest range? What do these numbers tell you about the player? Who would you prefer on your team? Why?'

Data may also be interpreted by describing the shape of the data in the graph. Corwin and Friel (1990) suggest first having students use informal language — such as 'clumps', 'holes' and 'spread out' — to describe features of the data. You may want to create a word wall, or visible word list, for students to refer to when describing data. Second, students should attempt to develop theories about why the data look the way they do. This second step encourages students to read beyond the data.

Another beneficial graphical interpretation task is to give students mystery graphs that are missing some of their labels and have students predict what the data might be. For example, groups of students may have each measured a body part such as arms, legs or distance around heads, necks and wrists. A graph that contains measurements from 14 to 20 centimetres is displayed. Students would be asked to hypothesise which body parts might be illustrated on the graph. This activity also reinforces the importance of using labels on graphs.

A variation of the mystery graph activity involves the use of a graphing calculator and a calculator-based laboratory (CBL). With a CBL motion detector, graphs may be created to show movements. Students may be given a CBL graph, and the challenge is to move so they can duplicate the graph. For example, figure 18.25 shows the graphs of 3 different people walking. Time is shown on the horizontal axis, and the distance from a motion detector is shown on the vertical axis. Examine parts (a), (b) and (c) and decide which of these graphs shows movement toward the motion detector. Part (c) is different because, instead of showing continuous movement away or toward, it shows a person leaving and then returning. These sorts of activities encourage students to consider the axes of the graphs and think carefully about how data are placed on a graph.

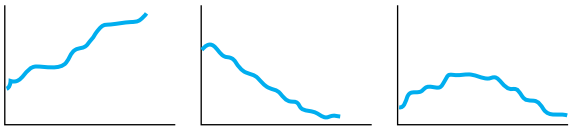


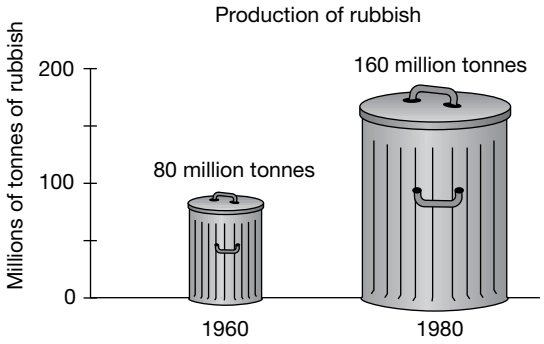
FIGURE 18.25 Graphs showing distance and time

Misleading graphs

Another important component in developing data sense is the ability to critically examine graphs and correctly interpret the data presented. As Harper (2004, p. 341) notes:

> Although viewing data graphically allows the reader to see the trend of the data easily, the choice of scaling along the *x*- and *y*-axes could influence their interpretation of the data.

Sometimes even simple graphs may be misleading. For example, consider the graph shown in figure 18.26. Year 8 students were asked to explain why this 200% graph was misleading.



The pictograph shown above is misleading. Explain why.

Answer: Both the width and the height have been doubled.

FIGURE 18.26 US-based Year 8 question on interpreting graphs

Although the captions in the graph indicate that the amount of rubbish has doubled in 2 decades, the visual elements reflect a doubling of both the width and height to produce a figure whose area is 4 times greater. People may focus on the visual graph and ignore the numerical data that accompany the graph. In fact, less than 10% of Year 8 students identified the critical problem associated with the graph in figure 18.27, which suggests that attention needs to be given to helping children examine graphs with a careful and suspicious eye.

Graphs may also be deceptive in other ways. For example, the graph in figure 18.27 (a) reports changes in allowances for 3 children. It shows that Ann's allowance was doubled, Bill's tripled and Chris's increased by one-half. Based on this information, Bill may be feeling philanthropic and Chris may complain of hard times.

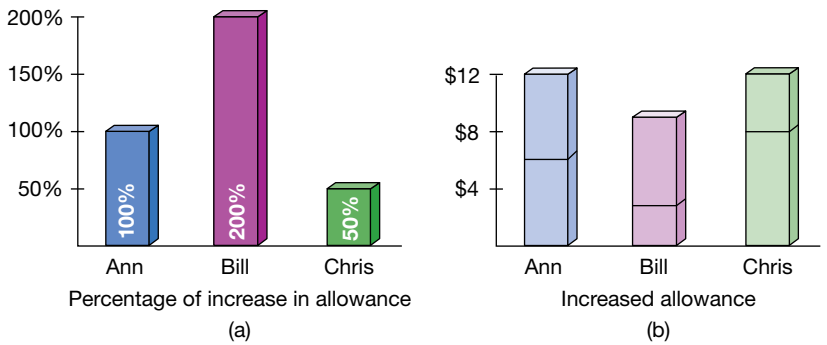


FIGURE 18.27 Example of distortion of data in graphs

What is wrong with the graph? Technically, it is correct; however, it doesn't tell the entire story because the original allowances were not the same.

Let's look at the data:

	Original allowance	Size of increase	Amount of increase	New allowance
Ann	$6.00	Double	$6.00	$12.00
Bill	$3.00	Triple	$6.00	$9.00
Chris	$8.00	Half	$4.00	$12.00

As figure 18.27(b) shows, a graph with a vertical axis labelled differently reflects the situation more accurately. These different graphs of the same data demonstrate how graphs can distort and sometimes misrepresent information. Developing a healthy skepticism of graphical displays is an important part of developing graphing skills. Harper (2004) suggests having students construct misleading graphs and then discussing the message being conveyed by the graphs. This can help students develop a better sense of misleading or inaccurate graphs, and what to look for when analysing a display. According to Harper, students are more likely to notice scaling features, but 'they are not as keen to observe pictorial embellishments or consider the source of the graph' (p. 343). Furthermore, having students look for misleading graphs in newspapers and magazines is a useful task. Data can be displayed in such a way that the exact opposite message comes across.

Communicating results

Once data have been collected, analysed and interpreted, it is important for students to communicate their findings. Just as in problem solving, students should be encouraged to look back at their results. Communication can help students clarify their ideas during this process. For example, students might complete regular journal entries throughout a research or investigation project. It is also useful to have large- or small-group discussions about the data.

Research supports having students go through a writing process when analysing data, similar to one used in language arts (Shaughnessy 2006). First, the students are involved in planning or prewriting. Second, a rough draft is created. Third, students revise their work. Finally, the work is published and shared with others. Students could draft business letters to a company or group that would be interested in the data or write to their school principal. This is an opportunity to integrate across subject areas.

It is valuable for students to learn to communicate their results with others. Their final presentation may include oral or written communication — or even a power point presentation. They must learn to report clearly the answers to their original questions and hypotheses and select the most appropriate way to communicate their findings in a graphic format or by using other statistical measures. Some projects may be interesting to share with local media. The presentation may be for other classmates; however, a presentation for someone outside of the classroom can be particularly meaningful. For example, one class calculated the area of classrooms in their building and compared the average area of the Year 5 classrooms to the Year 6 classrooms. Their findings were presented to the principal who made room assignments for the building (Scavo & Petraroja 1998). Another group of students collected data to find out which school lunches were the most and least popular. Once their findings were collected and displayed, they presented their findings to the tuckshop manager, who then used this information to plan the monthly specials.

18.5 Probability

LEARNING OUTCOME 18.5 Recognising the linkages between the studies of probability, data representation and data interpretation.

In daily conversations, it is common to speak of events in terms of their chances of occurring. *Probability* is used to predict the chance of something happening. The terms *chance* and *probability* are often applied to those situations where the outcome cannot completely be determined in advance. Here are some examples of common probabilistic statements:

'The chance of rain today is 40%.'
'The Sydney Swans are a 3-to-1 favourite to win.'
'The probability of an accident on the job is less than 1 in 100.'
'The patient has a 50–50 chance of recovering.'
'If I study, I will probably pass the test.'
'I am sure we will have a test Friday.'
'We will have milk in the tuckshop today.'

The first 4 statements are commonly heard and relate directly to probability. The last 3 illustrate a subtle but frequent use of probability in many everyday situations. In all of these cases, the utilitarian role of probability makes it an important basic skill. One way to increase students' awareness of the use of probability is to have them make a daily or weekly list of probability statements they have seen (in newspapers, in magazines or on television) or heard (on radio, television or in conversation).

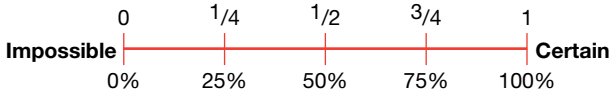
The study of probability is intertwined with the study of data representation and interpretation. According to the Australian Curriculum, 'Statistics and Probability initially develop in parallel and the curriculum then progressively builds the links between them' (ACARA 2012, p. 4). Probability should not be learned from formal definitions; rather, the presentation of varied examples and activities will help to illustrate and clarify important concepts. In the early year levels, the treatment of probability should be informal. At all stages of learning, the teacher can model the correct language to describe what is happening. This language helps children as they begin developing probability concepts and they will add new probability terms to their vocabulary. The following sections will now look at appropriate ways for primary students to experience some key concepts and terms.

Probability of an event

Look at these statements that involve probability:

The probability of tossing a head from a 2-sided coin is $\frac{1}{2}$.
The probability of rolling a 4 on a standard 6-sided die is $\frac{1}{6}$.
The probability of having a birthday on 30 February is 0.

In these examples, tossing a head, rolling a 4 and having a birthday on 30 February are *events*, or *outcomes*. Probability assigns a number (from 1 to 0) to an event. The more likely an event is to occur, the larger the number assigned to it. The probability is 1.0 when something is certain to happen. Probability can also be expressed as a percentage. For example, the probability of students in a Year 1 class having been born is 1, or 100%. On the other hand, the probability of something impossible happening is 0, or 0%. Likewise, the probability of students in the class having been born on 30 February is 0. Therefore, all probabilities lie between 0 and 1, or 0% and 100%. Initially, students should be able to identify situations in which the probability of an event can be determined. Later in school, experiments which involve sampling and simulations will help them quantify the likelihood of an outcome. Students will also use fractions, decimals and percentages from Year 6.



Long before children are ready to calculate probabilities of specific events, it is important that terms such as *certain, uncertain, impossible, likely* and *unlikely* are introduced and discussed. Most students, in the primary year levels, are familiar with the terms *impossible* and *certain* and can give meaningful examples. Although *likely* and *unlikely* are less familiar and require more careful development, using activities such as In the classroom 18.5 with children provides a good start. As each card is sorted, an explanation or argument for placing it in the specific box should be given. This rationale is essential in refining and developing a clear understanding of these important terms.

An excellent extension to In the classroom 18.5 is to have students write statements to be sorted into the same categories. Each student should write several original statements and then exchange papers so that someone else classifies them. Once these probability terms become familiar, more specific probabilities can be determined.

The activities from In the classroom 18.6 ask students to determine the likelihood of a particular event. Experiments involving materials such as blocks, spinners, dice and coins are often used to introduce probability concepts and symbols. Students should be given many opportunities to create and explore activities that ask them to answer questions about the likelihood of events, using the vocabulary of probability.

IN THE CLASSROOM 18.5

What are the chances?

Objective: Identify the likelihood of an event.

Year level: 2

Sort these statements into the best box.

Impossible	Unlikely	Likely	Certain

(a) The sun will rise in the west.

(b) The school tuckshop will serve chocolate milk.

(c) A boy in our class will be 2 metres tall.

(d) Everyone in this room is alive.

(e) People in our class have brown eyes.

(f) There are more right-handed people in this room than left-handed.

(g) The price of petrol will be higher next year.

(h) It will rain today.

Explain your reasons for each categorisation.

Find statements in daily life that will fit each of these categories.

IN THE CLASSROOM 18.6

What's more likely?

Objective: Determine the likelihood of an event.

Year level: 3–5

Activity 1

If you put 4 red blocks in a paper bag, can you be sure which colour block you will pull out? Why or why not?

If you put a green block in the bag with the red blocks, can you be sure which colour block you will pull out? Why or why not? Try this experiment. Record your results for 10 samples.

If you put 3 green blocks in a bag with 7 red blocks, is 1 colour more likely to be pulled out? Why or why not? Try this experiment. Record your results for 10 samples.

Perform a blocks-in-a-bag experiment of your own. Record your results.

Activity 2

Mary flipped a coin 4 times. It came up heads 4 times. She flipped the coin a fifth time. What is she likely to get on the fifth flip of the coin — heads, tails or is it equally likely to be heads or tails?

Explain your answer. How could you test your idea? Perform your experiment and record your results.

Sample space is a fundamental concept that must be established or at least understood before the probabilities of specific events can be determined. The sample space for a probability problem represents all possible outcomes.

Another suggestion to help children think about what outcomes are possible for a particular event is to have them examine some common probability situations:

> If I flip a coin, will it land heads up, on its edge, tails up or just float in the air?
> If I drop a glass, will it hit and break, float in the air or hit but not break?

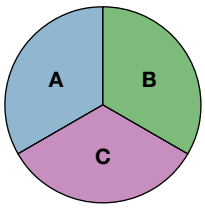
TABLE 18.2 **Sample spaces of some events and their probabilities**

Questions	Sample space	Number of successes	Probability
What is the probability of getting a head on a single toss of a coin?	H, T	1	
What is the probability of getting 2 heads when 2 coins are tossed?	HH, HT, TH, TT	1	
What is the probability of getting a 5 on a single roll of a die?	1, 2, 3, 4, 5, 6	1	
What is the probability of drawing a spade from a deck of 52 playing cards?	52 cards	13	or
If each letter of the alphabet is written on a piece of paper, what is the probability of drawing a vowel?	26 letters of the alphabet	5 (a, e, i, o, u)	

In each case, have the students discuss which events could happen. Consider, for example, the situation in which a coin is tossed. Some children may realise that a coin will not land on its edge, nor will it float. Thus only 2 outcomes can happen and these possible outcomes comprise the sample space. Asking children to create their own events and determine possible outcomes is a beneficial experience for Year 7 students.

Once the sample space is known, the calculation of specific probabilities usually follows naturally. When a coin is flipped, as described in table 18.2, the probability of a head is the number of ways a head can occur divided by the total number of outcomes (head or tail). Recording probabilities uses fractions, which provide a direct and convenient means of reporting and interpreting probabilities.

Discussion of possible outcomes helps identify the sample space and clarify notions of probability. Using the example of an equally likely spinner, students can be questioned about the probability of events occurring. Questions along these lines might get the discussion started:



> Can the spinner stop in region A? B? C? D?
> Can the spinner stop on a line?

Stopping in region D is impossible. Stopping on a line is unlikely, but it can happen. A plan of action such as spinning again should be specified if it does. (See Appendix C for spinner masters.)

Consider the sample space for tossing cups made of paper, plastic or foam as shown in In the classroom 18.7. Is a cup equally likely to land on its top, its side or its bottom? Without additional information, it would be difficult to decide. In the classroom 18.7 provides suggestions for a possible start.

In addition to helping children decide which outcomes are more or less likely, this exploration may lead to more precise statements.

IN THE CLASSROOM 18.7

What's the probability?

Objective: Collect data to decide more or less likely outcomes and determine experimental probabilities.

Year level: 3–5

Toss a cup to see how it lands:

Before you toss it again, circle how you think it will land:

Toss your cup 20 more times and tally how it lands!



Bottom	Top	Side

- I have decided that _____ is most likely.
- I have decided that _____ is least likely.

Here are some results from tossing the cup:

(a)

2	0	3

(b)

15	10	25

(c)

35	17	51

- Is it possible that all these tallies are results of tossing the same cup? Why?
- Which of these tables would you feel best about using? Why?

Decision time

The probability of landing on the side is about _____.

In the classroom 18.8 involves collecting data, graphing results and exploring patterns. It uses several valuable ideas of probability, including sample space and probability of an event, in a natural and interesting setting. As children are involved in this process, they are developing and practising basic facts, as was shown earlier in chapter 9. Such an activity further illustrates how mathematical topics are interrelated and how important connections can be made.

Once you have tried In the classroom 18.8 with children, another perspective can be obtained by examining figure 18.28, which summarises the results when 2 dice are added (a) and multiplied (b). One diagonal of figure 18.28 (a) shows all the ways that a sum of 7 can be obtained. Figure 18.28 (a) also shows the different ways that each of the other sums can result. Is the sum of 2 dice more likely to be even or odd? An examination of figure 18.28 (a) shows that the even sums will occur 18 out of 36 times, or half the time.

IN THE CLASSROOM 18.8

Rolling and recording

Objective: Conduct a probability experiment.

Year level: 3–6

- Try this:
  1. Choose a partner, and each of you make a chart like the one shown below. Predict which sum of 2 dice will come up most often.
  2. Each of you take turns rolling 2 dice.
  3. For each turn, find the sum of the spots on the 2 dice and place a tally mark in that column on your chart.

4. Continue rolling and recording until one of you has 10 tally marks in one of the columns. Compare your prediction with the actual result.



Putting it together

- Why doesn't the chart need a 1s column or a 13s column?
- In which column did you or your partner reach 10?
- Compare your results and discuss how they are similar or different.
- Explain why you would expect more sums of 7 than 2.
- Complete the following.

  'I would expect about the same number of sums of 4 as_____.'
- Would you expect to get about the same number of even sums as odd sums? Explain why.

Extending this activity

Suppose you multiplied the numbers on the dice instead of adding them.

- How would the values along the top of the chart change?
- How many values (i.e. different products) would be needed?
- Which values would be least likely?
- Would you expect to get about the same number of even values as odd values? Explain why.

Is the product of 2 dice more likely to be even or odd? Figure 18.28(b) shows that the even products are much more likely. In fact, an even product would be expected to occur 27 times out of 36, or three-quarters of the time. Analysing and discussing why this happens helps connect probability to properties and relationships between numbers and operations. For example, it is a reminder that the product of 2 even numbers is even and that the product of an odd number and an even number is also even.

(a) Results when dice are added

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

(b) Results when dice are multiplied

×	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

FIGURE 18.28 Results of operations with 2 dice

Randomness

Randomness is an important concept underlying all learning in probability. When something is random, it means that it is not influenced by any factors other than chance. In the classroom 18.9 could be used to build on In the classroom 18.5 and provides an opportunity to discuss randomness in a specific context. Here students are encouraged to think about events based on their classmates and decide about where these events would be placed on a probability number line that shows 0 and 1. Here are some starter questions:

Why is it important that the 'name will be randomly picked'?
Should the names be seen by the person doing the draw?
Would it matter if some people wrote their names on large pieces of paper and others on small pieces of paper?

If the names are seen or if people don't all write their names on the same size of paper, the draw might not be random. When this happens, some people would have an advantage and the notions of *fair* and *unfair* become important.

The term *fair* is often used in describing a situation. For example, to say 'a fair coin' or 'fair dice' makes it clear that no inherent biases exist that would affect randomness. A person may be asked to toss a die in such a way to ensure that no face is favoured. If ping-pong balls are drawn from a bowl, it is important that the balls be thoroughly mixed and the person doing the draw be blindfolded to ensure both randomness and fairness. The children's book *No Fair*! (Holtzman 1997) describes 2 children who are trying to determine when activities such as drawing an item from a bag are fair.

IN THE CLASSROOM 18.9

Are you a winner?

Objective: Explore the concept of randomness.

Year level: 4–7

Our class is having a lucky dip. Each person gets to place their name in the draw, just once. One name will be randomly picked, and that person will be the winner.

- Read each of the following statements.
- Think about the people in our class.
- Then check the number line below and decide about where the following statements should be placed:

(a) The winner will be left-handed.
(b) The winner will be a girl.
(c) The winner will be someone in our class.
(d) The number of letters in the first name of the winner will be less than the number of letters in their last name.
(e) The winner's first name will begin with a vowel.
(f) The winner will wear glasses.
(g) You will be the winner.
(h) The winner will be wearing socks.
(i) You will not be the winner.

Discussions of the consequences of unfairness and absence of randomness should be a regular part of developing probability. For example, would it be fair if 2 dice are rolled and player A wins if the product is even and player B wins if the product is odd? Figure 18.28(b) shows that A will win much more than B, so this game is not fair. On the other hand, if the game is based on the sum of 2 dice — see figure 18.28(a) — each player has an equal chance of winning and the game is fair. Suppose you modify the sum-of-2-dice game to play with 3 players:

Player A wins if the sum is 1, 2, 3 or 4.
Player B wins if the sum is 5, 6, 7 or 8.
Player C wins if the sum is 9, 10, 11 or 12.

Is this game fair for each of the players? Does each player have the same chance of winning? An analysis of figure 18.28(a) suggests that player B will win more often than either of the other players. As children explore this game, you might challenge them to explain how the game might be modified to make it fair for everyone.

Independence of events

Independence of events is an important concept in probability, but one that does not develop naturally from intuition. If 2 events are independent, one event in no way affects the outcome of the other. Thus, if a coin is tossed, lands heads and then is tossed again, it is still equally likely to land heads or tails. This sounds simple enough, but consider this question:

> Suppose 4 consecutive sixes have occurred on 4 rolls of a fair die. What is the probability of getting a six on the next roll?

A majority of middle years students find it hard to answer this question. Many students suggest that the die has a 'memory' and things would 'even out'. Most did not conclude that the probability was unchanged, regardless of what had already happened. If an event has occurred a number of times in a row, most people falsely presume that the 'law of averages' makes it unlikely that the event will occur on the next trial. The same thing happens when a coin is flipped multiple times, generating a string of the same outcomes. If the coin lands heads 4 times in a row, it has an equally likely chance of landing heads or tails on the fifth flip. This basic misunderstanding contradicts the notion of independence of certain events. Research suggests that experiences exploring independence should be encountered in upper primary and junior secondary school (Jones et al. 1999; Tarr 1997; Tarr & Jones 1997).

Having children collect data and discuss the results can help dispel some of this erroneous thinking:

> Working with real data allows students to appreciate the difference between empirical phenomena and probabilistic models; it shows them the usefulness of these models in explaining, predicting and controlling a variety of real phenomena beyond pure games of chance (Batanero & Sanchez 2005, p. 262).

When using In the classroom 18.10 with children, different students will produce different results, yet the answers to the questions will be very similar. Why? Because these events — the rolls of a die — are independent of one another.

Tossing a coin and recording the outcomes in sequence will likely generate some long runs of an occurrence, even though each outcome is independent of the others. Although the probability of a head is, children might flip a coin 10 times and get 8, 9 or even 10 heads in a row. Consider this record of 20 tosses of a coin:

T T T T T H T H H H H T T H H T H H H H

There are 2 sequences of 4 consecutive heads and 1 of 5 consecutive tails. Overall, 11 heads appeared. Such analysis and discussion helps children understand that things don't even out on each flip. As the number of trials becomes large, the ratio of heads to the total number of flips will become closer to the theoretical expected value of $\frac{1}{2}$. This latter point is very important, but it often baffles primary students. Using simulation programs to repeat an event many times helps children better understand the notion of the expression *in the long run*.

IN THE CLASSROOM 18.10

Can you make predictions?

Objective: Identify independent events.

Year level: 4–6

Roll a die 6 times and record the results:

1	2	3	4	5	6

- Did each face appear once? _____
- Does knowing what happened on the first roll help predict the second? _____ The third? _____

Roll a die 24 times and record the results.

1	2	3	4	5	6	7	8	9	10	11	12
13	14	15	16	17	18	19	20	21	22	23	24

- Did each face appear once? _____
- Which face appeared most? _____
- Does this mean the die is unfair? _____
- Does this record tell you what will occur on the next roll? _____
- What would change if you rolled the die 100 times?

There are, of course, times when 1 event may depend on another. For example, suppose you wanted to roll 2 dice and obtain a sum of 8. If a 1 is shown on the first die, it is impossible to get a sum of 8. This leads toward notions of *conditional probability*.

A variety of learning experiences are needed to develop the necessary techniques to solve simple probability problems. Furthermore, there are many complicated situations for which probabilities are difficult to calculate. What is the probability that the Australian cricket team wins back the Ashes? Of a woman being elected prime minister? Such questions do not lend themselves to simple solutions, but experts can approximate their probabilities. Regardless of who determines the numerical probabilities, the knowledge and interpretative skills developed in simpler probability situations can be successfully applied.

Misconceptions about probability

> If one intends to develop by instruction a strong, correct, coherent, formal, and intuitive background for probabilistic reasoning, one has to cope with a large variety of misunderstandings, misconceptions, biases and emotional tendencies (Fischbein, Nello & Marino 1991, p. 547).

Young children often hold common misconceptions about various aspects of probability. For example, they make predictions based on preference, such as their favourite colour or number (Jones & Langrall 2007). They also hold biases against certain numbers, believing, for example, that it may be hardest or luckiest to throw a 6 on a die. Young children are not surprised by extremely unlikely events and do not search for underlying causes. They may expect all outcomes in an experiment to be equally likely. It is also difficult for many children to make inferences from data.

So, do not be surprised when your own students reflect misconceptions in probability. The more opportunities you give them to explore a variety of probability notions through hands-on activities, the better they will be able to develop and evaluate inferences and predictions that are based on data and apply basic concepts of probability.

MAKING CONNECTIONS

Overall results for Australian students on 2 international assessments — TIMSS (2003 and 2007, at Year 4 and Year 8 levels) and PISA (2009 for 15-year-old students) — places them below several countries (Stacey & Stephens, 2008; Thomson et al. 2008; Thomson et al. 2009). However, Australian students produced better results in the Data domain than all other content domains. These good results for data (organisation, representation and interpretation) would indicate that the higher proportion of teacher time given to this content in the classroom might not be the case in all countries, particularly Asia.

Based on the reported use of the curriculum, many nations were not emphasising data analysis and probability in their curricula. This lack of emphasis is highlighted in Mullis et al. (2004), 'with 95% of students across countries having been taught the TIMSS number topics (strand) by eighth grade. This was followed by measurement (78%), geometry (69%), algebra (66%), and data (46%)' (p. 189). By Year 8, the top 5 nations ranged from 6% to 59% of the data strand covered, while Australia averaged 94% for data but 35% for probability (Thomson & Buckley 2009). Because data is not a key component of curricula in many parts of the world, students are scoring significantly lower on questions related to data.

Why is Australia spending more time on data and scoring much higher? Jones and Langrall (2007) suggest 2 major factors are the growing recognition of the importance of probabilistic thinking and the increasing amount of research. As a result, mathematics curricula material in Australia is devoting more attention to data analysis, statistics and probability, thereby increasing the students' opportunity to learn.

A GLANCE AT WHERE WE'VE BEEN

Data analysis, statistics and probability provide an opportunity for students to develop problem-solving and critical-thinking skills as well as to make connections to other mathematical topics and school subjects. Students engage in the process of data analysis by formulating questions that are meaningful to them, collecting and organising the data to answer the questions and displaying the data in a way that helps them analyse and interpret the results. Teachers and students pose questions about the data together, extend students' abilities to analyse their results and communicate their findings in a way that represents the data in a truthful manner.

In the primary and middle years of school students are exposed to many types of tables and graphs: line plots, stem-and-leaf plots, picture graphs, bar graphs, histograms, pie graphs, line graphs and box plots. Graphing calculators and other technological tools and programs allow students opportunities to experiment with different ways of displaying data, therefore developing multiple perspectives for the same set of data. This helps promote data sense.

Students in the primary years learn to analyse data using descriptive statistics, especially measures of variation and central tendency — mean, median and mode. The goal is to understand fundamental notions related to these statistics and to be able to choose averages that are most appropriate and describe the data in meaningful ways.

Ideas about probability develop in primary students through a variety of hands-on activities and experiments that challenge their intuitive notions of what is fair and unfair. It is important that children know that the probability of an event ranges between 0 and 1, and are able to calculate the probability of events. The ability to use the language of probability is an important skill in our daily lives and it can be explored in many ways that help build excitement for learning mathematics.

THINGS TO DO: FROM WHAT YOU'VE READ

1 Here are the results on 3 tests: 68, 78, 88. What are the mean and the median? Explain why the mode is of little value. What score would be needed on the next test to get an average of 81? Describe 2 different ways you could determine this score.

2 Arrange interlocking cubes together in lengths of 3, 6, 6 and 9.
(a) Describe how you could use the blocks to find the mean, median and mode.
(b) Suppose you introduce another length of 10 cubes. Has the mean changed? The median? The mode?

3 Answer each of the following questions and explain why you answered as you did.
(a) Is it possible for a set of data to have more than 1 mode? Give an example.
(b) Is it easier to find the median of 25 or 24 student scores?
(c) Could the mean be as large as the largest value in a set of data? Explain how.

4 Describe the differences between a bar graph and a histogram.

5 Describe 2 activities that could be used to discuss probabilities of 0 and 1.

6 Ten cards are marked 0, 1, 2, 3 . . . 9 and placed face down. If the cards are shuffled and then 1 card is drawn, explain why the following statements are true:
(a) The sample space has 10 events.
(b) The probability of drawing the 6 is $\frac{1}{10}$.
(c) The probability of drawing the 3 is the same as the probability of drawing the 7.

7 Many lotteries advertise with the slogan 'If you don't play, you can't win.' Does that mean if you do play, you will win? Explain.

8 If a lottery consists of picking 6 numbers from the numbers 1 through 45, which group of 6 numbers would you pick?
(a) 1, 2, 3, 4, 5, 6
(b) 3, 10, 17, 21, 35, 43
(c) Either (a) or (b) since both are equally likely to win.
Explain your choice.

THINGS TO DO: GOING BEYOND THIS TEXT

IN THE FIELD

1 Use In the classroom 18.1 with a small group of children as they plan and conduct a survey. Describe how it went.

2 Examine the graphs shown by Sakshaug (2000). Use these graphs and questions with some children and compare your results with those reported by Sakshaug and Wohlhuter (2001).

3 Use In the classroom 18.2 with some children in Years 3 to 5. Which key features of various graphs and statistics did you discuss with the children? How did you guide the children to create higher-level questions?

4 Select a middle-school student and use In the classroom 18.6 to guide some questions related to probability. What did you learn about the child's thinking?

5 Examine the scope-and-sequence chart for a primary textbook series. At what level is graphing first introduced? What kinds of graphing skills are highlighted? What important statistical topics are included? At what levels are they taught? Do the same for probability.

6 What would you say to a principal who accused you of allowing your students to 'play games' when they were conducting dice experiments?

WITH ADDITIONAL RESOURCES

7 Review the *Mathematics Curriculum and Teaching Program: Chance and Data Investigations*, Vols I & II. (Lovitt & Lowe 1993). Select an activity. Decide where and how you would use it in teaching a lesson.

8 Examine one of the NCTM's *Navigations Series* books for Data Analysis and Probability (preK–2, Sheffield et al., 2002; 3–5, Chapin et al. 2002; 6–8 probability, Bright, Frierson et al. 2003; or 6–8, data analysis, Bright, Brewer et al. 2003) and present one of the age-appropriate activities to a group of students. Discuss what happened.

9 Examine newspapers or magazines. Start a file of graphs students can examine. Make a list of the different kinds of graphs used. Select a few and explain why you think a particular graph (e.g. picture graph, pie graph, bar graph) was used in each situation.

10 Games are fun and can help develop a better understanding of probability. Play one of the games from *Dice Dilemmas: Activities to Promote Mental Computation and Develop Thinking About Chance Processes* (Swan 1997), *Card Capers: Developing Mathematics from Playing Cards* (Swan 1998) or *Dice Dazzlers: Short and Simple Dice Games to Promote Numeracy* (Swan 2003). These resources are available through the Australian Association of Mathematics Teachers (AAMT) website, www.aamt.edu.au. Alternatively, play the game Montana Red Dog, described in *Dealing with Data and Chance* (Zawojewski et al. 1991), play one of the games from *What Are My Chances?* (Shulte & Choate 1996) or play Cover Up from *Chance Encounters: Probability in Games and Simulation* (Brutlag 1996). Identify some of the mathematics learned in these games. Explain how you might use these games with students.

11 Read one of the research articles related to probability or statistics listed in Jones (2005), Shaughnessy (2006), or Friel, Curcio & Bright (2001). Discuss the nature of the research. Also identify an activity suggested by the research that you think would be effective in helping children learn a particular concept.

WITH TECHNOLOGY

12 Use technology such as a spreadsheet, graphing or survey software, or a graphing calculator to display and analyse data. Evaluate how these tools might be used with children. Better yet, try it out with children and then evaluate.

13 Visit the National Council of Teachers of Mathematics Illuminations website (http://illuminations.nctm.org) for an activity that can be used in helping primary school level children understand probability. If you can, try this activity with some students.

CHILDREN'S LITERATURE CONNECTIONS

Arnold, C. (1984). *Charts and Graphs: Fun, Facts, and Activities.* New York: Franklin Watts.
Photographs and drawings are used to illustrate real-world applications and examples using pie, bar and line graphs and pictographs. Illustrations are colourful and interesting and provide a multicultural focus for the book. The glossary is written in terms children can understand and is helpful in comparing various ways of presenting information.

Cushman, J. (1991). *Do You Wanna Bet? Your Chance to Find Out About Probability.* New York: Clarion Books.
Whether flipping coins to decide what television program to watch or analysing which events are 'certain', 'impossible' or 'maybe', Danny and Brian become involved in everyday situations — both in and out of school — that involve probability. Several important probability concepts are woven into an interesting storyline.

Holtzman, C. (1997). *No Fair!* New York: Scholastic.
When Kristy and David cannot agree on which game to play, David brings out a bag of marbles and says they can play Kristy's game if she chooses a blue marble. However, Kristy objects when she learns that only 1 marble is blue.

Leedy, L. (2006). *The Great Graph Contest*. New York: Holiday House.
The book shows the reader how to make and understand such things as: bar graphs, circle graphs, quantity graphs and Venn diagrams.

Murphy, S. J. (1997). *The Best Vacation Ever.* New York: HarperCollins.
This entry in the MathStart series demonstrates that collecting data and tabulating results can help the decision-making process. A girl with an active, overscheduled family puts mathematics to work to coordinate a family expedition. She uses charts to plot her family's preferences.

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APPENDIX A

Standards and Expectations, National Council of Teachers of Mathematics, 2000

Number and Operations

Standard	Pre-K–2	Grades 3–5	Grades 6–8
Instructional programs from prekindergarten through grade 12 should enable all students to —	**EXPECTATIONS** In prekindergarten through grade 2 all students should —	**EXPECTATIONS** In grades 3–5 all students should —	**EXPECTATIONS** In grades 6–8 all students should —
Understand numbers, ways of representing numbers, relationships among numbers, and number systems	• count with understanding and recognize "how many" in sets of objects; • use multiple models to develop initial understandings of place value and the base-ten number system; • develop understanding of the relative position and magnitude of whole numbers and of ordinal and cardinal numbers and their connections; • develop a sense of whole numbers and represent and use them in flexible ways, including relating, composing, and decomposing numbers; • connect number words and numerals to the quantities they represent, using various physical models and representations; • understand and represent commonly used fractions, such as $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{1}{2}$.	• understand the place-value structure of the base-ten number system and be able to represent and compare whole numbers and decimals; • recognize equivalent representations for the same number and generate them by decomposing and composing numbers; • develop understanding of fractions as parts of unit wholes, as parts of a collection, as locations on number lines, and as divisions of whole numbers; • use models, benchmarks, and equivalent forms to judge the size of fractions; • recognize and generate equivalent forms of commonly used fractions, decimals, and percents; • explore numbers less than 0 by extending the number line and through familiar applications; • describe classes of numbers according to characteristics such as the nature of their factors.	• work flexibly with fractions, decimals, and percents to solve problems; • compare and order fractions, decimals, and percents efficiently and find their approximate locations on a number line; • develop meaning for percents greater than 100 and less than 1; • understand and use ratios and proportions to represent quantitative relationships; • develop an understanding of large numbers and recognize and appropriately use exponential, scientific, and calculator notation; • use factors, multiples, prime factorization, and relatively prime numbers to solve problems; • develop meaning for integers and represent and compare quantities with them.
Understand meanings of operations and how they relate to one another	• understand various meanings of addition and subtraction of whole numbers and the relationship between the two operations; • understand the effects of adding and subtracting whole numbers; • understand situations that entail multiplication and division, such as equal groupings of objects and sharing equally.	• understand various meanings of multiplication and division; • understand the effects of multiplying and dividing whole numbers; • identify and use relationships between operations, such as division as the inverse of multiplication, to solve problems; • understand and use properties of operations, such as the distributivity of multiplication over addition.	• understand the meaning and effects of arithmetic operations with fractions, decimals, and integers; • use the associative and commutative properties of addition and multiplication and the distributive property of multiplication over addition to simplify computations with integers, fractions, and decimals; • understand and use the inverse relationships of addition and subtraction, multiplication and division, and squaring and finding square roots to simplify computations and solve problems.

Compute fluently and make reasonable estimates	• develop and use strategies for whole-number computations, with a focus on addition and subtraction; • develop fluency with basic number combinations for addition and subtraction; • use a variety of methods and tools to compute, including objects, mental computation, estimation, paper and pencil, and calculators.	• develop fluency with basic number combinations for multiplication and division and use these combinations to mentally compute related problems, such as 30×50; • develop fluency in adding, subtracting, multiplying, and dividing whole numbers; • develop and use strategies to estimate the results of whole-number computations and to judge the reasonableness of such results; • develop and use strategies to estimate computations involving fractions and decimals in situations relevant to students' experience; • use visual models, benchmarks, and equivalent forms to add and subtract commonly used fractions and decimals; • select appropriate methods and tools for computing with whole numbers from among mental computation, estimation, calculators, and paper and pencil according to the context and nature of the computation and use the selected method or tool.	• select appropriate methods and tools for computing with fractions and decimals from among mental computation, estimation, calculators or computers, and paper and pencil, depending on the situation, and apply the selected methods; • develop and analyze algorithms for computing with fractions, decimals, and integers and develop fluency in their use; • develop and use strategies to estimate the results of rational-number computations and judge the reasonableness of the results; • develop, analyze, and explain methods for solving problems involving proportions, such as scaling and finding equivalent ratios.

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Algebra

Standard	Pre-K–2	Grades 3–5	Grades 6–8
Instructional programs from prekindergarten through grade 12 should enable all students to —	**EXPECTATIONS** In prekindergarten through grade 2 all students should —	**EXPECTATIONS** In grades 3–5 all students should —	**EXPECTATIONS** In grades 6–8 all students should —
Understand patterns, relations, and functions	• sort, classify, and order objects by size, number, and other properties; • recognize, describe, and extend patterns such as sequences of sounds and shapes or simple numeric patterns and translate from one representation to another; • analyze how both repeating and growing patterns are generated.	• describe, extend, and make generalizations about geometric and numeric patterns; • represent and analyze patterns and functions, using words, tables, and graphs.	• represent, analyze, and generalize a variety of patterns with tables, graphs, words, and, when possible, symbolic rules; • relate and compare different forms of representation for a relationship; • identify functions as linear or nonlinear and contrast their properties from tables, graphs, or equations.
Represent and analyze mathematical situations and structures using algebraic symbols	• illustrate general principles and properties of operations, such as commutativity, using specific numbers; • use concrete, pictorial, and verbal representations to develop an understanding of invented and conventional symbolic notations.	• identify such properties as commutativity, associativity, and distributivity and use them to compute with whole numbers; • represent the idea of a variable as an unknown quantity using a letter or symbol; • express mathematical relationships using equations.	• develop an initial conceptual understanding of different uses of variables; • explore relationships between symbolic expressions and graphs of lines, paying particular attention to the meaning of intercept and slope; • use symbolic algebra to represent situations and to solve problems, especially those that involve linear relationships; • recognize and generate equivalent forms for simple algebraic expressions and solve linear equations.
Use mathematical models to represent and understand quantitative relationships	• model situations that involve the addition and subtraction of whole numbers, using objects, pictures, and symbols.	• model problem situations with objects and use representations such as graphs, tables, and equations to draw conclusions.	• model and solve contextualized problems using various representations, such as graphs, tables, and equations.
Analyze change in various contexts	• describe qualitative change, such as a student's growing taller; • describe quantitative change, such as a student's growing two inches in one year.	• investigate how a change in one variable relates to a change in a second variable; • identify and describe situations with constant or varying rates of change and compare them.	• use graphs to analyze the nature of changes in quantities in linear relationships.

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Geometry

Standard	Pre-K–2	Grades 3–5	Grades 6–8
Instructional programs from prekindergarten through grade 12 should enable all students to —	**EXPECTATIONS** In prekindergarten through grade 2 all students should —	**EXPECTATIONS** In grades 3–5 all students should —	**EXPECTATIONS** In grades 6–8 all students should —
Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships	• recognize, name, build, draw, compare, and sort two- and three-dimensional shapes; • describe attributes and parts of two- and three-dimensional shapes; • investigate and predict the results of putting together and taking apart two- and three-dimensional shapes.	• identify, compare, and analyze attributes of two- and three-dimensional shapes and develop vocabulary to describe the attributes; • classify two- and three-dimensional shapes according to their properties and develop definitions of classes of shapes such as triangles and pyramids; • investigate, describe, and reason about the results of subdividing, combining, and transforming shapes; • explore congruence and similarity; • make and test conjectures about geometric properties and relationships and develop logical arguments to justify conclusions.	• precisely describe, classify, and understand relationships among types of two- and three-dimensional objects using their defining properties; • understand relationships among the angles, side lengths, perimeters, areas, and volumes of similar objects; • create and critique inductive and deductive arguments concerning geometric ideas and relationships, such as congruence, similarity, and the Pythagorean relationship.
Specify locations and describe spatial relationships using coordinate geometry and other representational systems	• describe, name, and interpret relative positions in space and apply ideas about relative position; • describe, name, and interpret direction and distance in navigating space and apply ideas about direction and distance; • find and name locations with simple relationships such as “near to” and in coordinate systems such as maps.	• describe location and movement using common language and geometric vocabulary; • make and use coordinate systems to specify locations and to describe paths; • find the distance between points along horizontal and vertical lines of a coordinate system.	• use coordinate geometry to represent and examine the properties of geometric shapes; • use coordinate geometry to examine special geometric shapes, such as regular polygons or those with pairs of parallel or perpendicular sides.

(continued)

Standard	Pre-K–2	Grades 3–5	Grades 6–8
Apply transformations and use symmetry to analyze mathematical situations	• recognize and apply slides, flips, and turns; • recognize and create shapes that have symmetry.	• predict and describe the results of sliding, flipping, and turning two-dimensional shapes; • describe a motion or a series of motions that will show that two shapes are congruent; • identify and describe line and rotational symmetry in two- and three-dimensional shapes and designs.	• describe sizes, positions, and orientations of shapes under informal transformations such as flips, turns, slides, and scaling; • examine the congruence, similarity, and line or rotational symmetry of objects using transformations.
Use visualization, spatial reasoning, and geometric modeling to solve problems	• create mental images of geometric shapes using spatial memory and spatial visualization; • recognize and represent shapes from different perspectives; • relate ideas in geometry to ideas in number and measurement; • recognize geometric shapes and structures in the environment and specify their location.	• build and draw geometric objects; • create and describe mental images of objects, patterns, and paths; • identify and build a three-dimensional object from two-dimensional representations of that object; • identify and build a two-dimensional representation of a three-dimensional object; • use geometric models to solve problems in other areas of mathematics, such as number and measurement; • recognize geometric ideas and relationships and apply them to other disciplines and to problems that arise in the classroom or in everyday life.	• draw geometric objects with specified properties, such as side lengths or angle measures; • use two-dimensional representations of three-dimensional objects to visualize and solve problems such as those involving surface area and volume; • use visual tools such as networks to represent and solve problems; • use geometric models to represent and explain numerical and algebraic relationships; • recognize and apply geometric ideas and relationships in areas outside the mathematical classroom, such as art, science, and everyday life.

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Measurement

Standard	Pre-K–2	Grades 3–5	Grades 6–8
Instructional programs from prekindergarten through grade 12 should enable all students to —	**EXPECTATIONS** In prekindergarten through grade 2 all students should —	**EXPECTATIONS** In grades 3–5 all students should —	**EXPECTATIONS** In grades 6–8 all students should —
Understand measurable attributes of objects and the units, systems, and processes of measurement	• recognize the attributes of length, volume, weight, area, and time; • compare and order objects according to these attributes; • understand how to measure using nonstandard and standard units; • select an appropriate unit and tool for the attribute being measured.	• understand such attributes as length, area, weight, volume, and size of angle and select the appropriate type of unit for measuring each attribute; • understand the need for measuring with standard units and become familiar with standard units in the customary and metric systems; • carry out simple unit conversions, such as from centimeters to meters, within a system of measurement; • understand that measurements are approximations and understand how differences in units affect precision; • explore what happens to measurements of a two-dimensional shape such as its perimeter and area when the shape is changed in some way.	• understand both metric and customary systems of measurement; • understand relationships among units and convert from one unit to another within the same system; • understand, select, and use units of appropriate size and type to measure angles, perimeter, area, surface area, and volume.
Apply appropriate techniques, tools, and formulas to determine measurements	• measure with multiple copies of units of the same size, such as paper clips laid end to end; • use repetition of a single unit to measure something larger than the unit, for instance, measuring the length of a room with a single meterstick; • use tools to measure; • develop common referents for measures to make comparisons and estimates.	• develop strategies for estimating the perimeters, areas, and volumes of irregular shapes; • select and apply appropriate standard units and tools to measure length, area, volume, weight, time, temperature, and the size of angles; • select and use benchmarks to estimate measurements; • develop, understand, and use formulas to find the area of rectangles and related triangles and parallelograms; • develop strategies to determine the surface areas and volumes of rectangular solids.	• use common benchmarks to select appropriate methods for estimating measurements; • select and apply techniques and tools to accurately find length, area, volume, and angle measures to appropriate levels of precision; • develop and use formulas to determine the circumference of circles and the area of triangles, parallelograms, trapezoids, and circles and develop strategies to find the areas of more complex shapes; • develop strategies to determine the surface area and volume of selected prisms, pyramids, and cylinders; • solve problems involving scale factors, using ratio and proportion; • solve simple problems involving rates and derived measurements for such attributes as velocity and density.

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Data analysis and Probability

Standard	Pre-K–2	Grades 3–5	Grades 6–8
Instructional programs from prekindergarten through grade 12 should enable all students to —	**EXPECTATIONS** In prekindergarten through grade 2 all students should —	**EXPECTATIONS** In grades 3–5 all students should —	**EXPECTATIONS** In grades 6–8 all students should —
Formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them	• pose questions and gather data about themselves and their surroundings; • sort and classify objects according to their attributes and organize data about the objects; • represent data using concrete objects, pictures, and graphs.	• design investigations to address a question and consider how data-collection methods affect the nature of the data set; • collect data using observations, surveys, and experiments; • represent data using tables and graphs such as line plots, bar graphs, and line graphs; • recognize the differences in representing categorical and numerical data.	• formulate questions, design studies, and collect data about a characteristic shared by two populations or different characteristics within one population; • select, create, and use appropriate graphical representations of data, including histograms, box plots, and scatterplots.
Select and use appropriate statistical methods to analyze data	• describe parts of the data and the set of data as a whole to determine what the data show.	• describe the shape and important features of a set of data and compare related data sets, with an emphasis on how the data are distributed; • use measures of center, focusing on the median, and understand what each does and does not indicate about the data set; • compare different representations of the same data and evaluate how well each representation shows important aspects of the data.	• find, use, and interpret measures of center and spread, including mean and interquartile range; • discuss and understand the correspondence between data sets and their graphical representations, especially histograms, stem-and-leaf plots, box plots, and scatterplots.

Develop and evaluate inferences and predictions that are based on data	• discuss events related to students' experiences as likely or unlikely.	• propose and justify conclusions and predictions that are based on data and design studies to further investigate the conclusions or predictions.	• use observations about differences between two or more samples to make conjectures about the populations from which the samples were taken; • make conjectures about possible relationships between two characteristics of a sample on the basis of scatterplots of the data and approximate lines of fit; • use conjectures to formulate new questions and plan new studies to answer them.
Understand and apply basic concepts of probability		• describe events as likely or unlikely and discuss the degree of likelihood using such words as *certain, equally likely,* and *impossible;* • predict the probability of outcomes of simple experiments and test the predictions; • understand that the measure of the likelihood of an event can be represented by a number from 0 to 1.	• understand and use appropriate terminology to describe complementary and mutually exclusive events; • use proportionality and a basic understanding of probability to make and test conjectures about the results of experiments and simulations; • compute probabilities for simple compound events, using such methods as organized lists, tree diagrams, and area methods.

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Problem Solving

Standard

Instructional programs from prekindergarten through grade 12 should enable all students to —

- Build new mathematical knowledge through problem solving
- Solve problems that arise in mathematics and in other contexts
- Apply and adapt a variety of appropriate strategies to solve problems
- Monitor and reflect on the process of mathematical problem solving

Reasoning and Proof

Standard

Instructional programs from prekindergarten through grade 12 should enable all students to —

- Recognize reasoning and proof as fundamental aspects of mathematics
- Make and investigate mathematical conjectures
- Develop and evaluate mathematical arguments and proofs
- Select and use various types of reasoning and methods of proof

Communication

Standard

Instructional programs from prekindergarten through grade 12 should enable all students to —

- Organize and consolidate their mathematical thinking through communication
- Communicate their mathematical thinking coherently and clearly to peers, teachers, and others
- Analyze and evaluate the mathematical thinking and strategies of others
- Use the language of mathematics to express mathematical ideas precisely

Connections

Standard

Instructional programs from prekindergarten through grade 12 should enable all students to —

- Recognize and use connections among mathematical ideas
- Understand how mathematical ideas interconnect and build on one another to produce a coherent whole
- Recognize and apply mathematics in contexts outside of mathematics

Representation

Standard

Instructional programs from prekindergarten through grade 12 should enable all students to —

- Create and use representations to organize, record, and communicate mathematical ideas
- Select, apply, and translate among mathematical representations to solve problems
- Use representations to model and interpret physical, social, and mathematical phenomena

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APPENDIX B

Curriculum Focal Points, National Council of Teachers of Mathematics, 2006

Curriculum Focal Points for Mathematics in Prekindergarten through Grade 8

Three curriculum focal points are identified and described for each grade level, pre-K–8, along with connections to guide integration of the focal points at that grade level and across grade levels, to form a comprehensive mathematics curriculum. To build students' strength in the use of mathematical processes, instruction in these content areas should incorporate —

- the use of the mathematics to solve problems;
- an application of logical reasoning to justify procedures and solutions; and
- an involvement in the design and analysis of multiple representations to learn, make connections among, and communicate about the ideas within and outside of mathematics.

The purpose of identifying these grade-level curriculum focal points and connections is to enable students to learn the content in the context of a focused and cohesive curriculum that implements problem solving, reasoning, and critical thinking.

These curriculum focal points should be considered as major instructional goals and desirable learning expectations, not as a list of objectives for students to master. They should be implemented with the intention of building mathematical competency for all students, bolstered by the pedagogical understanding that not every student learns at the same rate or acquires concepts and skills at the same time.

Those who are involved in curriculum planning for grades 6–8 should note that this set of curriculum focal points has been designed with the intention of providing a three-year middle school program that includes a full year of general mathematics in each of grades 6, 7, and 8. Those whose programs order an algebra course in grade 8 (or earlier) should consider including the curriculum focal points that this framework calls for in grade 8 in grade 6 or grade 7. Alternatively, these topics could be incorporated into the high school program. Either way, curricula would not omit the important content that the grade 7 and grade 8 focal points order students in preparation for algebra and for their long-term mathematical knowledge.

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Curriculum Focal Points and Connections for Prekindergarten

The set of three curriculum focal points and related connections for mathematics in prekindergarten follow. These topics are the recommended content emphases for this grade level. It is essential that these focal points be addressed in contexts that promote problem solving, reasoning, communication, making connections, and designing and analyzing representations.

Prekindergarten Curriculum Focal Points	Connections to the Focal Points
Number and Operations: Developing an understanding of whole numbers, including concepts of correspondence, counting, cardinality, and comparison Children develop an understanding of the meanings of whole numbers and recognize the number of objects in small groups without counting and by counting — the first and most basic mathematical algorithm. They understand that number words refer to quantity. They use one-to-one correspondence to solve problems by matching sets and comparing number amounts and in counting objects to 10 and beyond. They understand that the last word that they state in counting tells "how many," they count to determine number amounts and compare quantities (using language such as "more than" and "less than"), and they order sets by the number of objects in them.	**Data Analysis:** Children learn the foundations of data analysis by using objects' attributes that they have identified in relation to geometry and measurement (e.g., size, quantity, orientation, number of sides or vertices, color) for various purposes, such as describing, sorting, or comparing. For example, children sort geometric figures by shape, compare objects by weight ("heavier," "lighter"), or describe sets of objects by the number of objects in each set. **Number and Operations:** Children use meanings of numbers to create strategies for solving problems and responding to practical situations, such as getting just enough napkins for a group, or mathematical situations, such as determining that any shape is a triangle if it has exactly three straight sides and is closed. **Algebra:** Children recognize and duplicate simple sequential patterns (e.g., square, circle, square, circle, square, circle,…).
Geometry: Identifying shapes and describing spatial relationships Children develop spatial reasoning by working from two perspectives on space as they examine the shapes of objects and inspect their relative positions. They find shapes in their environments and describe them in their own words. They build pictures and designs by combining two- and three-dimensional shapes, and they solve such problems as deciding which piece will fit into a space in a puzzle. They discuss the relative positions of objects with vocabulary such as "above," "below," and "next to."	
Measurement: Identifying measurable attributes and comparing objects by using these attributes Children identify objects as "the same" or "different," and then "more" or "less," on the basis of attributes that they can measure. They identify measurable attributes such as length and weight and solve problems by making direct comparisons of objects on the basis of those attributes.	

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Curriculum Focal Points and Connections for Kindergarten

The set of three curriculum focal points and related connections for mathematics in prekindergarten follow. These topics are the recommended content emphases for this grade level. It is essential that these focal points be addressed in contexts that promote problem solving, reasoning, communication, making connections, and designing and analyzing representations.

Kindergarten Curriculum Focal Points	Connections to the Focal Points
Number and Operations: Representing, comparing, and ordering whole numbers and joining and separating sets Children use numbers, including written numerals, to represent quantities and to solve quantitative problems, such as counting objects in a set, creating a set with a given number of objects, comparing and ordering sets or numerals by using both cardinal and ordinal meanings, and modeling simple joining and separating situations with objects. They choose, combine, and apply effective strategies for answering quantitative questions, including quickly recognizing the number in a small set, counting and producing sets of given sizes, counting the number in combined sets, and counting backward.	**Data Analysis:** Children sort objects and use one or more attributes to solve problems. For example, they might sort solids that roll easily from those that do not. Or they might collect data and use counting to answer such questions as, "What is our favorite snack?" They re-sort objects by using new attributes (e.g., after sorting solids according to which ones roll, they might re-sort the solids according to which ones stack easily). **Geometry:** Children integrate their understandings of geometry, measurement, and number. For example, they understand, discuss, and create simple navigational directions (e.g., "Walk forward 10 steps, turn right, and walk forward 5 steps"). **Algebra:** Children identify, duplicate, and extend simple number patterns and sequential and growing patterns (e.g., patterns made with shapes) as preparation for creating rules that describe relationships.
Geometry: Describing shapes and space Children interpret the physical world with geometric ideas (e.g., shape, orientation, spatial relations) and describe it with corresponding vocabulary. They identify, name, and describe a variety of shapes, such as squares, triangles, circles, rectangles, (regular) hexagons, and (isosceles) trapezoids presented in a variety of ways (e.g., with different sizes or orientations), as well as such three-dimensional shapes as spheres, cubes, and cylinders. They use basic shapes and spatial reasoning to model objects in their environment and to construct more complex shapes.	
Measurement: Ordering objects by measurable attributes Children use measurable attributes, such as length or weight, to solve problems by comparing and ordering objects. They compare the lengths of two objects both directly (by comparing them with each other) and indirectly (by comparing both with a third object), and they order several objects according to length.	

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Curriculum Focal Points and Connections for Grade 1

The set of three curriculum focal points and related connections for mathematics in grade 1 follow. These topics are the recommended content emphases for this grade level. It is essential that these focal points be addressed in contexts that promote problem solving, reasoning, communication, making connections, and designing and analyzing representations.

Grade 1 Curriculum Focal Points	Connections to the Focal Points
Number and Operations and Algebra: Developing understandings of addition and subtraction and strategies for basic addition facts and related subtraction facts Children develop strategies for adding and subtracting whole numbers on the basis of their earlier work with small numbers. They use a variety of models, including discrete objects, length-based models (e.g., lengths of connecting cubes), and number lines, to model "part-whole," "adding to," "taking away from," and "comparing" situations to develop an understanding of the meanings of addition and subtraction and strategies to solve such arithmetic problems. Children understand the connections between counting and the operations of addition and subtraction (e.g., adding two is the same as "counting on" two). They use properties of addition (commutativity and associativity) to add whole numbers, and they create and use increasingly sophisticated strategies based on these properties (e.g., "making tens") to solve addition and subtraction problems involving basic facts. By comparing a variety of solution strategies, children relate addition and subtraction as inverse operations.	**Number and Operations and Algebra:** Children use mathematical reasoning, including ideas such as commutativity and associativity and beginning ideas of tens and ones, to solve two-digit addition and subtraction problems with strategies that they understand and can explain. They solve both routine and nonroutine problems. **Measurement and Data Analysis:** Children strengthen their sense of number by solving problems involving measurements and data. Measuring by laying multiple copies of a unit end to end and then counting the units by using groups of tens and ones supports children's understanding of number lines and number relationships. Representing measurements and discrete data in picture and bar graphs involves counting and comparisons that provide another meaningful connection to number relationships. **Algebra:** Through identifying, describing, and applying number patterns and properties in developing strategies for basic facts, children learn about other properties of numbers and operations, such as odd and even (e.g., "Even numbers of objects can be paired, with none left over"), and 0 as the identity element for addition.
Number and Operations: Developing an understanding of whole number relationships, including grouping in tens and ones Children compare and order whole numbers (at least to 100) to develop an understanding of and solve problems involving the relative sizes of these numbers. They think of whole numbers between 10 and 100 in terms of groups of tens and ones (especially recognizing the numbers 11 to 19 as 1 group of ten and particular numbers of ones). They understand the sequential order of the counting numbers and their relative magnitudes and represent numbers on a number line.	
Geometry: Composing and decomposing geometric shapes Children compose and decompose plane and solid figures (e.g., by putting two congruent isosceles triangles together to make a rhombus), thus building an understanding of part-whole relationships as well as the properties of the original and composite shapes. As they combine figures, they recognize them from different perspectives and orientations, describe their geometric attributes and properties, and determine how they are alike and different, in the process developing a background for measurement and initial understandings of such properties as congruence and symmetry.	

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Curriculum Focal Points and Connections for Grade 2

The set of three curriculum focal points and related connections for mathematics in grade 2 follow. These topics are the recommended content emphases for this grade level. It is essential that these focal points be addressed in contexts that promote problem solving, reasoning, communication, making connections, and designing and analyzing representations.

Grade 2 Curriculum Focal Points	Connections to the Focal Points
Number and Operations: Developing an understanding of the base-ten numeration system and place-value concepts Children develop an understanding of the base-ten numeration system and place-value concepts (at least to 1000). Their understanding of base-ten numeration includes ideas of counting in units and multiples of hundreds, tens, and ones, as well as a grasp of number relationships, which they demonstrate in a variety of ways, including comparing and ordering numbers. They understand multidigit numbers in terms of place value, recognizing that place-value notation is a shorthand for the sums of multiples of powers of 10 (e.g., 853 as 8 hundreds + 5 tens + 3 ones).	**Number and Operations:** Children use place value and properties of operations to create equivalent representations of given numbers (such as 35 represented by 35 ones, 3 tens and 5 ones, or 2 tens and 15 ones) and to write, compare, and order multidigit numbers. They use these ideas to compose and decompose multidigit numbers. Children add and subtract to solve a variety of problems, including applications involving measurement, geometry, and data, as well as nonroutine problems. In preparation for grade 3, they solve problems involving multiplicative situations, developing initial understandings of multiplication as repeated addition. **Geometry and Measurement:** Children estimate, measure, and compute lengths as they solve problems involving data, space, and movement through space. By composing and decomposing two-dimensional shapes (intentionally substituting arrangements of smaller shapes for larger shapes or substituting larger shapes for many smaller shapes), they use geometric knowledge and spatial reasoning to develop foundations for understanding area, fractions, and proportions. **Algebra:** Children use number patterns to extend their knowledge of properties of numbers and operations. For example, when skip counting, they build foundations for understanding multiples and factors.
Number and Operations and Algebra: Developing quick recall of addition facts and related subtraction facts and fluency with multidigit addition and subtraction Children use their understanding of addition to develop quick recall of basic addition facts and related subtraction facts. They solve arithmetic problems by applying their understanding of models of addition and subtraction (such as combining or separating sets or using number lines), relationships and properties of number (such as place value), and properties of addition (commutativity and associativity). Children develop, discuss, and use efficient, accurate, and generalizable methods to add and subtract multidigit whole numbers. They select and apply appropriate methods to estimate sums and differences or calculate them mentally, depending on the context and numbers involved. They develop fluency with efficient procedures, including standard algorithms, for adding and subtracting whole numbers, understand why the procedures work (on the basis of place value and properties of operations), and use them to solve problems.	
Measurement: Developing an understanding of linear measurement and facility in measuring lengths Children develop an understanding of the meaning and processes of measurement, including such underlying concepts as partitioning (the mental activity of slicing the length of an object into equal sized units) and transitivity (e.g., if object A is longer than object B and object B is longer than object C, then object A is longer than object C). They understand linear measure as an iteration of units and use rulers and other measurement tools with that understanding. They understand the need for equal length units, the use of standard units of measure (centimeter and inch), and the inverse relationship between the size of a unit and the number of units used in a particular measurement (i.e., children recognize that the smaller the unit, the more iterations they need to cover a given length).	

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Curriculum Focal Points and Connections for Grade 3

The set of three curriculum focal points and related connections for mathematics in grade 3 follow. These topics are the recommended content emphases for this grade level. It is essential that these focal points be addressed in contexts that promote problem solving, reasoning, communication, making connections, and designing and analyzing representations.

Grade 3 Curriculum Focal Points	Connections to the Focal Points
Number and Operations and Algebra: Developing understandings of multiplication and division and strategies for basic multiplication facts and related division facts Students understand the meanings of multiplication and division of whole numbers through the use of representations (e.g., equal-sized groups, arrays, area models, and equal "jumps" on number lines for multiplication, and successive subtraction, partitioning, and sharing for division). They use properties of addition and multiplication (e.g., commutativity, associativity, and the distributive property) to multiply whole numbers and apply increasingly sophisticated strategies based on these properties to solve multiplication and division problems involving basic facts. By comparing a variety of solution strategies, students relate multiplication and division as inverse operations.	**Algebra:** Understanding properties of multiplication and the relationship between multiplication and division is a part of algebra readiness that develops at grade 3. The creation and analysis of patterns and relationships involving multiplication and division should occur at this grade level. Students build a foundation for later under-standing of functional relationships by describing relationships in context with such statements as, "The number of legs is 4 times the number of chairs." **Measurement:** Students in grade 3 strengthen their understanding of fractions as they confront problems in linear measurement that call for more precision than the whole unit allowed them in their work in grade 2. They develop their facility in measuring with fractional parts of linear units. Students develop measurement concepts and skills through experiences in analyzing attributes and properties of two-dimensional objects. They form an understanding of perimeter as a measurable attribute and select appropriate units, strategies, and tools to solve problems involving perimeter. **Data Analysis:** Addition, subtraction, multiplication, and division of whole numbers come into play as students construct and analyze frequency tables, bar graphs, picture graphs, and line plots and use them to solve problems. **Number and Operations:** Building on their work in grade 2, students extend their understanding of place value to numbers up to 10,000 in various contexts. Students also apply this understanding to the task of representing numbers in different equivalent forms (e.g., expanded notation). They develop their understanding of numbers by building their facility with mental computation (addition and subtraction in special cases, such as 2500 + 6000 and 9000 – 5000), by using computational estimation, and by performing paper-and-pencil computations.
Number and Operations: Developing an understanding of fractions and fraction equivalence Students develop an understanding of the meanings and uses of fractions to represent parts of a whole, parts of a set, or points or distances on a number line. They understand that the size of a fractional part is relative to the size of the whole, and they use fractions to represent numbers that are equal to, less than, or greater than 1. They solve problems that involve comparing and ordering fractions by using models, benchmark fractions, or common numerators or denominators. They understand and use models, including the number line, to identify equivalent fractions.	
Geometry: Describing and analyzing properties of two-dimensional shapes Students describe, analyze, compare, and classify two-dimensional shapes by their sides and angles and connect these attributes to definitions of shapes. Students investigate, describe, and reason about decomposing, combining, and transforming polygons to make other polygons. Through building, drawing, and analyzing two-dimensional shapes, students understand attributes and properties of two-dimensional space and the use of those attributes and properties in solving problems, including applications involving congruence and symmetry.	

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Curriculum Focal Points and Connections for Grade 4

The set of three curriculum focal points and related connections for mathematics in grade 4 follow. These topics are the recommended content emphases for this grade level. It is essential that these focal points be addressed in contexts that promote problem solving, reasoning, communication, making connections, and designing and analyzing representations.

Grade 4 Curriculum Focal Points	Connections to the Focal Points
Number and Operations and Algebra: Developing quick recall of multiplication facts and related division facts and fluency with whole number multiplication Students use understandings of multiplication to develop quick recall of the basic multiplication facts and related division facts. They apply their understanding of models for multiplication (i.e., equal-sized groups, arrays, area models, equal intervals on the number line), place value, and properties of operations (in particular, the distributive property) as they develop, discuss, and use efficient, accurate, and generalizable methods to multiply multidigit whole numbers. They select appropriate methods and apply them accurately to estimate products or calculate them mentally, depending on the context and numbers involved. They develop fluency with efficient procedures, including the standard algorithm, for multiplying whole numbers, understand why the procedures work (on the basis of place value and properties of operations), and use them to solve problems.	**Algebra:** Students continue identifying, describing, and extending numeric patterns involving all operations and nonnumeric growing or repeating patterns. Through these experiences, they develop an understanding of the use of a rule to describe a sequence of numbers or objects. **Geometry:** Students extend their understanding of properties of two-dimensional shapes as they find the areas of polygons. They build on their earlier work with symmetry and congruence in grade 3 to encompass transformations, including those that produce line and rotational symmetry. By using transformations to design and analyze simple tilings and tessellations, students deepen their understanding of two-dimensional space. **Measurement:** As part of understanding two-dimensional shapes, students measure and classify angles. **Data Analysis:** Students continue to use tools from grade 3, solving problems by making frequency tables, bar graphs, picture graphs, and line plots. They apply their understanding of place value to develop and use stem-and-leaf plots. **Number and Operations:** Building on their work in grade 3, students extend their understanding of place value and ways of representing numbers to 100 000 in various contexts. They use estimation in determining the relative sizes of amounts or distances. Students develop understandings of strategies for multidigit division by using models that represent division as the inverse of multiplication, as partitioning, or as successive subtraction. By working with decimals, students extend their ability to recognize equivalent fractions. Students' earlier work in grade 3 with models of fractions and multiplication and division facts supports their understanding of techniques for generating equivalent fractions and simplifying fractions.
Number and Operations: Developing an understanding of decimals, including the connections between fractions and decimals Students understand decimal notation as an extension of the base-ten system of writing whole numbers that is useful for representing more numbers, including numbers between 0 and 1, between 1 and 2, and so on. Students relate their understanding of fractions to reading and writing decimals that are greater than or less than 1, identifying equivalent decimals, comparing and ordering decimals, and estimating decimal or fractional amounts in problem solving. They connect equivalent fractions and decimals by comparing models to symbols and locating equivalent symbols on the number line.	
Measurement: Developing an understanding of area and determining the areas of two-dimensional shapes Students recognize area as an attribute of two-dimensional regions. They learn that they can quantify area by finding the total number of same-sized units of area that cover the shape without gaps or overlaps. They understand that a square that is 1 unit on a side is the standard unit for measuring area. They select appropriate units, strategies (e.g., decomposing shapes), and tools for solving problems that involve estimating or measuring area. Students connect area measure to the area model that they have used to represent multiplication, and they use this connection to justify the formula for the area of a rectangle.	

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Curriculum Focal Points and Connections for Grade 5

The set of three curriculum focal points and related connections for mathematics in grade 5 follow. These topics are the recommended content emphases for this grade level. It is essential that these focal points be addressed in contexts that promote problem solving, reasoning, communication, making connections, and designing and analyzing representations.

Grade 5 Curriculum Focal Points	Connections to the Focal Points
Number and Operations and Algebra: Developing an understanding of and fluency with division of whole numbers Students apply their understanding of models for division, place value, properties, and the relationship of division to multiplication as they develop, discuss, and use efficient, accurate, and generalizable procedures to find quotients involving multidigit dividends. They select appropriate methods and apply them accurately to estimate quotients or calculate them mentally, depending on the context and numbers involved. They develop fluency with efficient procedures, including the standard algorithm, for dividing whole numbers, understand why the procedures work (on the basis of place value and properties of operations), and use them to solve problems. They consider the context in which a problem is situated to select the most useful form of the quotient for the solution, and they interpret it appropriately.	**Algebra:** Students use patterns, models, and relationships as contexts for writing and solving simple equations and inequalities. They create graphs of simple equations. They explore prime and composite numbers and discover concepts related to the addition and subtraction of fractions as they use factors and multiples, including applications of common factors and common multiples. They develop an understanding of the order of operations and use it for all operations. **Measurement:** Students' experiences connect their work with solids and volume to their earlier work with capacity and weight or mass. They solve problems that require attention to both approximation and precision of measurement. **Data Analysis:** Students apply their understanding of whole numbers, fractions, and decimals as they construct and analyze double-bar and line graphs and use ordered pairs on coordinate grids. **Number and Operations:** Building on their work in grade 4, students extend their understanding of place value to numbers through millions and millionths in various contexts. They apply what they know about multiplication of whole numbers to larger numbers. Students also explore contexts that they can describe with negative numbers (e.g., situations of owing money or measuring elevations above and below sea level).
Number and Operations: Developing an understanding of and fluency with addition and subtraction of fractions and decimals Students apply their understandings of fractions and fraction models to represent the addition and subtraction of fractions with unlike denominators as equivalent calculations with like denominators. They apply their understandings of decimal models, place value, and properties to add and subtract decimals. They develop fluency with standard procedures for adding and subtracting fractions and decimals. They make reasonable estimates of fraction and decimal sums and differences. Students add and subtract fractions and decimals to solve problems, including problems involving measurement.	
Geometry and Measurement and Algebra: Describing three-dimensional shapes and analyzing their properties, including volume and surface area Students relate two-dimensional shapes to three-dimensional shapes and analyze properties of polyhedral solids, describing them by the number of edges, faces, or vertices as well as the types of faces. Students recognize volume as an attribute of three-dimensional space. They understand that they can quantify volume by finding the total number of same-sized units of volume that they need to fill the space without gaps or overlaps. They understand that a cube that is 1 unit on an edge is the standard unit for measuring volume. They select appropriate units, strategies, and tools for solving problems that involve estimating or measuring volume. They decompose three-dimensional shapes and find surface areas and volumes of prisms. As they work with surface area, they find and justify relationships among the formulas for the areas of different polygons. They measure necessary attributes of shapes to use area formulas to solve problems.	

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Curriculum Focal Points and Connections for Grade 6

The set of three curriculum focal points and related connections for mathematics in grade 6 follow. These topics are the recommended content emphases for this grade level. It is essential that these focal points be addressed in contexts that promote problem solving, reasoning, communication, making connections, and designing and analyzing representations.

Grade 6 Curriculum Focal Points	Connections to the Focal Points
Number and Operations: Developing an understanding of and fluency with multiplication and division of fractions and decimals Students use the meanings of fractions, multiplication and division, and the inverse relationship between multiplication and division to make sense of procedures for multiplying and dividing fractions and explain why they work. They use the relationship between decimals and fractions, as well as the relationship between finite decimals and whole numbers (i.e., a finite decimal multiplied by an appropriate power of 10 is a whole number), to understand and explain the procedures for multiplying and dividing decimals. Students use common procedures to multiply and divide fractions and decimals efficiently and accurately. They multiply and divide fractions and decimals to solve problems, including multistep problems and problems involving measurement.	**Number and Operations:** Students' work in dividing fractions shows them that they can express the result of dividing two whole numbers as a fraction (viewed as parts of a whole). Students then extend their work in grade 5 with division of whole numbers to give mixed number and decimal solutions to division problems with whole numbers. They recognize that ratio tables not only derive from rows in the multiplication table but also connect with equivalent fractions. Students distinguish multiplicative comparisons from additive comparisons. **Algebra:** Students use the commutative, associative, and distributive properties to show that two expressions are equivalent. They also illustrate properties of operations by showing that two expressions are equivalent in a given context (e.g., determining the area in two different ways for a rectangle whose dimensions are $x + 3$ by 5). Sequences, including those that arise in the context of finding possible rules for patterns of figures or stacks of objects, provide opportunities for students to develop formulas. **Measurement and Geometry:** Problems that involve areas and volumes, calling on students to find areas or volumes from lengths or to find lengths from volumes or areas and lengths, are especially appropriate. These problems extend the students' work in grade 5 on area and volume and provide a context for applying new work with equations.
Number and Operations: Connecting ratio and rate to multiplication and division Students use simple reasoning about multiplication and division to solve ratio and rate problems (e.g., "If 5 items cost $3.75 and all items are the same price, then I can find the cost of 12 items by first dividing $3.75 by 5 to find out how much one item costs and then multiplying the cost of a single item by 12"). By viewing equivalent ratios and rates as deriving from, and extending, pairs of rows (or columns) in the multiplication table, and by analyzing simple drawings that indicate the relative sizes of quantities, students extend whole number multiplication and division to ratios and rates. Thus, they expand the repertoire of problems that they can solve by using multiplication and division, and they build on their understanding of fractions to understand ratios. Students solve a wide variety of problems involving ratios and rates.	
Algebra: Writing, interpreting, and using mathematical expressions and equations Students write mathematical expressions and equations that correspond to given situations, they evaluate expressions, and they use expressions and formulas to solve problems. They understand that variables represent numbers whose exact values are not yet specified, and they use variables appropriately. Students understand that expressions in different forms can be equivalent, and they can rewrite an expression to represent a quantity in a different way (e.g., to make it more compact or to feature different information). Students know that the solutions of an equation are the values of the variables that make the equation true. They solve simple one-step equations by using number sense, properties of operations, and the idea of maintaining equality on both sides of an equation. They construct and analyze tables (e.g., to show quantities that are in equivalent ratios), and they use equations to describe simple relationships (such as $3x = y$) shown in a table.	

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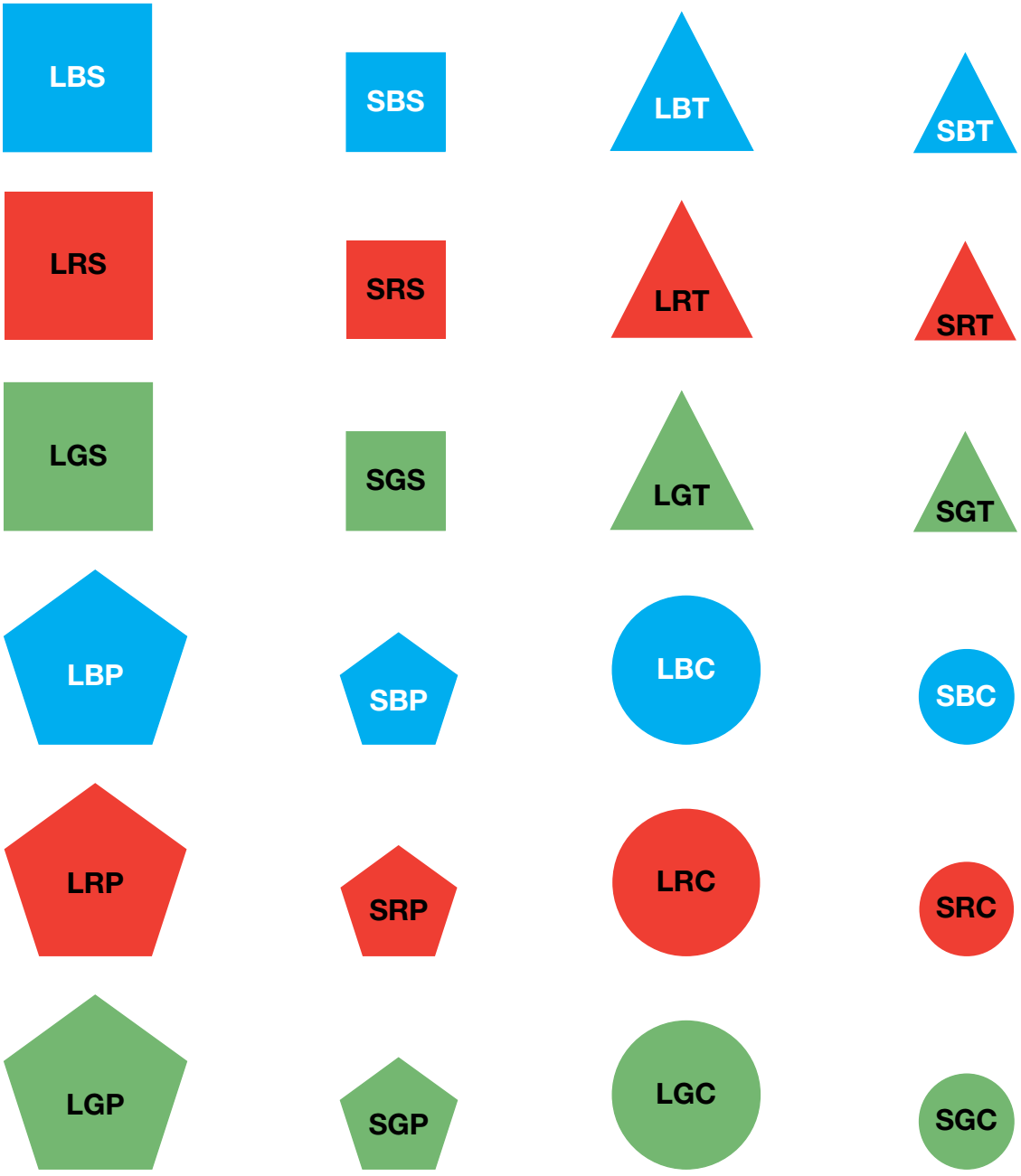
APPENDIX C

Masters

1. Attribute pieces
2. Cuisenaire rods
3. Base-ten blocks
4. Pattern blocks
5. Five- and ten-frames
6. Hundred charts
7. Variations of hundred charts
8. Basic addition and multiplication facts
9. 0–9 Cards
10. Blank place-value chart
11. Trading mat for different lands
12. Powers of 10 (place-value chart)
13. Decimal or percentage paper
14. Fraction bars
15A. Fraction models and spinners
15B. Fraction models and spinners
16. Rulers
17. Geoboard template
18. Geoboard recording paper
19. Centimetre dot paper
20. Isometric paper
21. Centimetre grid paper
22. Geometric design paper
23. Equilateral triangle paper
24. Tangram
25. Circle point paper

Reys et al., *Helping Children Learn Mathematics,* 9E, copyright © 2009 by John Wiley & Sons, Inc.

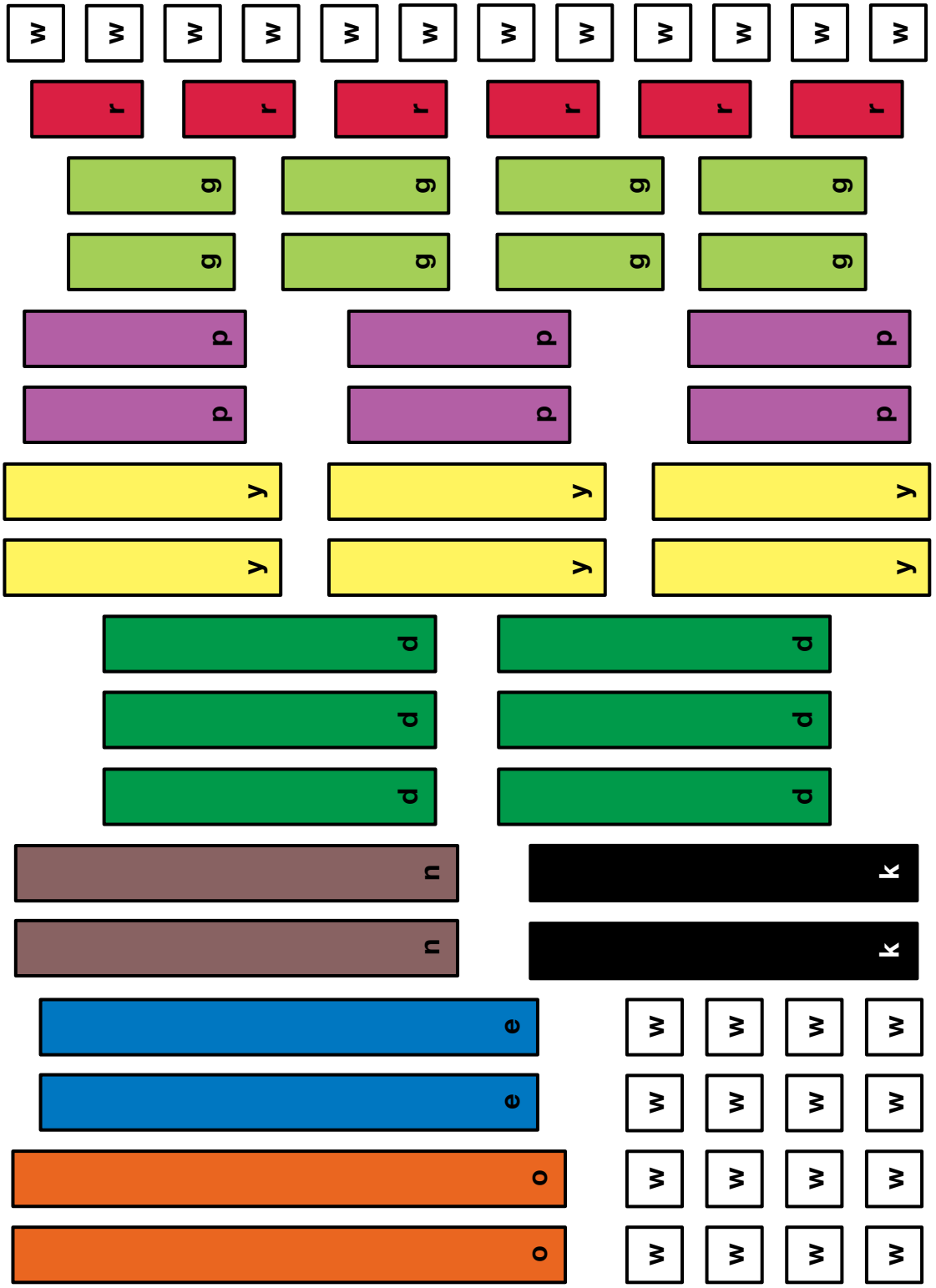
1. Attribute pieces



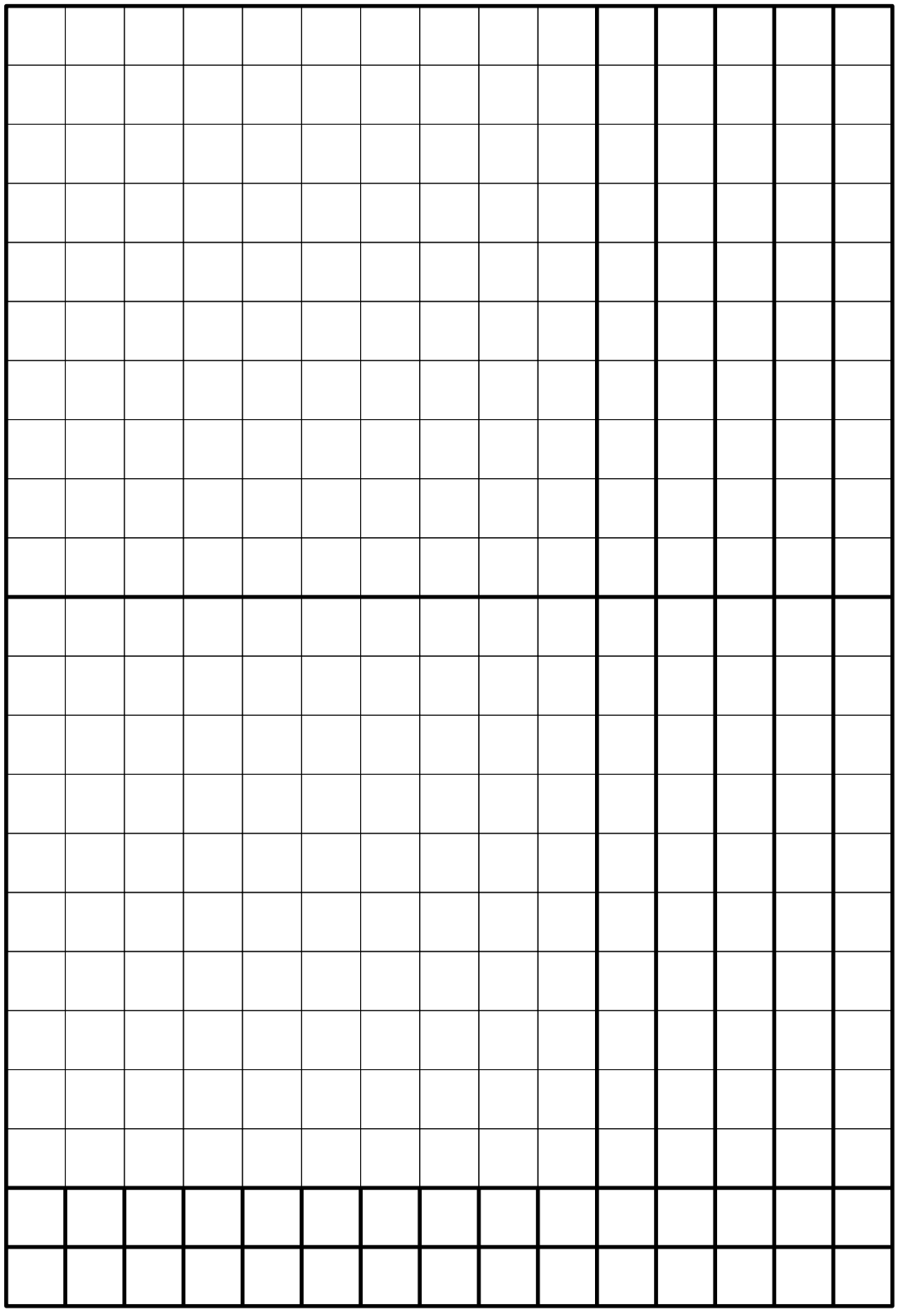
Code: L → Large S → Small B → Blue G → Green R → Red S → Square P → Pentagon T → Triangle C → Circle

LBS = Large Blue Square

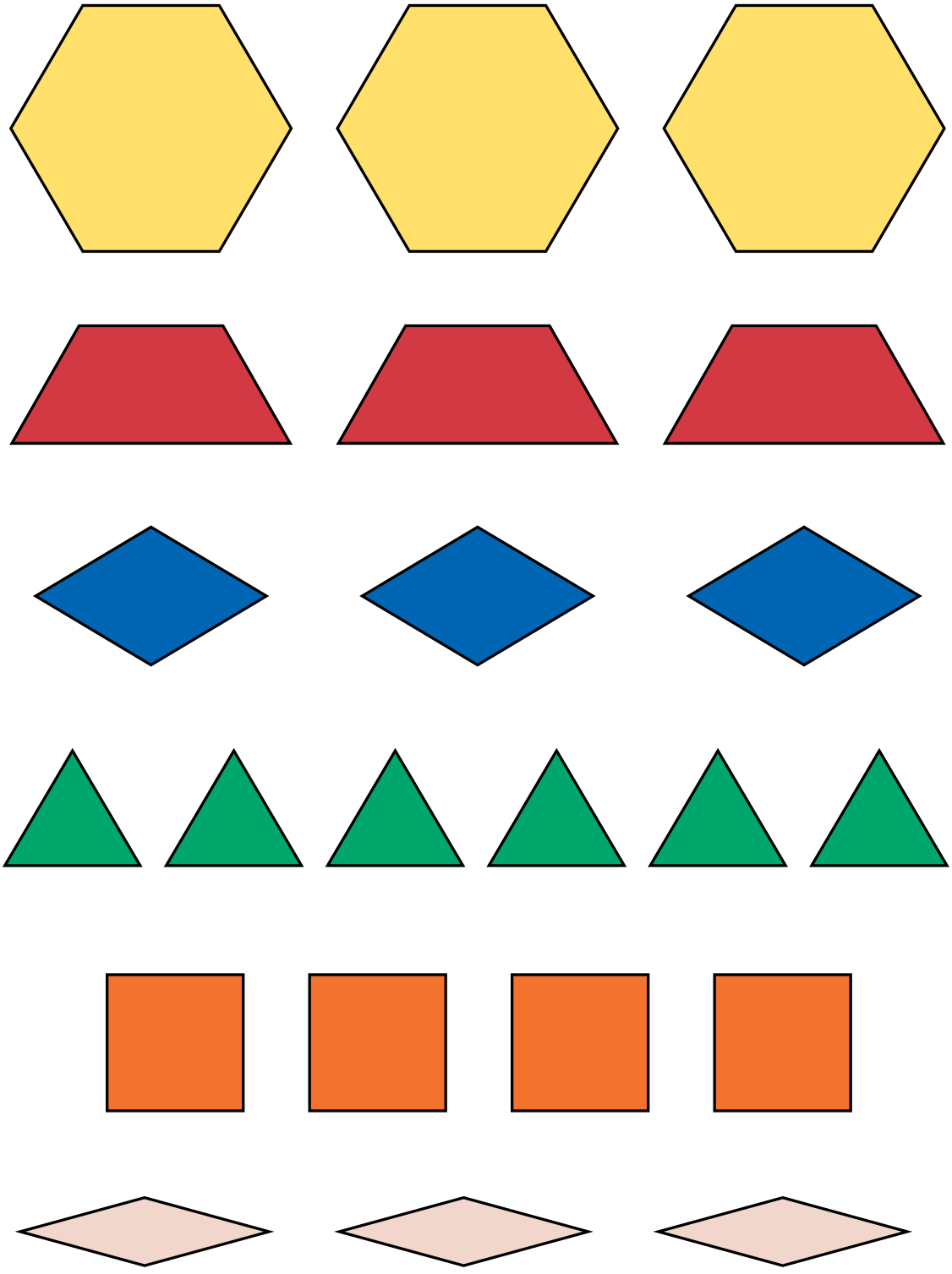
2. Cuisenaire rods



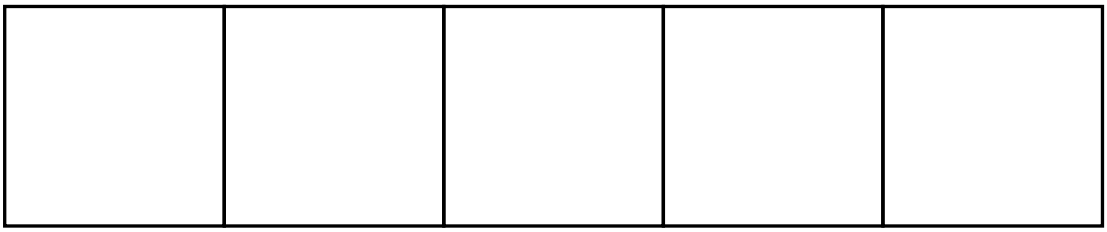
3. Base-ten blocks

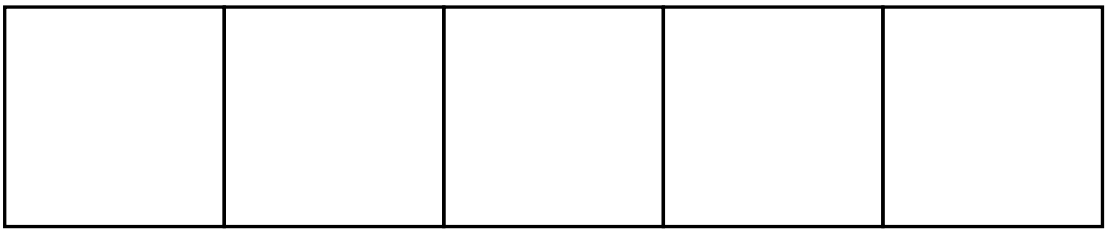


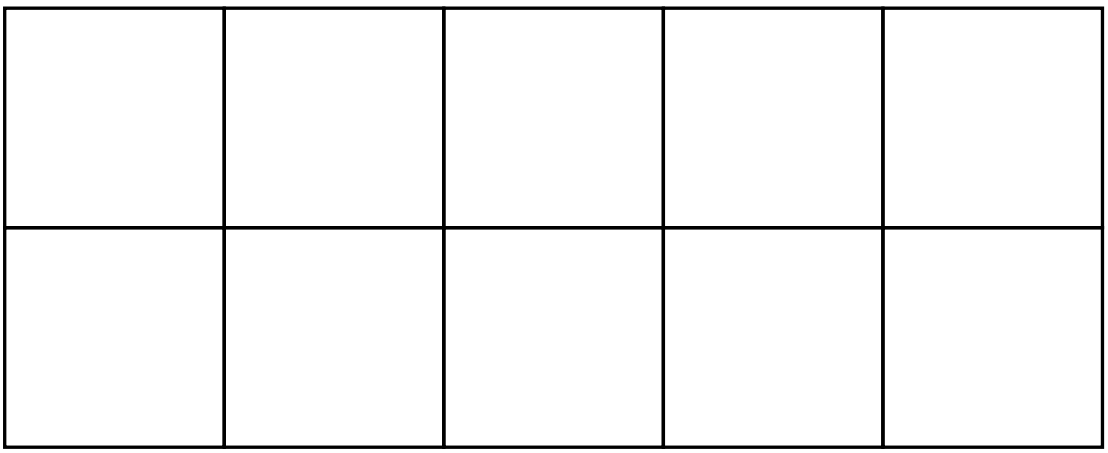
4. Pattern blocks

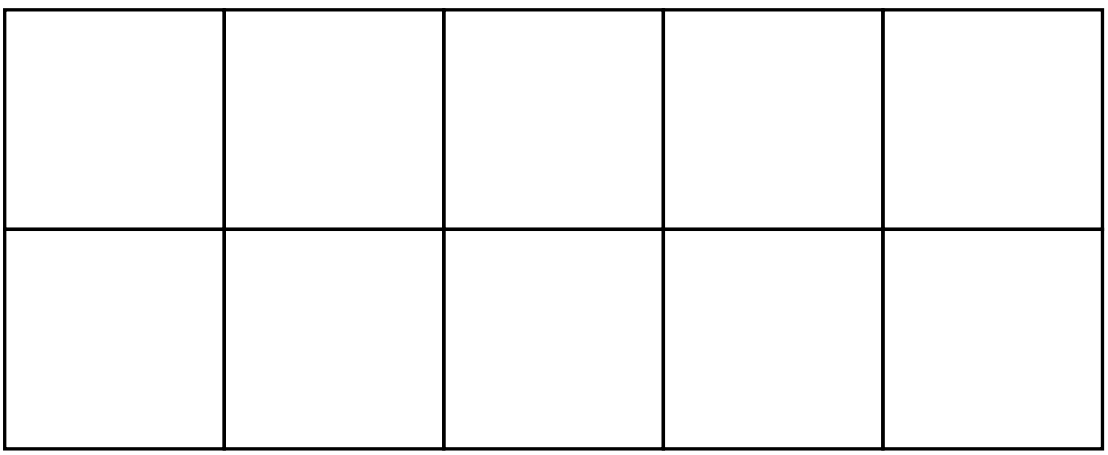


5. Five- and ten-frames







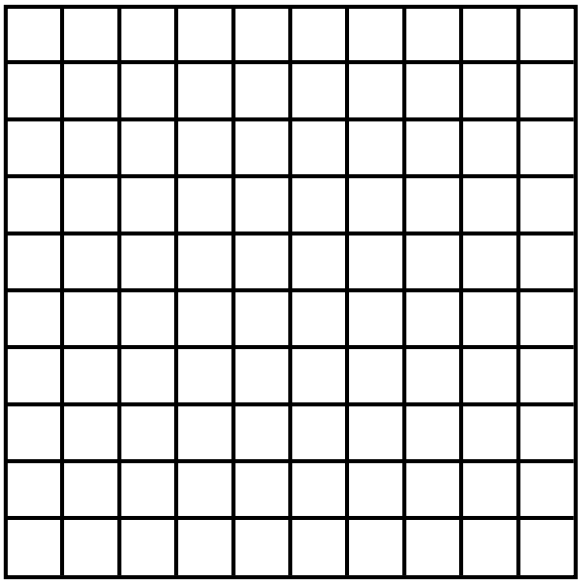


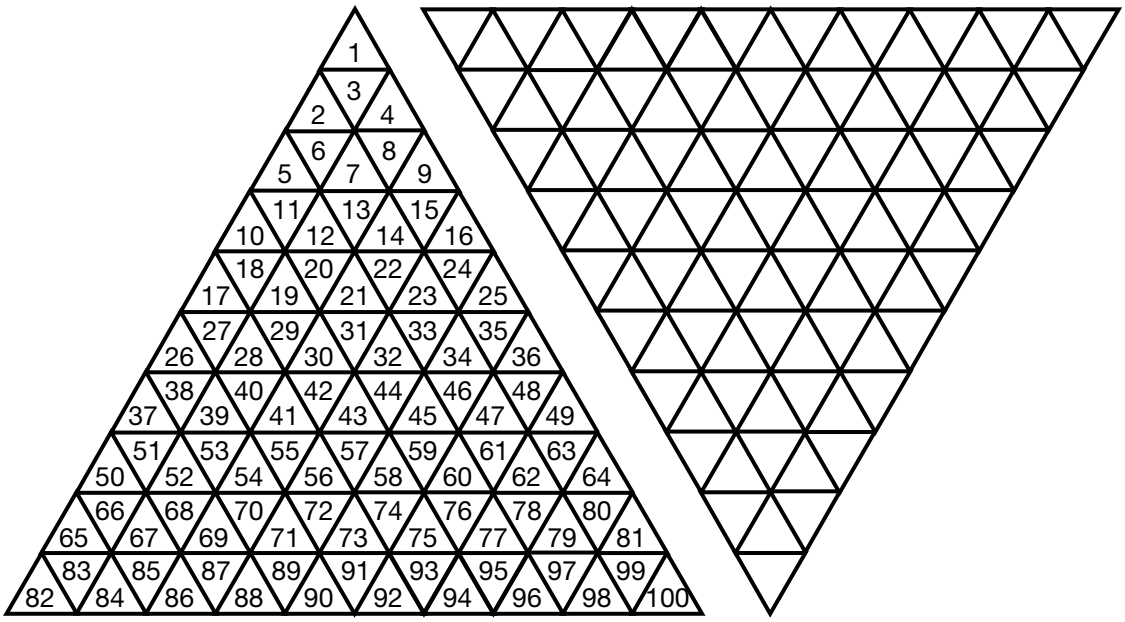
6. Hundred charts

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

0	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

7. Variations of hundred charts





8. Basic addition and multiplication facts

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18

- ☐ Commutativity
- ☐ Adding 0 and 1
- ☐ Doubles, +/–1
- ☐ Counting on (1,2)
- ☐ Combinations to 10 and adding to 10

×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

- ☐ Commutativity
- ☐ Multiplying by 1 and 0
- ☐ Skip counting
- ☐ Repeated addition
- ☐ Splitting product into known parts
- ☐ Patterns

9. 0–9 Cards

4	9
3	8
2	7
1	6
0	5

10. Blank place-value chart

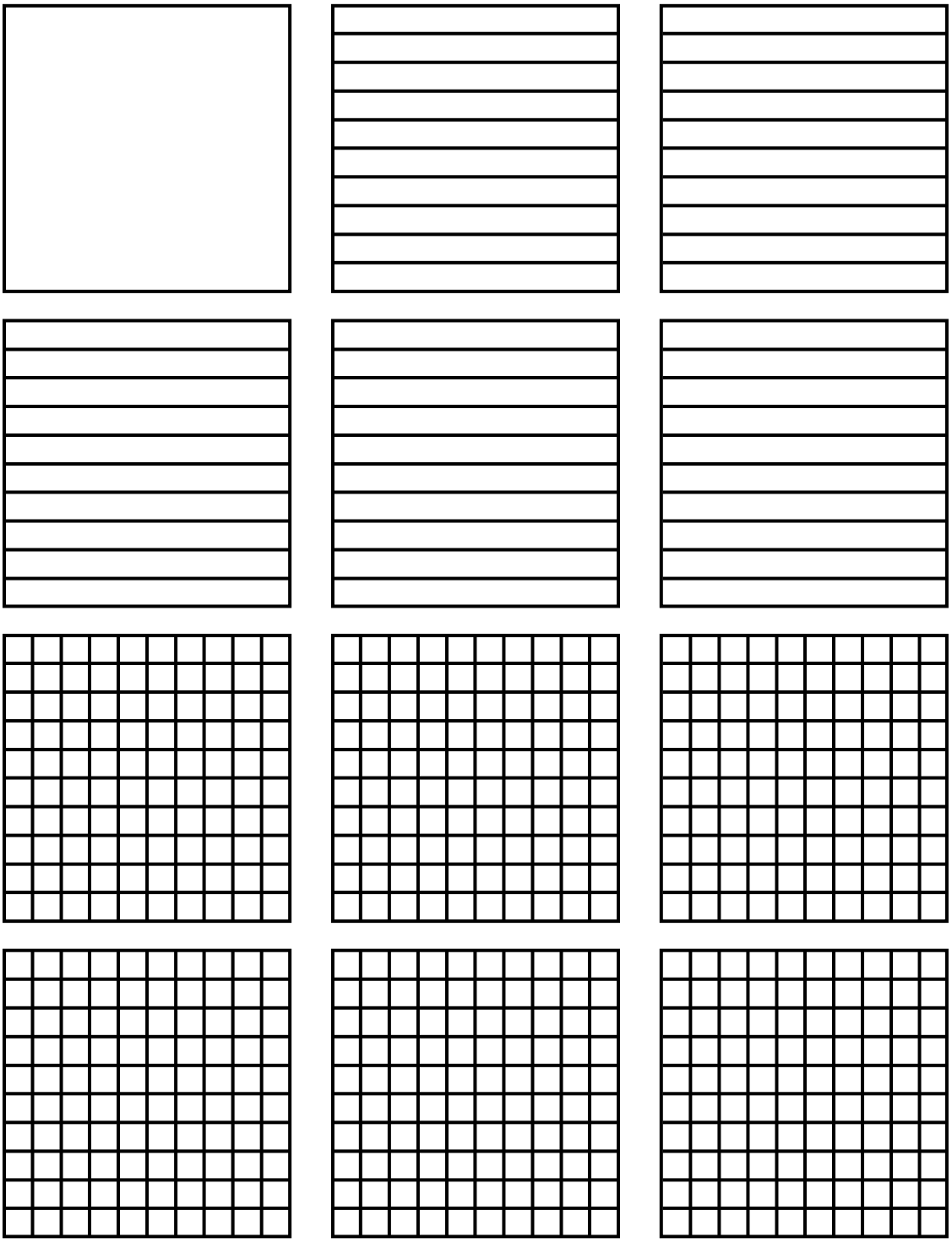
11. Trading mat for different lands

Red	Green	Blue	Yellow

12. Powers of 10 (place-value chart)

Thousands	Hundreds	Tens	Ones

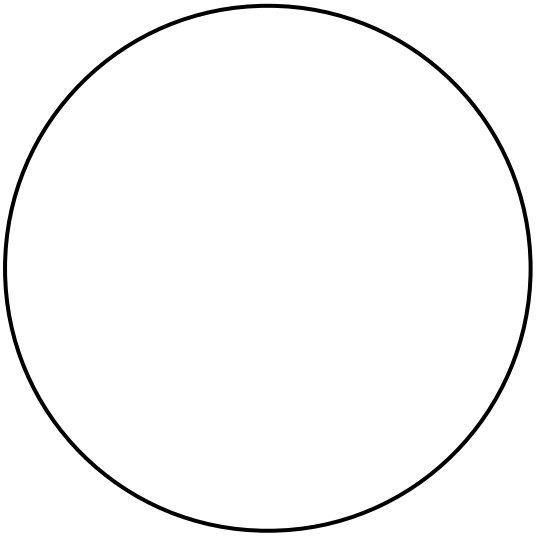
13. Decimal or percentage paper

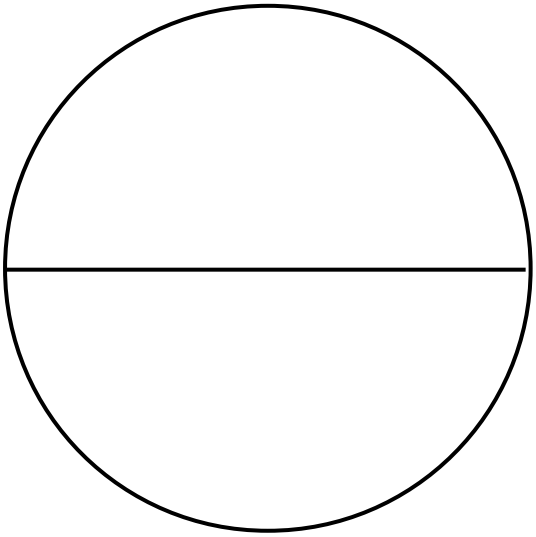


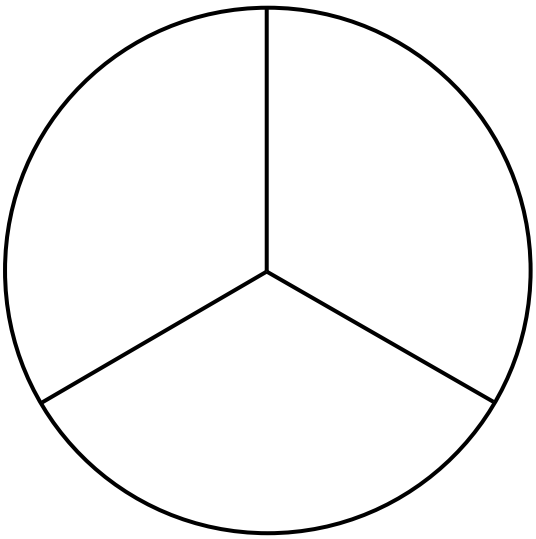
14. Fraction bars

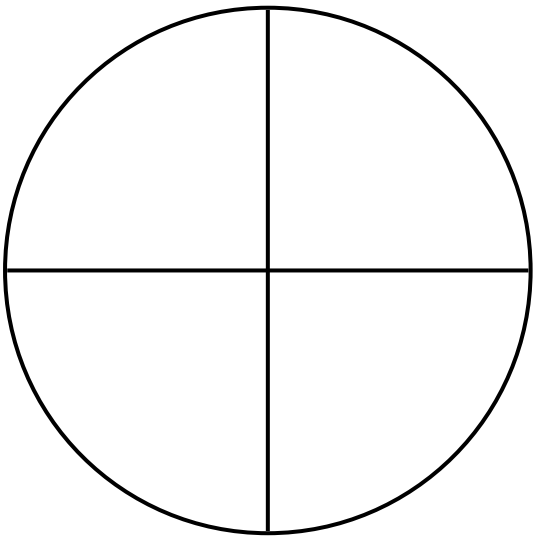
one	1										
halves	$\frac{1}{2}$									$\frac{2}{2}$	
thirds	$\frac{1}{3}$					$\frac{2}{3}$				$\frac{3}{3}$	
quarters	$\frac{1}{4}$			$\frac{2}{4}$			$\frac{3}{4}$			$\frac{4}{4}$	
fifths	$\frac{1}{5}$		$\frac{2}{5}$			$\frac{3}{5}$			$\frac{4}{5}$		$\frac{5}{5}$
sixths	$\frac{1}{6}$		$\frac{2}{6}$		$\frac{3}{6}$		$\frac{4}{6}$		$\frac{5}{6}$		$\frac{6}{6}$
sevenths	$\frac{1}{7}$		$\frac{2}{7}$		$\frac{3}{7}$		$\frac{4}{7}$		$\frac{5}{7}$	$\frac{6}{7}$	$\frac{7}{7}$
eighths	$\frac{1}{8}$	$\frac{2}{8}$		$\frac{3}{8}$		$\frac{4}{8}$		$\frac{5}{8}$		$\frac{6}{8}$	$\frac{7}{8}$	$\frac{8}{8}$
ninths	$\frac{1}{9}$	$\frac{2}{9}$		$\frac{3}{9}$		$\frac{4}{9}$		$\frac{5}{9}$	$\frac{6}{9}$	$\frac{7}{9}$	$\frac{8}{9}$	$\frac{9}{9}$
tenths	$\frac{1}{10}$	$\frac{2}{10}$		$\frac{3}{10}$		$\frac{4}{10}$		$\frac{5}{10}$	$\frac{6}{10}$	$\frac{7}{10}$	$\frac{8}{10}$	$\frac{9}{10}$	$\frac{10}{10}$
twelfths	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{3}{12}$	$\frac{4}{12}$	$\frac{5}{12}$	$\frac{6}{12}$	$\frac{7}{12}$	$\frac{8}{12}$	$\frac{9}{12}$	$\frac{10}{12}$	$\frac{11}{12}$	$\frac{12}{12}$

15A. Fraction models and spinners

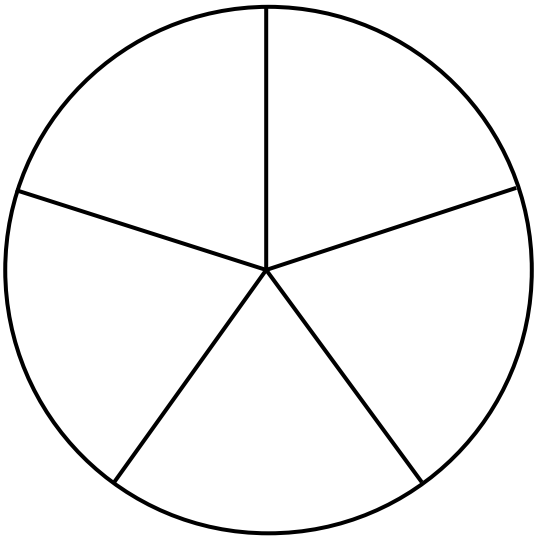


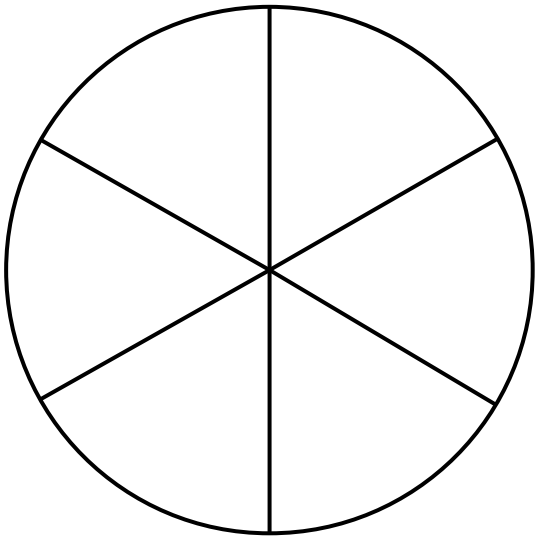


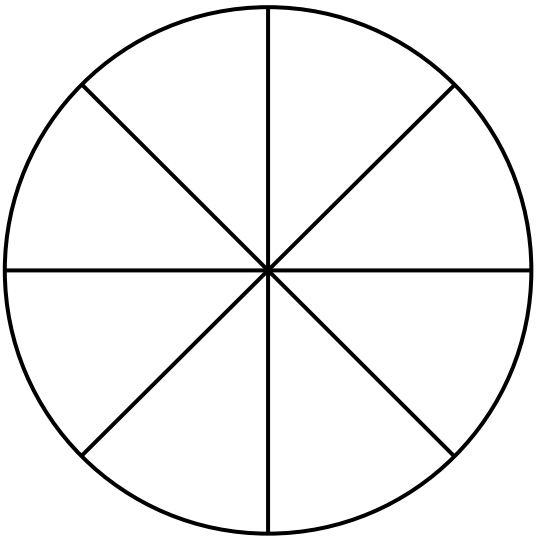


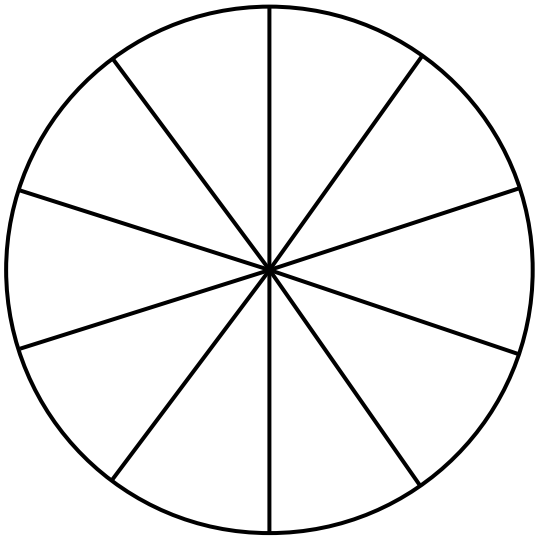


15B. Fraction models and spinners





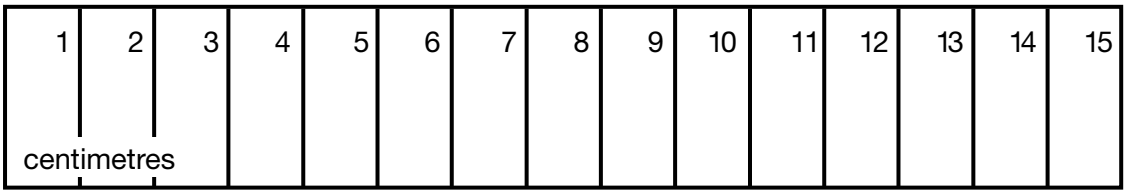




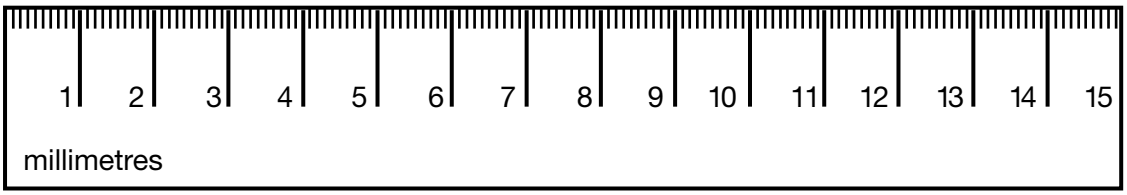




16. Rulers

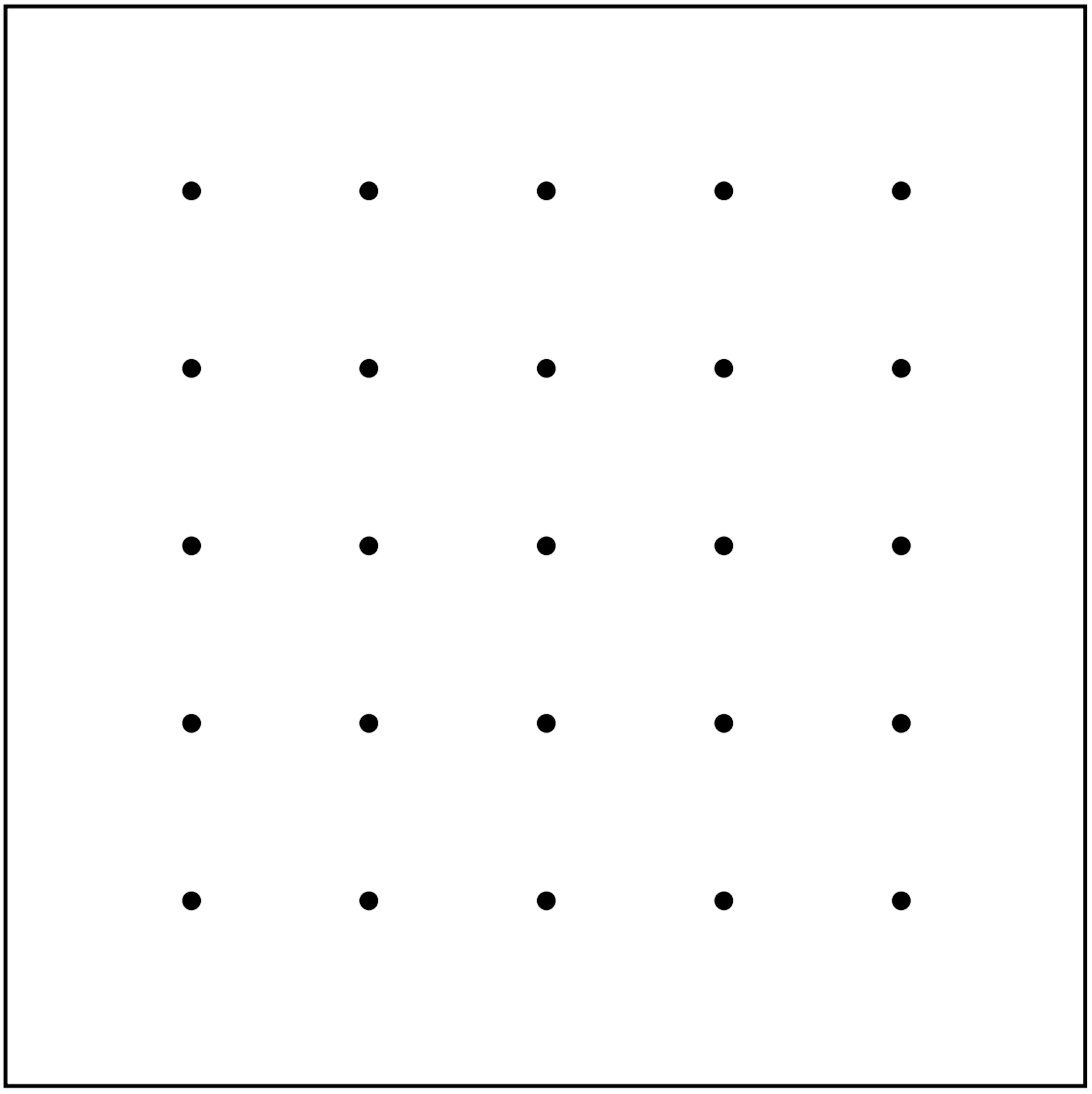


Primary centimetre ruler

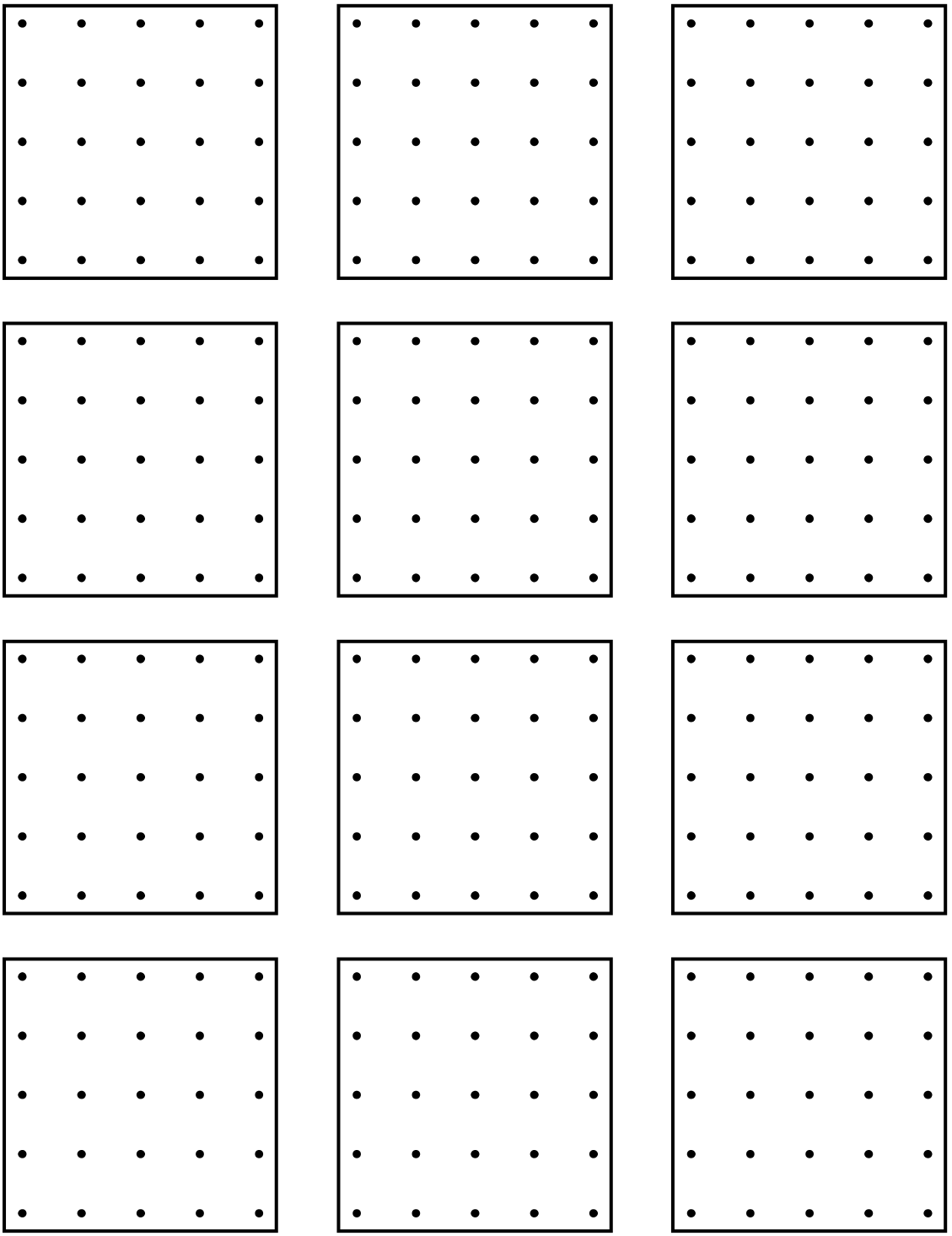


Millimetre ruler

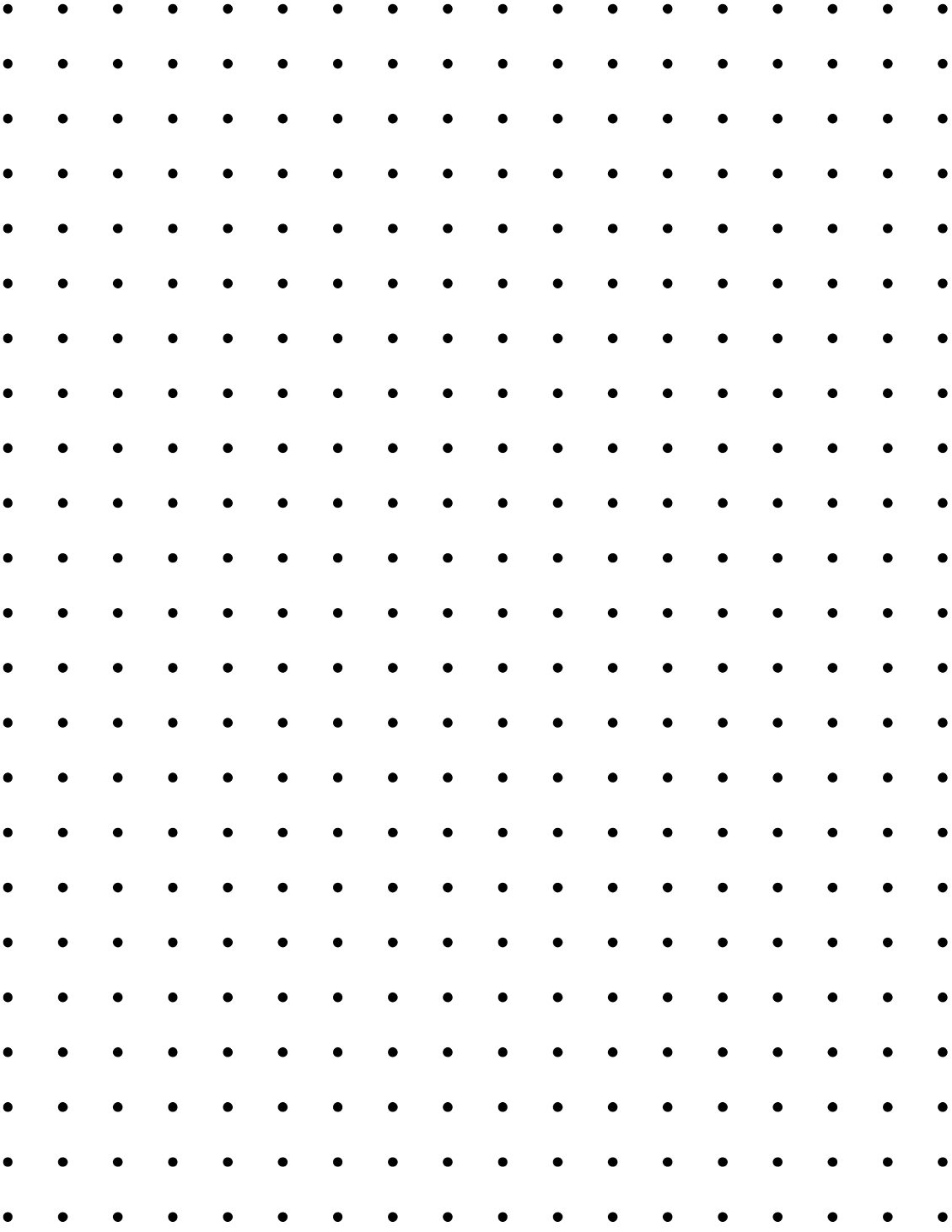
17. Geoboard template



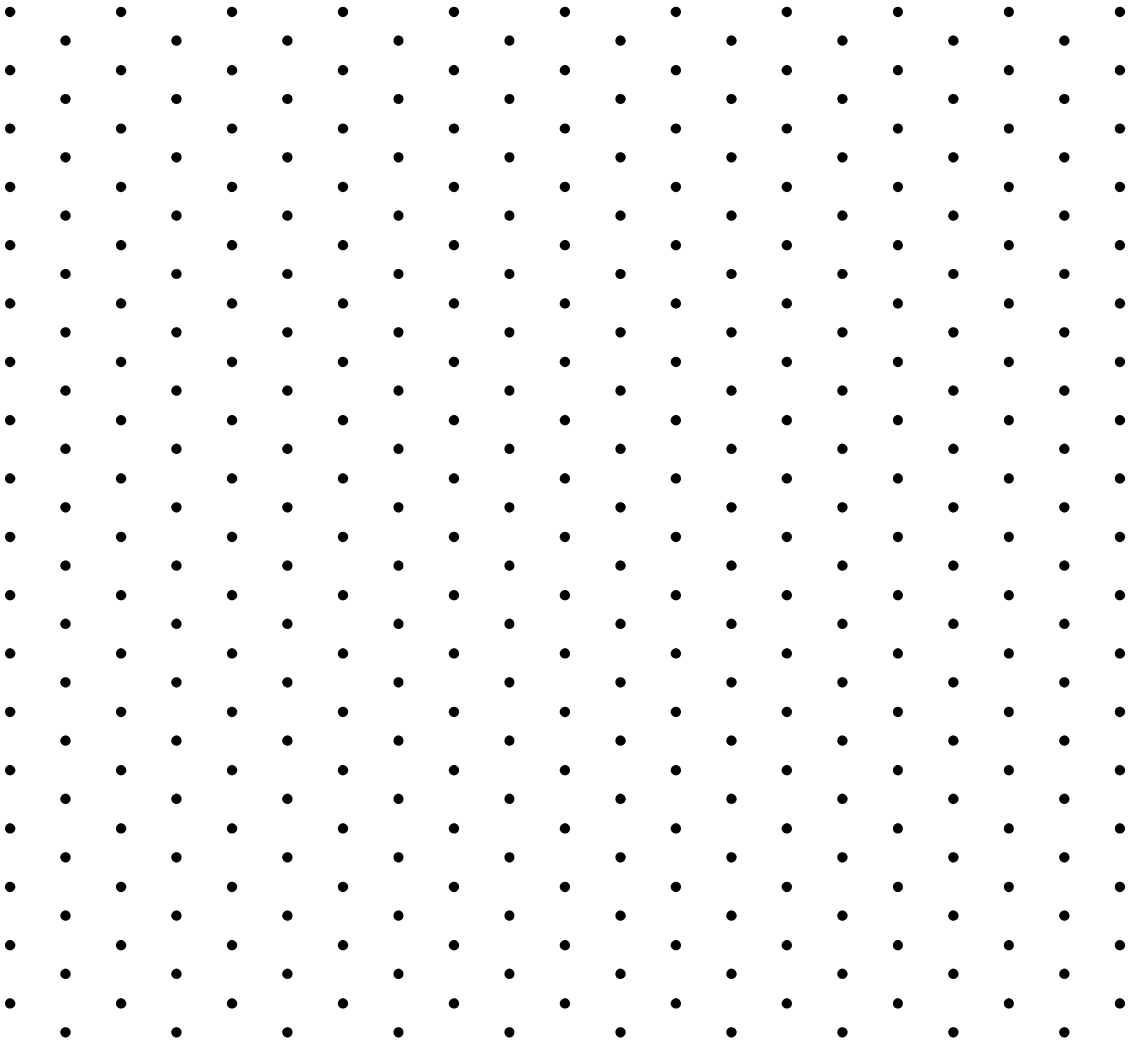
18. Geoboard recording paper



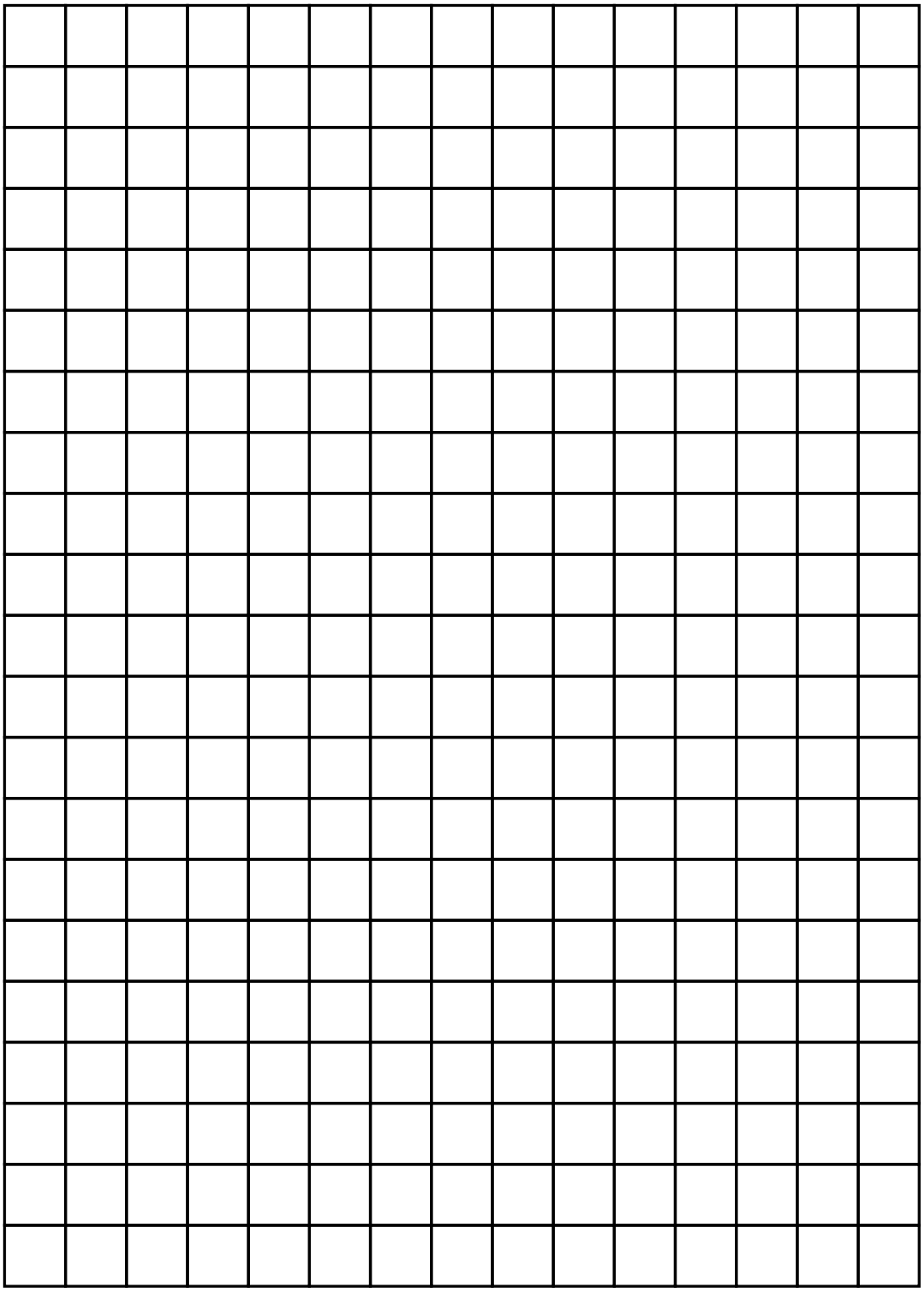
19. Centimetre dot paper



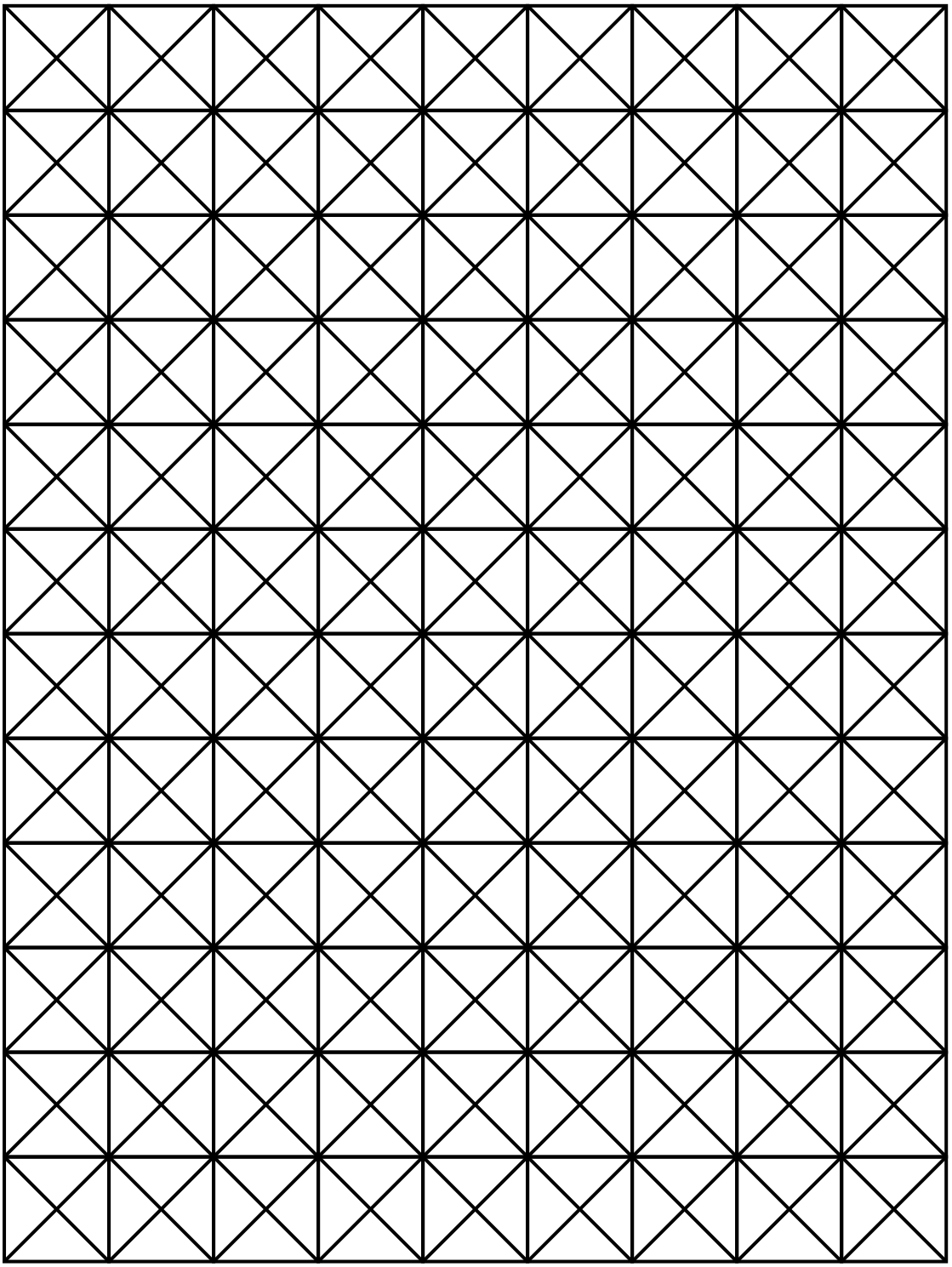
20. Isometric paper



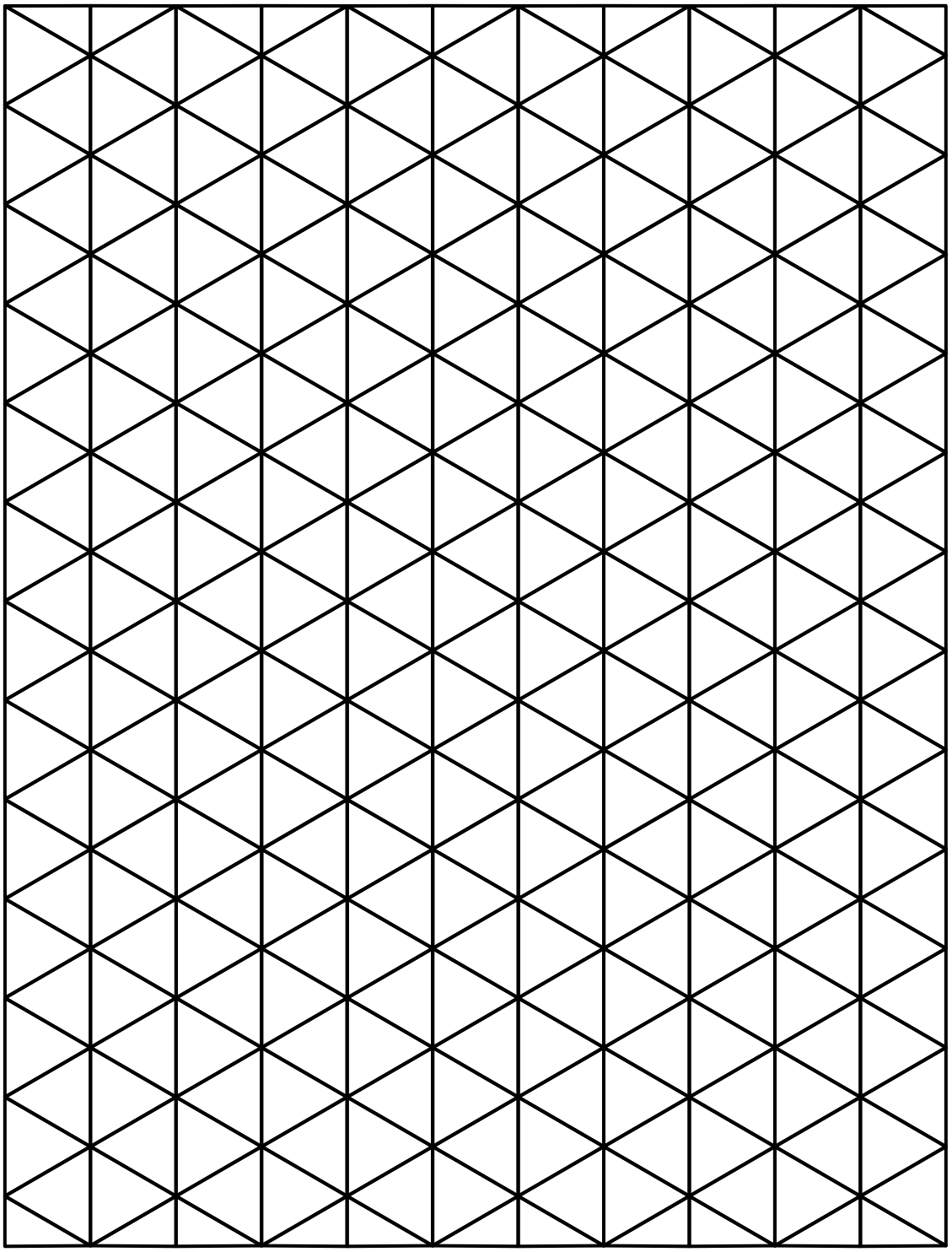
21. Centimetre grid paper



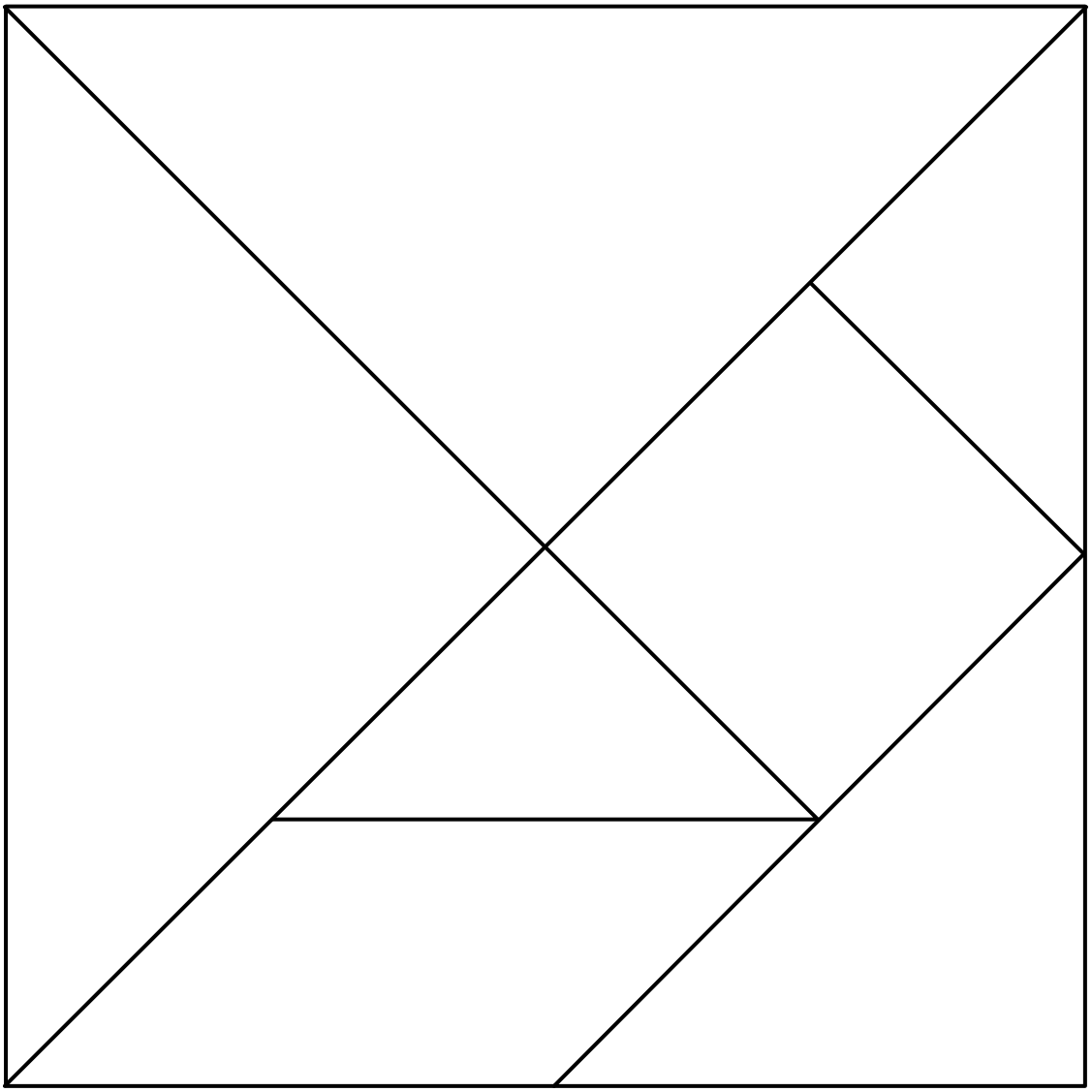
22. Geometric design paper



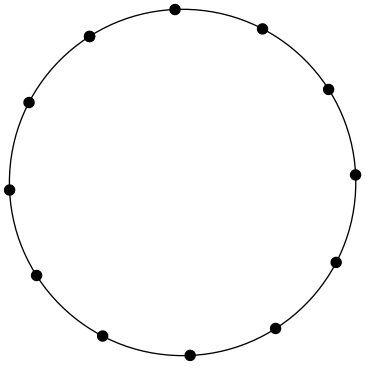
23. Equilateral triangle paper



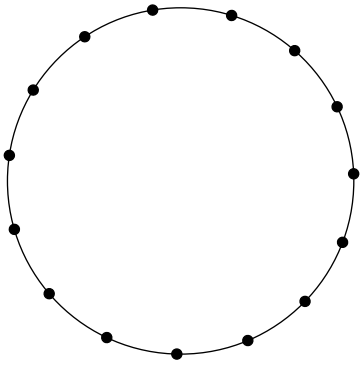
24. Tangram



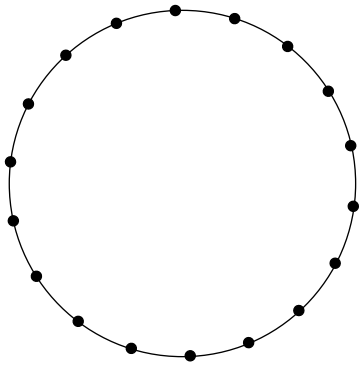
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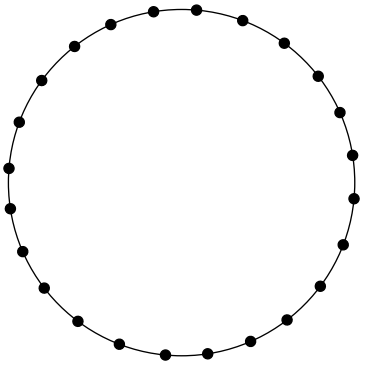
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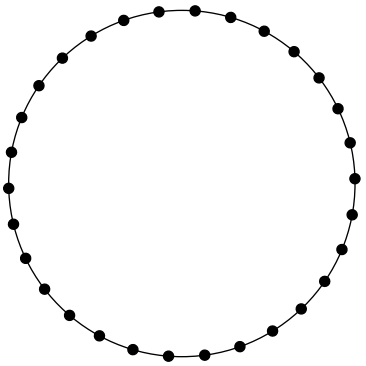
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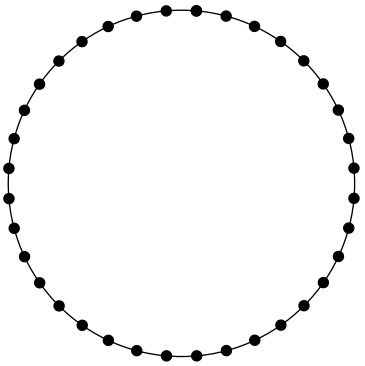
18 points



24 points



30 points



36 points

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