Mathematics for Elementary Teachers

A CONCEPTUAL APPROACH ■ NINTH EDITION



Albert B. Bennett, Jr. ■ Laurie J. Burton ■ L. Ted Nelson

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Ninth Edition

Mathematics for Elementary Teachers

A Conceptual Approach

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MATHEMATICS FOR ELEMENTARY TEACHERS: A CONCEPTUAL APPROACH, NINTH EDITION

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To Future Teachers

You are preparing to enter a very exciting and dynamic profession. As a teacher, you will be a role model for young people who will need your support and understanding to build their confidence. You will work with hundreds of students over your career and have a strong impact on their lives—a huge responsibility!

We wrote this book to give you the preparation you will need to become a teacher who can help students succeed in mathematics. You will be an effective teacher if you are able to

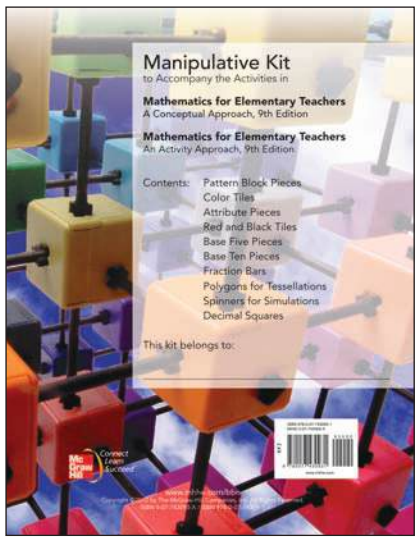
- Acquire a clear understanding of mathematical concepts.
- Learn problem-solving techniques.
- Familiarize yourself with NCTM Standards.
- Work with hands-on and Virtual Manipulatives to carry out activities.
- Apply calculator and computer technology to problem solving.
- See connections between your study of mathematics and the elementary school curriculum.
- Write about and discuss concepts and apply them to realistic classroom situations.
- Integrate online information into classroom teaching.

Your Manipulative Kit

You may recall that your elementary school classroom had colored materials for learning about numbers and geometry. For example, some of you may have used base-ten blocks. Many of those same materials are available on colored punch-out-ready cardstock in the Manipulative Kit that may be packaged with your text.

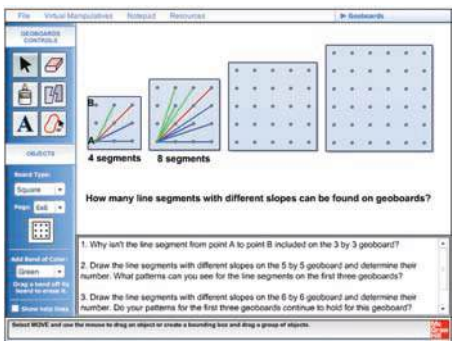


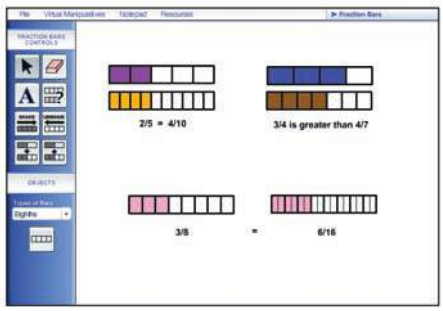
For each manipulative there is a resealable envelope that lists the numbers and types of pieces in the manipulative set. While these materials are designed for use with the one-page Math Activities that precede each section, they also can be used to form the beginning of a set of resources for your future elementary school classes.



Printable Virtual Manipulatives

The **Companion Website** (www.mhhe.com/bbn) has a Student Center that you can enter without a password. Among the site's resources are Virtual Manipulatives, electronic versions of the items in the Manipulative Kit. The following figure shows an example of the workspace for the Geoboard and Fraction Bar Virtual Manipulatives. Homework and projects done with virtual manipulatives can be printed. An overhead menu on the workspace for each type of virtual manipulative allows you to view the text's one-page Math Activities while working online.





Answers to Exercises

You can find answers to the odd-numbered exercises at the end of this book. However, these are brief answers without explanations. If you would like diagrams and careful explanations for the odd-numbered exercises, they are available in the *Student's Solutions Manual* ISBN 13: 978-0-07-743090-0 (ISBN 10: 0-07-743090-5).

Preface

The opening paragraph in the National Council of Teachers of Mathematics' *Principles and Standards for School Mathematics* states that its recommendations are grounded in the belief that all students should be taught in a way that fosters conceptual understanding of mathematical concepts and skills. This same belief has guided and influenced each edition of *Mathematics for Elementary Teachers: A Conceptual Approach,* which continues to emphasize the use of models and processes for providing insights into mathematical concepts.

PREPARING SUCCESSFUL TEACHERS

The primary objectives of this textbook are to provide (1) a conceptual understanding of mathematics, (2) a broad knowledge of basic mathematical skills, and (3) ideas and methods that generate enthusiasm for learning and teaching mathematics. Our approach to educating prospective teachers has been guided and influenced throughout by adherence to the standards set by the National Council of Teachers of Mathematics (NCTM) and by a special emphasis on problem solving and active student participation.

We have benefited from years of experience in the classroom and from the numerous workshops we have presented to teachers of elementary and middle school children. Our experiences have shown us that prospective teachers who engage with mathematics conceptually have a better chance of acquiring knowledge, solving problems, and gaining confidence in their ability to reason. We designed this text to effectively foster these conceptualizing and problem-solving skills.

Each section of the text begins with a one-page Math Activity and a Problem Opener. Both of these features involve problem solving and provide excellent opportunities for class discussions. We then proceed to develop the mathematical concepts of the section using models and diagrams before presenting students with abstractions. We also analyze a Problem-Solving Application within the section using Polya's four-step process. These applications serve to deepen and extend students' understanding of the content. In the exercise sets, we offer several categories of questions that are designed not only to reinforce critical knowledge, but also to strengthen students' reasoning and problem-solving skills.

At the end of each section there are Teaching Questions that enable students to practice their communication skills and Classroom Connections questions that ask them to explain the connections between the topics in each section, the special features of the section, and NCTM's Standards and Expectations. Many of these questions are posed within a classroom context to help undergraduates become better prepared for their teaching careers.

We believe that our approach gives future teachers a conceptual understanding of mathematics and a solid foundation in problem-solving and communication skills that they will be able to impart to their students.

SUGGESTIONS FOR ACTIVE STUDENT PARTICIPATION

NCTM's *Principles and Standards for School Mathematics* recommends that students develop their mathematical understanding by looking for patterns, making conjectures, and verifying hypotheses. Many instructors have been influenced by these recommendations

and are using instructional methods that involve more active student participation and less lecture time. Here are a few suggestions for utilizing the special features of the text that encourage active student participation.

Math Activities

The one-page Math Activities preceding each section of the text are augmented by the Manipulative Kit, a set of perforated color cardstock materials that may be packaged with the text at a nominal cost. These activities are designed to accomplish three major objectives: to develop students' conceptual understanding before they are given rules, definitions and procedures; to familiarize them with materials that are common in elementary schools and to present activities that can be adapted to the elementary school curriculum.

Virtual Manipulatives

Some instructors find that there is not sufficient time nor a suitable classroom setting for carrying out the one-page Math Activities with the cardstock materials. One solution is to use the Virtual Manipulatives on the companion website (www.mhhe.com/bbn), which are in an easily accessible Flash-based interface. There are Virtual Manipulatives for each colored cardstock material in the physical manipulative kit, and each type of virtual manipulative has a toolbar, a work area, and a "note pad" in which to type results. Moreover, many of the Virtual Manipulatives offer more variety than the cardstock manipulatives.

Problem Openers

Each section of the text opens with a problem statement related to the content of that section. Problem Openers can be used to prompt class discussions, to facilitate group work and problem solving, and to motivate interest in new topics. The solution to each Problem Opener and the problem-solving strategies required are contained in the *Instructor's Manual* on the companion website. The manual also offers one or more ideas for looking back and extending each Problem Opener for additional problem-solving practice in class or on assignments and tests.

Problem-Solving Applications

Each section of the text contains at least one Problem-Solving Application that is related to the section content and is analyzed using Polya's four-step strategy (introduced in Chapter 1). These problems can be posed to the class for small-group problem-solving activities. A suggested follow-up discussion might involve comparing students' plans for solving a particular problem and their solutions with those suggested in the text.

Technology and Laboratory Connections

Each section of the text has a Technology or Laboratory Connection featuring an in-depth investigation designed to enhance the mathematical content of the section. These investigations pose open-ended questions that require collecting data, looking for patterns, and forming and verifying conjectures. Many of the Technology and Laboratory Connections integrate the use of calculators while others use Geometer's Sketchpad® or the Mathematics Investigator software found at our companion website.

Interactive Math Applets

Eleven applets, available on the companion website, are designed to involve students in interactive explorations of some of the key concepts from each chapter.

connect™ MATHEMATICS

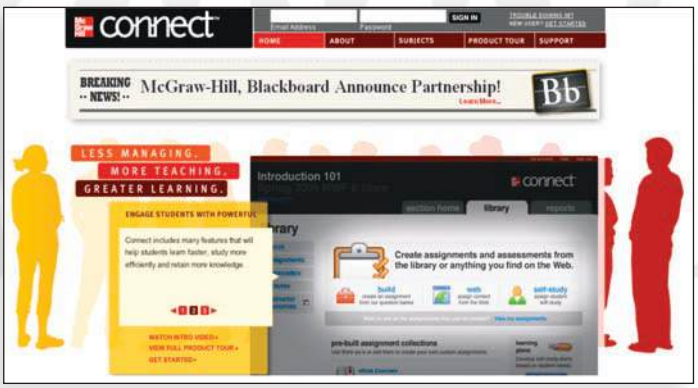
McGraw-Hill Connect Mathematics McGraw-Hill conducted in-depth research to create a new and improved learning experience that meets the needs of today's students and instructors. The result is a reinvented learning experience rich in information, visually engaging, and easily accessible to both instructors and students. McGraw-Hill's Connect is a Web-based assignment and assessment platform that helps students connect to their coursework and prepares them to succeed in and beyond the course.

Connect Mathematics enables math instructors to create and share courses and assignments with colleagues and adjuncts with only a few clicks of the mouse. All exercises, learning objectives, videos, and activities are directly tied to text-specific material.

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2 *Your students want an assignment page that is easy to use and includes lots of extra resources for help.*

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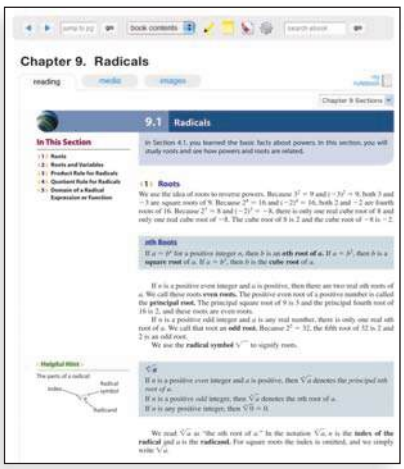


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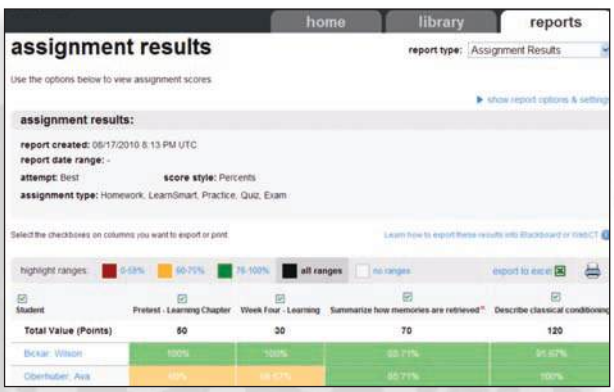
Assignment Creation Process



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Features of the Ninth Edition

Mathematics for Elementary Teachers: A Conceptual Approach has always contained features that effectively decrease future teachers' math anxiety and help them to see connections between their college mathematics courses and the mathematics they envision teaching to elementary school students. Many of these features are illustrated on the following pages.

- **Color photos** highlight key information and pedagogy and provide an appealing learning experience.

From ancient times tessellations have been used as patterns for rugs, fabrics, pottery, and architecture. The Moors, who settled in Spain in the eighth century, were masters of tessellating walls and floors with colored geometric tiles. Some of their work is shown in Figure 9.35, a photograph of a room and bath in the Alhambra, a fortress palace built in the middle of the fourteenth century for Moorish kings.

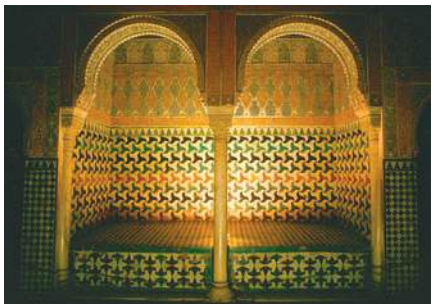


Figure 9.35
The Sala de las Camas (Room of the Beds), a beautifully tiled room in the Alhambra, the palace of fourteenth-century Moorish kings, in Granada, Spain.

The two tessellations in the center of the above photograph are made up of nonpolygonal (curved) figures. In the following paragraphs, however, we will concern ourselves only with polygons that tessellate. The triangle is an easy case to consider first. You can see

Section **1.2** PATTERNS AND PROBLEM SOLVING

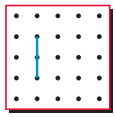


The graceful winding arms of the majestic spiral galaxy M51 look like a winding spiral staircase sweeping through space. This sharpest-ever image of the Whirlpool Galaxy was captured by the *Hubble Space Telescope* in January 2005 and released on April 24, 2005, to mark the 15th anniversary of *Hubble's* launch.

- **Teaching Questions** help develop students' critical-thinking, communication, and reasoning skills. Many of these problems encourage future teachers to consider and resolve questions from the elementary school classroom.

Teaching Questions

1. A class of middle school students was forming line segments and slopes on geoboards. The slope was intuitively explained by the teacher by referring to the slope of a roof or the slope of a road. One student asked if the line segment on this geoboard had the greatest possible slope. Explain how you would answer this question.



2. Suppose that you have discussed slopes with your class. Then, after graphing temperatures of cooling water (Example H in this section), a student asks: "Does the temperature graph have a slope?". Research this question and form a response you could give to your student.
3. The Standards quote on page 87 discusses the need for students to learn the relationship of slopes of lines to rates of change. Write an example that would make sense to a middle school student to explain what is meant by *rate of change* and why the slope of a line is a constant rate of change.

Classroom Connections

1. In the **PreK–2 Standards—Number and Operations** (see inside front cover) under *Understand Numbers . . .*, read the expectation that involves the use of multiple models. Name three models from Section 3.1 and explain how they satisfy this expectation.
2. Compare the **Standards** quote on page 131 with the **Research** statement on page 135. What conclusions can you draw from these two statements? Explain why you think that more than half the fifth and sixth grade students have this deficiency.
3. The **Standards** quote on page 133 notes that "concrete materials can help students learn to group and ungroup by tens." Use one of the models from this section to illustrate and explain how this can be done.
4. The one-page **Math Activity** at the beginning of this section introduces base-five pieces. Explain some of the advantages of using base five to help understand our base-ten numeration system.
5. The **Historical Highlight** on page 126 gives examples of number bases from different cultures. Check the Internet for at least two other examples of ancient numeration systems, other than those in this section, that used different number bases. If possible, offer a guess as to why each number base might have been selected.

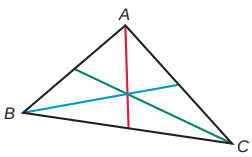
- **Classroom Connections** require students to relate or connect the section concepts to NCTM's Standards and Expectations and to special features such as the Historical Highlights, Research Quotes, Problem Openers, and sample Elementary School Text Pages.

- **Technology Connections** offer explorations utilizing calculators, computers, or Geometer's Sketchpad® and can be extended into broader problem-solving investigations to give students a deeper understanding of a topic. Some of the Connections utilize resources available on the companion website, including the Interactive Applets, the Mathematics Investigations, and the Virtual Manipulatives.

Technology Connection

Properties of Triangles

If each vertex of a triangle is connected to the midpoint of the opposite side of the triangle, will the areas of the six smaller triangles ever be equal? This and similar questions are explored using Geometer's Sketchpad® student modules available at the companion website.

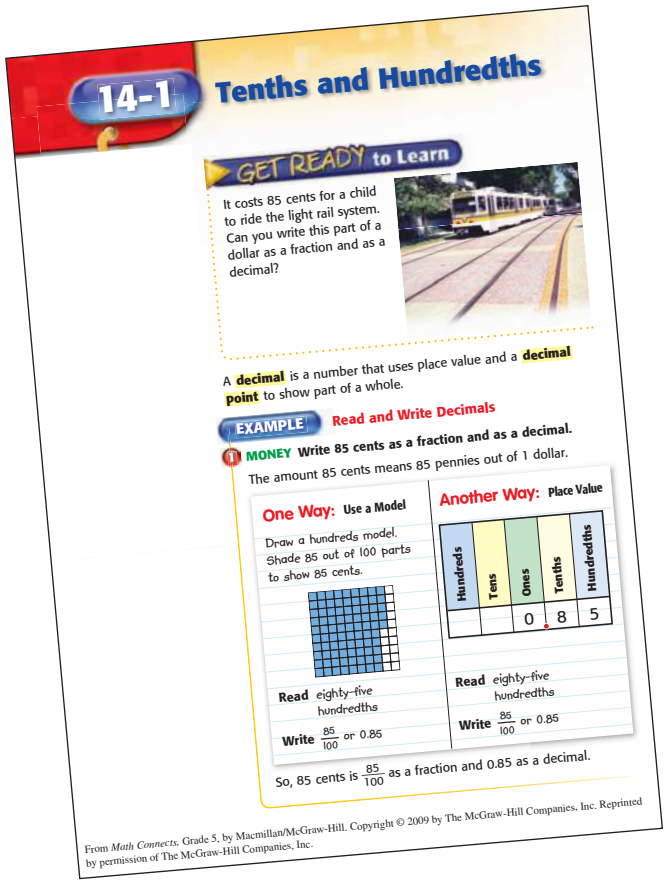


Technology Connection

Coin Toss Simulation

How many tosses of a coin on the average would be needed to obtain three consecutive heads? Four consecutive heads? You can experiment with a coin or use the online 8.1 Mathematics Investigation to simulate tossing a coin until the desired outcome is obtained. Explore this and related questions in this investigation.

- **Updated full-color Elementary School Text Pages** taken from current grade school textbooks show future teachers how key concepts from the section are presented to K–6 students. Questions corresponding to these School Text Pages are incorporated throughout the text.



14-1 Tenths and Hundredths

GET READY to Learn

It costs 85 cents for a child to ride the light rail system. Can you write this part of a dollar as a fraction and as a decimal?

A **decimal** is a number that uses place value and a **decimal point** to show part of a whole.

EXAMPLE 1 Read and Write Decimals

MONEY Write 85 cents as a fraction and as a decimal.

The amount 85 cents means 85 pennies out of 1 dollar.

One Way: Use a Model

Draw a hundreds model. Shade 85 out of 100 parts to show 85 cents.

Read eighty-five hundredths

Write $\frac{85}{100}$ or 0.85

Another Way: Place Value

Hundreds	Tens	Ones	Tenths	Hundredths
		0	8	5

Read eighty-five hundredths

Write $\frac{85}{100}$ or 0.85

So, 85 cents is $\frac{85}{100}$ as a fraction and 0.85 as a decimal.

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- **NCTM's Content Standards and Process Standards** for PreK–2, Grades 3–5, and Grades 6–8 are printed inside the front and back covers for quick reference and integrated via the Classroom Connections exercises.

- **Brightly Colored Cardstock Manipulatives** designed for use with the one-page Math Activities and selected exercises may be packaged with each text for a nominal fee. A Manipulative Kit holds resealable, labeled envelopes for each type of manipulative. Students will also be able to print additional copies of the manipulatives from the companion website. There are corresponding Virtual Manipulatives and work areas on the companion website (www.mhhe.com/bbn).

Chapter Level

- A **Spotlight on Teaching** opens each chapter with a selection from the NCTM Standards that relates to the chapter content.

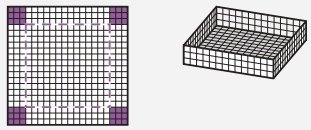
Spotlight on Teaching

Excerpts from NCTM's Standards 2 and 3 for Teaching Mathematics in Grades 5–8*

Reasoning is fundamental to the knowing and doing of mathematics. . . . To give more students access to mathematics as a powerful way of making sense of the world, it is essential that an emphasis on reasoning pervade all mathematical activity. Students need a great deal of time and many experiences to develop their ability to construct valid arguments in problem settings and evaluate the arguments of others.

. . . As students' mathematical language develops, so does their ability to reason about and solve problems. Moreover, problem-solving situations provide a setting for the development and extension of communication skills and reasoning ability. The following problem illustrates how students might share their approaches in solving problems:

The class is divided into small groups. Each group is given square pieces of grid paper and asked to make boxes by cutting out pieces from the corners. Each group is given a 20×20 sheet of grid paper. See figure [below]. Students cut and fold the paper to make boxes sized $18 \times 18 \times 1$, $16 \times 16 \times 2$, . . . , $2 \times 2 \times 9$. They are challenged to find a box that holds the maximum volume and to convince someone else that they have found the maximum. . . .



Building a grid-paper box.

- **Chapter Reviews** and **Chapter Tests** at the end of each chapter help students practice and reinforce their knowledge.

CHAPTER 8 REVIEW

1. Probability and Expected Value

a. Any activity such as spinning a spinner, tossing a coin, or rolling a die is called an **experiment.**

b. The different results that can occur from an experiment are called **outcomes.**

c. The set of all outcomes is called the **sample space.**

d. Probabilities determined from conducting experiments are called **experimental probabilities.**

e. Probabilities determined from ideal experiments are called **theoretical probabilities.**

f. If there are n equally likely outcomes, then the **probability of an outcome** is $\frac{1}{n}$.

g. If all the outcomes of a sample space S are equally likely, the **probability of an event** E is

b. The **odds against** an event are the ratio, m to n, of the number of unfavorable outcomes m to the number of favorable outcomes n. The probability of this event's not occurring is $\frac{m}{(n + m)}$.

c. **Simulations** (used in Chapter 7 for statistical experiments) are also used to obtain approximations to theoretical probabilities.

d. The **law of large numbers:** The more times a simulation is carried out, the closer the experimental probability is to the theoretical probability.

4. Single and multistage experiments

a. An experiment that is over after one step such as spinning a spinner, rolling a die, or tossing a coin is

CHAPTER 8 TEST

1. A box contains six tickets lettered A, B, C, D, E, and F. Two tickets will be randomly selected from the box (without replacement).

a. List all the outcomes of the sample space.

b. What is the probability of selecting tickets A and B?

c. What is the probability that one of the tickets will be ticket A?

2. A chip is selected at random from a box that contains 3 blue chips, 4 red chips, and 5 yellow chips. Determine the probabilities of selecting each of the following.

a. A red chip

b. A red chip or a yellow chip

7. Suppose that in exercise 6 the first marble that is selected is not replaced. Determine the probabilities of the events in 6a, b, and c.

8. A family has 4 children.

a. Draw a probability tree showing all possible combinations of boys and girls.

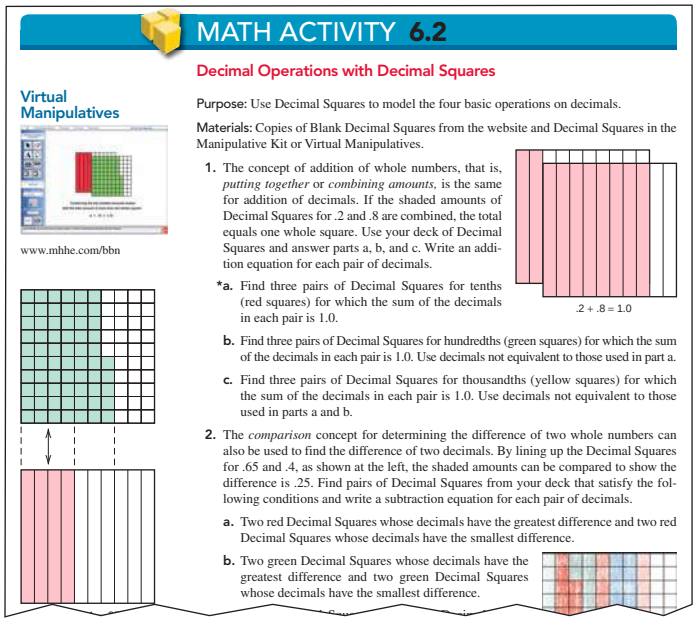
b. What is the probability of the family's having 2 boys and 2 girls?

c. What is the probability of the family's having at least 2 girls?

9. A contestant on a quiz show will choose 2 out of 7 envelopes (without replacement). If 2 of the 7 enve-

Section Level

- A one-page **Math Activity** precedes every section in the text and provides opportunities for hands-on problem solving and group discussions. Most of the Math Activities involve students in the use of either the physical or Virtual Manipulatives.



MATH ACTIVITY 6.2

Decimal Operations with Decimal Squares

Virtual Manipulatives

www.mhhe.com/bbn

Purpose: Use Decimal Squares to model the four basic operations on decimals.

Materials: Copies of Blank Decimal Squares from the website and Decimal Squares in the Manipulative Kit or Virtual Manipulatives.

1. The concept of addition of whole numbers, that is, *putting together* or *combining amounts,* is the same for addition of decimals. If the shaded amounts of Decimal Squares for .2 and .8 are combined, the total equals one whole square. Use your deck of Decimal Squares and answer parts a, b, and c. Write an addition equation for each pair of decimals.

   *a. Find three pairs of Decimal Squares for tenths (red squares) for which the sum of the decimals in each pair is 1.0.

   b. Find three pairs of Decimal Squares for hundredths (green squares) for which the sum of the decimals in each pair is 1.0. Use decimals not equivalent to those used in part a.

   c. Find three pairs of Decimal Squares for thousandths (yellow squares) for which the sum of the decimals in each pair is 1.0. Use decimals not equivalent to those used in parts a and b.

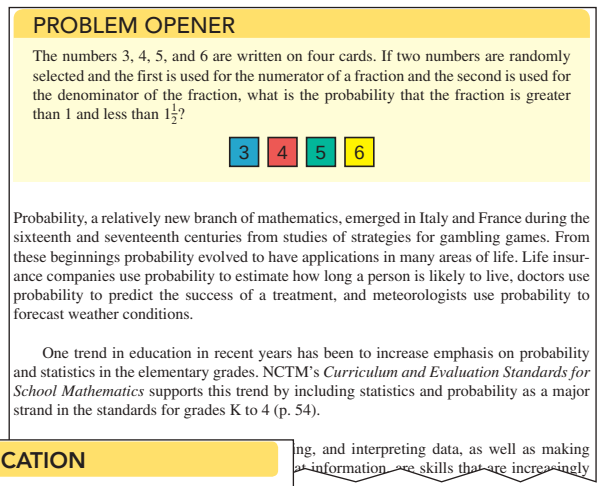
2. The *comparison* concept for determining the difference of two whole numbers can also be used to find the difference of two decimals. By lining up the Decimal Squares for .65 and .4, as shown at the left, the shaded amounts can be compared to show the difference is .25. Find pairs of Decimal Squares from your deck that satisfy the following conditions and write a subtraction equation for each pair of decimals.

   a. Two red Decimal Squares whose decimals have the greatest difference and two red Decimal Squares whose decimals have the smallest difference.

   b. Two green Decimal Squares whose decimals have the greatest difference and two green Decimal Squares whose decimals have the smallest difference.

.2 + .8 = 1.0

- **Problem Solving** is strongly and consistently emphasized throughout the text. Each section begins with a *Problem Opener* that poses an interesting problem to be solved and extended. *Problem-Solving Applications* within the sections utilize Polya's four-step approach and one or more problem-solving strategies to analyze a problem related to the concepts of the section. *Reasoning and Problem-Solving* exercises provide further opportunities to practice problem solving.



PROBLEM OPENER

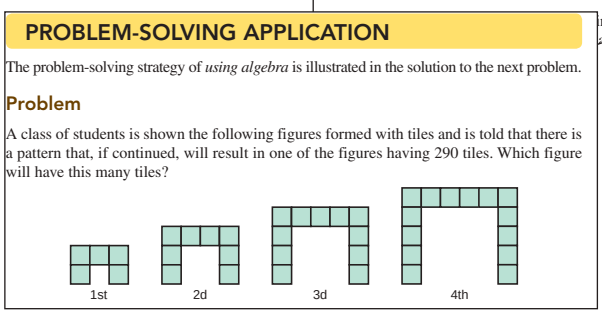
The numbers 3, 4, 5, and 6 are written on four cards. If two numbers are randomly selected and the first is used for the numerator of a fraction and the second is used for the denominator of the fraction, what is the probability that the fraction is greater than 1 and less than $1\frac{1}{2}$?

3 4 5 6

Probability, a relatively new branch of mathematics, emerged in Italy and France during the sixteenth and seventeenth centuries from studies of strategies for gambling games. From these beginnings probability evolved to have applications in many areas of life. Life insurance companies use probability to estimate how long a person is likely to live, doctors use probability to predict the success of a treatment, and meteorologists use probability to forecast weather conditions.

One trend in education in recent years has been to increase emphasis on probability and statistics in the elementary grades. NCTM's *Curriculum and Evaluation Standards for School Mathematics* supports this trend by including statistics and probability as a major strand in the standards for grades K to 4 (p. 54).

ng, and interpreting data, as well as making
at information are skills that are increasingly



PROBLEM-SOLVING APPLICATION

The problem-solving strategy of *using algebra* is illustrated in the solution to the next problem.

Problem

A class of students is shown the following figures formed with tiles and is told that there is a pattern that, if continued, will result in one of the figures having 290 tiles. Which figure will have this many tiles?

1st 2d 3d 4th

- **Worked-Out Examples** appear throughout each section.

These equations show that the answer to $1.504 \div .32$ is the same as that for $150.4 \div 32$. No further adjustment is needed as long as we shift the decimal points in both the divisor and the dividend by the same amount. Thus, division of a decimal by a decimal can always be carried out by dividing a decimal (or whole number) by a whole number.

EXAMPLE E Use the long division algorithm to compute each quotient.

1. $106.82 \div 7$ **2.** $.498 \div .6 = 4.98 \div 6$ **3.** $34.44 \div 1.4 = 344.4 \div 14$

Solution

1.
$$\begin{array}{r} 15.26 \\ 7\overline{)106.82} \\ \underline{7} \\ 36 \\ \underline{35} \\ 18 \\ \underline{14} \\ 42 \\ \underline{42} \end{array}$$

2.
$$\begin{array}{r} .83 \\ .6\overline{).498} \\ \underline{48} \\ 18 \\ \underline{18} \end{array}$$

3.
$$\begin{array}{r} 24.6 \\ 1.4\overline{)34.44} \\ \underline{28} \\ 64 \\ \underline{56} \\ 84 \\ \underline{84} \end{array}$$

- **Real-World Applications** are used extensively to illustrate the connections between mathematical concepts and many different aspects of the physical and natural world. These applications help engage students' interest by highlighting some fascinating examples of how studying mathematics enables us to better understand our world around us.

Electronic timers for athletic competition measure time to hundredths and thousandths of a second. This photo shows the disputed finish of a 100-meter final as both the United States' Gail Devers and Jamaica's Merlene Ottey (in lanes 3 and 4, respectively) had times of 10.94 seconds at the Summer Olympics in Atlanta. Officials determined that Devers won the race by .005 second over Ottey. (Notice the right foot of Devers has crossed the finish line.)



POLYHEDRA

The three-dimensional object with flat sides in Figure 9.43 is a crystal of pyrite that is embedded in rock. Its 12 flat pentagonal sides with their straight edges were not cut by people but were shaped by nature.



Figure 9.43
Crystal of pyrite

- **Quotes from NCTM Standards** appear frequently in the text and margins to link the content and pedagogy and to offer recommendations for teaching.

NCTM Standards

The NCTM K–4 Standard, *Fractions and Decimals* in the *Curriculum and Evaluation Standards for School Mathematics* (p. 59), advises that

> Physical materials should be used for exploratory work in adding and subtracting basic fractions, solving simple real-world problems, and partitioning sets of objects to find fractional parts of sets and relating this activity to division. For example, children learn that $\frac{1}{3}$ of 30 is equivalent to "30 divided by 3," which helps them relate operations with fractions to earlier operations with whole numbers.

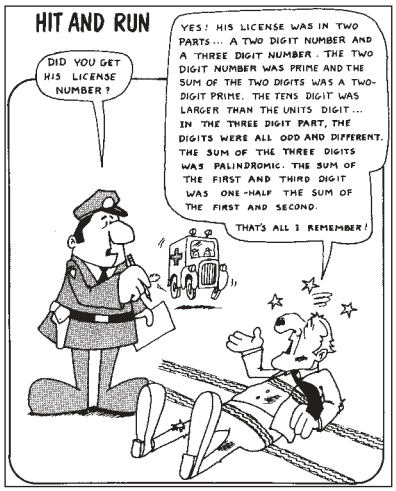
- **Research Statements** can be found in the margins and relate the topic being discussed to the performance of school students.

Research Statement

Teachers need to provide all students with experiences in which they identify the underlying rules for a variety of patterns that embody both constant and nonconstant rates of change.

Blume and Heckman

- **Cartoons** teach or emphasize mathematical concepts in a humorous and fun way.



- **Historical Highlights** familiarize students with the origins and evolution of key mathematical ideas and provide background on some of history's outstanding mathematicians.

HISTORICAL HIGHLIGHT



Emilie de Breteuil, 1706–1749

France, during the post-Renaissance period, offered little opportunity for the education of women. Emilie de Breteuil's precocity showed itself in many ways, but her true love was mathematics. One of her first scientific works was an investigation regarding the nature of fire, which was submitted to the French Academy of Sciences in 1738. It anticipated the results of subsequent research by arguing that both light and heat have the same cause or are both modes of motion. She also discovered that different-color rays do not give out an equal degree of heat. Her book *Institutions de physique* was originally intended as an essay on physics for her son. She produced instead a comprehensive textbook, not unlike a modern text, which traced the growth of physics, summarizing the thinking of the philosopher-scientists of her century. The work established Breteuil's competence among her contemporaries in mathematics and science.*

* L. M. Osen, *Women in Mathematics* (Cambridge, MA: The MIT Press, 1974), pp. 49–69.

- **Boxed Features** highlight key definitions, rules, properties, and theorems.

Sphere A **sphere** is the set of points in space that are the same distance from a fixed point, called the **center.** The union of a sphere and its interior is called a **solid sphere.**

If A is a subset of B and B is a subset of A, then both sets have exactly the same elements and they are **equal.** This relationship is written $A = B$. In this case, A and B are just different letters naming the same set. If set A is **not equal** to set B, we write $A \neq B$.

For any given set U, if two subsets A and B are disjoint and their union is U, then A and B are **complements** of each other. This is written: The complement of set B is A ($B' = A$); and the complement of set A is B ($A' = B$).

Exercise Sets

- **Skill and Concept Exercises** are at the end of each section, and answers to odd-numbered problems provided in the back of the text.

- **Special Exercise Types,** in addition to those previously described, include calculator exercises (marked ); exercises that require the use of mental calculating and estimating techniques (marked ); and "Reasoning and Problem Solving" exercises that reinforce Polya's four-step problem-solving approach (marked ).

Exercises and Problems 9.1



"City in Shards of Light" by Carolyn Hubbard-Ford has many examples of geometric figures and angles. Find at least one example of each of the following.

1. a. Acute angle
b. Trapezoid
c. Right angle
d. Convex pentagon

2. The photo of a growth structure in sapphire at the top of the next column shows angles that each have the same number of degrees.
a. Are these angles acute or obtuse?

16. a. 51 percent of 78.3 **b.** 23 percent of 1182
c. 11 percent of $19.99 **d.** 32 percent of $612.40

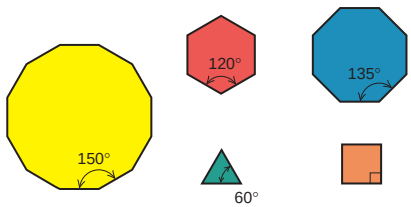
17. a. 9 percent of $30.75 **b.** 19 percent of 60
c. 4.9 percent of 128 **d.** 15 percent of 241

30. Compute each percent to the nearest tenth of a percent.
a. A down payment of $200 is what percent of the cost of $1460?
b. A cost of $3.63 in the year 2012 is what percent of a 2010 cost of $2.75?
c. The school has collected $744, which is 62 percent of its goal. What is the total amount of the school's goal?
d. During a flu epidemic, 17 percent of a school's 283 students were absent on a particular day. How many students were absent?

25. The Canadian nickel shown here is a regular dodecagon (12 sides). Assume that you have been asked to design a large posterboard model of this coin such that each side of the dodecagon has a length of 1 inch. Describe a method for constructing such a polygon.



27. Semiregular tessellations can be made by using two or more of the following regular polygons. Sketch a portion of a semiregular tessellation that is different from the one shown in Figure 9.39a and give the vertex point code for each. (*Hint:* Use the given measures of the vertex angles and Polygons for Tessellations in the Manipulative Kit or Virtual Manipulatives.)



NEW TO NINTH EDITION

This book consists of 11 chapters and 34 sections. Each chapter is preceded by a **Spotlight on Teaching** and ends with a **Chapter Review** and **Chapter Test**. Each section begins with a one-page **Math Activity** and a **Problem Opener** and is followed by questions that are classified under the following headings: Exercises; Reasoning and Problem Solving; Teaching Questions; and Classroom Connections. Based on extensive reviewer comments and classroom testing of the eighth edition, the following categories contain examples of some of the changes for the ninth edition.

- Highlighting by boxing and rewording many definitions

Several examples of highlighting definitions by boxing involve the divisibility tests. Two of these tests are shown here.

Test for Divisibility by 6

A number is **divisible by 6** if and only if the number is divisible by both 2 and 3.

Test for Divisibility by 4

A number is **divisible by 4** if the number represented by the last two digits is divisible by 4.

- Enhancing displays by improved formatting throughout the text

Arrows have been inserted to show key steps in obtaining the number properties for the rational numbers by using the number properties of the integers.

Identity for Multiplication The product of any fraction and 1 is the given fraction. This property is a result of the corresponding property for integers, which states that 1 times any integer is the given integer.

$$1 \times \frac{-4}{5} = \frac{1 \times -4}{5} = \frac{-4}{5}$$

Addition Is Commutative Two fractions that are being added can be interchanged (commuted) without changing the sum.

$$\frac{7}{8} + \frac{3}{5} = \frac{35}{40} + \frac{24}{40} = \frac{35 + 24}{40} = \frac{24 + 35}{40} = \frac{24}{40} + \frac{35}{40} = \frac{3}{5} + \frac{7}{8}$$

- Updating charts and statistical data

This example is one of many that have been updated from the *Statistical Abstracts of the United States*.

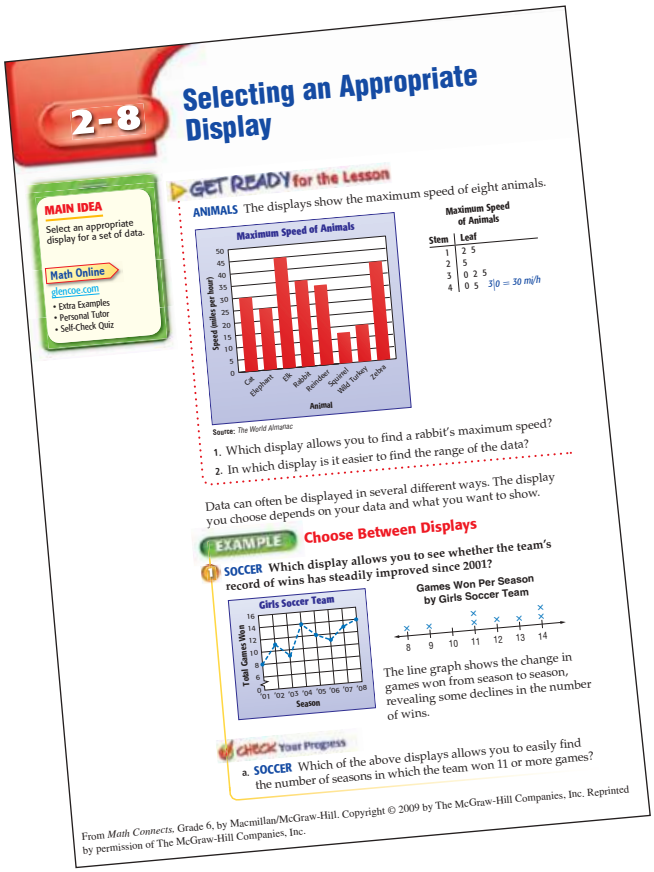
State	Students (1000)	State	Students (1000)
Alabama	536	Florida	1797
Alaska	95	Georgia	1075
Arizona	672	Hawaii	132
Arkansas	318	Idaho	171
California	4480	Illinois	1484
Colorado	529	Indiana	711
Connecticut	410	Iowa	330
Delaware	81	Kansas	322
District of Columbia	58	Kentucky	473
		Louisiana	537

(continues)

**Statistical Abstract of the United States,* 128th ed. (Washington, DC: Bureau of the Census, 2009), Table 1077.

The numbers of students in thousands in the public schools for a particular year in grades K–8.

- Replacing all eighth edition samples of **Elementary School Text Pages** with updated pages, and adding new questions to the **Classroom Connections** for these pages



2-8 Selecting an Appropriate Display

MAIN IDEA
Select an appropriate display for a set of data.

Math Online
glencoe.com
• Extra Examples
• Personal Tutor
• Self-Check Quiz

GET READY for the Lesson

ANIMALS The displays show the maximum speed of eight animals.

Maximum Speed of Animals

Speed (miles per hour)

Cat, Elephant, Elk, Rabbit, Reindeer, Squirrel, Wild Turkey, Zebra

Animal

Maximum Speed of Animals

Stem	Leaf
1	2 5
2	5
3	0 2 5
4	0 5

3|0 = 30 mi/h

Source: The World Almanac

1. Which display allows you to find a rabbit's maximum speed?
2. In which display is it easier to find the range of the data?

Data can often be displayed in several different ways. The display you choose depends on your data and what you want to show.

EXAMPLE 1 Choose Between Displays

SOCCER Which display allows you to see whether the team's record of wins has steadily improved since 2001?

Girls Soccer Team

Total Games Won

Season

Games Won Per Season by Girls Soccer Team

The line graph shows the change in games won from season to season, revealing some declines in the number of wins.

CHECK Your Progress

a. SOCCER Which of the above displays allows you to easily find the number of seasons in which the team won 11 or more games?

From *Math Connects*, Grade 6, by Macmillan/McGraw-Hill. Copyright © 2009 by The McGraw-Hill Companies, Inc. Reprinted by permission of The McGraw-Hill Companies, Inc.

5. On page 446 the example from the **Elementary School Text** illustrates how the same data can be displayed in different ways. For the Speed of Animals displays:
- **a.** What information can you obtain from the bar graph that you cannot get from the stem-and-leaf plot?
- **b.** What information you can easily obtain from the stem-and-leaf that you cannot get from the bar graph? Explain.

- Adding new exercises to include a greater variety and to expand the use of modeling mathematical concepts

Exercises 13 (shown below) and 14 both were increased from one to six examples. Exercises 25 through 28 were each increased from one to two examples.

For exercises 13 and 14, use a black and red chip model or use the given set of chips to illustrate each product or quotient. Explain your reasoning and complete each equation.

13. a. $^{-}2 \times 3 =$



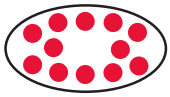
b. $^{-}4 \times {}^{-}2 =$



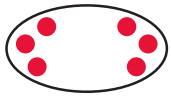
c. $3 \times 4 =$

d. $3 \times {}^{-}2 =$

e. $^{-}12 \div 4 =$

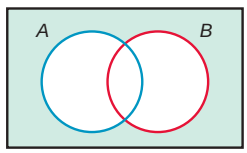


f. $^{-}6 \div {}^{-}2 =$

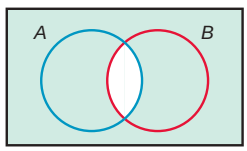


Use set notation to identify the shaded region in each of the sketches in exercises 25 through 28.

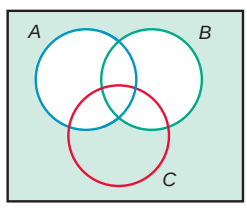
25. a.



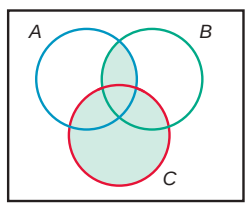
b.



27. a.



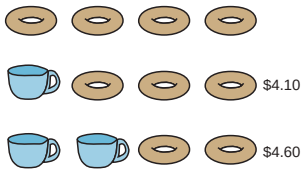
b.



- Updating many of the real-world questions in the **Exercises and Problems**

These are two of the many exercises involving prices that are subject to change from edition to edition.

23. At Joe's Cafe 1 cup of coffee and 3 doughnuts cost $4.10, and 2 cups of coffee and 2 doughnuts cost $4.60. What is the cost of 1 cup of coffee? 1 doughnut?



21. Merle spent $10.50 for DVDs and $8 for CDs. He purchased three more CDs than DVDs and the total amount of money he spent was less than $120. Let x represent the number of DVDs he purchased, and write an algebraic expression for each item in parts a through c.

a. The total cost in dollars of the DVDs
b. The number of CDs
c. The total cost in dollars of the CDs
d. The sum of the costs in parts a and c is less than $120. Write and solve an inequality to determine the possibilities for the number of DVDs Merle purchased.

- Revising and updating the many graphs, charts in the **Worked Examples** that involves date-sensitive information

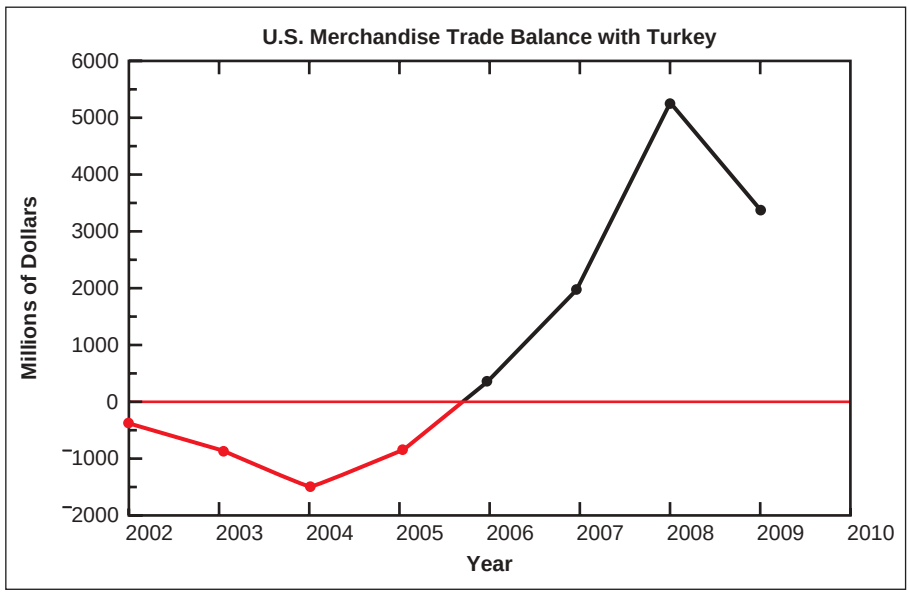
EXAMPLE B

1. Determine whether the U.S. trade balance with Turkey was positive or negative at the following times and interpret the results.

a. 2004 **b.** 2009

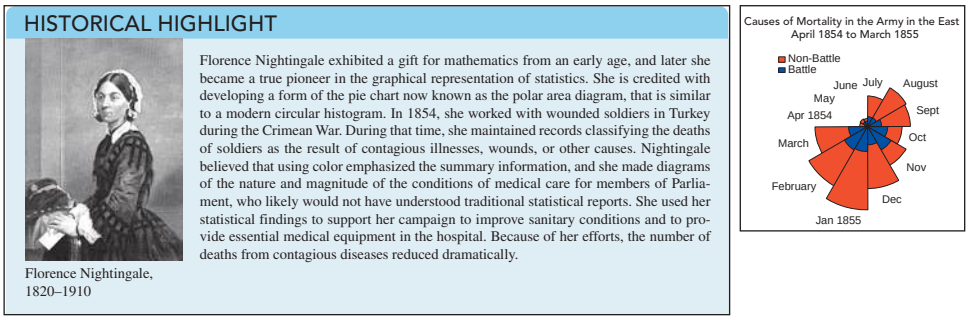
2. Determine whether the U.S. trade balance with Turkey was increasing or decreasing for the following periods:

a. 2002 to 2004 **b.** 2004 to 2006 **c.** 2008 to 2009

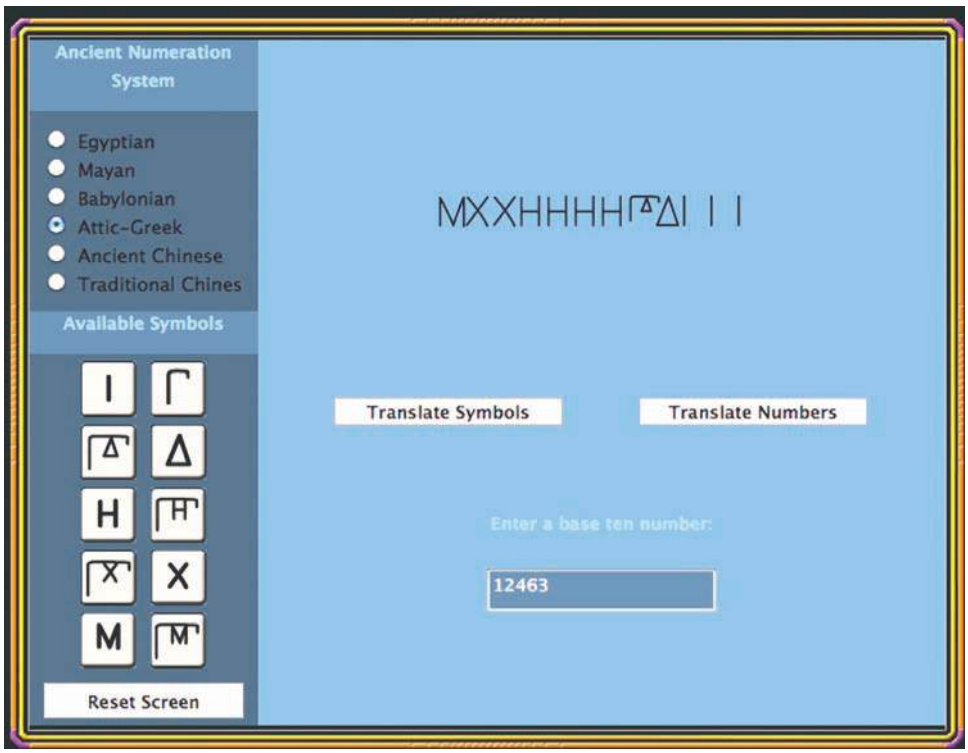


- Revising the **Historical Highlights**

Several of these have been updated and we have added some new ones to this edition. This Historical Highlight on Florence Nightingale is an example of one that we have added.



- Revising some of the 11 interactive Applets that occur in each of the 11 chapters



This is the screen capture for Deciphering Ancient Numeration Systems Applet. The applet was completely revised for the ninth edition.

FEATURES OF THE COMPANION WEBSITE

(www.mhhe.com/bbn)

The companion website for the ninth edition of *Mathematics for Elementary Teachers: A Conceptual Approach* features improved and updated resources. There are three major headings: Information Center, Instructor Center, and Student Center. Instructors can obtain a password from their local McGraw-Hill representative; the Student Center is accessible without a password.

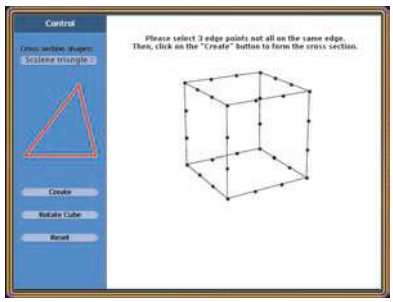
Student and Instructor Centers

NEW! Online Homework Access—McGraw-Hill's Connect web-based assignment and assessment platform (see Preface pages x–xii for more detail) is available via this companion website. This platform enables math and statistics instructors to create and share courses and assignments with colleagues and adjuncts. It allows students to connect to their course work and succeed in and beyond the course. All exercises, learning objectives, and activities are directly tied to text-specific material.

Interactive Mathematics Applets demonstrate key mathematical concepts in engaging contexts. The applets include Tower Puzzles, Hunting for Hidden Polygons, Deciphering Ancient Numeration Systems, Analyzing Star Polygons, Taking A Chance, Competing At Place Value, Distributions, Door Prizes, Cross Sections of a Cube, Filling 3-D Shapes, and Tessellations.

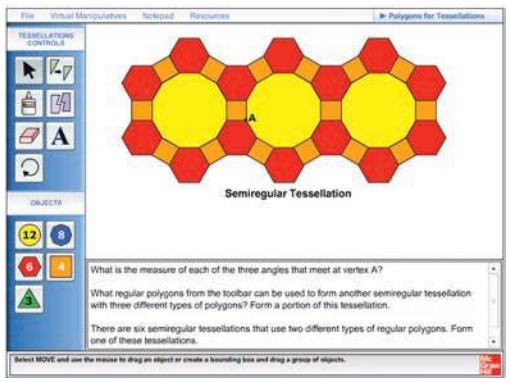
Technology Connection

How would you cut this cube into two parts with one straight slice so that the cross section is a triangle? A trapezoid? This applet lets you select points on the edges of the cube for your slices and then rotate the cube for a better perspective of the resulting cross section.



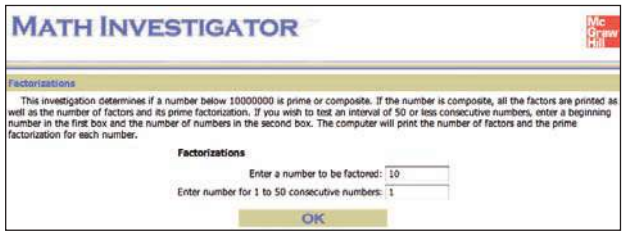
Cross-Sections of a Cube Applet, Chapter 9, Section 3
www.mhhe.com/bbn

Virtual Manipulatives, interactive versions of the cardstock Manipulative Kit pieces, are available for carrying out the one-page Math Activities in the text. Each manipulative piece has a toolbar, a work area, a note pad, and a menu for easy access to the corresponding activities.

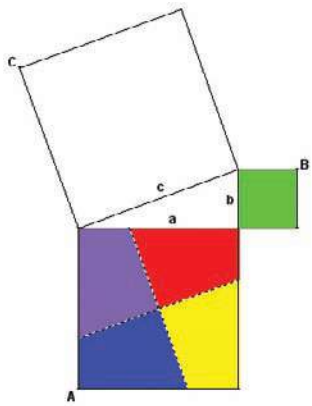


Virtual Manipulative Workspace

Math Investigations are offered for each section to promote a deeper understanding of topics from the text through the use of computers, calculators, and laboratory activities. Data for 14 of the investigations can be generated using the *Mathematics Investigator* software available on the site.



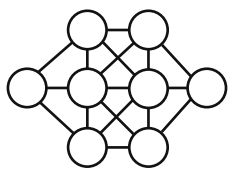
Geometer's Sketchpad Modules can be downloaded for eight of the Math Investigations. Each module is designed to facilitate open-ended student explorations while providing a gentle introduction to Geometer's Sketchpad®.



Puzzlers for each section can be used to engage and entertain students.

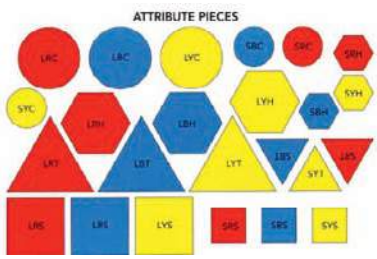
Puzzler 11.2

How can the whole numbers from 1 to 8 be placed in the circles of the figure shown here so that any two connected circles do not contain consecutive whole numbers? If we agree that all solutions that can be obtained through rotations and reflections of this diagram are the same, then there is only one solution. Find this unique solution.



Grid and Dot Paper are available in a variety of formats, along with black-and-white masters for geoboards, regular polygons, Decimal Squares, base-ten grids, the coordinate system, and the random-number chart.

Color Transparency Masters for the Manipulative Kit items can be downloaded for producing transparencies for in-class use.



Internet Resources for further reading are provided for each section of the text.

Logo Instruction on the site includes special commands, worked examples, and exercises. Answers for the odd-numbered Logo exercises are included in the Student Center of the companion website and answers for the even-numbered exercises are included in the Instructor Resources of the companion website.

Network Graphs Instruction on the site includes worked examples and exercises. Answers for the odd-numbered Network exercises are included on the companion website and answers for the even-numbered exercises are in the *Instructor's Manual.*

Instructor-Only Center

Daily Planning Guides have ideas for teaching every section of the *Conceptual Approach* text. The guide for each section is divided into two parts, First Class Meeting and Second Class Meeting, for classes of approximately 50 minutes each. There are three major purposes of these guides:

1. Providing a guide for the material and exercises of the sections, which promotes time for class discussions and questions
2. Providing comments, tips, and teaching suggestions
3. Providing suggestions for integrating Math Activities in your classroom

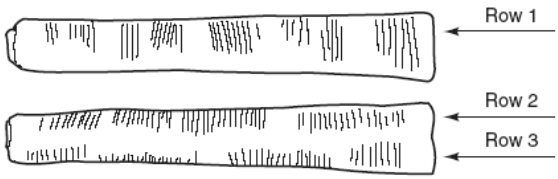
We believe that by using Math Activities consistently during mathematics classes, your students will find it a natural way to teach mathematics and also see how integrating hands-on and visual Math Activities aids their conceptual understanding of the mathematics they will be teaching.

SECTION 2.1 PLANNING GUIDE

Teaching Suggestions—First Class Meeting

Discussion Opener

Show the class a transparency of the two sides of the Ishango bone from Exercises and Problems 2.1. This bone was found on the shores of Lake Edward in the Congo (see photo at beginning of section). The marks on these bones are tallies made by people more than 8000 years ago. Ask the class for suggestions about what these people m ay have been recording. You m ay want to write the numbers of tallies for a few of the groups on the transparency.



After discussing, you m ay leave this question unresolved, as the answer occurs in the exercises. The use of tally m arks was the beginning of c ounting, and the relationship of counting to set theory was recognized and developed by Georg Cantor in the nineteenth century.

Class Activity

The one-page Math Activity at the beginning of Section 2.1 involves sorting and classifying attribute pieces differing in shape, color, and size.

Student Materials: Attribute pieces from the Manipulative Kit or Virtual attribute pieces.
Instructor Materials: Transparent attribute pieces for overhead d m operations (Col

Editable Chapter Tests are available for instructors to download for help with student assessment.

Complete Instructor's Manual with solutions to all **Problems and Exercises,** section Problem Openers, section one-page Math Activities, and website Math Investigations is available for instructor download.

ADDITIONAL SUPPLEMENTS FOR INSTRUCTORS

Instructor's Manual

The *Instructor's Manual,* found on the website at www.mhhe.com/bbn, contains extensions for all of the Problem Openers in the text, answers for the Problem Openers and extensions, answers for one-page Math Activities, answers for the Math Investigations on the website, and solutions to all exercises and problems.

Instructor's Testing Materials

A computerized test bank utilizing Brownstone Diploma testing software, available on the companion website, enables instructors to quickly and easily create customized exams. Instructors can edit existing questions, add new ones, and scramble questions and answer keys for multiple versions of a single test.

In addition to the testing software, there are three editable tests and solutions available for each chapter—each test containing roughly 25 questions. All of these testing resources can be found on the companion website at www.mhhe.com/bbn.

Create

Craft your teaching resources to match the way you teach! With McGraw-Hill Create™, www.mcgrawhillcreate.com, you can easily rearrange chapters, combine material from other content sources, and quickly upload content you have written like your course syllabus or teaching notes. Find the content you need in Create by searching through thousands of leading McGraw-Hill textbooks. Arrange your book to fit your teaching style. Create even allows you to personalize your book's appearance by selecting the cover and adding your name, school, and course information. Order a Create book, and you'll receive a complimentary print review copy in 3–5 business days or a complimentary electronic review copy (eComp) via e-mail in minutes. Go to www.mcgrawhillcreate.com today and register to experience how McGraw-Hill Create™ empowers you to teach *your* students *your* way.

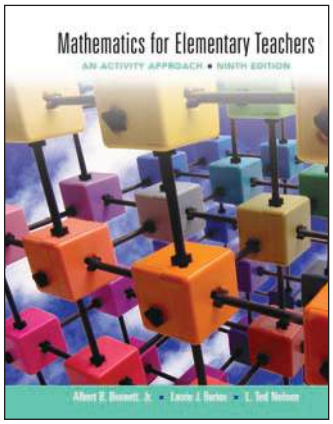
Tegrity



McGraw-Hill Tegrity Campus™ is a service that makes class time available all the time by automatically capturing every lecture in a searchable format for students to review when they study and complete assignments. With a simple one-click start-and-stop process, you capture all computer screens and corresponding audio. Students replay any part of any class with easy-to-use browser-based viewing on a PC or Mac. Educators know that the more students can see, hear, and experience class resources, the better they learn. With Tegrity Campus, students quickly recall key moments by using Tegrity Campus's unique search feature. This search helps students efficiently find what they need, when they need it, across an entire semester of class recordings. Help turn all your students' study time into learning moments immediately supported by your lecture.

To learn more about Tegrity watch a 2-minute Flash demo at http://tegritycampus.mhhe.com.

ADDITIONAL SUPPLEMENTS FOR STUDENTS



Mathematics for Elementary Teachers: An Activity Approach, Ninth Edition (ISBN-13: 978-07-743091-7 ISBN-10: 0-07-743091-3)

The *Activity Approach* contains Activity Sets that correspond to each section of the text and augment the ideas presented in the sections. Each Activity Set consists of a sequence of hands-on inductive activities and experiments that enable the student to build an understanding of mathematical ideas through the use of models and the discovery of patterns. In addition, over 35 Material Cards are included that complement the color cardstock materials in the Manipulative Kit (see description that follows). A section on *Ideas for the Elementary Classroom* at the end of each chapter includes a suggested Elementary School Activity that has been adapted from one of the chapter's Activity Sets.

Student's Solution Manual (ISBN-13: 978-0-07-7430900 ISBN-10: 0-07-743090-5)

The *Student's Solutions Manual* contains detailed solutions to the odd-numbered exercises and Chapter Tests. The introduction offers suggestions for solving problems and for answering the Teaching and Classroom Connections questions in the text. Additional questions and comments have been included at the ends of some of the solutions to give students opportunities to extend their learning.

Manipulative Kit Mathematics for Elementary Teachers, Ninth Edition (ISBN-13: 978-0-07-743093-1 ISBN-10: 0-07-743093-X)

A Manipulative Kit containing 10 colorful manipulatives commonly used in elementary schools is available for use with this text and the *Activity Approach*. This kit includes resealable, labeled envelopes for each type of manipulative.

COMMON PACKAGING OPTIONS

Mathematics for Elementary Teachers: A Conceptual Approach, Ninth Edition, packaged with *Manipulative Kit,* ninth edition
(ISBN-13: 978-0-07-796883-0 ISBN-10: 0-07-796883-2)

Mathematics for Elementary Teachers: A Conceptual Approach, Ninth Edition, packaged without *Manipulative Kit,* ninth edition
(ISBN-13: 978-0-07-744298-9 ISBN-10: 0-07-744298-9)

Mathematics for Elementary Teachers: An Activity Approach, Ninth Edition, packaged with *Manipulative Kit,* ninth edition
(ISBN-13: 978-0-07-796887-8 ISBN-10: 0-07-796887-5)

Mathematics for Elementary Teachers: A Conceptual Approach, Ninth Edition, and *Mathematics for Elementary Teachers: An Activity Approach,* ninth edition, packaged with *Manipulative Kit,* ninth edition
(ISBN-13: 978-0-07-796885-4 ISBN-10: 0-07-796885-9)

For additional packaging options please consult your McGraw-Hill Sales Representative. To find your rep, please visit http://www.mhhe.com/rep.

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CHAPTER

1 Problem Solving

Spotlight on Teaching

Excerpts from NCTM's Standards for School Mathematics Prekindergarten through Grade 12*

Problem solving can and should be used to help students develop fluency with specific skills. For example, consider the following problem, which is adapted from the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, p. 24):

> I have pennies, nickels, and dimes in my pocket. If I take 3 coins out of my pocket, how much money could I have taken?

This problem leads children to adopt a trial-and-error strategy. They can also act out the problem by using real coins. Children verify that their answers meet the problem conditions. Follow-up questions can also be posed: "Is it possible for me to have 4 cents? 11 cents? Can you list all the possible amounts I can have when I pick 3 coins?" The last question provides a challenge for older or more mathematically sophisticated children and requires them to make an organized list, perhaps like the one shown here.

Pennies	Nickels	Dimes	Total Value
0	0	3	30
0	1	2	25
0	2	1	20
0	3	0	15
1	0	2	21
⋮	⋮	⋮	⋮

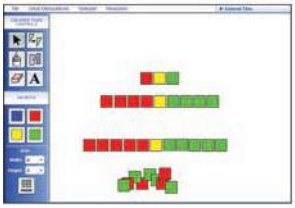
Working on this problem offers good practice in addition skills. But the important mathematical goal of this problem—helping students to think systematically about possibilities and to organize and record their thinking—need not wait until students can add fluently.

**Principles and Standards for School Mathematics*, p. 52.

MATH ACTIVITY 1.1

Peg-Jumping Puzzle

Virtual Manipulatives



www.mhhe.com/bbn

Purpose: Use four problem-solving strategies to solve a puzzle problem.

Materials: Color Tiles in the Manipulative Kit or Virtual Manipulatives.



Puzzle: There are four movable red pegs in the holes at one end of a board, four movable green pegs in the holes at the other end, and one empty hole in the center. The challenge is to interchange the pegs so that the red pegs occupy the positions of the green pegs and vice versa, in the fewest moves. Here are the legal moves: Any peg can move to an adjacent empty hole, pegs do not move backward, and a peg of one color can jump over a single peg of another color if there is a hole to jump into.

1. **Using a model:** Sketch nine 1-inch by 1-inch squares and place four red tiles on the left end and four green tiles on the right. Try solving this problem by moving the tiles according to the rules.



2. **Solving a simpler problem:** Sketch three squares and use one red tile and one green tile to solve this simpler problem. Then sketch five squares and solve the problem with two tiles of each color.



*3. **Making a table:** Sketch the following table and record the minimum number of moves and your strategy when there are three tiles on each side. For example, with one tile on each end you may have moved the red tile first (**R**), then jumped that with the green (**G**), and finally moved the red (**R**). So your strategy could be recorded **RGR.**

Tiles (Pegs) on a Side	Minimum Number of Moves	Strategy
1	3	RGR
2	8	RGGRRGGR
3		

4. **Finding patterns:** You may have noticed one or more patterns in your table. List at least one pattern in your strategies. There is also a pattern in the numbers of moves. Try finding this pattern and predict the number of moves for four tiles on a side. Then test the strategy for solving the Peg Puzzle with four tiles on a side.

*5. **Extending patterns:** Use one of the patterns you discovered to predict the fewest number of moves for solving the puzzle with five or more pegs on each side.

*Answer is given in answer section at back of book.

Section 1.1 INTRODUCTION TO PROBLEM SOLVING



There is no more significant privilege than to release the creative power of a child's mind. *Franz F. Hohn*

PROBLEM OPENER

Alice counted 7 cycle riders and 19 cycle wheels going past her house. How many tricycles were there?

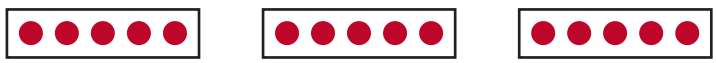
NCTM Standards

Problem solving is the hallmark of mathematical activity and a major means of developing mathematical knowledge. p. 116

"Learning to solve problems is the principal reason for studying mathematics."* This statement by the National Council of Supervisors of Mathematics represents a widespread opinion that problem solving should be the central focus of the mathematics curriculum.

A **problem** exists when there is a situation you want to resolve but no solution is readily apparent. **Problem solving** is the process by which the unfamiliar situation is resolved. A situation that is a problem to 1 person may not be a problem to someone else. For example, determining the number of people in 3 cars when each car contains 5 people may be a problem to some elementary school students. They might solve this problem by placing chips in boxes or by making a drawing to represent each car and each person (Figure 1.1) and then counting to determine the total number of people.

Figure 1.1



You may be surprised to know that some problems in mathematics are unsolved and have resisted the efforts of some of the best mathematicians to solve them. One such problem

*National Council of Supervisors of Mathematics, *Essential Mathematics for the 21st Century*.

NCTM Standards

Doing mathematics involves discovery. Conjecture—that is, informed guessing—is a major pathway to discovery. Teachers and researchers agree that students can learn to make, refine, and test conjectures in elementary school. p. 57

was discovered by Arthur Hamann, a seventh-grade student. He noticed that every even number could be written as the difference of two primes.* For example,

$$2 = 5 - 3 \qquad 4 = 11 - 7 \qquad 6 = 11 - 5 \qquad 8 = 13 - 5 \qquad 10 = 13 - 3$$

After showing that this was true for all even numbers less than 250, he predicted that every even number could be written as the difference of two primes. No one has been able to prove or disprove this statement. When a statement is thought to be true but remains unproved, it is called a **conjecture.**

Problem solving is the subject of a major portion of research and publishing in mathematics education. Much of this research is founded on the problem-solving writings of George Polya, one of the foremost twentieth-century mathematicians. Polya devoted much of his teaching to helping students become better problem solvers. His book *How to Solve It* has been translated into over 20 languages. In this book, he outlines the following four-step process for solving problems.

Understanding the Problem Polya suggests that a problem solver needs to become better acquainted with a problem and work toward a clearer understanding of it before progressing toward a solution. Increased understanding can come from rereading the statement of the problem, drawing a sketch or diagram to show connections and relationships, restating the problem in your own words, or making a reasonable guess at the solution to help become acquainted with the details.



Sometimes the main difficulty in solving a problem is knowing what question is to be answered.

Devising a Plan The path from understanding a problem to devising a plan may sometimes be long. Most interesting problems do not have obvious solutions. Experience and practice are the best teachers for devising plans. Throughout the text you will be introduced to strategies for devising plans to solve problems.

Carrying Out the Plan The plan gives a general outline of direction. Write down your thinking so your steps can be retraced. Is it clear that each step has been done correctly? Also, it's all right to be stuck, and if this happens, it is sometimes better to put aside the problem and return to it later.

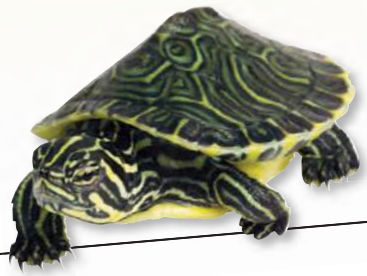
Looking Back When a result has been reached, verify or check it by referring to the original problem. In the process of reaching a solution, other ways of looking at the problem may become apparent. Quite often after you become familiar with a problem, new or perhaps more novel approaches may occur to you. Also, while solving a problem, you may find other interesting questions or variations that are worth exploring.

*M. R. Frame, "Hamann's Conjecture," *Arithmetic Teacher*.

13-3 Problem-Solving Strategy

MAIN IDEA I will solve problems by drawing a picture.

Brandi and her mom are at a pet store. The pet store has 15 reptiles. One third of the reptiles are turtles. Two are snakes, and the rest are lizards. How many of each reptile are there?



Understand

What facts do you know?

- There are 15 reptiles at the store.
- One third are turtles.
- Two are snakes.
- The rest are lizards.

What do you need to find?

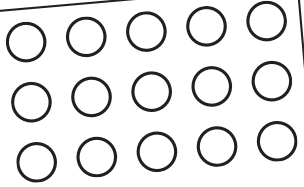
- Find the number of each reptile.

Plan

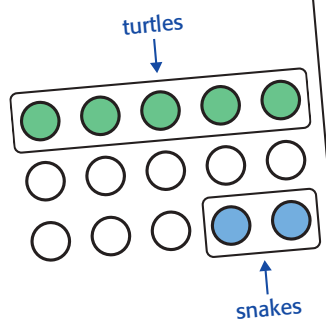
Draw a picture to solve the problem.

Solve

- Draw 15 circles. Since the fraction $\frac{1}{3}$ is used, place the circles in 3 equal groups.



- To show the turtles, shade $\frac{1}{3}$ of the circles. That is, one of the three equal groups. So, there are 5 turtles. There are 2 snakes, so shade 2 circles to show the snakes.



- There are 8 circles not shaded. This is the number of lizards.

So, there are 5 turtles, 2 snakes, and 8 lizards at the pet store.

Check

Look back. 5 turtles + 2 snakes + 8 lizards = 15 reptiles. The pet store has 15 reptiles. So, the answer is correct.

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Polya's problem-solving steps will be used throughout the text. The purpose of this section is to help you become familiar with the four-step process and to acquaint you with some of the common strategies for solving problems: *making a drawing, guessing and checking, making a table, using a model,* and *working backward.* Additional strategies will be introduced throughout the text.

MAKING A DRAWING

One of the most helpful strategies for understanding a problem and obtaining ideas for a solution is to *draw sketches and diagrams.* Most likely you have heard the expression "A picture is worth a thousand words." In the following problem, the drawings will help you to think through the solution.

Problem

For his wife's birthday, Mr. Jones is planning a dinner party in a large recreation room. There will be 22 people, and in order to seat them he needs to borrow card tables, the size that seats one person on each side. He wants to arrange the tables in a rectangular shape so that they will look like one large table. What is the smallest number of tables that Mr. Jones needs to borrow?

NCTM Standards

Of the many descriptions of problem-solving strategies, some of the best known can be found in the work of Polya (1957). Frequently cited strategies include using diagrams, looking for patterns, listing all possibilities, trying special values or cases, working backward, guessing and checking, creating an equivalent problem, and creating a simpler problem. p. 53

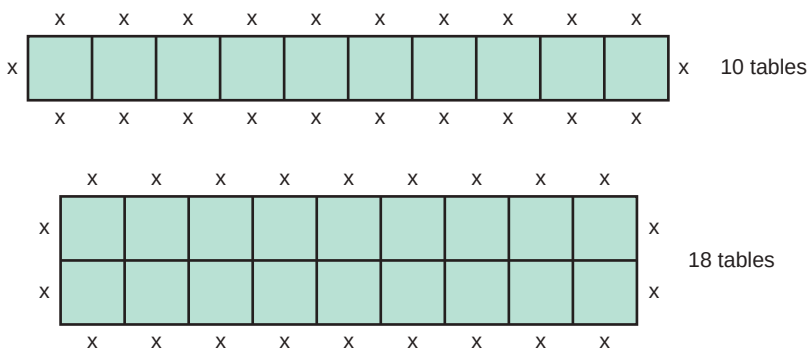
Understanding the Problem The tables must be placed next to each other, edge to edge, so that they form one large rectangular table. **Question 1:** If two tables are placed end to end, how many people can be seated?



One large table

Devising a Plan Drawing pictures of the different arrangements of card tables is a natural approach to solving this problem. There are only a few possibilities. The tables can be placed in one long row; they can be placed side by side with two abreast; etc. **Question 2:** How many people can be seated at five tables if they are placed end to end in a single row?

Carrying Out the Plan The following drawings show two of the five possible arrangements that will seat 22 people. The X's show that 22 people can be seated in each arrangement. The remaining arrangements—3 by 8, 4 by 7, and 5 by 6—require 24, 28, and 30 card tables, respectively. **Question 3:** What is the smallest number of card tables needed?



Looking Back The drawings show that a single row of tables requires the fewest tables because each end table has places for 3 people and each of the remaining tables has places for 2 people. In all the other arrangements, the corner tables seat only 2 people and the remaining tables seat only 1 person. Therefore, regardless of the number of people, a single row is the arrangement that uses the smallest number of card tables, provided the room is long enough. **Question 4:** What is the smallest number of card tables required to seat 38 people?

Answers to Questions 1–4 **1.** 6 **2.** 12 **3.** 10 **4.** There will be 3 people at each end table and 32 people in between. Therefore, 2 end tables and 16 tables in between will be needed to seat 38 people.

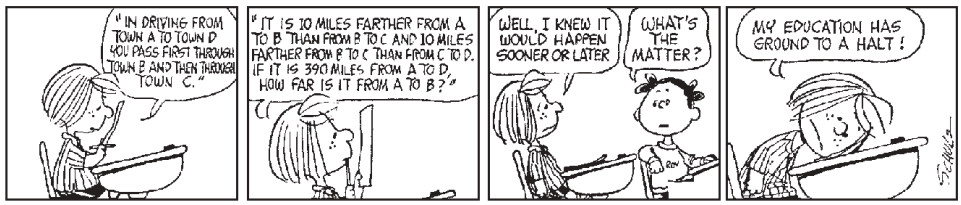


GUESSING AND CHECKING

Sometimes it doesn't pay to guess, as illustrated by the bus driver in this cartoon. However, many problems can be better understood and even solved by trial-and-error procedures. As Polya said, "Mathematics in the making consists of guesses." If your first guess is off, it may lead to a better guess. Even if guessing doesn't produce the correct answer, you may increase your understanding of the problem and obtain an idea for solving it. The *guess-and-check* approach is especially appropriate for elementary schoolchildren because it puts many problems within their reach.

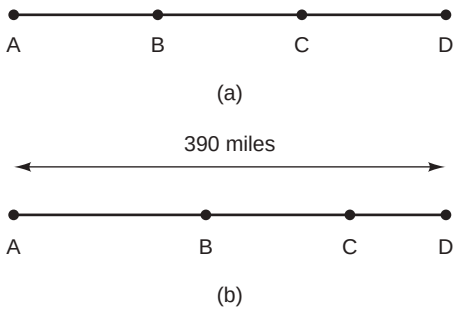
Problem

How far is it from town *A* to town *B* in this cartoon?



Peanuts: © United Feature Syndicate, Inc.

Understanding the Problem There are several bits of information in this problem. Let's see how Peppermint Patty could have obtained a better understanding of the problem with a diagram. First, let us assume these towns lie in a straight line, so they can be illustrated by points *A*, *B*, *C*, and *D*, as shown in (a). Next, it is 10 miles farther from *A* to *B* than from *B* to *C,* so we can move point *B* closer to point *C*, as in (b). It is also 10 miles farther from *B* to *C* than from *C* to *D*, so point *C* can be moved closer to point *D*. Finally, the distance from *A* to *D* is given as 390 miles. **Question 1:** The problem requires finding what distance?



NCTM Standards

Problem solving is not a distinct topic, but a process that should permeate the study of mathematics and provide a context in which concepts and skills are learned. p. 182

Devising a Plan One method of solving this problem is to make a reasonable guess and then use the result to make a better guess. If the 4 towns were equally spaced, as in (a), the distance between each town would be 130 miles (390 ÷ 3). However, the distance from town *A* to town *B* is the greatest. So let's begin with a guess of 150 miles for the distance from *A* to *B*. **Question 2:** In this case, what is the distance from *B* to *C* and *C* to *D*?

Carrying Out the Plan Using a guess of 150 for the distance from *A* to *B* produces a total distance from *A* to *D* that is greater than 390. If the distance from *A* to *B* is 145, then the *B*-to-*C* distance is 135 and the *C*-to-*D* distance is 125. The sum of these distances is 405, which is still too great. **Question 3:** What happens if we use a guess of 140 for the distance from *A* to *B*?

Looking Back One of the reasons for *looking back* at a problem is to consider different solutions or approaches. For example, you might have noticed that the first guess, which produced a distance of 420 miles, was 30 miles too great. **Question 4:** How can this observation be used to lead quickly to a correct solution of the original problem?

Answers to Questions 1–4 **1.** The problem requires finding the distance from *A* to *B*. **2.** The *B*-to-*C* distance is 140, and the *C*-to-*D* distance is 130. **3.** If the *A*-to-*B* distance is 140, then the *B*-to-*C* distance is 130 and the *C*-to-*D* distance is 120. Since the total of these distances is 390, the correct distance from *A* to *B* is 140 miles. **4.** If the distance between each of the 3 towns is decreased by 10 miles, the incorrect distance of 420 will be decreased to the correct distance of 390. Therefore, the distance between town *A* and town *B* is 140 miles.

MAKING A TABLE

A problem can sometimes be solved by listing some of or all the possibilities. A *table* is often convenient for organizing such a list.

Problem

Sue and Ann earned the same amount of money, although one worked 6 days more than the other. If Sue earned $36 per day and Ann earned $60 per day, how many days did each work?

Understanding the Problem Answer a few simple questions to get a feeling for the problem. **Question 1:** How much did Sue earn in 3 days? Did Sue earn as much in 3 days as Ann did in 2 days? Who worked more days?

Devising a Plan One method of solving this problem is to list each day and each person's total earnings through that day. **Question 2:** What is the first amount of total pay that is the same for Sue and Ann, and how many days did it take each to earn this amount?

Carrying Out the Plan The complete table is shown on page 9. There are three amounts in Sue's column that equal amounts in Ann's column. It took Sue 15 days to earn $540. **Question 3:** How many days did it take Ann to earn $540, and what is the difference between the numbers of days they each required?

Technology Connection

Four-Digit Numbers

If any four-digit number is selected and its digits reversed, will the sum of these two numbers be divisible by 11? Use your calculator to explore this and similar questions in this investigation.

Mathematics Investigation Chapter 1, Section 1 www.mhhe.com/bbn

Number of Days	Sue's Pay	Ann's Pay
1	36	60
2	72	120
3	108	180
4	144	240
5	180	300
6	216	360
7	252	420
8	288	480
9	324	540
10	360	600
11	396	660
12	432	720
13	468	780
14	504	840
15	540	900

Looking Back You may have noticed that every 5 days Sue earns \$180 and every 3 days Ann earns \$180. **Question 4:** How does this observation suggest a different way to answer the original question?

Answers to Questions 1–4 **1.** Sue earned \$108 in 3 days. Sue did not earn as much in 3 days as Ann did in 2 days. Sue must have worked more days than Ann to have earned the same amount. **2.** \$180. It took Sue 5 days to earn \$180, and it took Ann 3 days to earn \$180. **3.** It took Ann 9 days to earn \$540, and the difference between the numbers of days Sue and Ann worked is 6. **4.** When Sue has worked 10 days and Ann has worked 6 days (a difference of 4 days), each has earned \$360; when they have worked 15 days and 9 days (a difference of 6 days), respectively, each has earned \$540.

USING A MODEL

Models are important aids for visualizing a problem and suggesting a solution. The recommendations by the Conference Board of the Mathematical Sciences (CBMS) in their document, *The Mathematical Education of Teachers,* say: "Future teachers will need to connect fundamental concepts to a variety of situations, models, and representations."*

The next problem uses **whole numbers** 0, 1, 2, 3, . . . and is solved by *using a model.* It involves a well-known story about the German mathematician Karl Gauss. When Gauss was 10 years old, his schoolmaster gave him the problem of computing the sum of whole numbers from 1 to 100. Within a few moments the young Gauss wrote the answer on his slate and passed it to the teacher. Before you read the solution to the following problem, try to find a quick method for computing the sum of whole numbers from 1 to 100.

*Conference Board of the Mathematical Sciences (CBMS), *The Mathematical Education of Teachers,* "Chapter 7: The Preparation of Elementary Teachers."

Problem

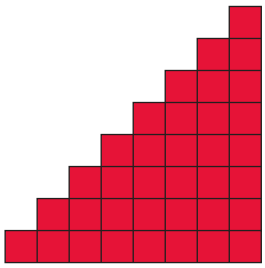
Find an easy method for computing the sum of consecutive whole numbers from 1 to any given number.

Understanding the Problem If the last number in the sum is 8, then the sum is 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8. If the last number in the sum is 100, then the sum is 1 + 2 + 3 + . . . + 100. **Question 1:** What is the sum of whole numbers from 1 to 8?

Devising a Plan One method of solving this problem is to cut staircases out of graph paper. The one shown in (a) is a 1-through-8 staircase: There is 1 square in the first step, there are 2 squares in the second step, and so forth, to the last step, which has a column of 8 squares. The total number of squares is the sum 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8. By using two copies of a staircase and placing them together, as in (b), we can obtain a rectangle whose total number of squares can easily be found by multiplying length by width. **Question 2:** What are the dimensions of the rectangle in (b), and how many small squares does it contain?

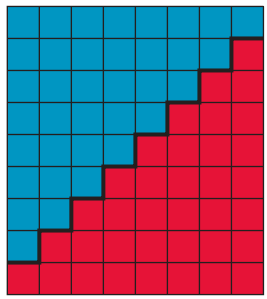
NCTM Standards

Problem solving is not only a goal of learning mathematics but also a major means of doing so. p. 52



1-through-8 staircase

(a)

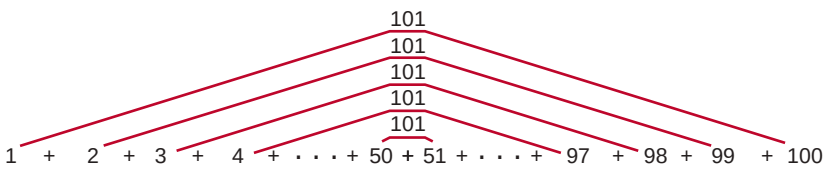


Two 1-through-8 staircases

(b)

Carrying Out the Plan Cut out two copies of the 1-through-8 staircase and place them together to form a rectangle. Since the total number of squares is 8 × 9, the number of squares in one of these staircases is $\frac{(8 \times 9)}{2} = 36$. So the sum of whole numbers from 1 to 8 is 36. By placing two staircases together to form a rectangle, we see that the number of squares in one staircase is just half the number of squares in the rectangle. This geometric approach to the problem suggests that the sum of consecutive whole numbers from 1 to any specific number is the product of the last number and the next number, divided by 2. **Question 3:** What is the sum of whole numbers from 1 to 100?

Looking Back Another approach to computing the sum of whole numbers from 1 to 100 is suggested by the following diagram, and it may have been the method used by Gauss. If the numbers from 1 to 100 are paired as shown, the sum of each pair of numbers is 101.



Question 4: How can this sum be used to obtain the sum of whole numbers from 1 to 100?

Answers to Questions 1–4 **1.** 36 **2.** The dimensions are 8 by 9, and there are $8 \times 9 = 72$ small squares. **3.** Think of combining two 1-through-100 staircases to obtain a rectangle with 100×101 squares. The sum of whole numbers from 1 to 100 is $\frac{100(101)}{2} = 5050$. **4.** Since there are 50 pairs of numbers and the sum for each pair is 101, the sum of numbers from 1 to 100 is $50 \times 101 = 5050$.

HISTORICAL HIGHLIGHT



Hypatia, 370–415

Athenaeus, a Greek writer (ca. 200), in his book *Deipnosophistae* mentions a number of women who were superior mathematicians. However, Hypatia in the fourth century is the first woman in mathematics of whom we have considerable knowledge. Her father, Theon, was a professor of mathematics at the University of Alexandria and was influential in her intellectual development, which eventually surpassed his own. She became a student of Athens at the school conducted by Plutarch the Younger, and it was there that her fame as a mathematician became established. Upon her return to Alexandria, she accepted an invitation to teach mathematics at the university. Her contemporaries wrote about her great genius. Socrates, the historian, wrote that her home as well as her lecture room was frequented by the most unrelenting scholars of the day. Hypatia was the author of several treatises on mathematics, but only fragments of her work remain. A portion of her original treatise *On the Astronomical Canon of Diophantus* was found during the fifteenth century in the Vatican library. She also wrote *On the Conics of Apollonius.* She invented an astrolabe and a planesphere, both devices for studying astronomy, and apparatuses for distilling water and determining the specific gravity of water.*

*L. M. Osen, *Women in Mathematics* (Cambridge, MA: MIT Press, 1974), pp. 21–32.

WORKING BACKWARD

Problem

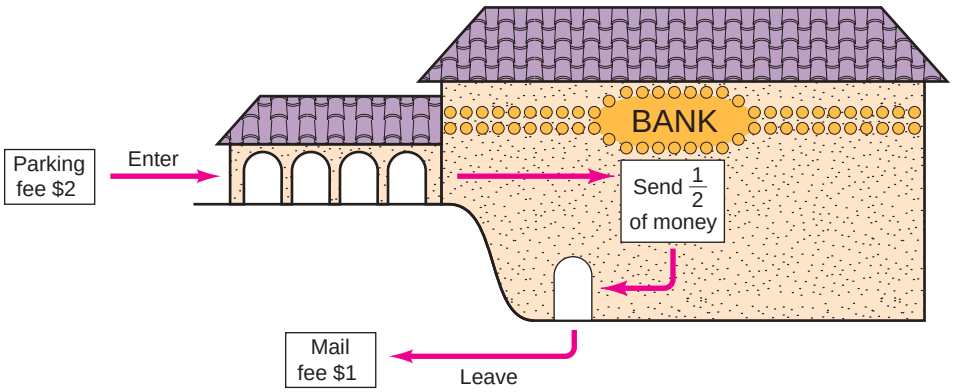
A businesswoman went to the bank and sent half of her money to a stockbroker. Other than a $2 parking fee before she entered the bank and a $1 mail fee after she left the bank, this was all the money she spent. On the second day she returned to the bank and sent half of her remaining money to the stockbroker. Once again, the only other expenses were the $2 parking fee and the $1 mail fee. If she had $182 left, how much money did she have before the trip to the bank on the first day?

Understanding the Problem Let's begin by guessing the original amount of money, say, $800, to get a better feel for the problem. **Question 1:** If the businesswoman begins the day with $800, how much money will she have at the end of the first day, after paying the mail fee?

Devising a Plan Guessing the original amount of money is one possible strategy, but it requires too many computations. Since we know the businesswoman has $182 at the end of the second day, a more appropriate strategy for solving the problem is to retrace her steps back through the bank (see the following diagram). First she receives $1 back from the mail fee. Continue to work back through the second day in the bank. **Question 2:** How much money did the businesswoman have at the beginning of the second day?

NCTM Standards

The goal of school mathematics should be for all students to become increasingly able and willing to engage with and solve problems. p. 182



Carrying Out the Plan The businesswoman had $368 at the beginning of the second day. Continue to work backward through the first day to determine how much money she had at the beginning of that day. **Question 3:** What was this amount?

Looking Back You can now check the solution by beginning with $740, the original amount of money, and going through the expenditures for both days to see if $182 is the remaining amount. The problem can be varied by replacing $182 at the end of the second day by any amount and working backward to the beginning of the first day. **Question 4:** For example, if there was $240 at the end of the second day, what was the original amount of money?

Answers to Questions 1–4 **1.** $398 **2.** The following diagram shows that the businesswoman had $368 at the beginning of the second day.

End of day 2 $182 → $183 → $366 → Beginning of day 2 $368

Receive $1 mail fee; Receive $\frac{1}{2}$ of money sent; Receive $2 parking fee

3. The diagram shows that the businesswoman had $740 at the beginning of the first day, so this is the original amount of money.

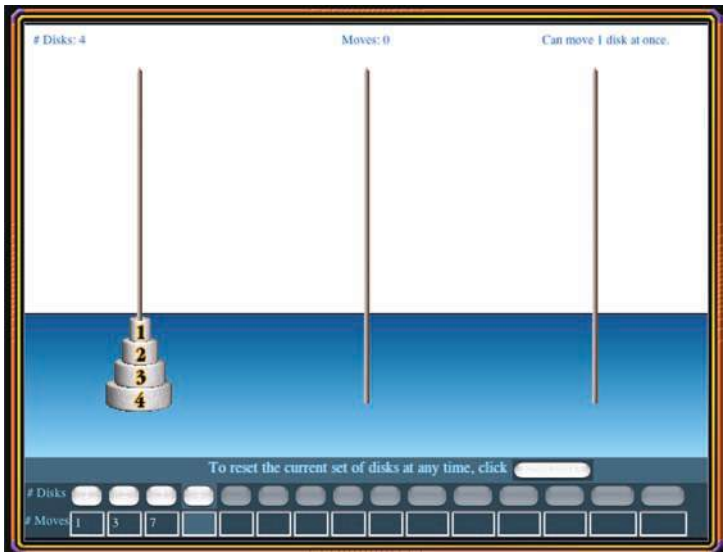
End of day 1 $368 → $369 → $738 → Beginning of day 1 $740

Receive $1 mail fee; Receive $\frac{1}{2}$ of money sent; Receive $2 parking fee

4. $972

Technology Connection

What is the least number of moves to transfer four disks from one tower to another if only one disk can be moved at a time and a disk cannot be placed on top of a smaller disk? In this applet, you will solve an ancient problem by finding patterns to determine the minimum number of moves for transferring an arbitrary number of disks.



Tower Puzzle Applet, Chapter 1
www.mhhe.com/bbn

Exercises and Problems 1.1

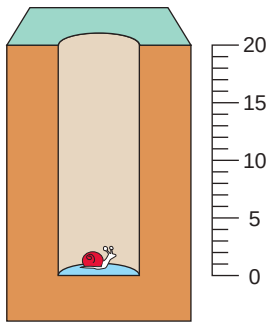
PS Problems 1 through 20 involve strategies that were presented in this section. Some of these problems are analyzed by Polya's four-step process. See if you can solve these problems before answering parts a, b, c, and d. Other strategies may occur to you, and you are encouraged to use the ones you wish. Often a good problem requires several strategies.

Making a Drawing (1–4)

1. A well is 20 feet deep. A snail at the bottom climbs up 4 feet each day and slips back 2 feet each night. How many days will it take the snail to reach the top of the well?

a. Understanding the Problem. What is the greatest height the snail reaches during the first 24 hours? How far up the well will the snail be at the end of the first 24 hours?

b. Devising a Plan. One plan that is commonly chosen is to compute $\frac{20}{2}$, since it appears that the snail gains 2 feet each day. However, 10 days is not the correct answer. A second plan is to *make a drawing* and plot the snail's daily progress. What is the snail's greatest height during the second day?



c. Carrying Out the Plan. Trace out the snail's daily progress, and mark its position at the end of each day. On which day does the snail get out of the well?

d. Looking Back. There is a "surprise ending" at the top of the well because the snail does not slip back on the ninth day. Make up a new snail problem by changing the numbers so that there will be a similar surprise ending at the top of the well.

2. Five people enter a racquetball tournament in which each person must play every other person exactly once. Determine the total number of games that will be played.

3. When two pieces of rope are placed end to end, their combined length is 130 feet. When the two pieces are placed side by side, one is 26 feet longer than the other. What are the lengths of the two pieces?

4. There are 560 third- and fourth-grade students in King Elementary School. If there are 80 more third-graders than fourth-graders, how many third-graders are there in the school?

Making a Table (5–8)

5. A bank that has been charging a monthly service fee of \$2 for checking accounts plus 15 cents for each check announces that it will change its monthly fee to \$3 and that each check will cost 8 cents. The bank claims the new plan will save the customer money. How many checks must a customer write per month before the new plan is cheaper than the old plan?

a. Understanding the Problem. Try some numbers to get a feel for the problem. Compute the cost of 10 checks under the old plan and under the new plan. Which plan is cheaper for a customer who writes 10 checks per month?

b. Devising a Plan. One method of solving this problem is to make a table showing the cost of 1 check, 2 checks, etc., such as that shown here. How much more does the new plan cost than the old plan for 6 checks?

Checks	Cost for Old Plan, \$	Cost for New Plan, \$
1	2.15	3.08
2	2.30	3.16
3	2.45	3.24
4	2.60	3.32
5	2.75	3.40
6		
7		
8		

c. Carrying Out the Plan. Extend the table until you reach a point at which the new plan is cheaper than the old plan. How many checks must be written per month for the new plan to be cheaper?

d. Looking Back. For customers who write 1 check per month, the difference in cost between the old plan and the new plan is 93 cents. What happens to the difference as the number of checks increases? How many checks must a customer write per month before the new plan is 33 cents cheaper?

6. Sasha and Francisco were selling lemonade for 25 cents per half cup and 50 cents per full cup. At the end of the day they had collected \$15 and had used 37 cups. How many full cups and how many half cups did they sell?

7. Harold wrote to 15 people, and the cost of postage was \$5.64. If it cost 28 cents to mail a postcard and 44 cents to mail a letter, how many letters did he write?

8. I had some pennies, nickels, dimes, and quarters in my pocket. When I reached in and pulled out some change, I had less than 10 coins whose value was 42 cents. What are all the possibilities for the coins I had in my hand?

Guessing and Checking (9–12)

9. There are two 2-digit numbers that satisfy the following conditions: (1) Each number has the same digits, (2) the sum of the digits in each number is 10, and (3) the difference between the 2 numbers is 54. What are the two numbers?

a. Understanding the Problem. The numbers 58 and 85 are 2-digit numbers that have the same digits, and the sum of the digits in each number is 13. Find two 2-digit numbers such that the sum of the digits is 10 and both numbers have the same digits.

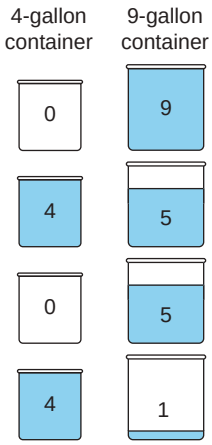
b. Devising a Plan. Since there are only nine 2-digit numbers whose digits have a sum of 10, the problem can be easily solved by guessing. What is the difference of your two 2-digit numbers from part a? If this difference is not 54, it can provide information about your next guess.

c. Carrying Out the Plan. Continue to guess and check. Which pair of numbers has a difference of 54?

d. Looking Back. This problem can be extended by changing the requirement that the sum of the two digits equals 10. Solve the problem for the case in which the digits have a sum of 12.

10. When two numbers are multiplied, their product is 759; but when one is subtracted from the other, their difference is 10. What are these two numbers?

11. When asked how a person can measure out 1 gallon of water with only a 4-gallon container and a 9-gallon container, a student used this "picture."



a. Briefly describe what the student could have shown by this sketch.

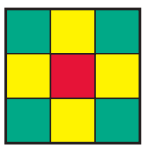
b. Use a similar sketch to show how 6 gallons can be measured out by using these same containers.

12. Carmela opened her piggy bank and found she had \$15.30. If she had only nickels, dimes, quarters, and half-dollars and an equal number of coins of each kind, how many coins in all did she have?

Using a Model (13–16)

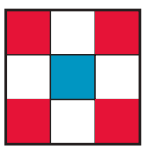
13. Suppose that you have a supply of red, blue, green, and yellow square tiles. What is the fewest number of different colors needed to form a 3×3 square of tiles so that no tile touches another tile of the same color at any point?

a. Understanding the Problem. Why is the square arrangement of tiles shown here not a correct solution?



b. Devising a Plan. One plan is to choose a tile for the center of the grid and then place others around it so that no two of the same color touch. Why must the center tile be a different color than the other eight tiles?

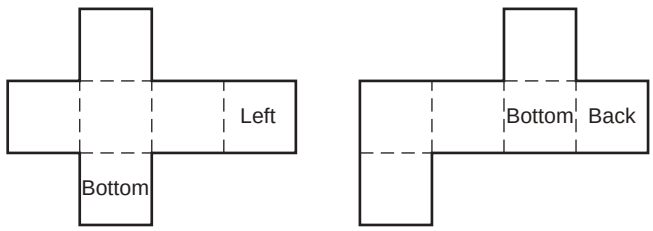
c. Carrying Out the Plan. Suppose that you put a blue tile in the center and a red tile in each corner, as shown here. Why will it require two more colors for the remaining openings?



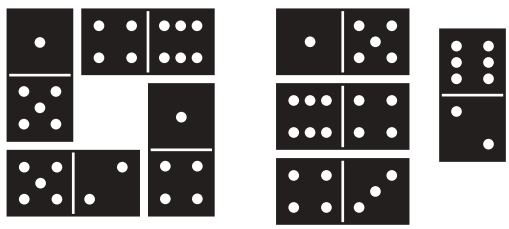
d. Looking Back. Suppose the problem had asked for the smallest number of colors to form a square of nine tiles so that no tile touches another tile of the same color along an entire edge. Can it be done in fewer colors; if so, how many?

14. What is the smallest number of different colors of tile needed to form a 4×4 square so that no tile touches another of the same color along an entire edge?

15. The following patterns can be used to form a cube. A cube has six faces: the top and bottom faces, the left and right faces, and the front and back faces. Two faces have been labeled on each of the following patterns. Label the remaining four faces on each pattern so that when the cube is assembled with the labels on the outside, each face will be in the correct place.



16. At the left in the following figure is a domino doughnut with 11 dots on each side. Arrange the four single dominoes on the right into a domino doughnut so that all four sides have 12 dots.



Domino doughnut

Working Backward (17–20)

17. Three girls play three rounds of a game. On each round there are two winners and one loser. The girl who loses on a round has to double the number of chips that each of the other girls has by giving up some of her own chips. Each girl loses one round. At the end of three rounds, each girl has 40 chips. How many chips did each girl have at the beginning of the game?

a. Understanding the Problem. Let's select some numbers to get a feel for this game. Suppose girl *A*, girl *B*, and girl *C* have 70, 30, and 20 chips,

respectively, and girl *A* loses the first round. Girl *B* and girl *C* will receive chips from girl *A*, and thus their supply of chips will be doubled. How many chips will each girl have after this round?

b. Devising a Plan. Since we know the end result (each girl finished with 40 chips), a natural strategy is to work backward through the three rounds to the beginning. Assume that girl *C* loses the third round. How many chips did each girl have at the end of the second round?

	A	B	C
Beginning			
End of first round			
End of second round			
End of third round	40	40	40

c. Carrying Out the Plan. Assume that girl *B* loses the second round and girl *A* loses the first round. Continue working back through the three rounds to determine the number of chips each of the girls had at the beginning of the game.

d. Looking Back. Check your answer by working forward from the beginning. The girl with the most chips at the beginning of this game lost the first round. Could the girl with the fewest chips at the beginning of the game have lost the first round? Try it.

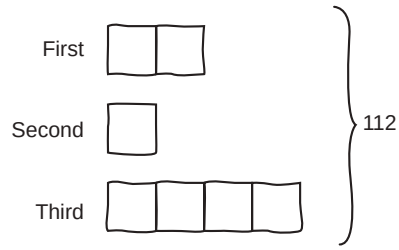
18. Sue Ellen and Angela have both saved $51 for their family trip to the coast. They each put money in their piggy banks on the same day but Sue Ellen started with $7 more than Angela. From then on Sue Ellen added $1 to her piggy bank each week and Angela put $2 in her piggy bank each week. How much money did Sue Ellen put in her piggy bank when they started?

19. Ramon took a collection of color tiles from a box. Amelia took 13 tiles from his collection, and Keiko took half of those remaining. Ramon had 11 left. How many did he start with?

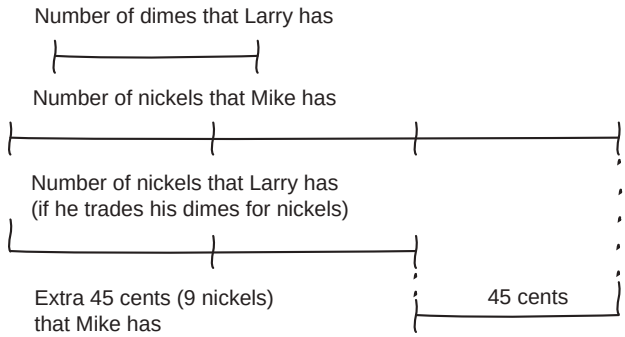
20. Keiko had 6 more red tiles than yellow tiles. She gave half of her red tiles to Amelia and half of her yellow tiles to Ramon. If Ramon has 7 yellow tiles, how many tiles does Keiko have now?

PS Each of problems 21 through 24 is accompanied by a sketch or diagram that was used by a student to solve it. Describe how you think the student used the diagram, and use this method to solve the problem.

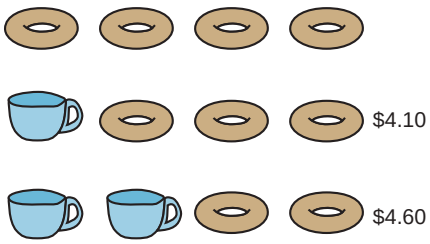
21. There are three numbers. The first number is twice the second number. The third is twice the first number. Their sum is 112. What are the numbers?



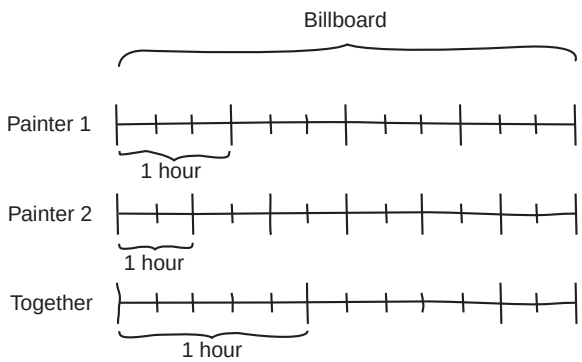
22. Mike has 3 times as many nickels as Larry has dimes. Mike has 45 cents more than Larry. How much money does Mike have?



23. At Joe's Cafe 1 cup of coffee and 3 doughnuts cost $4.10, and 2 cups of coffee and 2 doughnuts cost $4.60. What is the cost of 1 cup of coffee? 1 doughnut?



24. One painter can letter a billboard in 4 hours and another requires 6 hours. How long will it take them together to letter the billboard?



PS Problems 25 through 34 can be solved by using strategies presented in this section. While you are problem-solving, try to record the strategies you are using. If you are using a strategy different from those of this section, try to identify and record it.

25. There were ships with 3 masts and ships with 4 masts at the Tall Ships Exhibition. Millie counted a total of 30 masts on the 8 ships she saw. How many of these ships had 4 masts?

26. When a teacher counted her students in groups of 4, there were 2 students left over. When she counted them in groups of 5, she had 1 student left over. If 15 of her students were girls and she had more girls than boys, how many students did she have?

27. The movie club to which Lin belongs allows her to receive a free DVD for every three DVDs she rents. If she pays \$3 for each movie and paid \$132 over a 4-month period, how many free movie DVDs did she obtain?

28. Linda picked a basket of apples. She gave half of the apples to a neighbor, then 8 apples to her mother, then half of the remaining apples to her best friend, and she kept the 3 remaining apples for herself. How many apples did she start with in the basket?

29. Four people want to cross the river. There is only one boat available, and it can carry a maximum of 200 pounds. The weight of the four people are 190, 170, 110, and 90 pounds. How can they all manage to get across the river, and what is the minimum number of crossings required for the boat?

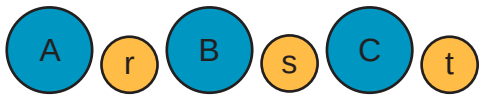
30. A farmer has to get a fox, a goose, and a bag of corn across a river in a boat that is only large enough for her and one of these three items. She does not want to leave the fox alone with the goose nor the goose alone with the corn. How can she get all these items across the river?

31. Three circular cardboard disks have numbers written on the front and back sides. The front sides have the numbers shown here.

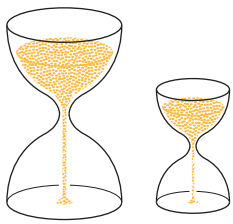


By tossing all three disks and adding the numbers that show face up, we can obtain these totals: 15, 16, 17, 18, 19, 20, 21, and 22. What numbers are written on the back sides of these disks?

32. By moving adjacent disks two at a time, you can change the arrangement of large and small disks shown below to an arrangement in which 3 big disks are side by side followed by the 3 little disks. Describe the steps.



33. How can a chef use an 11-minute hourglass and a 7-minute hourglass to time vegetables that must steam for 15 minutes?



34. The curator of an art exhibit wants to place security guards along the four walls of a large auditorium so that each wall has the same number of guards. Any guard who is placed in a corner can watch the two adjacent walls, but each of the other guards can watch only the wall by which she or he is placed.

a. Draw a sketch to show how this can be done with 6 security guards.

b. Show how this can be done for each of the following numbers of security guards: 7, 8, 9, 10, 11, and 12.

c. List all the numbers less than 100 that are solutions to this problem.

35. Trick questions like the following are fun, and they can PS help improve problem-solving ability because they require that a person listen and think carefully about the information and the question.

a. Take 2 apples from 3 apples, and what do you have?

b. A farmer had 17 sheep, and all but 9 died. How many sheep did he have left?

c. I have two U.S. coins that total 30 cents. One is not a nickel. What are the two coins?

d. A bottle of cider costs \$2.86. The cider costs \$2.60 more than the bottle. How much does the bottle cost?

e. How much dirt is in a hole 3 feet long, 2 feet wide, and 2 feet deep?

f. A hen weighs 3 pounds plus half its weight. How much does it weigh?

g. There are nine brothers in a family and each brother has a sister. How many children are in the family?

h. Which of the following expressions is correct? (1) The whites of the egg are yellow. (2) The whites of the egg is yellow.

Teaching Questions

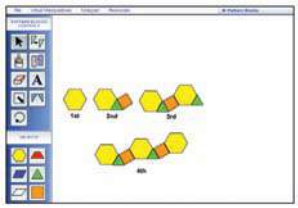
1. Suppose one of your elementary school students was having trouble solving the following problem and asked for help: "Tauna gave half of her marbles away. If she gave some to her sister and twice as many to her brother, and had 6 marbles left, how many marbles did she give to her brother?" List a few suggestions you can give to this student to help her solve this problem.
2. When an elementary schoolteacher who had been teaching problem solving introduced the strategy of *making a drawing,* one of her students said that he was not good at drawing. Give examples of three problems you can give this student that would illustrate that artistic ability is not required. Accompany the problems with solution sketches.
3. In years past, it was a common practice for teachers to tell students not to draw pictures or sketches because "you can't prove anything with drawings." Today it is common for teachers to encourage students to form sketches to solve problems. Discuss this change in approach to teaching mathematics. Give examples of advantages and disadvantages of solving problems by *making drawings.*
4. At one time, teachers scolded students for guessing the answers to problems. In recent years, mathematics educators have recommended that *guessing and checking* be taught to school students. Write a few sentences to discuss the advantages of teaching students to "guess and check." Include examples of problems for which this strategy may be helpful.
5. Write a definition of what it means for a question to involve "problem solving." Create a problem that is appropriate for middle school students and explain how it satisfies your definition of problem solving.

Classroom Connections

1. The **Spotlight on Teaching** at the beginning of Chapter 1 poses the following problem: I have pennies, nickels, and dimes in my pocket. If I take three coins out of my pocket, how much money could I have taken? The solution in this spotlight involves *forming a table.* Explain and illustrate how a different organized list can lead to a solution by noting that the greatest value of the coins is 30 cents and the least value is 3 cents.
2. On page 5, the example from the **Elementary School Text** poses a problem and solves it by the strategy of *making a drawing*. (a) Find another solution if the question states $\frac{1}{5}$ are snakes (instead of there are 2 snakes). (b) Name a strategy from the **Standards** quote on page 6 that is helpful in solving this problem a different way and explain why the strategy is helpful.
3. The **Standards** quote on page 8 says that problem solving should "provide a context in which concepts and skills are learned." Explain how the staircase model, page 10, provides this context.
4. In the **Process Standard on Problem Solving** (see inside front cover), read the fourth expectation and explain several ways in which Polya's fourth problem-solving step addresses the fourth expectation.
5. The **Historical Highlight** on page 11 has some examples of the accomplishments of Hypatia, one of the first women mathematicians. Learn more about her by researching history of math books or searching the Internet. Record some interesting facts or anecdotes about Hypatia that you could use to enhance your elementary school teaching.
6. The **Problem Opener** on page 3 of this section says that "Alice counted 7 cycle riders and 19 cycle wheels" and it asks for the number of tricycles. Use one or more of the problem-solving strategies in this section to find all the different answers that are possible if the riders might have been using unicycles, bicycles, or tricycles.

MATH ACTIVITY 1.2

Virtual Manipulatives



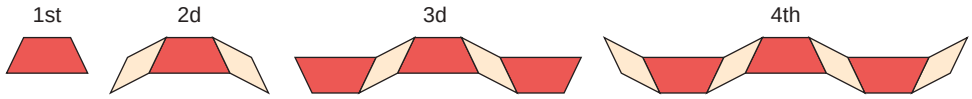
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Pattern Block Sequences

Purpose: Identify and extend patterns in pattern block sequences.

Materials: Pattern Blocks in the Manipulative Kit or Virtual Manipulatives.

1. Here are the first four pattern block figures of a sequence composed of trapezoids (red) and parallelograms (tan).



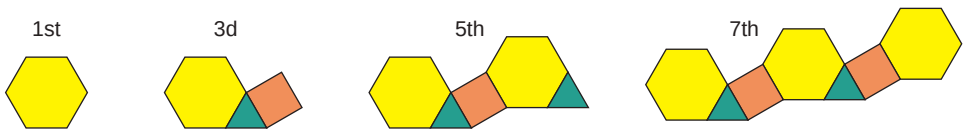
***a.** Find a pattern and use your pattern blocks to build a fifth figure. Sketch this figure.

***b.** If the pattern is continued, how many trapezoids and parallelograms will be in the 10th figure?

c. What pattern blocks are on each end of the 35th figure in the sequence, and how many of each shape are in that figure?

d. Determine the total number of pattern blocks in the 75th figure, and write an explanation describing how you reached your conclusion.

2. Figures 1, 3, 5, and 7 are shown from a sequence using hexagons, squares, and triangles.

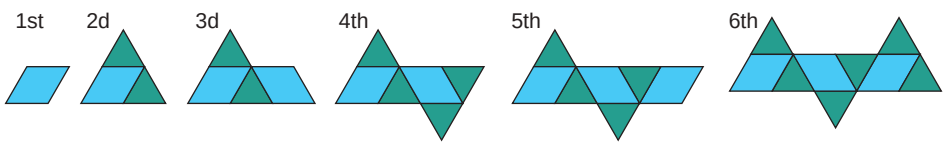


a. Find a pattern and use your pattern blocks to build the eighth and ninth figures.

***b.** Write a description of the 20th figure.

c. Write a description of the 174th, 175th, and 176th figures, and include the number of hexagons, squares, and triangles in each.

3. Use your pattern blocks to build figures 8 and 9 of the following sequence.



***a.** Describe the pattern by which you extend the sequence. Determine the number of triangles and parallelograms in the 20th figure.

b. How many pattern blocks are in the 45th figure?

c. The 5th figure in the sequence has a total of 7 pattern blocks. Which figure has a total of 87 pattern blocks? Explain your reasoning.

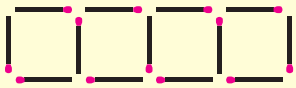
Section 1.2 PATTERNS AND PROBLEM SOLVING



The graceful winding arms of the majestic spiral galaxy M51 look like a winding spiral staircase sweeping through space. This sharpest-ever image of the Whirlpool Galaxy was captured by the *Hubble Space Telescope* in January 2005 and released on April 24, 2005, to mark the 15th anniversary of *Hubble's* launch.

PROBLEM OPENER

This matchstick track has 4 squares. If the pattern of squares is continued, how many matches will be needed to build a track with 60 squares?



FINDING A PATTERN

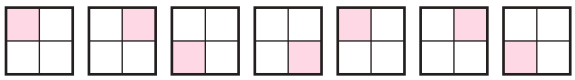
Patterns play a major role in the solution of problems in all areas of life. Psychologists analyze patterns of human behavior; meteorologists study weather patterns; astronomers seek patterns in the movements of stars and galaxies; and detectives look for patterns among clues. Finding a pattern is such a useful problem-solving strategy in mathematics that some have called it the *art of mathematics.*

To find patterns, we need to compare and contrast. We must compare to find features that remain constant and contrast to find those that are changing. Patterns appear in many forms. There are number patterns, geometric patterns, word patterns, and letter patterns, to name a few.

EXAMPLE A Consider the sequence 1, 2, 4, ... Find a pattern and determine the next term.

Solution One possibility: Each term is twice the previous term. The next term is 8.

EXAMPLE B Consider the sequence of figures. Find a pattern and determine the next figure.



Solution One possibility: In each block of four squares, one square is shaded. The upper left, upper right, lower left, and lower right corners are shaded in order. The next term in this sequence has the shaded block in the lower right corner.

EXAMPLE C Consider the sequence of names. Find a pattern and determine the next name.

Al, Bev, Carl, Donna

Solution One possibility: The first letters of the names are consecutive letters of the alphabet. The next name begins with E.

NCTM Standards

Historically, much of the mathematics used today was developed to model real-world situations, with the goal of making predictions about those situations. Students in grades 3–5 develop the idea that a mathematical model has both descriptive and predictive power. p. 162

Finding a pattern requires making educated guesses. You are guessing the pattern based on some observation, and a different observation may lead to another pattern. In Example A, the difference between the first and second terms is 1, and the difference between the second and third terms is 2. So using differences between consecutive terms as the basis of the pattern, we would have a difference of 3 between the third and fourth terms, and the fourth term would be 7 rather than 8. In Example C, we might use the pattern of alternating masculine and feminine names or of increasing numbers of letters in the names.

PATTERNS IN NATURE

The spiral is a common pattern in nature. It is found in spiderwebs, seashells, plants, animals, weather patterns, and the shapes of galaxies. The frequent occurrence of spirals in living things can be explained by different growth rates. Living forms curl because the faster-growing (longer) surface lies outside and the slower-growing (shorter) surface lies inside. An example of a living spiral is the shell of the mollusk chambered nautilus (Figure 1.2). As it grows, the creature lives in successively larger compartments.



Figure 1.2
Chambered nautilus

A variety of patterns occur in plants and trees. Many of these patterns are related to a famous sequence of numbers called the **Fibonacci numbers.** After the first two numbers of this sequence, which are 1 and 1, each successive number can be obtained by adding the two previous numbers.

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, . . .

NCTM Standards

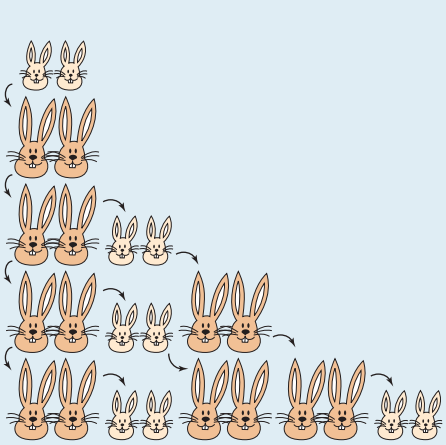
Initially, students may describe the regularity in patterns verbally rather than with mathematical symbols (English and Warren 1998). In grades 3–5, they can begin to use variables and algebraic expressions as they describe and extend patterns. p. 38

The seeds in the center of a daisy are arranged in two intersecting sets of spirals, one turning clockwise and the other turning counterclockwise. The number of spirals in each set is a Fibonacci number. Also, the number of petals will often be a Fibonacci number. The daisy in Figure 1.3 has 21 petals.



Figure 1.3

HISTORICAL HIGHLIGHT



Fibonacci numbers were discovered by the Italian mathematician Leonardo Fibonacci (ca. 1175–1250) while studying the birthrates of rabbits. Suppose that a pair of baby rabbits is too young to produce more rabbits the first month, but produces a pair of baby rabbits every month thereafter. Each new pair of rabbits will follow the same rule. The pairs of rabbits for the first 5 months are shown here. The numbers of pairs of rabbits for the first 5 months are the Fibonacci numbers 1, 1, 2, 3, 5. If this birthrate pattern is continued, the numbers of pairs of rabbits in succeeding months will be Fibonacci numbers. The realization that Fibonacci numbers could be applied to the science of plants and trees occurred several hundred years after the discovery of this number sequence.

NUMBER PATTERNS

Number patterns have fascinated people since the beginning of recorded history. One of the earliest patterns to be recognized led to the distinction between **even numbers**

$$0, 2, 4, 6, 8, 10, 12, 14, \ldots$$

and **odd numbers**

$$1, 3, 5, 7, 9, 11, 13, 15, \ldots$$

The game Even and Odd has been played for generations. To play this game, one person picks up some stones, and a second person guesses whether the number of stones is odd or even. If the guess is correct, the second person wins.

NCTM Standards

The recognition, comparison, and analysis of patterns are important components of a student's intellectual development. p. 91

Pascal's Triangle The triangular pattern of numbers shown in Figure 1.4 is Pascal's triangle. It has been of interest to mathematicians for hundreds of years, appearing in China as early as 1303. This triangle is named after the French mathematician Blaise Pascal (1623–1662), who wrote a book on some of its uses.

Row 0					1				
Row 1				1		1			
Row 2			1		2		1		
Row 3		1		3		3		1	
Row 4	1		4		6		4		1

Figure 1.4

EXAMPLE D

1. Find a pattern that might explain the numbering of the rows as 0, 1, 2, 3, etc.
2. In the fourth row, each of the numbers 4, 6, and 4 can be obtained by adding the two adjacent numbers from the row above it. What numbers are in the fifth row of Pascal's triangle?

Solution **1.** Except for row 0, the second number in each row is the number of the row. **2.** 1, 5, 10, 10, 5, 1.

Arithmetic Sequence Sequences of numbers are often generated by patterns. The sequences 1, 2, 3, 4, 5, . . . and 2, 4, 6, 8, 10, . . . are among the first that children learn. In such sequences, each new number is obtained from the previous number in the sequence by adding a selected number throughout. This selected number is called the **common difference,** and the sequence is called an **arithmetic sequence.**

EXAMPLE E

7, 11, 15, 19, 23, . . .

172, 256, 340, 424, 508, . . .

The first arithmetic sequence has a common difference of 4. What is the common difference for the second sequence? Write the next three terms in each sequence.

Solution The next three terms in the first sequence are 27, 31, and 35. The common difference for the second sequence is 84, and the next three terms are 592, 676, and 760.

Geometric Sequence In a geometric sequence, each new number is obtained by multiplying the previous number by a selected number. This selected number is called the **common ratio,** and the resulting sequence is called a **geometric sequence.**

EXAMPLE F

3, 6, 12, 24, 48, . . .

1, 5, 25, 125, 625, . . .

The common ratio in the first sequence is 2. What is the common ratio in the second sequence? Write the next two terms in each sequence.

Technology Connection

Triangular Numbers

There is an interesting pattern in the units digits of the triangular numbers 1, 3, 6, 10, 15, ... Look for this pattern and others using the online 1.2 Mathematics Investigation to quickly gather and display data for the triangular numbers.

Mathematics Investigation Chapter 1, Section 2 www.mhhe.com/bbn

Solution The next two terms in the first sequence are 96 and 192. The common ratio for the second sequence is 5, and the next two terms are 3125 and 15,625.

Triangular Numbers The sequence of numbers illustrated in Figure 1.5 is neither arithmetic nor geometric. These numbers are called **triangular numbers** because of the arrangement of dots that is associated with each number. Since each triangular number is the sum of whole numbers beginning with 1, the formula for the sum of consecutive whole numbers can be used to obtain triangular numbers (1, 1 + 2, 1 + 2 + 3, etc).*

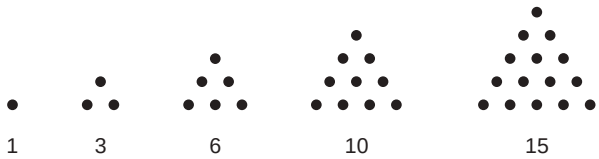


Figure 1.5

EXAMPLE G

The first triangular number is 1, and the fifth triangular number is 15. What is the sixth triangular number?

Solution The sixth triangular number is $1 + 2 + 3 + 4 + 5 + 6 = \frac{(6 \times 7)}{2} = 21$.

HISTORICAL HIGHLIGHT



Karl Friedrich Gauss, 1777–1855

Archimedes, Newton, and the German mathematician Karl Friedrich Gauss are considered to be the three greatest mathematicians of all time. Gauss exhibited a cleverness with numbers at an early age. The story is told that at age 3, as he watched his father making out the weekly payroll for laborers of a small bricklaying business, Gauss pointed out an error in the computation. Gauss enjoyed telling the story later in life and joked that he could figure before he could talk. Gauss kept a mathematical diary, which contained records of many of his discoveries. Some of the results were entered cryptically. For example,

$$\text{Num} = \Delta + \Delta + \Delta$$

is an abbreviated statement that every whole number greater than zero is the sum of three or fewer triangular numbers.†

†H. W. Eves, *In Mathematical Circles* (Boston: Prindle, Weber, and Schmidt, 1969), pp. 111–115.

There are other types of numbers that receive their names from the numbers of dots in geometric figures (see 28–30 in Exercises and Problems 1.2). Such numbers are called **figurate numbers,** and they represent one kind of link between geometry and arithmetic.

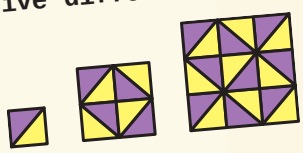
*The online 1.2 Mathematics Investigation, Triangular Numbers, prints sequences of triangular numbers.

13-5 Problem-Solving Investigation

MAIN IDEA I will choose the best strategy to solve a problem.

P.S.I. TEAM +

EMILIO: To make a quilt pattern, I pieced together triangles to make squares of different sizes. The first square has 2 triangles, the second square has 8 triangles, and the third square has 18 triangles. The quilt will have squares of five different sizes.





YOUR MISSION: Find how many triangles are in the fifth square.

Understand	You know how many triangles are in the first, second, and third squares. You need to find how many triangles are in the fifth square.
Plan	Look for a pattern to find the number of triangles.
Solve	Each square has twice as many triangles as small squares. First square 2×1 or 2 triangles Second square 2×4 or 8 triangles Third square 2×9 or 18 triangles Continuing the pattern, the fourth square has 2×16 or 32 triangles. The fifth square has 2×25 or 50 triangles.
Check	Draw the fifth square and count the number of triangles. Since there are 50 triangles in the fifth square, the answer is correct. ✓

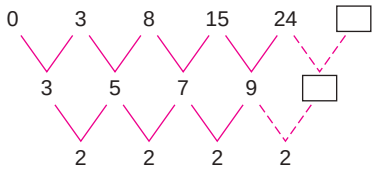
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Finite Differences Often sequences of numbers don't appear to have a pattern. However, sometimes number patterns can be found by looking at the differences between consecutive terms. This approach is called the method of **finite differences.**

EXAMPLE H

Consider the sequence 0, 3, 8, 15, 24, ... Find a pattern and determine the next term.

Solution Using the method of finite differences, we can obtain a second sequence of numbers by computing the differences between numbers from the original sequence, as shown below. Then a third sequence is obtained by computing the differences from the second sequence. The process stops when all the numbers in the sequence of differences are equal. In this example, when the sequence becomes all 2s, we stop and work our way back from the bottom row to the original sequence. Assuming the pattern of 2s continues, the next number after 9 is 11, so the next number after 24 is 35.



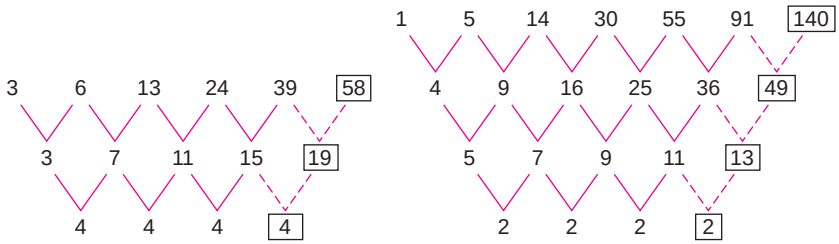
EXAMPLE I

Use the method of finite differences to determine the next term in each sequence.

1. 3, 6, 13, 24, 39

2. 1, 5, 14, 30, 55, 91

Solution **1.** The next number in the sequence is 58. **2.** The next number in the sequence is 140.



INDUCTIVE REASONING

The process of forming conclusions on the basis of patterns, observations, examples, or experiments is called **inductive reasoning.** The NCTM *Curriculum and Evaluation Standards for School Mathematics* (p. 82) describes this type of reasoning in the following statement.

NCTM Standards

Identifying patterns is a powerful problem-solving strategy. It is also the essence of inductive reasoning. As students explore problem situations appropriate to their grade level, they can often consider or generate a set of specific instances, organize them, and look for a pattern. These, in turn, can lead to conjectures about the problem.

EXAMPLE J

Each of these sums of three consecutive whole numbers is divisible by 3.

$$4 + 5 + 6 = 15 \qquad 2 + 3 + 4 = 9 \qquad 7 + 8 + 9 = 24$$

If we conclude, on the basis of these sums, that the sum of any three consecutive whole numbers is divisible by 3, we are using inductive reasoning.

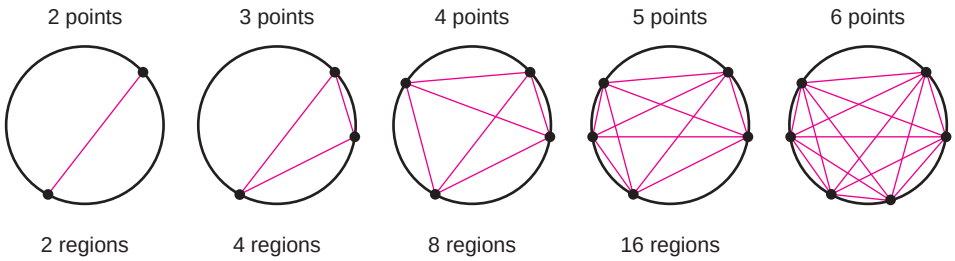
Inductive reasoning can be thought of as making an "informed guess." Although this type of reasoning is important in mathematics, it sometimes leads to incorrect results.

EXAMPLE K

Consider the number of regions that can be obtained in a circle by connecting points on the circumference of the circle. Connecting 2 points produces 2 regions, connecting 3 points produces 4 regions, etc. Each time a new point on the circle is used, the number of regions appears to double.

NCTM Standards

Because many elementary and middle school tasks rely on inductive reasoning, teachers need to be aware that students might develop an incorrect expectation that patterns always generalize in ways that would be expected on the basis of the regularities found in the first few terms. p. 265



The numbers of regions in the circles shown here are the beginning of the geometric sequence 2, 4, 8, 16, . . . , and it is tempting to conclude that 6 points will produce 32 regions. However, no matter how the 6 points are located on the circle, there will not be more than 31 regions.

Counterexample An example that shows a statement to be false is called a **counterexample.** If you have a general statement, test it to see if it is true for a few special cases. You may be able to find a counterexample to show that the statement is not true, or that a conjecture cannot be proved.

EXAMPLE L

Find two whole numbers for which the following statement is false: The sum of any two whole numbers is divisible by 2.

Solution It is not true for 7 and 4, since $7 + 4 = 11$, and 11 is not divisible by 2. There are pairs of whole numbers for which the statement is true. For example, $3 + 7 = 10$, and 10 is divisible by 2. However, the counterexample of the sum of 7 and 4 shows that the statement is not true for all pairs of whole numbers.

Counterexamples can help us to restate a conjecture. The statement in Example L is false, but if it is changed to read "The sum of two odd numbers is divisible by 2," it becomes a true statement.

EXAMPLE M

For which of the following statements is there a counterexample? If a statement is false, change a condition to produce a true statement.

1. The sum of any four whole numbers is divisible by 2.
2. The sum of any two even numbers is divisible by 2.
3. The sum of any three consecutive whole numbers is divisible by 2.

Solution **1.** The following counterexample shows that statement 1 is false: $4 + 12 + 6 + 3 = 25$, which is not divisible by 2. If the condition "four whole numbers" is replaced by "four even numbers," the statement becomes true. **2.** Statement 2 is true. **3.** The following counterexample shows that statement 3 is false: $8 + 9 + 10 = 27$, which is not divisible by 2. If the condition "three consecutive whole numbers" is replaced by "three consecutive whole numbers beginning with an odd number," the statement becomes true.

HISTORICAL HIGHLIGHT



Leaning Tower of Pisa
Pisa, Italy.

Aristotle (384–322 B.C.E.), Greek scientist and philosopher, believed that heavy objects fall faster than lighter ones, and this principle was accepted as true for hundreds of years. Then in the sixteenth century, Galileo produced a counterexample by dropping two pieces of metal from the Leaning Tower of Pisa. In spite of the fact that one was twice as heavy as the other, both hit the ground at the same time.

SOLVING A SIMPLER PROBLEM

Simplifying a problem or solving a related but easier problem can help in understanding the given information and devising a plan for the solution. Sometimes the numbers in a problem are large or inconvenient, and finding a solution for smaller numbers can lead to a plan or reveal a pattern for solving the original problem.

PROBLEM-SOLVING APPLICATION

The strategies of **solving a simpler problem** and **finding a pattern** are used in the following problem. Read this problem and try to solve it. Then read the following four-step solution and compare it to your solution.

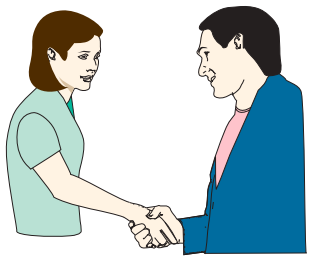
Problem

There are 15 people in a room, and each person shakes hands exactly once with everyone else. How many handshakes take place?

NCTM Standards

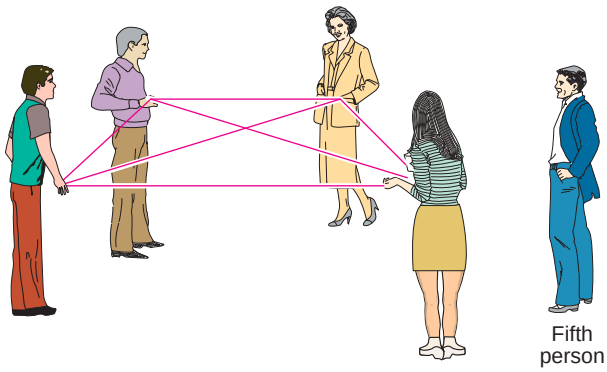
During grades 3–5, students should be involved in an important transition in their mathematical reasoning. Many students begin this grade band believing that something is true because it has occurred before, because they have seen examples of it, or because their experience to date seems to confirm it. During these grades, formulating conjectures and assessing them on the basis of evidence should become the norm. p. 188

Understanding the Problem For each pair of people, there will be 1 handshake. For example, if Sue and Paul shake hands, this is counted as 1 handshake. Thus, the problem is to determine the total number of different ways that 15 people can be paired. **Question 1:** How many handshakes will occur when 3 people shake hands?



Devising a Plan Fifteen people are a lot of people to work with at one time. Let's simplify the problem and count the number of handshakes for small groups of people. Solving these special cases may give us an idea for solving the original problem. **Question 2:** What is the number of handshakes in a group of 4 people?

Carrying Out the Plan We have already noted that there is 1 handshake for 2 people, and you can see there are 3 handshakes for 3 people. The following figure illustrates how 6 handshakes will occur among 4 people. Suppose a fifth person joins the group. This person will shake hands with each of the first 4 people, accounting for 4 more handshakes.



Similarly, if we bring in a 6th person, this person will shake hands with the first 5 people, and so there will be 5 new handshakes. Suddenly we can see a pattern developing: The 5th person adds 4 new handshakes, the 6th person adds 5 new handshakes, the 7th person adds 6 new handshakes, and so on until the 15th person adds 14 new handshakes. **Question 3:** How many handshakes will there be for 15 people?

Looking Back By looking at special cases with numbers smaller than 15, we obtained a better understanding of the problem and an insight for solving it. The pattern we found suggests a method for determining the number of handshakes for any number of people: Add the whole numbers from 1 to the number that is 1 less than the number of people. You may recall from Section 1.1 that staircases were used to develop a formula for computing such a sum. **Question 4:** How can this formula be used to determine the number of handshakes for 15 people?

Answers to Questions 1–4 **1.** 3 **2.** 6 **3.** $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 = 105$ **4.** The sum of whole numbers from 1 to 14 is $\frac{(14 \times 15)}{2} = 105$.

Exercises and Problems 1.2

PS In the *Curriculum and Evaluation Standards for School Mathematics* (p. 61), NCTM's K–4 Standard *Patterns and Relationships* notes that identifying the *core* of a pattern helps children become aware of the structure. For example, in some patterns there is a core that repeats, as in exercise 1a. In some patterns there is a core that grows, as in exercise 2b. Classify each of the sequences in 1 and 2 as having a core that repeats or that grows, and determine the next few elements in each sequence.

1. a. ▲+○▲+○▲+○▲+○ · · ·

b. × × + × × × + × × × + × · · ·

c. ○✳○○○✳○○○○✳○○○○ · · ·

2. a. ⌜ ⌜ ⌝ ⌟ ⌞ ⌜ ⌜ ⌝ ⌟ ⌞ · · ·

b. 1, 2, 1, 1, 2, 3, 2, 1, 1, 2, 3, 4, 3, 2, 1, . . .

c. 2, 3, 5, 7, 2, 3, 5, 7, 2, 3, 5, 7, . . .

PS Some sequences have a pattern, but they do not have a core. Determine the next three numbers in each of the sequences in exercises 3 and 4.

3. a. 2, 5, 8, 11, 14, 17, 20, 23, . . .

b. 13, 16, 19, 23, 27, 32, 37, 43, . . .

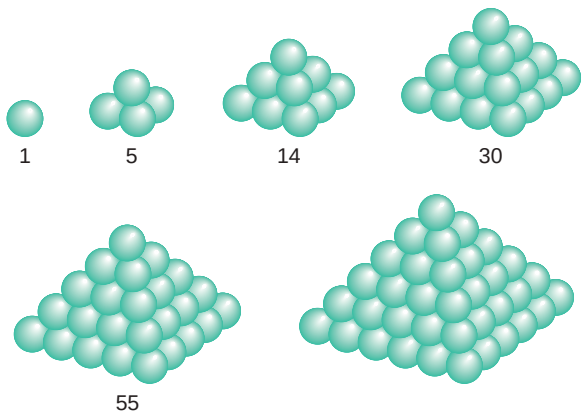
c. 17, 22, 20, 25, 23, 28, 26, 31, . . .

4. a. 31, 28, 25, 22, 19, 16, . . .

b. 46, 48, 50, 54, 58, 64, 70, 78, 86, . . .

c. 43, 46, 49, 45, 41, 44, 47, 43, 39, . . .

PS One method of stacking cannonballs is to form a pyramid with a square base. The first six such pyramids are shown. Use these figures in exercises 5 and 6.



5. a. How many cannonballs are in the sixth figure?

b. Can the method of finite differences be used to find the number of cannonballs in the sixth figure?

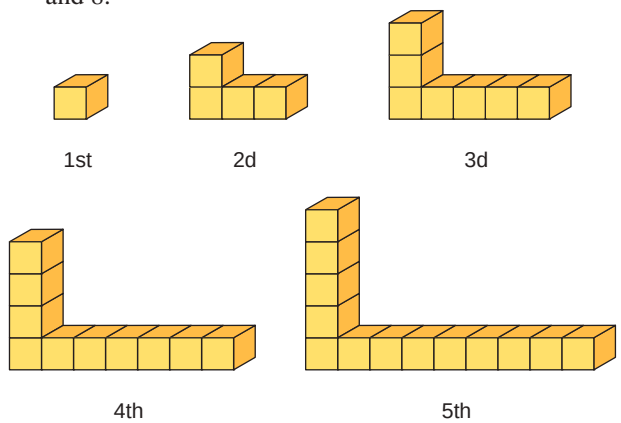
c. Describe the 10th pyramid, and determine the number of cannonballs.

6. a. Describe the seventh pyramid, and determine the number of cannonballs.

b. Do the numbers of cannonballs in successive figures form an arithmetic sequence?

c. Write an expression for the number of cannonballs in the 20th figure. (*Note:* It is not necessary to compute the number.)

PS Use the following sequence of figures in exercises 7 and 8.



7. a. What type of sequence is formed by the numbers of cubes in successive figures?

b. Describe the 20th figure and determine the number of cubes in the figure.

8. a. Can the method of finite differences be used to determine the number of cubes in the 6th figure?

b. Describe the 100th figure and determine the number of cubes in the figure.

c. Write an expression for the number of cubes in the *n*th figure, for any whole number *n*.

There are many patterns and number relationships that can be easily discovered on a calendar. Some of these patterns are explored in exercises 9 through 11.

NOVEMBER 2013

Sun	Mon	Tue	Wed	Thu	Fri	Sat
					1	2
3	4	5	6	7	8	9
10	11	12	13	14	15	16
17	18	19	20	21	22	23
24	25	26	27	28	29	30

9. The sum of the three circled numbers on the preceding calendar is 45. For any sum of three consecutive numbers (from the rows), there is a quick method for determining the numbers. Explain how this can be done. Try your method to find three consecutive numbers whose sum is 54.

10. If you are told the sum of any three adjacent numbers from a column, it is possible to determine the three numbers. Explain how this can be done, and use your method to find the numbers whose sum is 48.

11. The sum of the 3×3 array of numbers outlined on the preceding calendar is 99. There is a shortcut method for using this sum to find the 3×3 array of numbers. Explain how this can be done. Try using your method to find the 3×3 array with sum 198.

12. Here are the first few Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55. Compute the sums shown below, and compare the answers with the Fibonacci numbers. Find a pattern and explain how this pattern can be used to find the sums of consecutive Fibonacci numbers.

$$1 + 1 + 2 =$$
$$1 + 1 + 2 + 3 =$$
$$1 + 1 + 2 + 3 + 5 =$$
$$1 + 1 + 2 + 3 + 5 + 8 =$$
$$1 + 1 + 2 + 3 + 5 + 8 + 13 =$$
$$1 + 1 + 2 + 3 + 5 + 8 + 13 + 21 =$$

13. The sums of the squares of consecutive Fibonacci numbers form a pattern when written as a product of two numbers.

a. Complete the missing sums and find a pattern.

b. Use your pattern to explain how the sum of the squares of the first few consecutive Fibonacci numbers can be found.

$$1^2 + 1^2 = 1 \times 2$$
$$1^2 + 1^2 + 2^2 = 2 \times 3$$
$$1^2 + 1^2 + 2^2 + 3^2 = 3 \times 5$$
$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 =$$
$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 =$$
$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 + 13^2 =$$

A Fibonacci-type sequence can be started with any two numbers. Then each successive number is formed by adding the two previous numbers. Each number after 3 and 4 in the sequence 3, 4, 7, 11, 18, 29, etc. was obtained by adding the previous two numbers. Find the missing numbers among the first 10 numbers of the Fibonacci-type sequences in exercises 14 and 15.

14. a. 10, _____, 24, _____, _____, 100, _____, _____, _____, 686

b. 2, _____, _____, 16, 25, _____, _____, _____, _____, 280

c. The sum of the first 10 numbers in the sequence in part a is equal to 11 times the seventh number, 162. What is this sum?

d. Can the sum of the first 10 numbers in the sequence in part b be obtained by multiplying the seventh number by 11?

e. Do you think the sum of the first 10 numbers in any Fibonacci-type sequence will always be 11 times the seventh number? Try some other Fibonacci-type sequences to support your conclusion.

15. a. 1, _____, _____, 11, _____, _____, _____, _____, 118, _____

b. 14, _____, 20, 26, _____, _____, 118, _____, _____, 498

c. The sum of the first 10 numbers in part a is equal to 11 times the seventh number. Is this true for the sequence in part b?

d. Is the sum of the first 10 numbers in the Fibonacci sequence equal to 11 times the seventh number in that sequence?

e. Form a conjecture based on your observations in parts c and d.

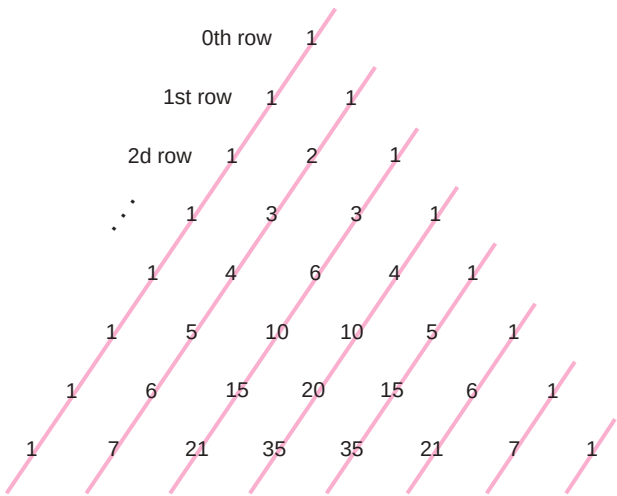
16. The products of 1089 and the first few digits produce some interesting number patterns. Describe one of these patterns. Will this pattern continue if 1089 is multiplied by 5, 6, 7, 8, and 9?

$$1 \times 1089 = 1089$$
$$2 \times 1089 = 2178$$
$$3 \times 1089 = 3267$$
$$4 \times 1089 = 4356$$
$$5 \times 1089 =$$

17. a. Find a pattern in the following equations, and use your pattern to write the next equation.
b. If the pattern in the first three equations is continued, what will be the 20th equation?

$$1 + 2 = 3$$
$$4 + 5 + 6 = 7 + 8$$
$$9 + 10 + 11 + 12 = 13 + 14 + 15$$

In Pascal's triangle, which is shown here, there are many patterns. Use this triangle of numbers in exercises 18 through 21.



18. Add the first few numbers in the first diagonal of Pascal's triangle (diagonals are marked by lines), starting from the top. This sum will be another number from the triangle. Will this be true for the sums of the first few numbers in the other diagonals? Support your conclusion with examples.

19. The third diagonal in Pascal's triangle has the numbers 1, 3, 6, ...
a. What is the 10th number in this diagonal?
b. What is the 10th number in the fourth diagonal?

20. Compute the sums of the numbers in the first few rows of Pascal's triangle. What kind of sequence (arithmetic or geometric) do these sums form?

21. What will be the sum of the numbers in the 12th row of Pascal's triangle?

Identify each of the sequences in exercises 22 and 23 as arithmetic or geometric. State a rule for obtaining each number from the preceding number. What is the 12th number in each sequence?

22. a. 280, 257, 234, 211, . . .
b. 17, 51, 153, 459, . . .
c. 32, 64, 128, 256, . . .
d. 87, 102, 117, 132, . . .

23. a. 4, 9, 14, 19, . . .
b. 15, 30, 60, 120, . . .
c. 24, 20, 16, 12, . . .
d. 4, 12, 36, 108, . . .

The method of finite differences is used in exercises 24 and 25. This method will sometimes enable you to find the next number in a sequence, but not always.

24. a. Write the first eight numbers of a geometric sequence, and try using the method of finite differences to find the ninth number. Will this method work?
b. Repeat part a for an arithmetic sequence. Support your conclusions.

25. a. Will the method of finite differences produce the next number in the diagonals of Pascal's triangle? Support your conclusions with examples.
b. The sums of the numbers in the first few rows of Pascal's triangle are 1, 2, 4, 8, ... Will the method of finite differences produce the next number in this sequence?

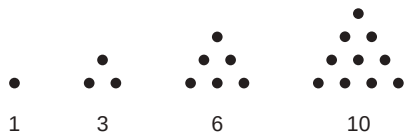
Use the method of finite differences in exercises 26 and 27 to find the next number in each sequence.

26. a. 3, 7, 13, 21, 31, 43, . . .
b. 215, 124, 63, 26, 7, . . .

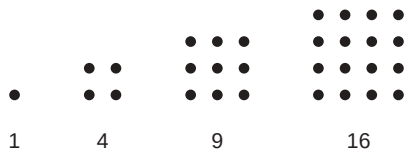
27. a. 1, 2, 7, 22, 53, 106, . . .
b. 1, 3, 11, 25, 45, 71, . . .

As early as 500 B.C.E., the Greeks were interested in numbers associated with patterns of dots in the shape of geometric figures. Write the next three numbers and the 100th number in each sequence in exercises 28 through 30.

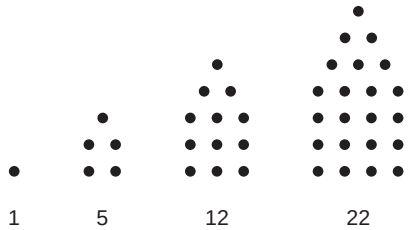
28. Triangular numbers:



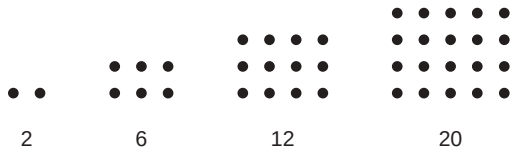
29. Square numbers:



30. Pentagonal numbers. (After the first figure these are five-sided figures composed of figures for triangular numbers and square numbers.)



The Greeks called the numbers represented by the following arrays of dots **oblong numbers.** Use this pattern in exercises 31 and 32.



31. **a.** What is the next oblong number?
b. What is the 20th oblong number?

32. **a.** Can the method of finite differences be used to obtain the number of dots in the 5th oblong number?
b. What is the 25th oblong number?

33. The numbers in the following sequence are the first six pentagonal numbers: 1, 5, 12, 22, 35, 51.
a. If the method of finite differences is used, what type of sequence is produced by the first sequence of differences?
b. Can the method of finite differences be used to obtain the next few pentagonal numbers from the first six?

34. Use the method of finite differences to create a new sequence of numbers for the following sequence of square numbers.

1, 4, 9, 16, 25, 36, 49, 64, 81

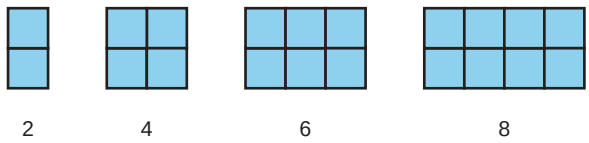
a. What kind of a sequence do you obtain?
b. How can square arrays of dots (see exercise 29) be used to show that the difference of two consecutive square numbers will be an odd number?

What kind of reasoning is used to arrive at the conclusions in the studies in exercises 35 and 36?

35. In a research study involving 600 people, there was a 30 percent reduction in the severity of colds by using less vitamin C than previously recommended. The researchers concluded that smaller amounts of vitamin C are as effective in reducing colds as large amounts.

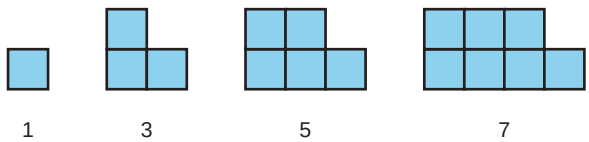
36. A large survey of hospitals found there is an increase in cancer and other disease rates in operating room personnel. The researchers conducting the survey concluded that exposure to anesthetic agents causes health hazards.

37. Continue the pattern of even numbers illustrated here.



a. The fourth even number is 8. Sketch the figure for the ninth even number and determine this number.
b. What is the 45th even number?

38. Continue the pattern of odd numbers illustrated here.



a. The fourth odd number is 7. Sketch the figure for the 12th odd number.
b. What is the 35th odd number?

39. If we begin with the number 6, then double it to get 12, and then place the 12 and 6 side by side, the result is 126. This number is divisible by 7. Try this procedure for some other numbers. Find a counterexample that shows that the result is not always evenly divisible by 7.

Find a counterexample for each of the statements in exercises 40 and 41.

40. **a.** Every whole number greater than 4 and less than 20 is the sum of two or more consecutive whole numbers.
b. Every whole number between 25 and 50 is the product of two whole numbers greater than 1.

41. **a.** The product of any two whole numbers is evenly divisible by 2.
b. Every whole number greater than 5 is the sum of either two or three consecutive whole numbers, for example, $11 = 5 + 6$ and $18 = 5 + 6 + 7$.

Determine which statements in exercises 42 and 43 are false, and show a counterexample for each false statement. If a statement is false, change one of the conditions to obtain a true statement.

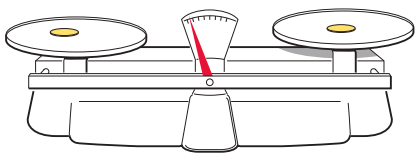
42. **a.** The product of any three consecutive whole numbers is divisible by 2.
b. The sum of any two consecutive whole numbers is divisible by 2.

43. **a.** The sum of any four consecutive whole numbers is divisible by 4.
b. Every whole number greater than 0 and less than 15 is either a triangular number or the sum of two or three triangular numbers.

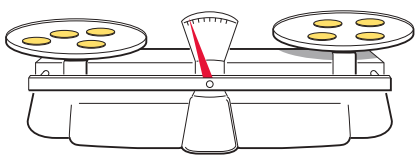
Reasoning and Problem Solving

44. Featured Strategy: Solving a Simpler Problem. You are given eight coins and a balance scale. The coins look alike, but one is counterfeit and lighter than the others. Find the counterfeit coin, using just two weighings on the balance scale.

a. Understanding the Problem. If there were only two coins and one was counterfeit and lighter, the bad coin could be determined in just one weighing. The balance scale here shows this situation. Is the counterfeit coin on the left or right side of the balance scale?



b. Devising a Plan. One method of solving this problem is to guess and check. It is natural to begin with four coins on each side of the balance scale. Explain why this approach will not produce the counterfeit coin in just two weighings. Another method is to simplify the problem and try to solve it for fewer coins.



c. Carrying Out the Plan. Explain how the counterfeit coin can be found with one weighing if there are only three coins and with two weighings if there are six coins. By now you may have an idea for solving the original problem. How can the counterfeit coin be found in two weighings?

d. Looking Back. Explain how the counterfeit coin can be found in two weighings when there are nine coins.

45. Kay started a computer club, and for a while she was the only member. She planned to have each member find two new members each month. By the end of the first month she had found two new members. If her plan is carried out, how many members will the club have at the end of the following periods?
a. 6 months **b.** 1 year

46. For several years Charlie has had a tree farm where he grows blue spruce. The trees are planted in a square array (square arrays are shown in exercise 29). This year he planted 87 new trees along two adjacent edges of the square to form a larger square. How many trees are in the new square?

47. In the familiar song "The Twelve Days of Christmas," the total number of gifts received each day is a triangular number. On the first day there was 1 gift, on the second day there were 3 gifts, on the third day 6 gifts, etc., until the 12th day of Christmas.
a. How many gifts were received on the 12th day?
b. What is the total number of gifts received during all 12 days?

48. One hundred eighty seedling maple trees are to be set out in a straight line such that the distance between the centers of two adjacent trees is 12 feet. What is the distance from the center of the first tree to the center of the 180th tree?

49. In a long line of railroad cars, an Agco Refrigeration car is the 147th from the beginning of the line, and by counting from the end of the line, the refrigeration car is the 198th car. How many railroad cars are in the line?

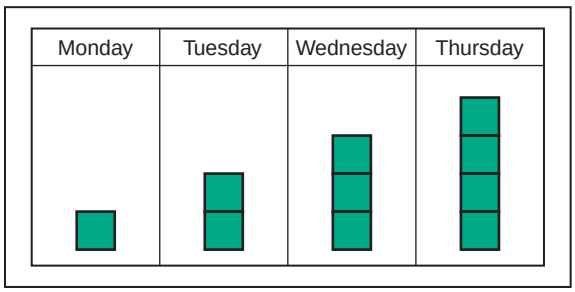
50. If 255 square tiles with colors of blue, red, green, or yellow are placed side by side in a single row so that two tiles of the same color are not next to each other, what is the maximum possible number of red tiles?

51. A card is to be selected at random from 500 cards that are numbered with whole numbers from 1 to 500. How many of these cards have at least one 6 printed on them?

52. A deck of 300 cards is numbered with whole numbers from 1 to 300, with each card having just one number. How many of these cards do not have a 4 printed on them?

Teaching Questions

1. Suppose you were teaching an elementary school class and for the first four days of a week you put the following tiles on the calendar for Monday through Thursday, as shown here. If the pattern you had in mind for Friday was five tiles in a column, and one student formed a different arrangement, what would you say to this student? Is it possible that this student might be "right"? Explain.



2. An elementary school student discovered a way to get from one square number to the next square number and wanted to know why this was true. For example, if you know that 7^2 is 49, then 8^2 is just $49 + 7 + 8$, or 64. Similarly, 9^2 is $8^2 + 8 + 9$, or 81. Write an explanation with a diagram that illustrates why this relationship holds for all consecutive pairs of square numbers.

3. The beginning of the number pattern, 1, 2, 4, 8, was used by two teachers in separate classes. Teacher *A* asked, "What is the next number in this pattern?" Teacher *B* asked, "What are some possibilities for the next number in this pattern?" List more than one way this number pattern can be continued and explain your reasoning for each way. Discuss the difference between the two questions in terms of the expected student responses.

4. It has been said that mathematics is the study of patterns. How would you explain this point of view to the parents of the children in your classroom? Provide examples to support your position.

5. The NCTM Standards inference at the beginning of exercise set 1.2 refers to the importance of the "core" of a pattern to help children become aware of structure. Write examples of number sequences for the following cases: a pattern with a core that repeats; a pattern with a core that grows; a pattern that is a geometric sequence; and a pattern that is an arithmetic sequence.

Classroom Connections

1. Compare the **NCTM Standards** quotes on pages 4 and 27. (a) Explain why these two statements do not contradict each other. (b) If conjectures and inductive reasoning sometimes lead to false statements, explain why these methods of reasoning are taught in schools.

2. The **NCTM Standards** quote on page 27 speaks of students' "incorrect expectations" when generalizing patterns. Give some examples of possible incorrect student expectations that may result from their generalizing patterns.

3. On page 25 the example from the **Elementary School Text** shows a sequence of squares containing triangles. (a) Which method of finding additional terms in a sequence from this section can you use to find the number of triangles in the fifth square? (b) Explain in general how the number of triangles in any given square is related to the number of the squares in the sequence.

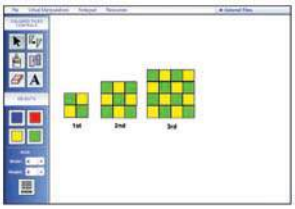
4. Read the three expectations in the **PreK–2 Standards—Algebra** (see front inside cover) under *Understand patterns, relations* . . . , and explain with examples how the third expectation is satisfied in the Exercises and Problems 1.2.

5. The origin of Fibonacci numbers is explained in the **Historical Highlight** on page 22. Use the bibliography and the links for section 1.2 on the companion website and/or browse the Internet to find further applications of Fibonacci numbers. Describe some of these applications.

6. The **Historical Highlight** on page 24 has information on Karl Friedrich Gauss, one of the greatest mathematicians of all time. Gauss kept a mathematical diary and one of his notes claims that every whole number greater than zero can be written as the sum of three or fewer triangular numbers. Verify this statement for numbers less than 20, or find a counterexample.

MATH ACTIVITY 1.3

Virtual Manipulatives



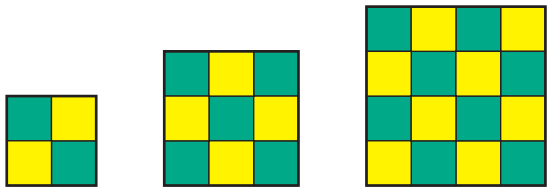
www.mhhe.com/bbn

Extending Tile Patterns

Purpose: Use algebraic thinking to identify and extend patterns in color tile sequences.

Materials: Color Tiles in the Manipulative Kit or Virtual Manipulatives.

1. Here are the first three figures in a sequence. Find a pattern and build the fourth figure.



***a.** For each of the first five figures, determine how many tiles there are of each color.

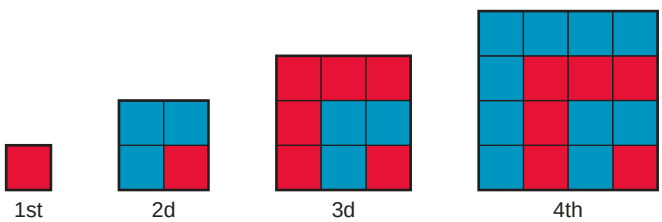
b. Find a pattern and determine the number of tiles of each color for the 10th figure.

c. What is the total number of tiles for the 10th figure?

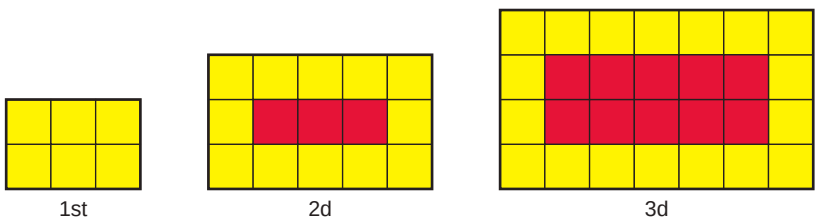
d. Write a description of the 25th figure so that someone reading it could build the figure. Include in your description the number of tiles with each of the different colors and the total number of tiles in the figure.

2. Extend each of the following sequences to the 5th figure, and record the numbers of different color tiles in each figure. Find a pattern that enables you to determine the numbers of different color tiles in the 10th and 25th figures of each sequence. Describe your reasoning.

***a.**



b.



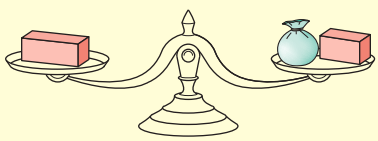
Section 1.3 PROBLEM SOLVING WITH ALGEBRA



"If he could only think in abstract terms"

PROBLEM OPENER

A whole brick is balanced with $\frac{3}{4}$ pound and $\frac{3}{4}$ brick. What is the weight of the whole brick?



NCTM Standards

By viewing algebra as a strand in the curriculum from prekindergarten on, teachers can help students build a solid foundation of understanding and experience as a preparation for more-sophisticated work in algebra in the middle grades and high school. p. 37

Algebra is a powerful tool for representing information and solving problems. It originated in Babylonia and Egypt more than 4000 years ago. At first there were no equations, and words rather than letters were used for variables. The Egyptians used words that have been translated as *heap* and *aha* for unknown quantities in their word problems. Here is a problem from the Rhind Papyrus, written by the Egyptian priest Ahmes about 1650 B.C.E.:

Heap and one-seventh of heap is 19. What is heap?

Today we would use a letter for the unknown quantity and express the given information in an equation.

$$x + \frac{1}{7}x = 19$$

You may wish to try solving this equation. Its solution is in Example D on page 42.

HISTORICAL HIGHLIGHT



Amalie Emmy Noether, 1882–1935

Germany's Amalie Emmy Noether is considered to be the greatest woman mathematician of her time. She studied mathematics at the University of Erlangen, where she was one of only two women among nearly a thousand students. In 1907 she received her doctorate in mathematics from the University of Erlangen. In 1916, the legendary David Hilbert was working on the mathematics of a general relativity theory at the University of Göttingen and invited Emmy Noether to assist him. Although Göttingen had been the first university to grant a doctorate degree to a woman, it was still reluctant to offer a teaching position to a woman, no matter how great her ability and learning. When her appointment failed, Hilbert let her deliver lectures in courses that were announced under his name. Eventually she was appointed to a lectureship at the University of Göttingen. Noether became the center of an active group of algebraists in Europe, and the mathematics that grew out of her papers and lectures at Göttingen made her one of the pioneers of modern algebra. Her famous papers "The Theory of Ideals in Rings" and "Abstract Construction of Ideal Theory in the Domain of Algebraic Number Fields" are the cornerstones of modern algebra courses now presented to mathematics graduate students.*

*D. M. Burton, *The History of Mathematics,* 7th ed. (New York: McGraw-Hill, 2010), pp. 727–732.

NCTM Standards

Research indicates a variety of students have difficulties with the concept of a variable (Kuchmann 1978; Kieran 1983; Wafner and Parker 1993) . . . A thorough understanding of a variable develops over a long time, and it needs to be grounded in extensive experience. p. 39

VARIABLES AND EQUATIONS

A letter or symbol that is used to denote an unknown number is called a **variable.** One method of introducing variables in elementary schools is with geometric shapes such as ☐ and △. For example, students might be asked to find the number for ☐ such that ☐ + 7 = 12, or to find some possibilities for △ and ☐ such that △ + ☐ = 15. These geometric symbols are less intimidating than letters. Students can replace a variable with a number by writing the numeral inside the geometric shape, as if they were filling in a blank.

To indicate the operations of addition, subtraction, and division with numbers and variables, we use the familiar signs for these operations; for example, $3 + x$, $x - 5$, $x \div 4$, and $\frac{x}{4}$. A product is typically indicated by writing a numeral next to a variable. For example, $6x$ represents 6 times x, or 6 times whatever number is used as a replacement for x. An expression containing algebraic symbols, such as $2x + 3$ or $(4x)(7x) - 5$, is called an **algebraic expression.**

EXAMPLE A

Evaluate the following algebraic expressions for $x = 14$ and $n = 28$.

1. $15 + 3x$

2. $4n - 6$

3. $\frac{n}{7} + 20$

4. $6x \div 12$

Solution **1.** $15 + 3(14) = 15 + 42 = 57$. Notice that when the variable is replaced, parentheses are used; 3(14) means 3 times 14. **2.** $4(28) - 6 = 112 - 6 = 106$. **3.** $\frac{28}{7} + 20 = 4 + 20 = 24$. **4.** $6(14) \div 12 = 84 \div 12 = 7$.

12-3 Solving Addition Equations

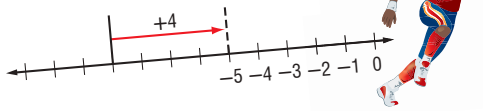
MAIN IDEA
Solve addition equations.

New Vocabulary
inverse operations
Subtraction
Property of Equality

Math Online
glencoe.com
- Extra Examples
- Personal Tutor
- Self-Check Quiz
- Reading in the Content Area

GET READY for the Lesson

FOOTBALL After gaining 4 yards on a play, the football team was still 5 yards short of the goal line. The number line represents this situation.



1. Write an expression to represent the gain of 4 yards.
2. Write an addition equation you could use to find the yards needed before gaining 4 yards.
3. You could solve the addition equation by counting back on the number line. What operation does counting back suggest?

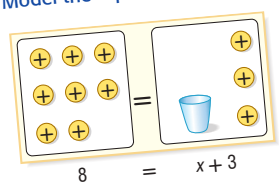
In Lesson 1-8, you mentally solved equations. Another way is to use **inverse operations**, which *undo* each other. For example, to solve an addition equation, use subtraction.

EXAMPLE 1 Solve an Equation By Subtracting

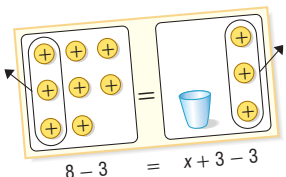
Solve $8 = x + 3$.

METHOD 1 Use models.

Model the equation.



Remove 3 counters from each side.



METHOD 2 Use symbols.

$8 = x + 3$ Write the equation.
$-3 = \quad -3$ Subtract 3 from each side to "undo" the addition of 3 on the right.
$5 = x$

Check $8 = 5 + 3$ ✔

The solution is 5.

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The elementary ideas of algebra can be presented early in school mathematics. Consider Example B.

EXAMPLE B Eleanor wins the jackpot in a marble game and doubles her number of marbles. If later she wins 55 more, bringing her total to 127, how many marbles did she have at the beginning?

Solution One possibility is to work backward from the final total of 127 marbles. Subtracting 55 leaves 72, so we need to find the number that yields 72 when doubled. This number is 36. A second approach is to work forward to obtain 127 by guessing. A guess of 20 for the original number of marbles will result in $2(20) + 55 = 95$, which is less than 127. Guesses of increasingly larger numbers eventually will lead to a solution of 36 marbles.

Example B says that if some unknown number of marbles is doubled and 55 more are added, the total is 127. This numerical information is stated in the following *equation* in which the variable x represents the original number of marbles.

$$2x + 55 = 127$$

An **equation** is a statement of the equality of mathematical expressions; it is a sentence in which the verb is *equals* (=).

A *balance scale* is one model for introducing equations in the elementary school. The idea of *balance* is related to the concept of *equality.* A balance scale with its corresponding equation is shown in Figure 1.6. If each chip on the scale has the same weight, the weight on the left side of the scale *equals* (is the same as) the weight on the right side. Similarly, the sum of numbers on the left side of the equation *equals* the number on the right side.

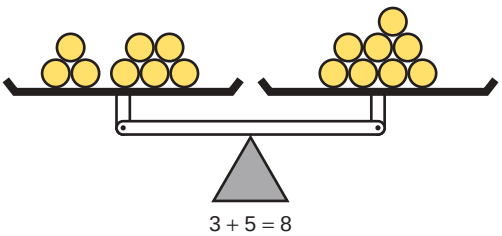


Figure 1.6

The balance scale in Figure 1.7 models the *missing-addend* form of subtraction, that is, what number must be added to 5 to obtain 11. The box on the scale may be thought of as *taking the place of,* or *hiding,* the chips needed to balance the scale.

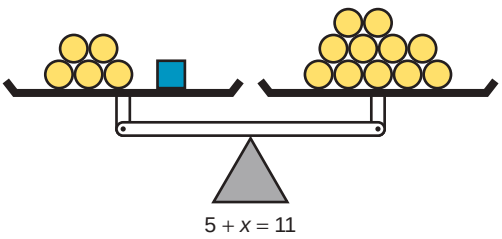


Figure 1.7

One approach to determining the number of chips needed to balance the scale is to *guess and check.* Another approach is to notice that by removing 5 chips from both sides of the scale in Figure 1.7, we obtain the scale shown in Figure 1.8. This scale shows that the box must be replaced by (or is hiding) 6 chips.

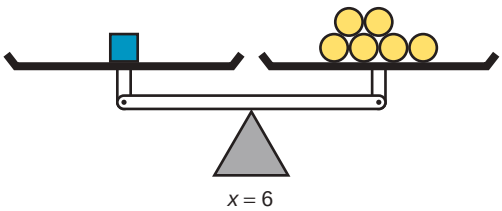


Figure 1.8

Similarly, the equation $5 + x = 11$ can be simplified by subtracting 5 from both sides to obtain $x = 6$. This simpler equation shows that the variable must be replaced by 6.

SOLVING EQUATIONS

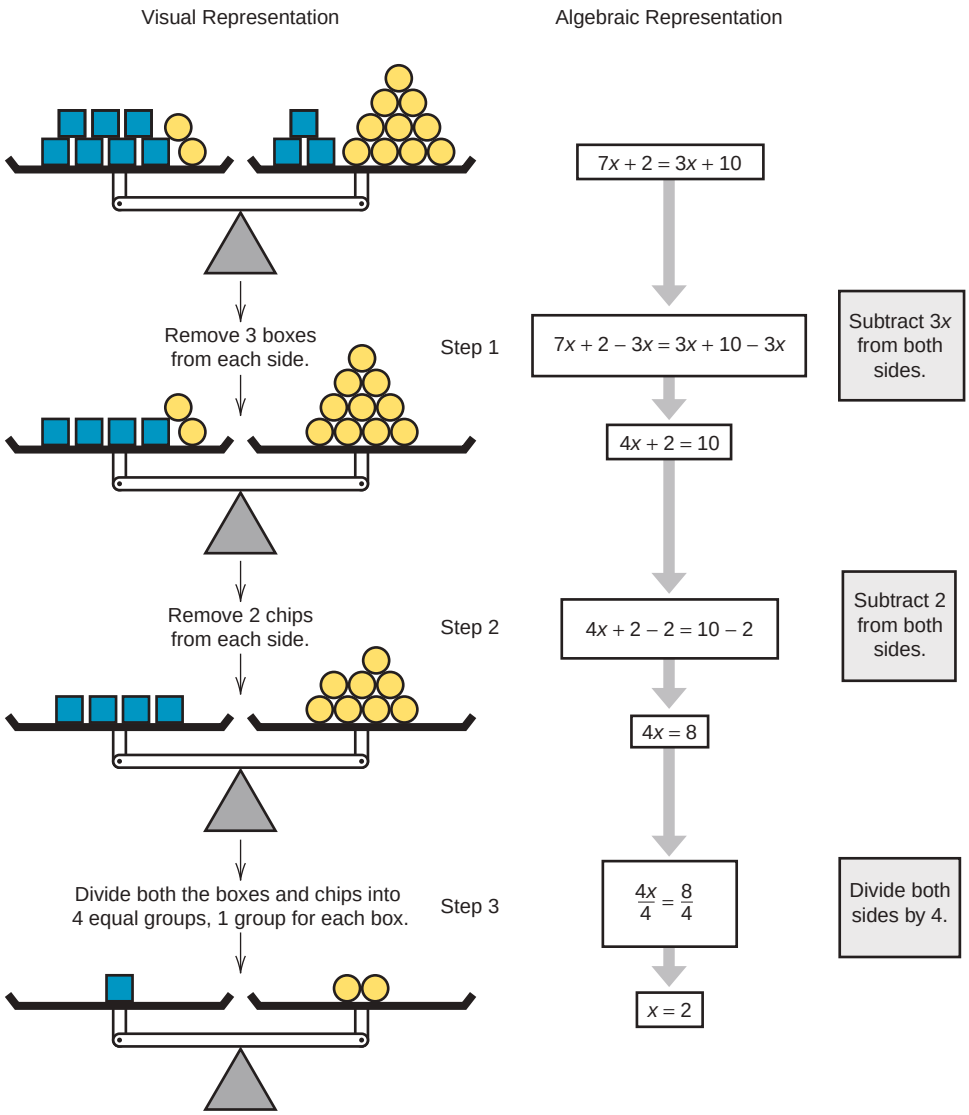
To **solve an equation** or **find the solution(s)** means to find all replacements for the variable that make the equation true. The usual approach to solving an equation is to replace it by a simpler equation whose solutions are the same as those of the original equation. Two equations that have exactly the same solution are called **equivalent equations.**

The *balance-scale model* is used in Example C to illustrate solving an equation. Each step in simplifying the balance scale corresponds to a step in solving the equation.

EXAMPLE C

Solve $7x + 2 = 3x + 10$, using the balance-scale model and equations.

Solution



Check: If each box on the first scale is replaced by 2 chips, the scale will balance with 16 chips on each side. Replacing x by 2 in the equation $7x + 2 = 3x + 10$ makes the equation a true statement and shows that 2 is a solution to this equation.

NCTM Standards

The notion of equality also should be developed throughout the curriculum. They [students] should come to view the equals sign as a symbol of equivalence and balance. p. 39

When the balance-scale model is used, the same amount must be *put on* or *removed from* each side to maintain a balance. Similarly, with an equation, the *same operation* must be performed on each side to maintain an equality. In other words, *whatever is done to one side of an equation must be done to the other side.* Specifically, three methods for obtaining equivalent equations are stated next as properties of equality.

Properties of Equality

1. **Addition or Subtraction Property of Equality:** Add the same number or subtract the same number from both sides of an equation.
2. **Multiplication or Division Property of Equality:** Multiply or divide both sides of an equation by the same nonzero number.
3. **Simplification:** Replace an expression in an equation by an equivalent expression.

The preceding methods of obtaining equivalent equations are illustrated in Example D.

EXAMPLE D

Solve these equations.

1. $5x - 9 = 2x + 15$

2. $x + \frac{1}{7}x = 19$ (This is the problem posed by the Egyptian priest Ahmes, described on the opening page of this section.)

Research Statement

Students' difficulties in constructing equations stem in part from their inability to grasp the notion of the equivalence between the two expressions in the left and right sides of the equation.

MacGregor

Solution **1.**

$5x - 9 = 2x + 15$	
$5x - 9 - 2x = 2x + 15 - 2x$	subtraction property of equality; subtract $2x$ from both sides
$3x - 9 = 15$	simplification
$3x - 9 + 9 = 15 + 9$	addition property of equality; add 9 to both sides
$3x = 24$	simplification
$\frac{3x}{3} = \frac{24}{3}$	division property of equality; divide both sides by 3
$x = 8$	simplification

Check: When x is replaced by 8 in the original equation (or in any of the equivalent equations), the equation is true.

$$5(8) - 9 = 2(8) + 15$$
$$31 = 31$$

2. $x + \frac{1}{7}x = 19$

$7\left(x + \frac{1}{7}x\right) = 7(19)$	multiplication property of equality; multiply both sides by 7
$8x = 133$	simplification; $7(x + \frac{1}{7}x) = 7x + \frac{7}{7}x = 8x$. This is an example of the **distributive property.***
$\frac{8x}{8} = \frac{133}{8}$	division property of equality; divide both sides by 8
$x = 16\frac{5}{8} = 16.625$	simplification

*For examples of the distributive property, as well as several other number properties, see the subsection Number Properties in Section 3.3.

Check: When x is replaced by 16.625 in the original equation, the equation is true.

$$16.625 + \frac{1}{7}(16.625) = 16.625 + 2.375 = 19$$

SOLVING INEQUALITIES

Not all algebra problems are solved by equations. Consider Example E.

EXAMPLE E

John has \$19 to spend at a carnival. After paying the entrance fee of \$3, he finds that each ride costs \$2. What are the possibilities for the number of rides he can take?

Solution This table shows John's total expenses with different numbers of rides. John can take any number of rides from 0 to 8 and not spend more than \$19.

Number of Rides	Expense
0	\$ 3
1	5
2	7
3	9
4	11
5	13
6	15
7	17
8	19

Example E says that \$3 plus some number of \$2 rides must be less than or equal to \$19. This numerical information is stated in the following *inequality,* where x represents the unknown number of rides:

$$3 + 2x \leq 19$$

An **inequality** is a statement that uses one of the following phrases: *is less than* ($<$), *is less than or equal to* ($\leq$), *is greater than* ($>$), *is greater than or equal to* ($\geq$), or *is not equal to* ($\neq$).

The *balance-scale model* can also be used for illustrating inequalities. Figure 1.9 illustrates the inequality in Example E. The box can be replaced by any number of chips as long as the scale doesn't tip down on the left side. Some elementary schoolteachers who use the balance-scale model have students tip their arms to imitate the balance scale. Sometimes the teacher places a heavy weight in one hand of a student and a light weight in the other. This helps students become accustomed to the fact that the amount on the side of the scale that is tipped down is *greater than* the amount on the other side of the scale.

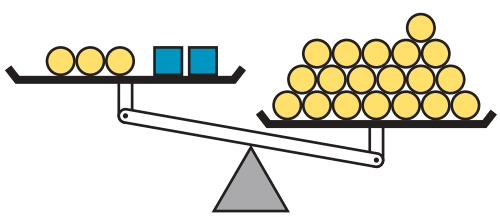


Figure 1.9

One method of finding the number of chips that can be used in place of the box in Figure 1.9 is to think of replacing each box on the scale by the same number of chips, keeping the total number of chips on the left side of the scale less than or equal to 19. Another method is to simplify the scale to determine the possibilities for the number of chips for the box. First, we can remove 3 chips from both sides to obtain the scale setting in Figure 1.10.

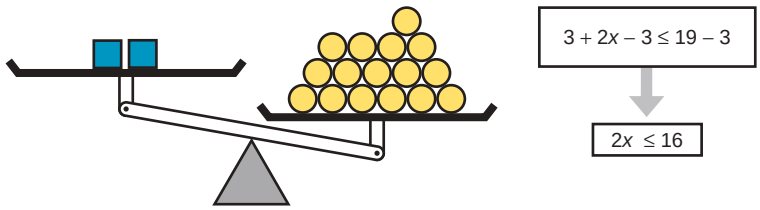


Figure 1.10

Next, we can divide the chips on the right side of the scale into two groups, one group for each box on the left side of the scale. The simplified scale in Figure 1.11 shows that replacing the box by 7 or fewer chips will keep the scale tipped down on the right side and if the box is replaced by 8 chips the scale will be balanced.

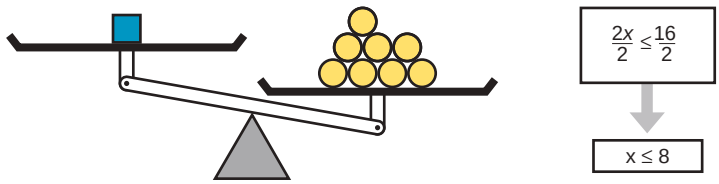


Figure 1.11

To the right of each balance scale above, there is a corresponding inequality. These inequalities are replaced by simpler inequalities to obtain $x \leq 8$. To make this inequality true, we must replace the variable by a number less than or equal to 8.

To **solve an inequality** means to find all the replacements for the variable that make the inequality true. The replacements that make the inequality true are called **solutions.** Like an equation, an inequality is solved by replacing it by simpler inequalities. Two inequalities that have exactly the same solution are called **equivalent inequalities.**

Equivalent inequalities can be obtained using the same steps as those for obtaining *equivalent equations* (performing the same operation on both sides and replacing an expression by an equivalent expression), with one exception: Multiplying or dividing both sides of an inequality by a negative number *reverses the inequality*. For example, $8 > 3$; but if we multiply both sides of the inequality by $^-1$, we obtain $^-8$ and $^-3$, and $^-8$ is less than $^-3$ ($^-8 < ^-3$). These inequalities are illustrated in Figure 1.12.

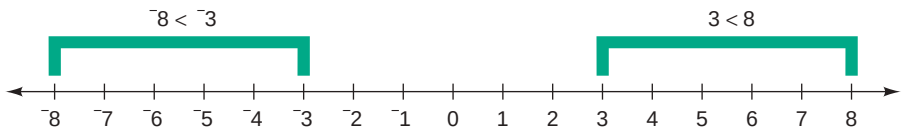


Figure 1.12

Three methods for obtaining equivalent inequalities are stated next as the properties of inequality. (These properties also apply to the inequalities $\leq$, $>$, and $\geq$.)

Properties of Inequality

1. **Addition or Subtraction Property of Inequality:** Add the same number or subtract the same number from both sides of an inequality.
2. **Multiplication or Division Property of Inequality:** Multiply or divide both sides of an inequality by the same nonzero number; and if the number is negative, reverse the inequality sign.
3. **Simplification:** Replace an expression in an inequality by an equivalent expression.

The three methods of obtaining equivalent inequalities are illustrated in Examples F and G.

EXAMPLE F

Solve the inequality $4(3x) + 16 < 52$.

NCTM Standards

In the middle grades it is essential that students become comfortable in relating symbolic expressions containing variables to verbal, tabular, and graphical representations or numerical and quantitative relationships. p. 223

Solution

$4(3x) + 16 < 52$	
$12x + 16 < 52$	simplification
$12x + 16 - 16 < 52 - 16$	subtraction property for inequality; subtract 16 from both sides
$12x < 36$	simplification
$\frac{12x}{12} < \frac{36}{12}$	division property for inequality; divide both sides by 12
$x < 3$	simplification

Check: We can get some indication of whether the inequality was solved correctly by trying a number less than 3 to see if it is a solution. When we replace x in the original inequality by 2 we can see that the inequality holds.

$$\begin{aligned} 4[3(2)] + 16 &= 4(6) + 16 \\ &= 24 + 16 \\ &= 40 \end{aligned}$$

and 40 is less than 52.

Number Line Display: The solutions for an inequality in one variable can be visualized on a number line. The solutions for the inequality are displayed in Figure 1.13. The circle about the point for 3 indicates that this point is not part of the solution. So the solution includes all the points on the half-line extending to the left of the point for 3.

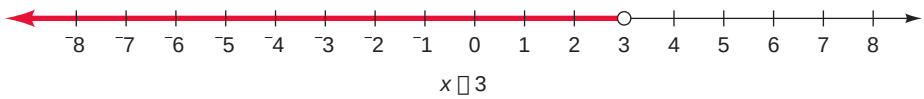


Figure 1.13

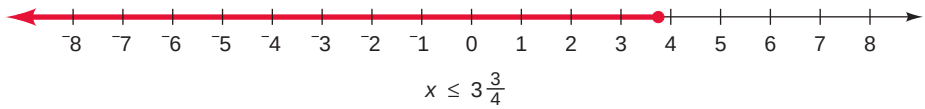
EXAMPLE G

Solve the inequality $11x - 7 \leq 3x + 23$, and illustrate its solution by using a number line.

Solution

$11x - 7 \leq 3x + 23$	
$11x - 7 + 7 \leq 3x + 23 + 7$	addition property for inequality; add 7 to both sides
$11x \leq 3x + 30$	simplification
$11x - 3x \leq 3x - 3x + 30$	subtraction property for inequality; subtract $3x$ from both sides
$8x \leq 30$	simplification
$\frac{8x}{8} \leq \frac{30}{8}$	division property for inequality; divide both sides by 8
$x \leq 3\frac{3}{4}$	simplification

Every number less than or equal to $3\frac{3}{4}$ is a solution for the original inequality, and these solutions can be shown on a number line by shading $3\frac{3}{4}$ with a solid circle and shading the half-line to the left of $3\frac{3}{4}$.



$x \leq 3\frac{3}{4}$

USING ALGEBRA AS A PROBLEM-SOLVING STRATEGY

One application of algebra is solving problems whose solutions involve equations and inequalities.

EXAMPLE H

The manager of a garden center wants to order a total of 138 trees consisting of two types: Japanese red maple and flowering pears. Each maple tree costs \$156 and each pear tree costs \$114. If the manager has a budget of \$18,000, which must all be spent for the trees, how many maple trees will be in the order?

Solution If x equals the number of maple trees, then $138 - x$ will equal the number of pear trees. The following equation shows that the total cost of both types of trees is \$18,000. Notice the use of the distributive property in going from the first to the second equation.

$$\begin{aligned} 156x + 114(138 - x) &= 18{,}000 \\ 156x + 15{,}732 - 114x &= 18{,}000 \\ 42x &= 2268 \\ x &= 54 \end{aligned}$$

There will be 54 Japanese red maple trees in the order.

Example I is a variation of Example H, but its solution requires an inequality.

EXAMPLE I

The manager of a garden center wants to place an order for Hawthorne trees and Service Berry trees so that the number of Service Berry trees is 6 times the number of Hawthorne trees. Each Hawthorne tree costs \$250 and each Service Berry tree costs \$125. If the budget requires that the total cost of the trees be less than \$30,000 and that there be at least 20 Hawthorne trees, what are the different possibilities for the number of Hawthorne trees in the order?

Solution If x equals the number of Hawthorne trees, then $6x$ is the number of Service Berry trees, and the following inequality shows that the total cost of the two types of trees is less than \$30,000.

$$\begin{aligned} 250x + 125(6x) &< 30{,}000 \\ 250x + 750x &< 30{,}000 \\ 1000x &< 30{,}000 \\ x &< 30 \end{aligned}$$

Since there is a requirement that the order contain at least 20 Hawthorne trees, the possibilities for the number of Hawthorne trees is 20, 21, 22, . . . 29.

Technology Connection

Palindromic Sums

Notice that $48 + 84 = 132$ and $132 + 231 = 363$, and 363 is a palindromic number. Will all two-digit numbers eventually go to palindromic numbers using this process of reversing digits and adding? The online 1.3 Mathematics Investigation will help you explore this and similar questions.

Mathematics Investigation
Chapter 1, Section 3
www.mhhe.com/bbn

Another application of algebra is in analyzing number tricks and so-called magic formulas. Select any number and perform the following operations:

> Add 4 to any number; multiply the result by 6; subtract 9; divide by 3; add 13; divide by 2; and then subtract the number you started with.

If you performed these operations correctly, your final answer will be 9, regardless of the number you started with. This can be proved using the variable x to represent the number selected and performing the following algebraic operations:

1. Select any number	x
2. Add 4	$x + 4$
3. Multiply by 6	$6(x + 4) = 6x + 24$
4. Subtract 9	$6x + 24 - 9 = 6x + 15$
5. Divide by 3	$\frac{6x + 15}{3} = 2x + 5$
6. Add 13	$2x + 5 + 13 = 2x + 18$
7. Divide by 2	$\frac{2x + 18}{2} = x + 9$
8. Subtract the number x	$x + 9 - x = 9$

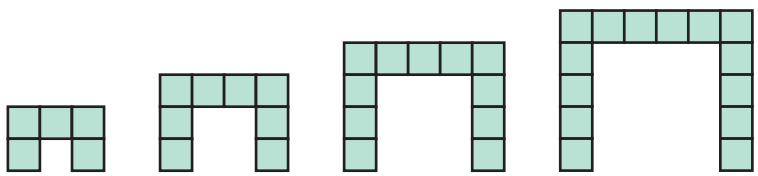
The preceding steps show that it doesn't matter what number is selected for x, for in the final step x is subtracted and the end result is always 9.

PROBLEM-SOLVING APPLICATION

The problem-solving strategy of *using algebra* is illustrated in the solution to the next problem.

Problem

A class of students is shown the following figures formed with tiles and is told that there is a pattern that, if continued, will result in one of the figures having 290 tiles. Which figure will have this many tiles?



NCTM Standards

Two central themes of algebraic thinking are appropriate for young students. The first involves making generalizations and using symbols to represent mathematical ideas, and the second is representing and solving problems (Carpenter and Levi). p. 93

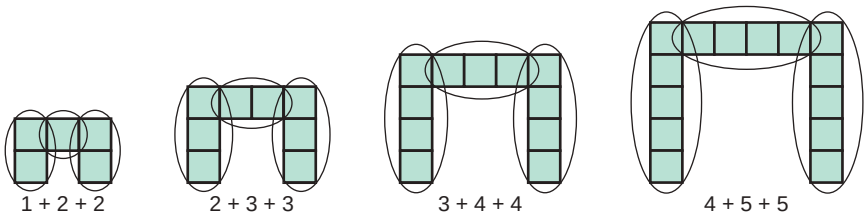
Understanding the Problem The fourth figure has 14 tiles. Find a pattern in the formation of the first few figures, and sketch the fifth and sixth figures. **Question 1:** How many tiles are in the fifth and sixth figures?

Devising a Plan One approach to solving this problem is to use a variable and write an algebraic expression for the nth term. This expression can then be used to determine which figure has 290 tiles.

Notice that the third figure has 3 tiles in each "leg," 3 tiles in the middle of the top row, and 2 corner tiles. The fourth figure has 4 tiles in each leg, 4 in the middle of the top row, and 2 corner tiles. **Question 2:** By extending this reasoning, how many tiles are in the 20th figure? the 100th?

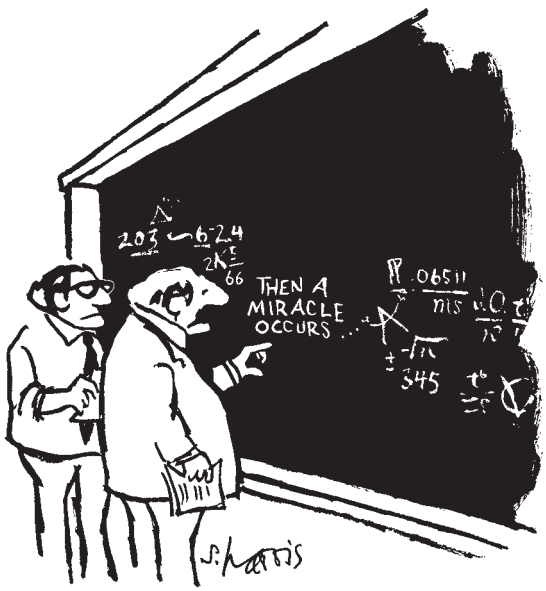
Carrying Out the Plan The *n*th figure will have *n* tiles in each leg, *n* tiles in the middle of the top row, and 2 corner tiles. So the algebraic expression for the number of tiles in the *n*th figure is $n + n + n + 2$, or $3n + 2$. **Question 3:** What number for *n* gives the expression $3n + 2$ a value of 290?

Looking Back Perhaps you saw a different way to group the tiles in the first four figures. **Question 4:** If you saw the pattern developing as follows, what would be the algebraic expression for the *n*th figure?



Answers to Questions 1–4 **1.** The fifth figure has 17 tiles and the sixth has 20. **2.** The 20th figure has $20 + 20 + 20 + 2 = 62$ tiles, and the 100th figure has $100 + 100 + 100 + 2 = 302$ tiles. **3.** $n = 96$. **4.** $n + (n + 1) + (n + 1)$ or $n + 2(n + 1)$.

Exercises and Problems 1.3



"I THINK YOU SHOULD BE MORE EXPLICIT HERE IN STEP TWO."

1. a. At a depth of *x* feet under water, the pressure in pounds per square inch is $.43x + 14.7$. What is the pressure in pounds per square inch for a depth of 10 feet? 100 feet? 0 feet (surface of the water)?

b. The temperature (Fahrenheit) can be approximated by $\frac{x}{4} + 40$, where *x* is the number of cricket chirps in 1 minute. What is the temperature for 20 chirps per minute? 100 chirps per minute?

c. A person's normal blood pressure increases with age and is approximated by $\frac{x}{2} + 110$, where *x* is the person's age. The blood pressure for people between 20 and 30 years old should be between what two numbers?

2. a. A woman's shoe size is given by $3x - 22$, where *x* is the length of her foot in inches. What is a woman's shoe size for a length of 9 inches? 11 inches?

b. The number of words in a child's vocabulary for children between 20 and 50 months is $60x - 900$, where *x* is the child's age in months. What is the number of vocabulary words for a child whose age is 20 months? 35 months? 4 years?

c. A person's maximum heart rate is $220 - x$, where *x* is the person's age, and the heart rate for aerobic activity should be between $.7(220 - x)$ and $.8(220 - x)$. A 20-year-old person's heart rate for aerobic activity should be between what two numbers?

3. Tickets for the historical review of ballroom dancing at the Portsmouth Music Hall cost $28 each for the main-floor seats and $19 each for the balcony seats. Let *m*

represent the number of tickets sold for main-floor seats and let b represent the number of tickets sold for balcony seats. Write an algebraic expression for the following amounts.

a. The cost in dollars of all the main-floor seats that were sold

b. The total number of seats that were sold for the performance

c. The difference in dollars in the total amount of money paid for all main-floor seats and the total amount paid for all balcony seats, if the total for all main-floor seats was the greater of the two amounts

4. At the Saturday farmers' market, melons cost \$1.20 each and coconuts cost \$1.45 each. Let m represent the number of melons sold during the day, and let c represent the number of coconuts sold. Write an algebraic expression for each of the following.

a. The total number of melons and coconuts sold

b. The cost of all the coconuts sold

c. The total cost of all the melons and coconuts sold

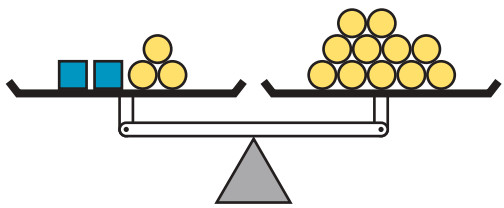
5. In research conducted at the University of Massachusetts, Peter Rosnick found that $37\frac{1}{3}$ percent of a group of 150 engineering students were unable to write the correct equation for the following problem.* Write an equation using variables s and p to represent the following statement: "At this university there are 6 times as many students as professors." Use s for the number of students and p for the number of professors.

a. What is the correct equation?

b. The most common erroneous answer was $6s = p$. Give a possible explanation for this.

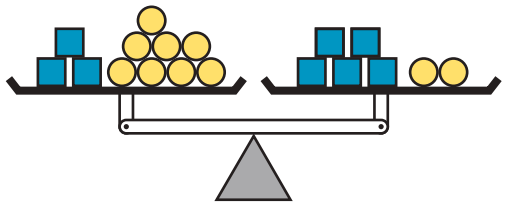
Determine the number of chips needed to replace each box in order for the scales in exercises 6 and 7 to balance. Then using x to represent the number of chips for each box, write the corresponding equation that represents each scale and solve the equation.

6. a.

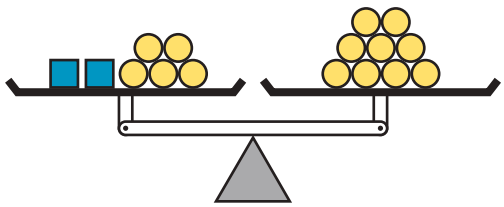


*Peter Rosnick, "Some Misconceptions Concerning the Concept of a Variable," *The Mathematics Teacher* 74: 418–420.

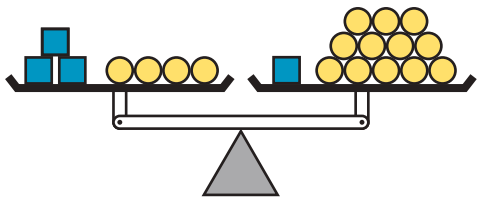
b.



7. a.



b.



Each of the equations in exercises 8 and 9 has been replaced by a similar equivalent equation. Write the property of equality that has been used in each step.

8. a.

$$\begin{aligned} 6x - 14 &= 2x \\ 6x - 14 + 14 &= 2x + 14 \quad \text{(step 1)} \\ 6x &= 2x + 14 \quad \text{(step 2)} \\ 6x - 2x &= 2x + 14 - 2x \quad \text{(step 3)} \\ 4x &= 14 \quad \text{(step 4)} \\ \frac{4x}{4} &= \frac{14}{4} \quad \text{(step 5)} \\ x &= 3\frac{1}{2} \quad \text{(step 6)} \end{aligned}$$

b.

$$\begin{aligned} 42x + 102 &= 6(3x + 45) \\ 42x + 102 &= 18x + 270 \quad \text{(step 1)} \\ 42x + 102 - 18x &= 18x + 270 - 18x \quad \text{(step 2)} \\ 24x + 102 &= 270 \quad \text{(step 3)} \\ 24x + 102 - 102 &= 270 - 102 \quad \text{(step 4)} \\ 24x &= 168 \quad \text{(step 5)} \\ \frac{24x}{24} &= \frac{168}{24} \quad \text{(step 6)} \\ x &= 7 \quad \text{(step 7)} \end{aligned}$$

9. a.

$$\begin{aligned} 6(2x - 5) &= 7x + 15 \\ 12x - 30 &= 7x + 15 \quad \text{(step 1)} \\ 12x - 30 + 30 &= 7x + 15 + 30 \quad \text{(step 2)} \\ 12x &= 7x + 45 \quad \text{(step 3)} \\ 12x - 7x &= 7x + 45 - 7x \quad \text{(step 4)} \\ 5x &= 45 \quad \text{(step 5)} \\ \frac{5x}{5} &= \frac{45}{5} \quad \text{(step 6)} \\ x &= 9 \quad \text{(step 7)} \end{aligned}$$

b.
$$\begin{aligned} 11(3x) + 2 &= 35 \\ 33x + 2 &= 35 \quad \text{(step 1)} \\ 33x + 2 - 2 &= 35 - 2 \quad \text{(step 2)} \\ 33x &= 33 \quad \text{(step 3)} \\ \frac{33x}{33} &= \frac{33}{33} \quad \text{(step 4)} \\ x &= 1 \quad \text{(step 5)} \end{aligned}$$

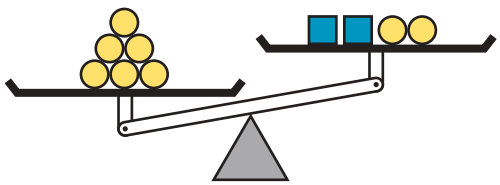
Solve each equation in exercises 10 and 11.

10. a. $2x + 30 = 18 + 5x$
b. $3x - 17 = 22$
c. $13(2x) + 20 = 6(5x + 2)$
d. $8\left(\frac{x}{2} - 5\right) = 2x - 6$

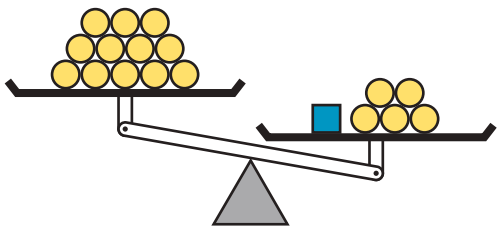
11. a. $43x - 281 = 17x + 8117$
b. $17(3x - 4) = 25x + 218$
c. $56(x + 1) + 7x = 45{,}353$
d. $3x + 5 = 2(2x - 7)$

Determine the number of chips for each box that will keep the scales in exercises 12 and 13 tipped as shown. Then, using x for a variable, write the corresponding inequality for each scale and solve the inequality.

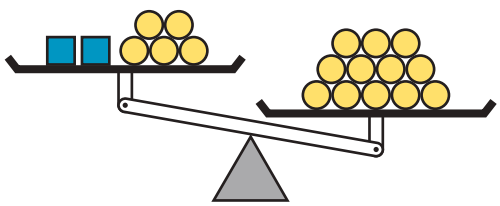
12. a.



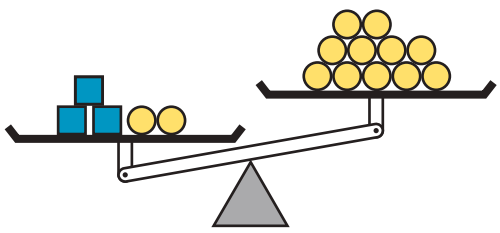
b.



13. a.



b.



Each of the inequalities in exercises 14 and 15 has been replaced by a similar equivalent inequality. Write the property of inequality that has been used in each step.

14. a.
$$\begin{aligned} 3x + 14 &< 55 \\ 3x + 14 - 14 &< 55 - 14 \quad \text{(step 1)} \\ 3x &< 41 \quad \text{(step 2)} \\ \frac{3x}{3} &< \frac{41}{3} \quad \text{(step 3)} \\ x &< 13\tfrac{2}{3} \quad \text{(step 4)} \end{aligned}$$

b. $10x < 55$

$$\begin{aligned} \frac{10x}{10} &< \frac{55}{10} \quad \text{(step 1)} \\ x &< 5\tfrac{1}{2} \quad \text{(step 2)} \end{aligned}$$

15. a.
$$\begin{aligned} 6x + 11 &> 2x + 19 \\ 6x + 11 - 2x &> 2x + 19 - 2x \quad \text{(step 1)} \\ 4x + 11 &> 19 \quad \text{(step 2)} \\ 4x + 11 - 11 &> 19 - 11 \quad \text{(step 3)} \\ 4x &> 8 \quad \text{(step 4)} \\ \frac{4x}{4} &> \frac{8}{4} \quad \text{(step 5)} \\ x &> 2 \quad \text{(step 6)} \end{aligned}$$

b.
$$\begin{aligned} 2x &> 3x - 12 \\ 2x - 3x &> 3x - 12 - 3x \quad \text{(step 1)} \\ {}^{-}x &> {}^{-}12 \quad \text{(step 2)} \\ ({}^{-}1)({}^{-}x) &< ({}^{-}1)({}^{-}12) \quad \text{(step 3)} \\ x &< 12 \quad \text{(step 4)} \end{aligned}$$

Solve each inequality in exercises 16 and 17, and illustrate the solution by using a number line.

16. a. $3x + 5 < x + 17$
b. $3(2x + 7) > 36$

17. a. $6(x + 5) > 11x$
b. $5(x + 8) - 6 > 44$

18. Mr. Dawson purchased some artichokes for 80 cents each and twice as many pineapples for 95 cents each. Altogether he spent $18.90. Let x represent the number of artichokes, and write an algebraic expression for each item in parts a through c.
a. The total cost in dollars of the artichokes
b. The number of pineapples
c. The total cost in dollars of the pineapples
d. The sum of the costs in parts a and c is $18.90. Write and solve an equation to determine the number of artichokes Mr. Dawson bought.

19. It cost Marci 28 cents to mail a postcard and 44 cents to mail a letter. She sent either a postcard or a letter to each of 18 people and spent $6.00. Let x represent the

number of postcards she wrote, and write an algebraic expression for each item in parts a through c.

a. The total cost in dollars of the postcards
b. The number of letters
c. The total cost in dollars of the letters
d. The sum of the costs in parts a and c is $6.00. Write and solve an equation to determine the number of postcards Marci mailed.

20. Teresa purchased pens for 50 cents each and pencils for 25 cents each. She purchased 10 more pencils than pens and gave the clerk a five-dollar bill, which was more than enough to pay the total cost. Let x represent the number of pens, and write an algebraic expression for each item in parts a through c.

a. The total cost in dollars of the pens
b. The number of pencils
c. The total cost in dollars of the pencils
d. The sum of the costs in parts a and c is less than $5.00. Write and solve an inequality to determine the possibilities for the numbers of pens and pencils that Teresa purchased.

21. Merle spent $10.50 for DVDs and $8 for CDs. He purchased three more CDs than DVDs and the total amount of money he spent was less than $120. Let x represent the number of DVDs he purchased, and write an algebraic expression for each item in parts a through c.

a. The total cost in dollars of the DVDs
b. The number of CDs
c. The total cost in dollars of the CDs
d. The sum of the costs in parts a and c is less than $120. Write and solve an inequality to determine the possibilities for the number of DVDs Merle purchased.

Reasoning and Problem Solving

Solve word problems 22 through 24 by writing an equation with a variable to represent the given information and then solving the equation.

22. Jeri spends $60 of her paycheck on clothes and then spends one-half of her remaining money on food. If she has $80 left, what was the amount of her paycheck?

23. Marcia has 350 feet of fence. After fencing in a square region, she has 110 feet of fence left. What is the length of one side of the square?

24. Rico noticed that if he began with his age, added 24, divided the result by 2, and then subtracted 6, he got his age back. What is his age?

Solve word problems 25 and 26 by writing an inequality with a variable and then solving the inequality.

25. If you add 14 to a certain number, the sum is less than 3 times the number. For what numbers is this true?

26. The length of the first side of a triangle is a whole number greater than 3. The second side is 3 inches longer than the first, and the third side is 3 inches longer than the second. How many such triangles have perimeters less than 36 inches?

27. How can 350 be written as the sum of four consecutive whole numbers?

a. Understanding the Problem. If 75 is the first of four consecutive numbers, then the others are 76, 77, and 78. If n is the first number, write an algebraic expression for each of the other three.
b. Devising a Plan. One plan is to let n be the first of four consecutive numbers, write an algebraic expression for their sum, and determine which number for n gives that expression a value of 350. If n is the first of four consecutive numbers, what is an algebraic expression for their sum?
c. Carrying Out the Plan. What number for n gives a sum of 350?
d. Looking Back. Can 350 be written as the sum of other consecutive whole numbers? Use algebraic expressions to determine if 350 can be written as the sum of three consecutive numbers or five consecutive numbers.

28. The teacher asks the class to select a 4×4 array of numbers from a 10×10 number chart and to use only those numbers for the four-step process described on the next page.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1. Circle any number and cross out the remaining numbers in its row and column.
2. Circle another number that has not been crossed out and cross out the remaining numbers in its row and column.
3. Repeat step 2 until there are four circled numbers.
4. Add the four circled numbers.

No matter what sum a student comes up with, the teacher will be able to predict the number in the upper-left corner of the student's square by subtracting 66 from the sum and dividing the result by 4. Show why this formula works.

a. **Understanding the Problem.** Let's carry out the steps on the 4×4 array shown on the previous page. The first circled number is 46, and the remaining numbers in its row and column have been crossed out. The next circled number is 38. Continue the four-step process. Does the teacher's formula produce the number in the upper-left corner of this 4×4 array?

b. **Devising a Plan.** This problem can be solved by algebra. If we represent the number in the upper-left corner by x, the remaining numbers can be represented in terms of x. Complete the next two rows of the 4×4 array of algebraic expressions shown here.

x	$x + 1$	$x + 2$	$x + 3$
$x + 10$	$x + 11$	$x + 12$	$x + 13$

c. **Carrying Out the Plan.** Carry out the four-step process on the 4×4 array of algebraic expressions you completed in part b. Use the results to show that the teacher's formula works.

d. **Looking Back.** One variation on this number trick is to change the size of the array that the student selects. For example, what formula will the teacher use if a 3×3 array is selected from the 10×10 number chart? (*Hint:* Use a 3×3 algebraic array.) Another variation is to change the number chart. Suppose that a 3×3 array is selected from a calendar. What is the formula in this case?

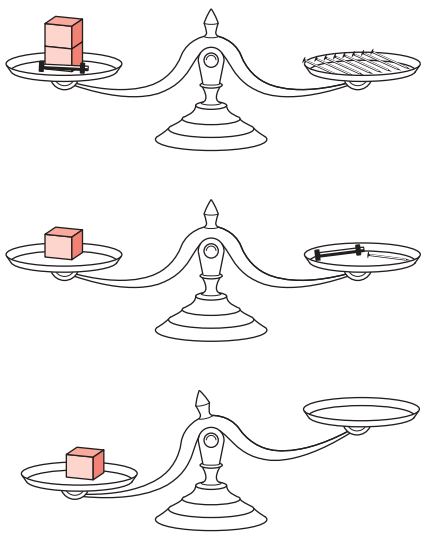
29. PS Many number tricks can be explained by algebra. Select any number and perform the steps here to see what number you obtain. Add 221 to the number, multiply by 2652, subtract 1326, divide by 663, subtract 870, divide by 4, and subtract the original number. Use algebra to show that the steps always result in the same number.

30. PS Ask a person to write the number of the month of his or her birth and perform the following operations: Multiply by 5, add 6, multiply by 4, add 9, multiply by 5, and add the number of the day of birth. When 165 is subtracted from this number, the result is a number that represents the person's month and day of birth. Try it.

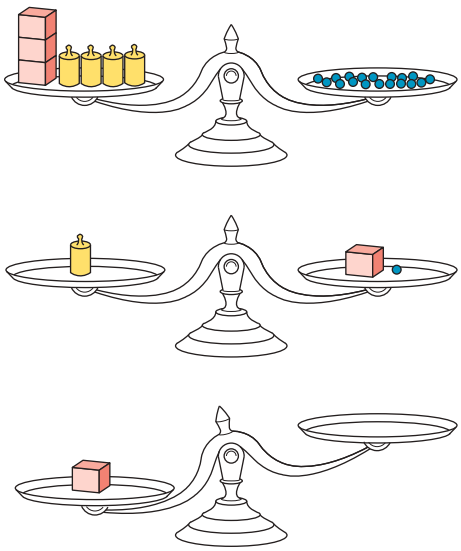
Analysis: Let d and m equal the day and month, respectively. The first three of the preceding steps are represented by the following algebraic expressions. Continue these expressions to show that the final expression is equal to $100m + d$. This shows that the units digit (or tens and units digits) of the final result is the day and the remaining digits are the month.

$$5m$$
$$5m + 6$$
$$4(5m + 6) = 20m + 24$$
$$\vdots$$

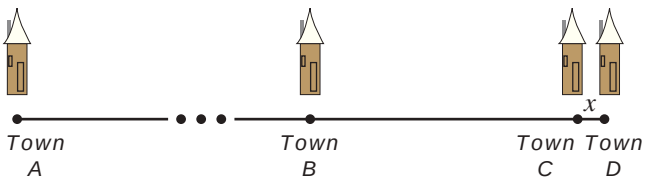
31. PS Use the information from the balance scales shown here to determine the number of nails needed to balance one cube. (*Note:* There are 2 cubes and 1 bolt in the pan on the left side and 8 nails in the pan on the right side of the first balance scale and 1 bolt and 1 nail in the pan on the right side of the second balance scale.)



32. Use the information from the first two balance scales below to determine the number of marbles needed to balance a cube. (*Note:* There are 18 marbles in the pan on the right.)

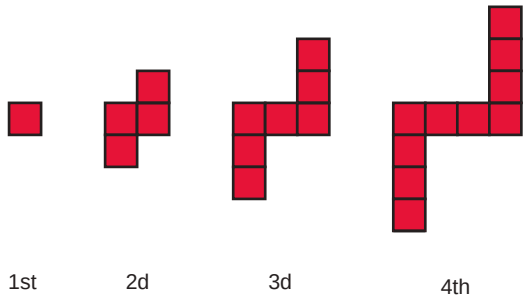


33. In driving from town A to town D, you pass first through town B and then through town C. It is 10 times farther from town A to town B than from towns B to C and 10 times farther from towns B to C than from towns C to D. If it is 1332 miles from towns A to D, how far is it from towns A to B? (Let the variable x represent the distance from towns C to D.)

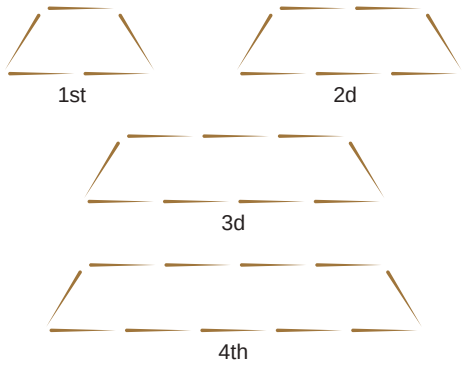


34. The cost of a bottle of perfume, \$28.90, was determined from the cost of the bottle plus the cost of the perfume. If the perfume costs \$14.10 more than the bottle, how much does the bottle cost? (Let the variable x represent the cost of the bottle.)

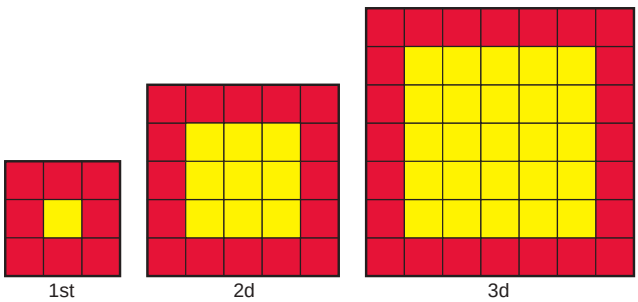
35. If the following tile figures are continued, will there be a figure with 8230 tiles? If so, which figure will it be? If not, what figure has the number of tiles closest to 8230? (*Hint:* Write an algebraic expression for the nth figure and set it equal to 8230.)



36. Find a pattern to extend the toothpick figures. Determine the number of toothpicks in the 50th figure, and write an algebraic expression for the number of toothpicks in the nth figure.



37. Here are the first three figures of a sequence formed by color tiles.



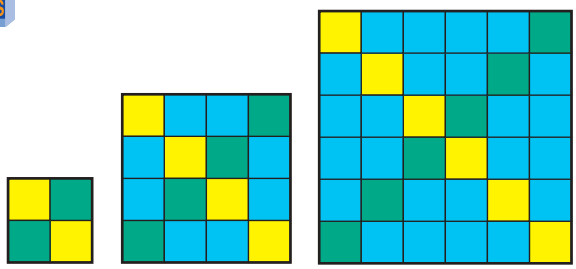
a. Find a pattern and describe the next two figures in the sequence. Then determine how many tiles of each color are used in the fifth figure. Answer this question for each of the first four figures.

b. Describe the 100th figure. Include the number of each color of tile and the total number of tiles in the figure.

c. Write algebraic expressions for the nth figure for (1) the number of yellow tiles, (2) the number of red tiles, and (3) the total number of tiles.

38. These three figures were formed by color tiles.

PS



a. Find a pattern and describe the next two figures in the sequence. How many tiles of each different color are in the fifth figure? Answer this question for each of the first four figures.

b. Describe the 100th figure and determine (1) the number of yellow tiles, (2) the number of green tiles, (3) the number of blue tiles, and (4) the total number of tiles.

c. Write algebraic expressions for the nth figure for (1) the number of yellow tiles, (2) the number of green tiles, (3) the number of blue tiles, and (4) the total number of tiles.

Teaching Questions

1. Suppose one of your students wrote the following statement in his math journal: "I do not understand what an equation is." Explain how you would help him with his understanding.

2. Suppose that your elementary school math text had a page labeled ALGEBRA with exercises for the students like the following:

$$\square + \Delta = 15, \quad \square + 4 = 9, \quad 3 \times \square = 12, \quad 14 - \square = 8$$

If your students asked why this was algebra, how would you answer their question?

3. The following type of question has been used to assess algebraic thinking: "On a certain field trip there were 8 times as many students as chaperones. If c represents the number of chaperones and s represents the number of students, what equation represents the given information?" When answering this question, students are sometimes confused over which two equations, $8c = s$ or $8s = c$, to use. Write an activity or series of questions that would help middle school students understand which of these two equations is correct.

Classroom Connections

1. On page 39 the example from the **Elementary School Text** illustrates the solution of an equation involving a variable. (a) How is this method of solution similar to finding solutions with the balance-scale model from this section? (b) Does either the model on this elementary text page or the balance-scale model in this section help students overcome the difficulties cited in the **Standards** and **Research** statements on page 42? Explain.

2. The **Standards** quote on page 47 discusses two central themes of algebraic thinking. Give an example of each of these themes. Explain.

3. Examine the **Algebra Standards** for the three different levels (see inside front and back covers) to see whether or not any standard suggests using models, like the balance scale, for algebra concepts. If so at what grade levels does this occur?

4. The **Historical Highlight** on page 38 features Amalie Emmy Noether, one of the greatest women mathematicians of her time and a pioneer in modern algebra. Search the Internet for more details on her life and accomplishments and write a paragraph with details that would be interesting to school students.

5. The **Standards** quote on page 38 notes that an understanding of variable needs to be grounded in experience, and the **Standards** quote on page 45 lists several types of experiences in which students should encounter expressions with variables. Give examples of a few of these experiences.

6. Explain how the weight of the whole brick on the balance scale in the **Problem Opener** for this section can be solved visually by using this model. Then solve the problem using algebra. Discuss the connections and similarity between the steps in the two solutions.

7. The **Standards** quote on page 37 notes that algebra is a strand in the curriculum from prekindergarten through high school. Give examples of what algebraic thinking is for children in the early grades. Write an explanation to convince a parent that algebra is appropriate for the early grades.

CHAPTER 1 REVIEW

1. Problem Solving

a. Problem solving is the process by which an unfamiliar situation is resolved.

b. Polya's Four-Step Process

Understanding the problem
Devising a plan
Carrying out the plan
Looking back

c. Problem-Solving Strategies

Making a drawing
Guessing and checking
Making a table
Using a model
Working backward
Finding a pattern
Solving a simpler problem
Using algebra

2. Unsolved Problems

a. There are many unsolved problems in mathematics.

b. A **conjecture** is a statement that has not been proved, yet is thought to be true.

3. Patterns and Sequences

a. There are many kinds of patterns. They are found by comparing and contrasting information.

b. The numbers in the sequence 1, 1, 2, 3, 5, 8, 13, 21, . . . are called **Fibonacci numbers.** The growth patterns of plants and trees frequently can be described by Fibonacci numbers.

c. Pascal's triangle is a triangle of numbers with many patterns. One pattern enables each row to be obtained from the previous row.

d. An **arithmetic sequence** is a sequence in which each term is obtained by adding a **common difference** to the previous term.

e. A **geometric sequence** is a sequence in which each term is obtained by multiplying the previous term by a **common ratio.**

f. The numbers in the sequence 1, 3, 6, 10, 15, 21, . . . are called **triangular numbers.**

g. Finite differences is a method of finding patterns by computing differences of consecutive terms.

4. Inductive Reasoning

a. Inductive reasoning is the process of forming conclusions on the basis of observations, patterns, or experiments.

b. A **counterexample** is an example that shows that a statement is false.

5. Variables

a. A letter or symbol that is used to denote an unknown number is called a **variable.**

b. An expression containing algebraic symbols is called an **algebraic expression.**

6. Equations

a. An **equation** is a sentence in which the verb is "equals" (=). It is a statement of the equality of mathematical expressions. The following are examples of equations: no variables, $17 + 5 = 22$; and one variable, $15x + 3 = 48$.

b. To **solve an equation** means to find values for the variable which make the equation true.

c. Two equations that have exactly the same solutions are called **equivalent equations.**

d. Properties of Equality

(1) Addition or Subtraction Property of Equality: Add the same number to or subtract the same number from both sides of an equation.

(2) Multiplication or Division Property of Equality: Multiply or divide both sides of an equation by the same nonzero number.

(3) Simplification: Replace an expression in an equation by an equivalent expression.

7. Inequalities

a. An **inequality** is a sentence that contains $<$, $\leq$, $>$, $\geqslant$, or $\neq$. It is a statement of the inequalities of mathematical expressions. The following are examples of inequalities: no variables, $12 < 50$; and one variable, $13x + 5 \geq 28$.

b. To **solve an inequality** means to find all the values for the variable that make the inequality true.

c. Two inequalities that have exactly the same solutions are called **equivalent inequalities.**

d. Properties of Inequality

(1) Addition or Subtraction Property of Inequality: Add the same number to or subtract the same number from both sides of an inequality.

(2) Multiplication or Division Property of Equality: Multiply or divide both sides of an inequality by the same nonzero number; and if this number is negative, reverse the inequality sign.

(3) Simplification: Replace an expression in an inequality by an equivalent expression.

CHAPTER 1 TEST

1. List Polya's four steps in problem solving.

2. List the eight problem-solving strategies that were introduced in this chapter.

3. The numbers in the following sums were obtained by using every other Fibonacci number (circled).

(1) 1 (2) 3 (5) 8 (13) 21 (34)

$$1 + 2 =$$
$$1 + 2 + 5 =$$
$$1 + 2 + 5 + 13 =$$
$$1 + 2 + 5 + 13 + 34 =$$

Compute these sums. What is the relationship between these sums and the Fibonacci numbers?

4. What is the sum of numbers in row 9 of Pascal's triangle, if row 2 has the numbers 1, 2, and 1?

5. Find a pattern in the following sequences, and write the next term.

a. 1, 3, 9, 27, 81
b. 3, 6, 9, 12, 15
c. 0, 6, 12, 18, 24
d. 1, 4, 9, 16, 25
e. 3, 5, 11, 21, 35

6. Classify each sequence in problem 5 as arithmetic, geometric, or neither.

7. Use the method of finite differences to find the next three terms in each sequence.

a. 1, 5, 14, 30, 55
b. 2, 9, 20, 35

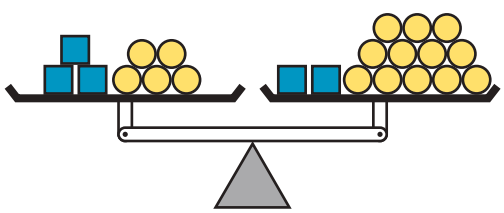
8. What is the fifth number in each of the following sequences of numbers?

a. Triangular numbers
b. Square numbers
c. Pentagonal numbers

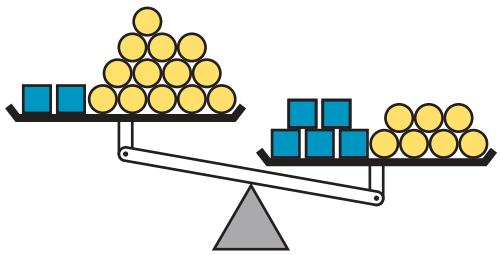
9. Find a counterexample for this conjecture: The sum of any seven consecutive whole numbers is evenly divisible by 4.

10. Determine the number of chips needed for each box in order for the scale to stay in the given position.

a.



b.



11. Solve each equation.

a. $3(x - 40) = x + 16$
b. $4x + 18 = 2(441 - 34x)$

12. Solve each inequality.

a. $7x - 3 < 52 + 2x$
b. $6x - 46 > 79 - 4x$

13. Name the property of equality or inequality used to obtain the new expression in each step.

a.
$$15x - 217 = 2x + 17$$
$$15x - 217 - 2x = 2x + 17 - 2x \text{ (step 1)}$$
$$13x - 217 = 17 \text{ (step 2)}$$
$$13x - 217 + 217 = 17 + 217 \text{ (step 3)}$$
$$13x = 234 \text{ (step 4)}$$
$$\frac{13x}{13} = \frac{234}{13} \text{ (step 5)}$$
$$x = 18 \text{ (step 6)}$$

b.
$$38x < 5(23 + 3x)$$
$$38x < 115 + 15x \text{ (step 1)}$$
$$38x - 15x < 115 + 15x - 15x \text{ (step 2)}$$
$$23x < 115 \text{ (step 3)}$$
$$\frac{23x}{23} < \frac{115}{23} \text{ (step 4)}$$
$$x < 5 \text{ (step 5)}$$

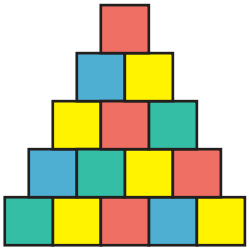
Solve problems 14 through 19, and identify the strategy or strategies you use.

14. A 2000-foot-long straight fence has posts that are set 10 feet on center; that is, the distance between the centers of two adjacent poles is 10 feet. If the fence begins with a post and ends with a post, determine the number of posts in the entire fence.

15. In a game of chips, Pauli lost half her chips in the first round, won 50 chips, then lost half her total, and finally won 80 chips. She finished with 170 chips. How many chips did she have at the beginning of the game?

16. The following tower has 5 tiles along its base and 5 rows of tiles. How many tiles will be required to build

a tower like this with 25 tiles along its base and 25 rows of tiles? Write an algebraic expression for the number of tiles in a tower with n tiles along its base and n rows of tiles.



17. Shown below are the first three squares in a pattern. Each square has one more dot on each side than the previous square.

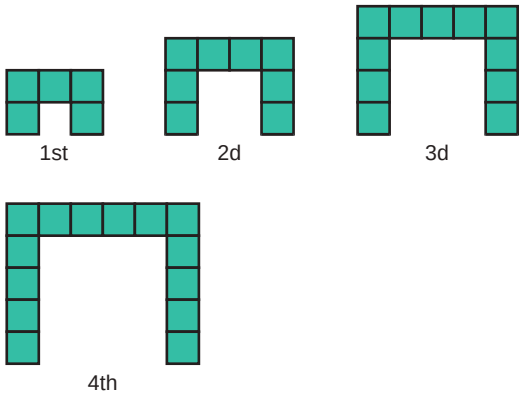


a. How many dots are there in the fourth square?
b. How many dots are there in the 50th square?
c. Write an algebraic expression for the number of dots in the nth square.

18. There are 78 people around a table. Each person shakes hands with the people to his or her immediate right and left. How many handshakes take place? Write an algebraic expression for the number of handshakes, if there are n people around the table.

19. Two men and two boys want to cross a river, using a small canoe. The canoe can carry two boys or one man. What is the least number of times the canoe must cross the river to get everyone to the other side?

20. Find a pattern to extend the following figures of tiles.
a. How many tiles will there be in the 5th figure? The 150th figure?
b. Write an algebraic expression for the number of tiles in the nth figure.



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CHAPTER

2 Sets, Functions, and Reasoning

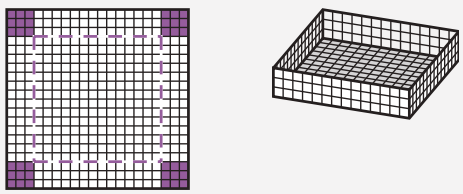
Spotlight on Teaching

Excerpts from NCTM's Standards 2 and 3 for Teaching Mathematics in Grades 5–8*

Reasoning is fundamental to the knowing and doing of mathematics. . . . To give more students access to mathematics as a powerful way of making sense of the world, it is essential that an emphasis on reasoning pervade all mathematical activity. Students need a great deal of time and many experiences to develop their ability to construct valid arguments in problem settings and evaluate the arguments of others.

. . . As students' mathematical language develops, so does their ability to reason about and solve problems. Moreover, problem-solving situations provide a setting for the development and extension of communication skills and reasoning ability. The following problem illustrates how students might share their approaches in solving problems:

The class is divided into small groups. Each group is given square pieces of grid paper and asked to make boxes by cutting out pieces from the corners. Each group is given a 20×20 sheet of grid paper. See figure [below]. Students cut and fold the paper to make boxes sized $18 \times 18 \times 1$, $16 \times 16 \times 2$, . . . , $2 \times 2 \times 9$. They are challenged to find a box that holds the maximum volume and to convince someone else that they have found the maximum. . . .



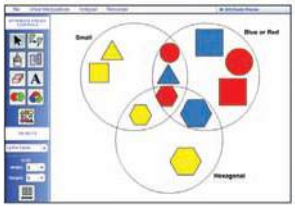
Building a grid-paper box.

Curriculum and Evaluation Standards for School Mathematics, p. 80.

MATH ACTIVITY 2.1

Sorting and Classifying Attribute Pieces

Virtual Manipulatives

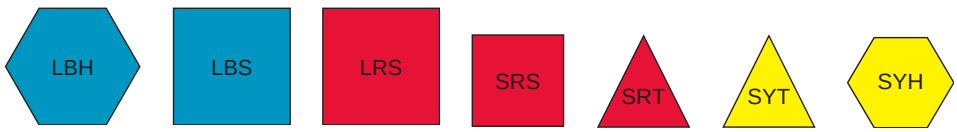


www.mhhe.com/bbn

Purpose: Use attribute pieces to sort and classify according to shared characteristics.

Materials: Attribute Pieces in the Manipulative Kit or Virtual Manipulatives.

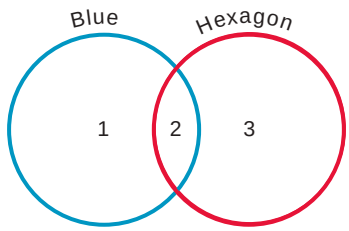
*1. Each attribute piece in the following sequence differs from the preceding piece by exactly one of the attributes of shape, color, or size. As examples, the large blue square differs from the large blue hexagon by shape; the large red square differs from the small red square by size; and so forth. Place five of your attribute pieces in a sequence that continues the one-difference pattern below and does not repeat any piece that has been used. List your five pieces in order. (You may wish to abbreviate large blue hexagon as LBH, small red triangle as SRT, etc.)



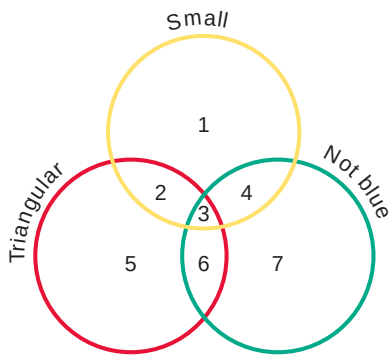
2. In the following sequence, each attribute piece differs from the preceding one by exactly two attributes. Use as many of your attribute pieces as possible to continue this sequence. List your pieces in order, and note the ones, if any, that did not fit at the end of the sequence.



*3. Draw two large circles on paper (or form them with string), and label them as shown here. The circle labeled *Blue* should have only blue attribute pieces inside. The circle labeled *Hexagon* should contain only hexagonal pieces. The overlapping region should have the attribute pieces that are both blue and hexagonal. Place your attribute pieces in the appropriate regions of the circles. List the attribute pieces for each region. Place the remaining pieces outside the circles and list them.

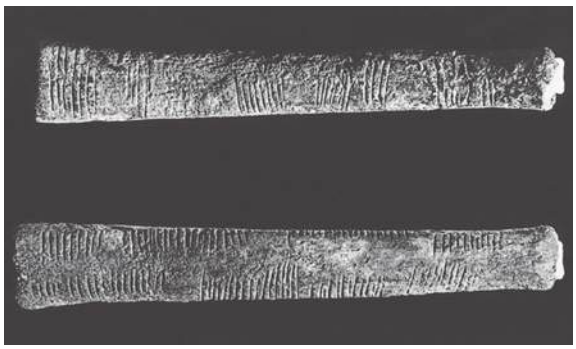


4. Draw three large circles, labeled as shown. Place the appropriate attribute pieces in the seven numbered regions and the remaining pieces in the region outside the circles. List the attribute pieces for each of the eight regions.



Section **2.1**

SETS AND VENN DIAGRAMS



Two views of the Ishango bone, found on the shores of Lake Edward in the Congo

PROBLEM OPENER

Melvin has 12 blue buttons and 16 buttons with four holes in his button box. What are the possibilities for the number of buttons Melvin has in his box?

Long before there were systematic ways of representing numbers, numerical records were kept by means of tallies. Archaeologists have unearthed thousands of animal bones marked with groups of notches, which date from prehistoric times. Some anthropologists conjecture that many of these ancient bones are records of days, months, and seasons. One example is the 8000-year-old Ishango bone, which was discovered in East Africa. The marks on this bone occur in several groups that are arranged in columns (see exercises 3 and 4 in Exercises and Problems 2.1).

HISTORICAL HIGHLIGHT



Art symbols dating from 12,000 B.C.E. found in the El Castillo caves in Spain

During the Old Stone Age (10,000–15,000 B.C.E.), figures of people and animals and abstract symbols were painted in caves in Spain and France. The symbols were composed of many geometric forms: straight lines, spirals, circles, ovals, and dots. The rows of dots and rectangular figures in this photograph were discovered on the walls of the El Castillo caves, in Spain, and date from 12,000 B.C.E. Some scholars conjecture that these symbols made up a system for recording the days of the year. It seems likely that numbers were in existence by this time. At first, it may only have been necessary to distinguish among one, two, and many objects. The first words for numbers were probably associated with specific things. This influence can be seen in the expressions we have for *two,* such as a *couple* of people, a *brace* of hens, and a *pair* of shoes. Eventually the concepts of twoness, threeness, etc., were separated from physical objects, and the abstract notion of *number* developed.

NCTM Standards

Young children are motivated to count everything . . . and through their repeated experience with the counting process, they learn many fundamental number concepts. p. 79

Keeping a tally involves matching sets of objects and marks and is the beginning of the idea of counting. The importance of counting to sets was first recognized by the nineteenth-century mathematician Georg Cantor. He created a new field of mathematics called **set theory.** Cantor used sets to define numbers and, in particular, to develop the theory of infinite sets. Today, sets are one of the major unifying ideas in mathematics, and set terminology is commonly found in elementary school texts.

SETS AND THEIR ELEMENTS

There are many words for sets: a *flock* of birds, a *herd* of cattle, a *collection* of paintings, a *bunch* of grapes, a *group* of people, and a *pride* of lions, to name a few. Intuitively, we understand a **set** to be a collection of objects called **elements.**

There are two common methods of specifying a set. One is to **describe the elements** of the set with words, as in Example A.

EXAMPLE A

1. "The capitals of the six New England states"
2. "The multiples of 10 from 10 to 500"

The other method of specifying a set is to **list the elements** of the set. When this is done, the elements of the set are written between braces, as in Example B.

EXAMPLE B

1. {Augusta, Concord, Boston, Hartford, Providence, Montpelier}
2. {10, 20, 30, 40, 50, 60, 70, . . . , 490, 500}

If the set of elements is large, as in the set of numbers in Example B, we sometimes begin the list and then use three dots to show that the pattern continues.

It is possible to have a set with no elements. This set is called the **empty set** or **null set** and is denoted by the set braces with no elements between them, { }, or by the null set symbol $\varnothing$.

EXAMPLE C

The set of all whole numbers between 16 and 28 that can be divided evenly by 15 has no elements, so it can be denoted by { } or $\varnothing$.

It is customary to name sets using uppercase letters and to denote elements of sets by lowercase letters. If k is an **element** of set S, we write $k \in S$, and if it is **not an element of** S, we write $k \notin S$. As an example, if we use T to denote the set of numbers in Example B, $60 \in T$ and $55 \notin T$.

Venn Diagrams Sets are often pictured by using rectangles, circles, or other convenient figures. For example, in Figure 2.1, all whole numbers less than 100 are represented by the region inside the rectangle. All even whole numbers less than 100 are represented by the

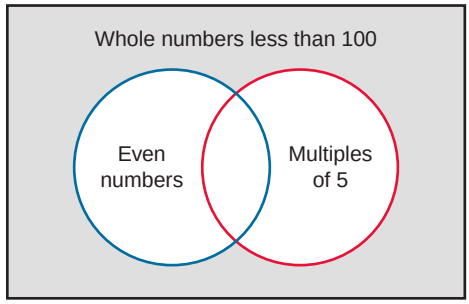


Figure 2.1

region inside the blue circle, and all whole numbers less than 100 that are multiples of 5 are represented by the region inside the red circle. Notice the overlapping regions of the two circles represent the set of all even whole numbers less than 100 that are multiples of 5. That is, the numbers 0, 10, 20, 30, . . . , 90 are common to both sets. Such figures for representing sets were first used by the Englishman John Venn (1834–1923) and are called **Venn diagrams.**

NCTM Standards

Sorting, classifying, and ordering facilitate work with patterns, geometric shapes, and data. Given a package of assorted stickers, children quickly notice many differences among the items. They can sort the stickers into groups having similar traits such as color, size, or design and order them from smallest to largest. p. 91

Attribute Pieces Attribute pieces are geometric models of various shapes, sizes, and colors that are commonly used in elementary schools for illustrating sets. As an example, the attribute pieces in Figure 2.2 have three attributes: size, shape, and color. There are three different shapes: triangular, square, and hexagonal. There are two sizes: large and small. There are two colors: blue and yellow.

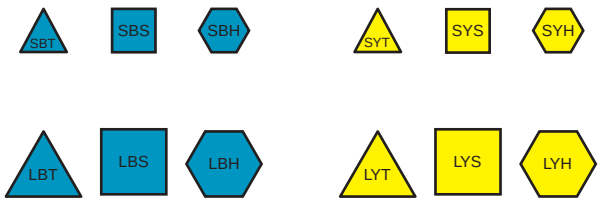


Figure 2.2

These objects can be classified into sets in many different ways. Here are a few possibilities:

EXAMPLE D

S is the set of small attribute pieces.

L is the set of large attribute pieces.

T is the set of triangles.

H is the set of hexagons.

Y is the set of yellow attribute pieces.

BT is the set of blue triangles.

The attribute pieces will be used in the following paragraphs to illustrate set relationships and operations. You may find it helpful to use the Attribute Pieces from your Manipulative Kit with the examples.

RELATIONSHIPS BETWEEN SETS

There are several ways in which sets may be related to one another. For example, two sets may have no elements in common (disjoint) or all elements in common (equal), or one set may be contained in another (subset). Let's look at some examples.

Disjoint Sets The two sets of attribute pieces in Figure 2.3 have no elements in common. When two sets do not have any of the same elements, we say the two sets are **disjoint.** Thus, the sets in Figure 2.3 are disjoint. The two sets in Figure 2.1, the even numbers and the multiples of 5, are **not disjoint** because they have elements in common.

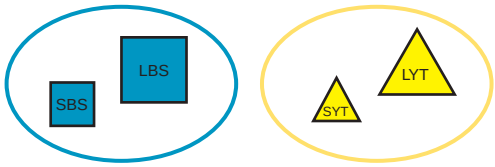


Figure 2.3

EXAMPLE E

Which of the following pairs of sets of attribute pieces are disjoint? (*Hint:* List the elements of each set, and check to see if they have any elements in common.)

1. *L* (large pieces), *S* (small pieces)
2. *S* (small pieces), *Y* (yellow pieces)
3. *SH* (small hexagons), *BT* (blue triangles)

Solution **1.** Sets *L* and *S* are disjoint. **2.** Sets *S* and *Y* have small yellow pieces in common; they are not disjoint. **3.** Sets *SH* and *BT* are disjoint.

Subsets Figure 2.4 shows that every attribute piece in set *BT* (blue triangles) is also in set *T* (triangles). In this case we say that *BT* is a **subset** of *T*.

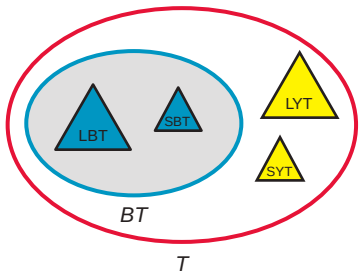


Figure 2.4

If every element of set *A* is also an element of set *B*, then set *A* is a **subset** of *B*. This relationship is written $A \subseteq B$. If set *A* is **not a subset** of set *B*, we write $A \not\subseteq B$.

EXAMPLE F

In which of the following pairs of sets of attribute pieces is the first set a subset of the second?

1. *LSQ* (large squares), *SQ* (squares)
2. *T* (triangles), *S* (small pieces)

Solution **1.** *LSQ* is a subset of *SQ*, $LSQ \subseteq SQ$. **2.** *T* is not a subset of *S* because *T* has both large and small triangles, $T \not\subseteq S$.

According to the definition of subset, every set is a subset of itself. For example, $BT \subseteq BT$, $T \subseteq T$, and $H \subseteq H$, because every element in the first set is also in the second set. If we know that $A \subseteq B$ and that one or more elements of B are not in A, then A is called a **proper subset** of B and we write $A \subset B$. For example, BT, the set of blue triangles in Figure 2.4, is a proper subset of T, the set of blue triangles and yellow triangles.

Equal Sets Sets that contain the same elements are called **equal sets.** Sometimes two sets may look different or have different descriptions, but they may represent the same elements. Consider the set E of even whole numbers and the set D of whole numbers that are divisible by 2. Since every even whole number is divisible by 2, the numbers in set E are contained in set D. Conversely, since every whole number that is divisible by 2 is an even number, the numbers in set D are contained in set E. Thus, the two sets have the same elements and are equal.

If A is a subset of B and B is a subset of A, then both sets have exactly the same elements and they are **equal.** This relationship is written $A = B$. In this case, A and B are just different letters naming the same set. If set A is **not equal** to set B, we write $A \neq B$.

One-to-One Correspondence It is possible to match the elements in the set SB (small blue) in Figure 2.5 with those in the set LY (large yellow) so that for each element in SB there is exactly one element in LY and, conversely, for each element in LY there is exactly one element in SB. We refer to this fact by saying that the two sets can be put into **one-to-one correspondence,** or that they are **equivalent sets.**

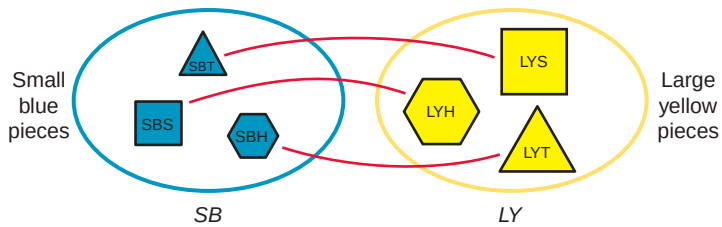


Figure 2.5

Technology Connection

Consecutive Numbers
Notice that 18 can be written as the sum of nonzero consecutive whole numbers in two ways: $3 + 4 + 5 + 6$ and $5 + 6 + 7$. Can all whole numbers greater than 1 be written as the sum of nonzero consecutive whole numbers? Use the online 2.1 Mathematics Investigation to gather data and help you to answer this question.

Mathematics Investigation
Chapter 2, Section 1
www.mhhe.com/bbn

The concepts of number and counting are extensions of the idea of one-to-one correspondence. If two sets can be put into one-to-one correspondence, we say they have the **same number** of elements. To **count** the elements of a set, we match these elements with the whole numbers 1, 2, 3, 4, ... Adults will often point to the objects being counted, and children will sometimes touch each object as they match the objects and whole numbers.

The number of elements in the set of small blue pieces in Figure 2.5 is 3, because this set can be put into one-to-one correspondence with $\{1, 2, 3\}$. If there is a whole number n such that the elements of a set can be matched one to one with the whole numbers $1, 2, 3, \ldots, n$, the set has ***n* elements** and the set is called **finite.** For example, the set of small blue attribute pieces is a finite set. If a set is not finite, it is called **infinite.** Informally, a set is infinite if its elements go on without end. The set of all whole numbers is an example of an infinite set.

A set is **finite** if it is empty or if it can be put into one-to-one correspondence with set $\{1, 2, 3, \ldots, n\}$, where n is a whole number. A set is **infinite** if it is not finite.

OPERATIONS ON SETS

There are operations that replace two sets by a third set, just as there are operations on numbers that replace two numbers by a third number. *Addition* and *multiplication* are examples of operations on whole numbers; *intersection* and *union* are operations on sets.

Intersection of Sets Figure 2.6 shows that the set of small attribute pieces and the set of blue attribute pieces have three elements in common. If we form a third set containing these common elements, it is called the **intersection** of the two sets.

NCTM Standards

Teachers should try to uncover students' thinking as they work with concrete materials by asking questions that elicit students' thinking and reasoning. p. 80

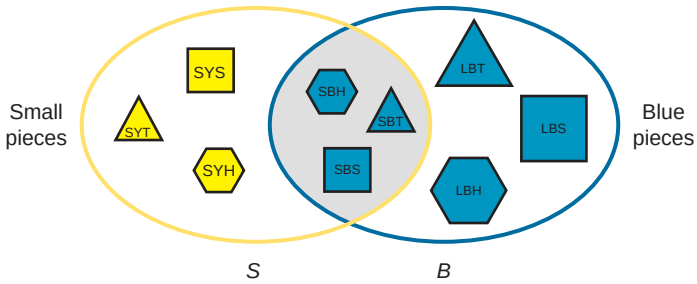


Figure 2.6

> The **intersection** of two sets A and B is the set of all elements that are in both A **and** B. This operation is written $A \cap B$.

The intersection of the two sets in Figure 2.6 is the set of attribute pieces that are small and blue. This new set is indicated by shading the common region inside the curves. We write the intersection of these two sets as

$$S \cap B = \{SBS, SBT, SBH\}$$

EXAMPLE G

Find the intersection of these sets of attribute pieces. (You may find it helpful to form two large overlapping circles and place attribute pieces inside the appropriate regions.)

1. L (large pieces), H (hexagons)
2. ST (small triangles), BH (blue hexagons)
3. SSQ (small squares), S (small pieces)

Solution **1.** $L \cap H = \{LBH, LYH\}$ **2.** $ST \cap BH = \varnothing$, because these sets are disjoint. **3.** $SSQ \cap S = SSQ$, because SSQ is a subset of S.

The key word in the definition of intersection is *and.* In everyday use, as well as in mathematics, the word ***and*** means that two conditions must be satisfied. For example, if you are required to take the Graduate Record Examination (GRE) *and* the Miller Analogies Test (MAT), you must take both tests.

Research Statement

Developing meaning for mathematical symbols is essential for using these symbols effectively.

Hiebert and Carpenter

Union of Sets The set of small attribute pieces and the set of blue attribute pieces have three pieces in common (Figure 2.7). A new set containing these three pieces and all other pieces within either set is called the **union** of the two sets.

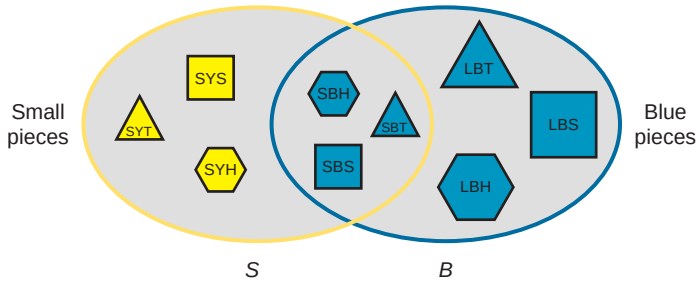


Figure 2.7

> The **union** of two sets A and B is the set of all elements that are in A **or** in B **or** in both A and B. This operation is written $A \cup B$.

The union of the two sets in Figure 2.7 is the set of all attribute pieces that are small or blue or both small and blue. This new set is indicated by shading the total region inside the two curves. We write the union of these two sets as

$$S \cup B = \{SYS, SYH, SYT, SBT, SBS, SBH, LBT, LBS, LBH\}$$

EXAMPLE H

Find the union of these sets of attribute pieces.

1. L (large pieces), H (hexagons)
2. ST (small triangles), BH (blue hexagons)

Solution 1. $\{LBT, LYT, LBS, LYS, LBH, LYH, SBH, SYH\}$ 2. $\{SBT, SYT, LBH, SBH\}$

Notice that the solution for 1 in Example H contains *LBH* and *LYH* only once, even though these two attribute pieces are contained in both sets.

The key word in the definition of union is *or.* This word has two different meanings. In everyday use, *or* usually means that it is necessary to satisfy one condition or the other, *but not both.* For example, "You must take the course *or* pass the qualifying exam" means that you must do one of these two things but not necessarily both. This is called the **exclusive *or.*** In mathematics, however, the word *or* usually means that one condition or the other condition, or both, may be satisfied. This is called the **inclusive or.** The inclusive *or* is used in defining the union of sets because an element in the union of two sets may be in the first set or in the second set or in both sets.

Sometimes we wish to consider more than two sets at a time. The Venn diagram in Figure 2.8 shows three sets of attribute pieces: Y (yellow pieces), S (small pieces), and H (hexagonal pieces). This diagram can be used to determine the combinations of operations in Example I.

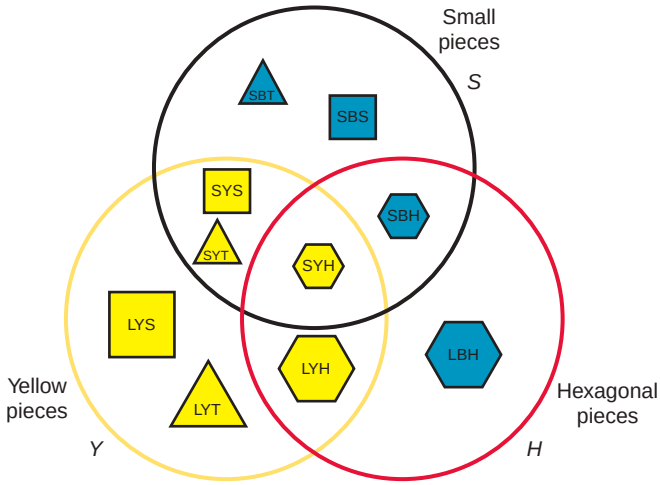


Figure 2.8

EXAMPLE 1

Use Figure 2.8 to list the elements in each of the following sets. (First determine the set in parentheses. You may find it helpful to shade the regions of the diagram in Figure 2.8.)

1. $(Y \cup S) \cup H$ **2.** $(Y \cap S) \cap H$

3. $(Y \cup S) \cap H$ **4.** $Y \cup (S \cap H)$

Solution **1.** {*LYS, LYT, LYH, LBH, SYS, SYT, SYH, SBS, SBT, SBH*} **2.** {*SYH*} **3.** {*LYH, SYH, SBH*} **4.** {*LYS, LYT, LYH, SYS, SYT, SYH, SBH*}

Complement of a Set The word *complement* has the same meaning in mathematics as in everyday use. If you know that 11 people are on their way to your house and only 7 arrive, you might ask, "Where is the complement?" That is, where are the rest of the people who make up the whole group? Consider the set of small blue attribute pieces (grey shaded region) and the set of remaining pieces (red shaded region) in Figure 2.9. The small blue pieces are inside the circle, and the others are outside. These two subsets (the grey shaded region and the red shaded region) are called **complements** of each other because together they make up the whole set. That is, their union is the whole set.

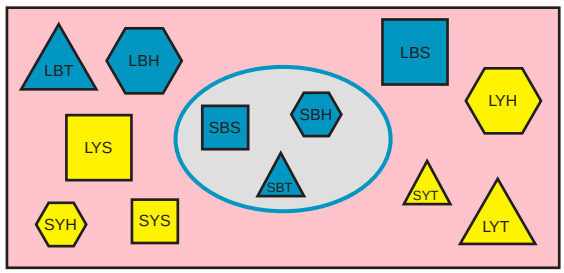


Figure 2.9

For any given set U, if two subsets A and B are disjoint and their union is U, then A and B are **complements** of each other. This is written: The complement of set B is A ($B' = A$); and the complement of set A is B ($A' = B$).

EXAMPLE J

Set *SB* is the set of small blue attribute pieces. The set of all the other attribute pieces—that is, those that are not (small and blue)—is the complement of *SB* (Figure 2.9).

$$SB = \{SBS, SBT, SBH\}$$

$$\text{Complement of } SB\ (SB') = \{LBT, LBH, LBS, LYT, LYH, LYS, SYT, SYS, SYH\}$$

The "given set *U*" referred to in the previous definition is sometimes called the **universal set.** We have been using a universal set of 12 attribute pieces in Examples D through J. In word problems involving whole numbers, the universal set is often the set of whole numbers.

EXAMPLE K

Use the set of whole numbers as the universal set to determine the following complements.

1. What is the complement of the set of even whole numbers?
2. What is the complement of the set of whole numbers that are less than 10?

Solution 1. The set of odd whole numbers. 2. The set of whole numbers greater than or equal to 10.

The universal set can be any set, but once it is established, each subset has a unique (one and only one) complement. In other words, *complement* is an operation that assigns each set to another set, namely, its complement.

PROBLEM-SOLVING APPLICATION

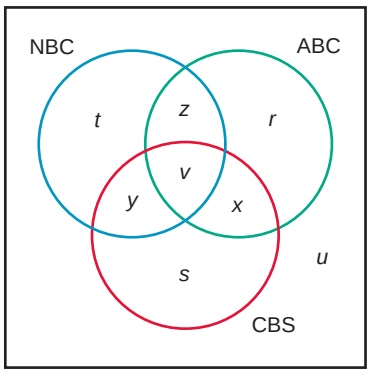
Drawing Venn diagrams is a problem-solving strategy for sorting and classifying information. Try solving the following problem by using the information given in the table and drawing three overlapping circles, one for each of the three networks.

Problem

A survey of 120 people was conducted to determine the numbers who watched three different television networks. The results are shown in the following table. How many of the 120 people did not watch any of the three networks?

Networks	Number of People
ABC	55
NBC	30
CBS	40
ABC and CBS	10
ABC and NBC	12
NBC and CBS	8
NBC and CBS and ABC	5

Understanding the Problem The Venn diagram in the following figure shows three circles, one to represent each of the three networks. Each of the seven regions inside the circles represents a different category of viewers. For example, people in region y watched NBC and CBS but not ABC. **Question 1:** What region represents the people who did not watch any of the three networks? We need to find the number of people in this region.



Devising a Plan We can find the number of people who did not watch any of the three networks by first finding the numbers for the seven regions inside the circles and then subtracting this total from 120. It is generally useful to begin with the innermost region and work outward. For example, v is the intersection of all three circles, and the table shows that $v = 5$. Using this number and the fact that there are eight people in the intersection of NBC and CBS, we can determine the value of y. **Question 2:** What is the value of y?

Carrying Out the Plan Continuing the process described in the previous paragraph, we can determine that $z = 7$ and $x = 5$. Now since there are 40 people represented inside the CBS circle and $v + y + x = 13$, we know that $s = 40 - 13 = 27$. In a similar manner we can determine that $r = 38$ and $t = 15$. So the total number of people represented by the seven regions is

$$\begin{matrix} v & y & z & x & s & r & t \\ 5 & + 3 & + 7 & + 5 & + 27 & + 38 & + 15 = 100 \end{matrix}$$

Question 3: How many people did not watch any of the three networks?

Looking Back We solved this problem by finding the number of people in the union of three sets and then finding the number of people in the complement. In addition to solving the original problem, the Venn diagram provides much more information. For example, since $s = 27$, we know 27 people watched only CBS. **Question 4:** How many people watched both NBC and ABC but not CBS?

Answers to Questions 1–4 **1.** The region labeled u, which is inside the rectangle but outside the union of the three circles. **2.** $y = 3$ ($y + v = 8$, so $y + 5 = 8$) **3.** 20 ($120 - 100 = 20$) **4.** 7 ($12 - 5 = 7$)

8-3 Problem-Solving Strategy

MAIN IDEA I will solve problems by using logical reasoning.

Of the students in a fifth grade class, 15 play basketball and 18 play soccer. Three of those students play both sports. How many students play only basketball? only soccer?



Understand

What facts do you know?

- We know how many students play basketball, play soccer, and play both sports.

What do you need to find?

- The number of students that play only basketball and only soccer.

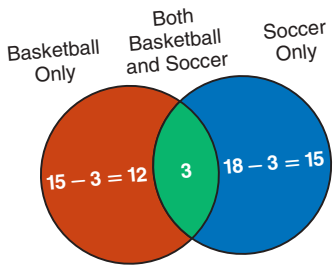
Plan

One way to solve the problem is to make a Venn diagram. A **Venn diagram** uses overlapping circles to show how common elements among sets of objects are related. The overlapping section shows the objects that are in both groups.

Solve

Draw two overlapping circles to represent the two different sports. Since 3 students are involved in both sports, place a 3 in the overlapping section. Use subtraction to determine the number for each of the other sections.

only basketball: $15 - 3 = 12$
only soccer: $18 - 3 = 15$



Check

Look back. Check each circle to make sure that the correct number of students is represented. ✓

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HISTORICAL HIGHLIGHT



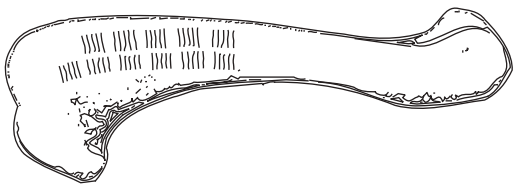
Grace Chisholm Young, 1868–1944

Grace Chisholm Young was born in England at a time when education for women was restricted to Bible reading and training in the homely arts. Mathematics and the classics were considered unsuitable subjects for women. Young's only formal education was the tutoring she received at home, but this was sufficient for her to pass the Cambridge Senior Examination. In 1893, she completed her final examinations and qualified for a first-class degree at Cambridge. Since women were not yet admitted to graduate schools in England, Young went to the University of Göttingen, Germany, the major center for mathematics in Germany, where Felix Klein was her adviser. Her outstanding work earned her a doctorate in mathematics, the first official degree granted to a woman in Germany on any subject. Her subsequent mathematical work was productive and creative. With her husband she coauthored the first textbook on set theory, a classic work in its field. Her *First Book of Geometry,* although published in 1905, looks surprisingly contemporary. She advocated that three-dimensional geometry be taught earlier in schools and that students fold patterns to form solids as an aid in visualizing theorems in solid geometry.*

*T. Perl, *Math Equals* (Reading, MA: Addison-Wesley Publishing Company, 1978), pp. 149–171.

Exercises and Problems 2.1

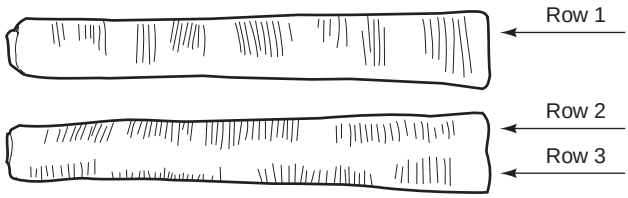
The notches in the 30,000-year-old Czechoslovakian wolf bone are arranged in two groups. There are 25 notches in one group and 30 in the other. Within each series the notches are in groups of 5. Use this information in exercises 1 and 2, and give reasons for your conclusions.



1. Could this system have been devised without number names?

2. Could this system have been devised without number symbols?

Both sides of the 8000-year-old Ishango bone are sketched in the next column. There is one row of marks on one side of the bone, and there are two rows of marks on the other side. Anthropologists have questioned the significance of the number of marks: Could they be records of game killed or of belongings? Maybe they are intended to show a relationship between numbers. Write the number of marks in each group in these rows. Use the results to answer exercises 3 and 4.



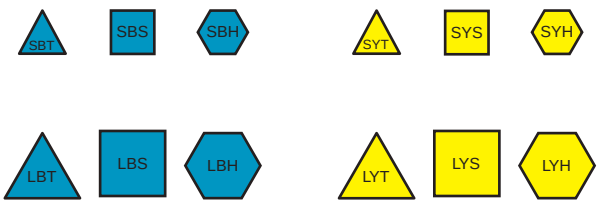
3. a. Which of these rows suggests a knowledge of multiplication by 2?

b. Find some other number relationships.

4. In his book *The Roots of Civilization,* Alexander Marshack correlates these marks† with the phases of the moon and the days of a lunar calendar. Using 28 for the number of days in a lunar month, how many months are represented by the total number of marks on this bone?

†For a discussion of these marks, see A. Marshack, *The Roots of Civilization.*

Use the universal set of 12 attribute pieces below and the following sets to answer questions 5 through 14: Y, yellow attribute pieces; H, hexagonal attribute pieces; SY, small yellow pieces; SB, small blue pieces; and L, large pieces. (You may find it helpful to use the Attribute Pieces from your Manipulative Kit and place them in Venn diagrams.)



5. Which pairs of sets, if any, can be put into one-to-one correspondence?

6. Which pairs of sets, if any, are equal?

7. Which pairs of sets, if any, are disjoint?

8. Which set is a proper subset of another?

9. Which attribute pieces are in $Y \cap L$?

10. Which attribute pieces are in $Y \cup L$?

Which of the statements in exercises 11 and 12 are true?

11. **a.** $LBH \in L \cap Y$
b. $LYT \in H \cup L$
c. $SYS \in H'$

12. **a.** $SYH \in Y \cap H$
b. $LYT \in L'$
c. $LBS \in SB \cup Y$

List the pieces described in exercises 13 and 14. (Use the *inclusive or.*)

13. **a.** Hexagonal and small
b. Not (hexagonal and small)
c. Small or yellow

14. **a.** Triangular or large
b. Yellow and triangular
c. Not (small or yellow)

Given the universal set $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and sets $A = \{0, 2, 4, 6, 8\}$, $B = \{1, 3, 5, 7\}$, and $C = \{3, 4, 5, 6\}$, list the elements in the sets described in exercises 15 and 16.

15. **a.** $A \cap C$ **b.** $C' \cup B$

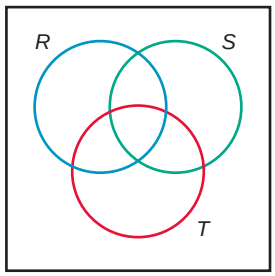
16. **a.** $C' \cap A$ **b.** $(A \cap C) \cup B$

Draw a Venn diagram for each part of exercises 17 and 18 so that for sets A, B, and C, all the given conditions are satisfied.

17. **a.** $A \subseteq B, B \subseteq C$
b. $C \cap B = \varnothing, A \subseteq C$
c. $(B \cup C) \subseteq A, B \cap C = \varnothing$

18. **a.** $A \cap B \neq \varnothing, B \cap C \neq \varnothing, A \cap C = \varnothing$
b. $(B \cup C) \subseteq A, B \cap C \neq \varnothing$
c. $C \subseteq A, (B \cap C) \subseteq A, A' \cap B \neq \varnothing$

Sketch a three-circle Venn diagram like the one shown here for each of the sets in exercises 19 and 20, and shade the region represented by the set.



19. **a.** $R \cap S$
b. $T \cup R$
c. $(R \cup S) \cap T$

20. **a.** $(T \cap S) \cup R$
b. $(R \cup T)'$
c. $(R \cap T) \cap S'$

Given that set A has 15 elements and set B has 13 elements, answer exercises 21 and 22. Draw a sketch of each set.

21. Wh.at is the maximum number of elements in $A \cup B$? in $A \cap B$?

22. What is the minimum number of elements in $A \cup B$? in $A \cap B$?

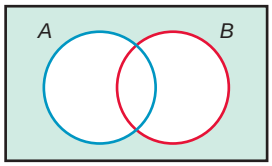
Illustrate the sets given in each part of exercises 23 and 24 by sketching a two- or three-circle Venn diagram and shading the figure to show the set.

23. **a.** $A \cap B'$
b. $A' \cup B$
c. $A \cup (C' \cup B)$
d. $(A \cap B') \cap C$

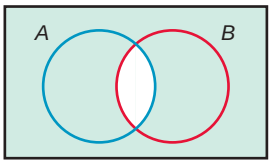
24. **a.** $A' \cup B'$
b. $A' \cap B'$
c. $A' \cup (C' \cap B)$
d. $(A \cap B')' \cup C$

Use set notation to identify the shaded region in each of the sketches in exercises 25 through 28.

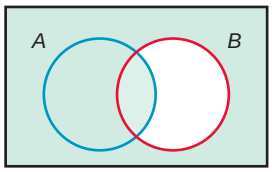
25. a.



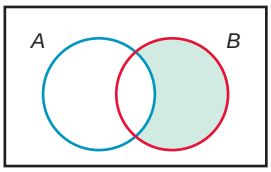
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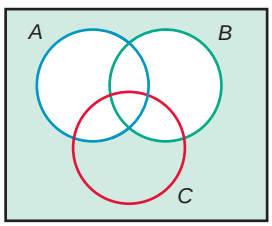
26. a.



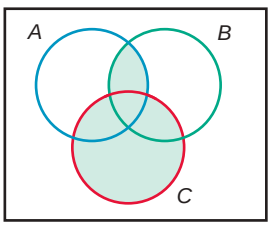
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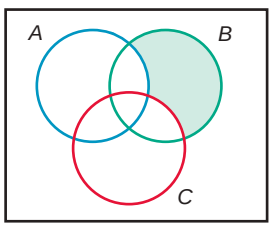
27. a.



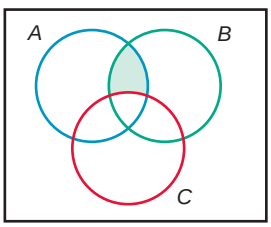
b.



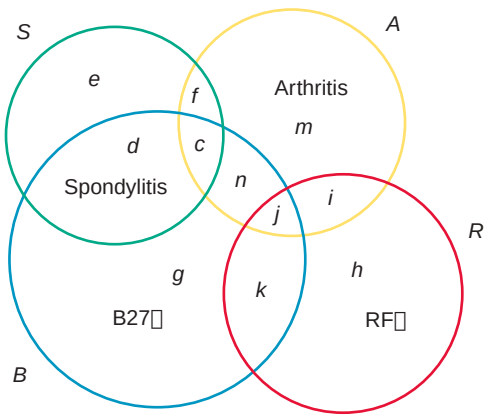
28. a.



b.



The following diagram of human populations was used in investigations correlating the presence or absence of B27+ (a human antigen), RF+ (an antibody protein), spondylitis (an inflammation of the vertebrae), and arthritis (an inflammation of the joints) with the incidence of various rheumatic diseases.



Find the letter(s) of the region(s) in the preceding diagram corresponding to each set in exercises 29 and 30.

29. a. $S \cap B$ **b.** $A \cap R$

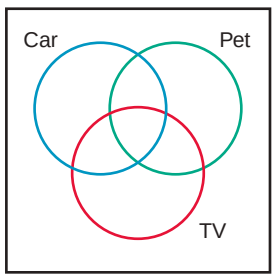
30. a. $(S \cap B) \cap A$ **b.** $(R \cup A) \cap B$

Reasoning and Problem Solving

31. In a music club with 15 members, 7 people played piano, 6 people played guitar, and 4 people didn't play either of these two instruments. How many people played both piano and guitar?

32. There were 55 people at a high school class reunion. If 16 people had college degrees, 12 people had college degrees and were married, and 14 people were single and did not have college degrees, how many people were married and did not have college degrees?

Use the following information in problems 33 and 34. In a survey of 6500 people, 5100 had a car, 2280 had a pet, 5420 had a television set, 4800 had a TV and a car, 1500 had a TV and a pet, 1250 had a car and a pet, and 1100 had a TV, a car, and a pet.



33. How many people had a TV and a pet, but did not have a car?

34. How many people did not have a pet or a TV or a car?

Use the following information in problems 35 and 36. A class survey found that 25 students watched television on Monday, 20 on Tuesday, and 16 on Wednesday. Of those who watched TV on only one of these days, 11 chose Monday, 7 chose Tuesday, and 6 chose Wednesday. Every student watched TV on at least one of these days, and 7 students watched on all three days.

35. If 12 students watched TV on both Monday and Tuesday, find the number of students in the class.

36. If 8 students watched TV on Tuesday and Wednesday, how many students watched TV on Monday or Tuesday but not on Wednesday?

37. An elementary school class polled 150 people at a shopping center to determine how many read the *Daily News* and how many read the *Sun Gazette.* They found the following information: 129 read the *Daily News,* 34 read both, and 12 read neither. How many read the *Sun Gazette*?

38. Of the 22 fast-food businesses in a small city, the numbers that have a drive-up window, outside seating, or delivery service are summarized as follows: 7 have delivery service; 15 have outside seating; 13 have a drive-up window; 9 have a drive-up window and outside seating; 3 have outside seating and delivery service; 3 have delivery service and a drive-up window; and 2 have all three services. How many of these businesses have only a drive-up window?

39. During spring registration at a midwestern liberal arts college, 442 students registered for English, 187 registered for history, and 234 registered for mathematics. What is the greatest possible total number of different students who could have registered for these courses, if it is known that only 96 registered for both English and mathematics?

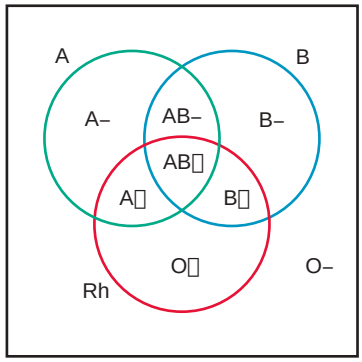
40. The police records of a city contain the following statistics on offenses for the month of May: 430 assaults, 146 robberies, and 131 drug sales. The records also show that 26 people were involved in both assault and robbery and that 33 people were involved in assault and drug sales. What is the greatest possible number of offenders for May who satisfy these statistics?

Use the following information in problems 41 and 42. There are eight blood types, shown by the Venn diagram. Each circle represents one of three antigens: A, B, or Rh. If A and B are both absent, the blood is type O. If Rh is present, the blood type is positive; otherwise, it is negative. The following table represents the blood types of 150 people.

Antigens	Number of People
A	60
B	27
Rh	123
A and B	12
B and Rh	17
A and Rh	46
A and B and Rh	9

How many people have the blood types given in 41 and 42?

41. a. B+ **b.** A+ **42. a.** O+ **b.** O−



Teaching Questions

1. A teacher asked a group of fourth-graders to find all the pieces from the 24-piece attribute set (4 shapes, 3 colors, 2 sizes) that are red or small. One student in the group said they should just find all the red pieces and then find all the small pieces that are left. The other three students in the group disagreed and said they should find all the red pieces that are not small and all the small pieces that are not red. Who was right? Explain how you would resolve this disagreement using attribute pieces.

2. Write an explanation of the similarities and differences between the subset symbols $\subseteq$ and $\subset$ to the inequality symbols $\leq$ and $<$ that would make sense to middle school students.

3. Do you think it is important for prospective schoolteachers to learn about set terminology and set operations? Make a case as to why or why not. If possible, check some elementary or middle school mathematics textbooks to obtain an idea of the extent to which you may be teaching the topic of working with sets.

4. It is always a convenient connection when mathematical words have the same meaning in everyday discourse. Which of the following words from this section have the same meaning in both mathematics and everyday discourse? For each word that has a different usage, write a definition of the math usage and at least one definition of the everyday usage. "Complement," "intersection," "union," "equal," "disjoint," "null," "and," "or."

Classroom Connections

1. On page 71 the problem from the **Elementary School Text** involves a Venn diagram and the intersection of sets. (a) Once the regions of the Venn diagram are filled

in with numbers, other questions can be posed and easily solved. Write another question involving the data. (b) If this problem is revised to include the additional information that there are 35 students in the class, how many students would play neither sport?

2. In the **PreK–2 Standards—Algebra** (see inside front cover) under *Understand patterns, relations* . . . , read the first expectation and cite examples of how this expectation is satisfied in Section 2.1.

3. A brief description of the life and accomplishments of Grace Chisholm Young is contained in the **Historical Highlight** on page 72. Research her life and list some of her other accomplishments in mathematics. Write a few facts about her life that would be of interest to elementary school students.

4. The **Standards** quote on page 63 notes that given a set of objects, children will naturally sort them by color, shape, and/or other attributes. Attribute pieces similar to those in the One-Page Math Activity at the beginning of this section are frequently found in schools. One common school activity involves placing the attribute pieces in rows to form one-difference trains and two-difference trains as in activities 1 and 2. Form and record the pieces for a three-difference train using as many pieces as possible.

5. Old Stone Age art (10,000–15,000 B.C.E.) from caves in Spain and France show many geometric figures, including rows of dots and other marks as shown in the **Historical Highlight** on page 61, that may have been systems for recording days of the year. Write and discuss some of the reasons why keeping track of the seasons may have been important in ancient times.

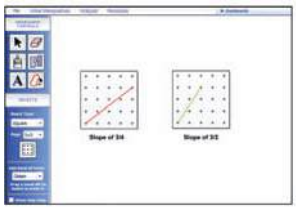
6. There are many triangular patterns of numbers, and one of these is shown below. If zero is considered as the first row in this triangle, find at least two different patterns that can be used to determine the numbers in the 50th row.

$$0$$
$$1 + 2$$
$$3 + 4 + 5$$
$$6 + 7 + 8 + 9$$
$$10 + 11 + 12 + 13 + 14$$

MATH ACTIVITY 2.2

Slopes of Geoboard Line Segments

Virtual Manipulatives

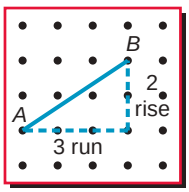


www.mhhe.com/bbn

Purpose: Explore slopes of line segments using a rectangular geoboard.

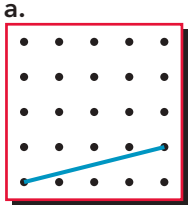
Materials: Copy Rectangular Geoboards from the website or use Virtual Manipulatives.

***1.** The slope of the line from point A to B on the geoboard below at the left is $\frac{2}{3}$. Notice that you can move from A to B by moving horizontally 3 spaces (called the **run**) and vertically 2 spaces (called the **rise**). These distances are the lengths of the legs of a right triangle. The **slope** of a line is the *rise* (vertical distance) divided by the *run* (horizontal distance) in moving from one point to another on the line. Sketch the following line segments on geoboard paper. Label a run, rise, and slope for each line segment. As the lines get "steeper," what happens to their numerical slopes?

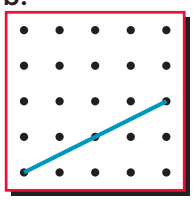


Slope of $\frac{2}{3}$

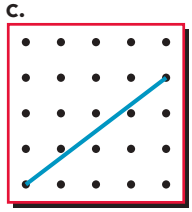
a.



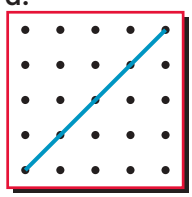
b.



c.

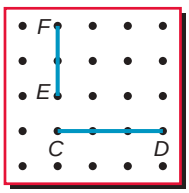


d.



2. Sketch line segments on geoboard paper for the following slopes: $\frac{1}{3}$, $\frac{3}{2}$, 2, $\frac{4}{3}$.

3. The line through points C and D on the geoboard at the left is horizontal, that is, has no "steepness." Using points C and D, we find its run is 3 and its rise is 0, and since $\frac{0}{3} = 0$, the slope of this line is 0. The line through points E and F has a rise of 2 and a run of 0 (there is no horizontal movement from point E to F). Since the formula for the slope is $\frac{\text{rise}}{\text{run}}$ and $\frac{2}{0}$ is not defined, we say that the slope of such line segments is not defined. Sketch line segments on geoboard paper that satisfy the following conditions.

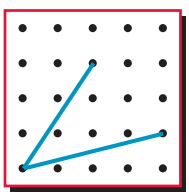


a. Length of 4 and slope of 0

b. Length of 3 and undefined slope

c. Length of 1 and undefined slope

d. Length of 5 and slope of $\frac{3}{4}$



***4.** There are 24 line segments, with 12 distinct slopes, that have the lower left point of the geoboard as an end point (two such segments are shown at the left). Sketch 12 line segments with different slopes, each with one end point the lower left point on the geoboard. Label each segment with its slope. You may find it helpful to use more than one geoboard for your sketches. (Remember, vertical line segments cannot be used since they have no slope.) Make a list of the 12 slopes, and write them in increasing order from smallest to greatest.

Section 2.2 FUNCTIONS, COORDINATES, AND GRAPHS



Eight-person skydivers' star

PROBLEM OPENER

In a guessing game called What's My Rule? team *A* makes up a rule, such as "double the number and add 1," and team *B* tries to guess the rule. To obtain information about the rule, team *B* selects a number *x* and members of team *A* use their rule on the number to obtain a second number *y*. Find a rule for each table of numbers.

x	1	2	8	5	0
y	8	13	43	28	3

x	5	1	6	2	9
y	26	2	37	5	82

x	20	8	3	7	1
y	59	23	8	20	2

NCTM Standards

Two concepts underlie every branch of mathematics: One is the *set,* and the other, which will be defined in this section, is the *function.* The *Curriculum and Evaluation Standards for School Mathematics* (p. 98) comments on the importance of functions:

> One of the central themes of mathematics is the study of patterns and functions. This study requires students to recognize, describe, and generalize patterns and build mathematical models to predict the behavior of real-world phenomena that exhibit the observed pattern. The widespread occurrence of regular and chaotic pattern behavior makes the study of patterns and functions important.

FUNCTIONS

The distance a skydiver falls is related to the time that elapses during the jump. By the end of 1 second, a skydiver has fallen 16 feet, and after 2 seconds, the distance is 62 feet. The distances for the first 10 seconds are shown in the following table.

Distance Fallen in Free-Fall Stable-Spread Position

Seconds	Distance, Feet	Seconds	Distance, Feet
1	16	6	504
2	62	7	652
3	138	8	808
4	242	9	971
5	306	10	1138

Figure 2.10

NCTM Standards

Students' observations and discussions of how quantities relate to one another lead to initial experiences with function relationships, and their representations of mathematical situations using concrete objects, pictures, and symbols are the beginnings of mathematical modeling. p. 91

The table in Figure 2.10 matches each time from 1 to 10 seconds with a unique (one and only one) distance. Since the distance fallen depends on time, distance is said to be a *function* of time. A **function** is *two sets* and a *rule* that assigns each element of the first set to exactly one element of the second set.

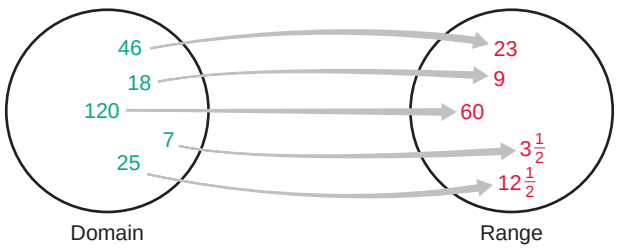
The two sets for a function have names. The first set is called the **domain,** and the second set is called the **range.** In the skydiving example, the *domain* is the set of whole numbers from numeral 1 to 10, and the *range* is the set of whole numbers from 16 to 1138.

One visual method of illustrating the assignment of elements from the domain to their corresponding elements in the range is with **arrow diagrams,** as shown in Example A. Such diagrams indicate the dynamic relationship between the elements of the two sets and show why we sometimes speak of an element of the range that gets "hit" by an element of the domain.

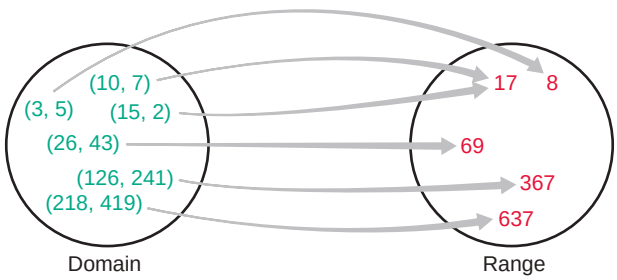
EXAMPLE A

Describe a rule for assigning each element of the domain to an element of the range for the following functions.

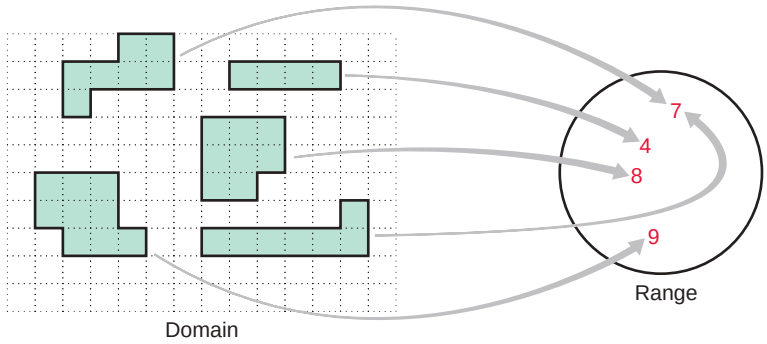
1.



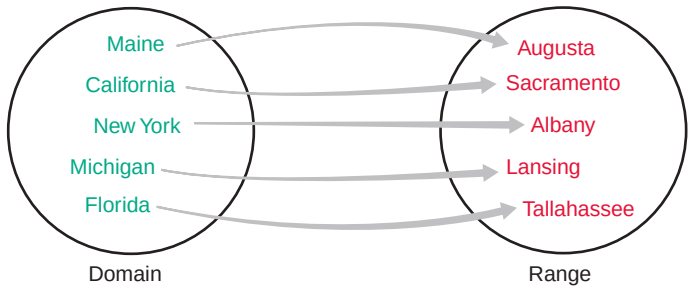
2.



3.



4.



Solution **1.** Each number in the domain is assigned to one-half its value in the range. **2.** Each pair of numbers in the domain is assigned to its sum in the range. **3.** Each figure in the domain is assigned to its area in the range. **4.** Each state in the domain is assigned to its capital city in the range.

Example A shows that the elements in the domain and range of a function may be different types of objects: numbers, geometric figures, etc. This example also shows that sometimes two or more elements in the domain can be assigned to the same element in the range. The important requirement for a function is that each element in the domain be assigned to not more than one element of the range. Example B will help you become familiar with this requirement. For each part of this example, ask yourself: Can the first element be assigned to more than one second element? If so, the correspondence of elements is not a function.

EXAMPLE B

Determine which of the following rules for the given sets are functions. If the rule is not a function, explain why.

1. Each person is assigned to his or her social-security number.

2. Each amount of money is assigned to the object it will buy.

3. Each person is assigned to a person who is older.

4. Each pencil is assigned to its length.

Solution **1.** Function. **2.** Not a function because two or more objects may cost the same amount. **3.** Not a function because many people are older than a given person. **4.** Function.

The rule for a function is often defined by an algebraic formula. It is customary to refer to an arbitrary element of the domain by a variable, such as x, and the corresponding element of the range by $f(x)$, as shown in Figure 2.11 [or $g(x)$, $s(x)$, etc.]. The symbol $f(x)$ is read as "f of x." Note that $f(x)$ does *not* mean f times x. For a domain element x, the corresponding range element is also denoted by the variable y.



Figure 2.11

Consider the rule that assigns each x from the set of whole numbers to $3x + 1$ in the range. This rule can also be written as $f(x) = 3x + 1$. Thus, the equation is the rule that specifies what each element of the domain is assigned to: $f(5) = 3(5) + 1 = 16$, so 5 is assigned to 16; $f(0) = 3(0) + 1 = 1$, so 0 is assigned to 1; etc.

EXAMPLE C

Write an algebraic rule for each of the following functions, where the domain is all whole numbers and x represents an element in the domain.

1. $f(x)$ is an element in the range, and each element in the domain is assigned to 3 more than twice its value.
2. $g(x)$ is an element in the range, and each element in the domain is assigned to 1 more than 4 times its value.
3. $h(x)$ is an element in the range, and each element in the domain is assigned to 10 times its value.
4. Evaluate $f(45)$, $g(56)$, and $h(84)$.

Solution **1.** $f(x) = 2x + 3$ **2.** $g(x) = 4x + 1$ **3.** $h(x) = 10x$ **4.** $f(45) = 93$, $g(56) = 225$, $h(84) = 840$

RECTANGULAR COORDINATES

Graphs provide a visual method for illustrating functions. A **horizontal axis,** called the *x axis,* is used for the elements of the domain, and a **vertical axis,** called the *y axis,* is used for the elements of the range. Each point on a graph is located by two numbers; their order is significant. The first number is called the *x coordinate* and indicates the distance to the right or left of the vertical axis. The second number is called the *y coordinate* and indicates the distance above or below the horizontal axis. These numbers are called the **coordinates** of the point. Figure 2.12 on page 83 illustrates the coordinates of four points. The intersection of the two axes is called the **origin** and has coordinates (0, 0).

This method of locating and describing points is called the **rectangular** (or **Cartesian**) **coordinate system.** The name *Cartesian* is in honor of René Descartes, the French mathematician and philosopher who first used this system for graphing geometric figures.

11-7 The Coordinate Plane

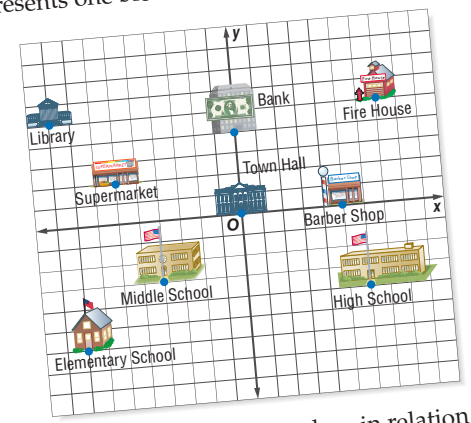
MAIN IDEA
Locate and graph ordered pairs on a coordinate plane.

New Vocabulary
quadrants

Math Online
glencoe.com
- Extra Examples
- Personal Tutor
- Self-Check Quiz

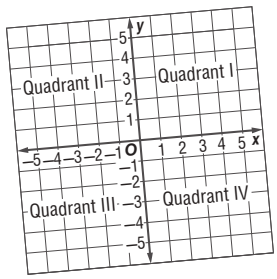
GET READY for the Lesson

MAPS The map shows the layout of a small town. The locations of buildings are described in respect to the town hall. Each unit on the grid represents one block.



1. Describe the location of the barber shop in relation to the town hall.
2. What building is located 7 blocks east and 5 blocks north of the town hall?
3. Violeta is at the library. Describe how many blocks and in what direction she should travel to get to the supermarket.
4. Let north and east directions be represented by positive integers. Let west and south directions be represented by negative integers. Describe the location of the high school as an ordered pair using integers.
5. Describe the location of the bank as an ordered pair using integers.

Recall that a coordinate plane is formed when the x-axis and y-axis intersect at their zero points. The x-axis and y-axis extend forever in each direction and separate the coordinate plane into four regions called **quadrants**. Each quadrant has an area of infinite size. In Lesson 4-9, you graphed and named ordered pairs in Quadrant I.



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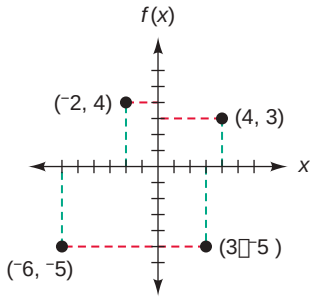


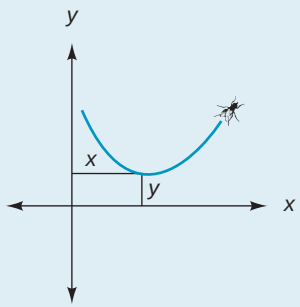
Figure 2.12

HISTORICAL HIGHLIGHT



René Descartes, 1596–1650

The French mathematician René Descartes is sometimes referred to as the father of modern mathematics. Although he made important contributions in the fields of chemistry, physics, physiology, and psychology, he is perhaps best known for his creation of the rectangular coordinate system. Legend has it that the idea of coordinates in geometry came to Descartes while he lay in bed and watched a fly crawling on the ceiling. Noting that each position of the fly could be expressed by two distances from the edges of the ceiling where the walls and ceiling met, Descartes realized that these distances could be related by an equation. That is, each point on a curve has coordinates that are solutions to an equation, and conversely, every two numbers x and y that are solutions to an equation correspond to a point on a curve. This discovery made it possible to study geometric figures by using equations and algebra. This link between geometry and algebra is one of the greatest mathematical achievements of all time.*



* H. W. Eves, *In Mathematical Circles* (Boston: Prindle, Weber, and Schmidt, 1969), pp. 127–130.

LINEAR FUNCTIONS AND SLOPE

NCTM Standards

As they progress from preschool through high school, students should develop a repertoire of functions. In the middle grades, students should focus on understanding linear relationships. p. 38

Consider the function that relates time and distance as sound travels through the air. An observer can estimate the distance to an approaching thunderstorm by counting the seconds between a flash of lightning and the resulting sound of thunder. Every 5 seconds sound travels approximately 1 mile. If you count up to 10 seconds before hearing the thunder, the storm is approximately 2 miles away. In this example, *distance is a function of time.* Here are a few times in seconds and their corresponding distances in miles using function notation.

x	5	10	15	20	25
$f(x)$	1	2	3	4	5

Since each member in the domain is multiplied by $\frac{1}{5}$ to obtain the corresponding number for the range, the equation for this function is $f(x) = \frac{1}{5}x$, and its graph is the line shown in Figure 2.13.

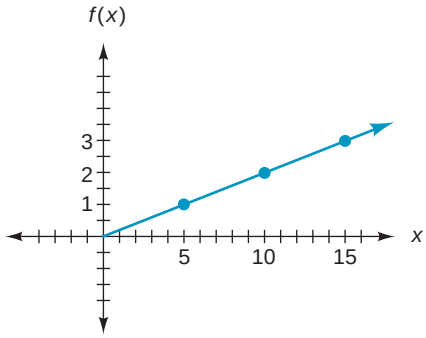


Figure 2.13

Sound travels faster in water than in air. In water it travels about 1 mile per second. In 2 seconds it travels 2 miles; in 3 seconds, 3 miles; etc. This is another example in which distance is a function of time. The equation for this function is $f(x) = x$, and its graph is the line shown in Figure 2.14.

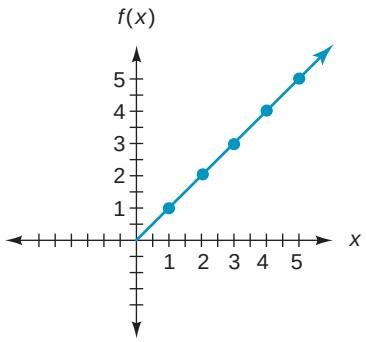


Figure 2.14

Notice that the graph of $f(x) = x$ has a greater slope than the graph of $f(x) = \frac{1}{5}x$. In general, the equation of a line through the origin is $f(x) = mx$, where the constant m is the slope of the line.

The concept of slope occurs in many applications of mathematics. For example, highway engineers measure the slope of a road by comparing the vertical rise to each 100 feet of horizontal distance. The Federal Highway Administration recommends a maximum vertical rise of 12 feet for each 100 feet of horizontal distance (Figure 2.15). Many secondary roads and streets are much steeper. Filbert Street in San Francisco has a vertical rise of approximately 1 foot for each 3 feet of horizontal distance. By comparison, the walls at the

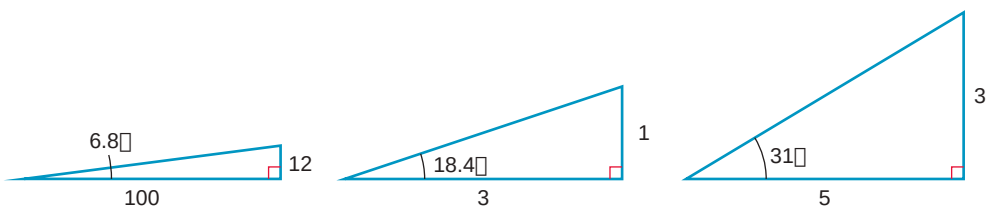


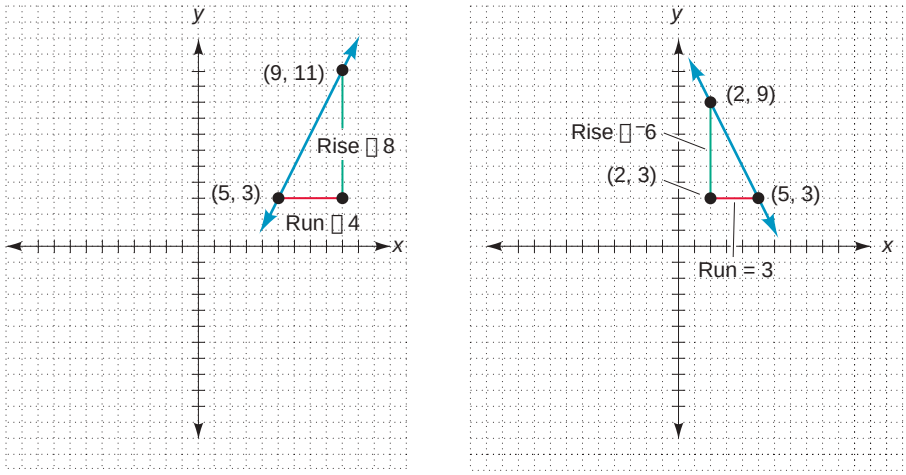
Figure 2.15

ends of the Daytona International Speedway have a vertical rise of 3 feet for each 5 feet of horizontal distance.

The slope of a line or line segment is defined in much the same way as the steepness of highways: Two points on a line or line segment are selected, and the **slope** of the line connecting these points is the difference between the two y coordinates (the **rise**) divided by the difference between the two x coordinates (the **run**). Two examples are shown in Figure 2.16. In part a, the ordered pairs (5, 3) and (9, 11) are used to compute the slope.

$$\text{Rise: } 11 - 3 = 8 \qquad \text{Run: } 9 - 5 = 4 \qquad \text{Slope: } \frac{8}{4} = 2$$

Notice that we started with the coordinates of (9, 11) and subtracted the coordinates of (5, 3). The same slope will be obtained by starting with (5, 3) and subtracting the coordinates of (9, 11).



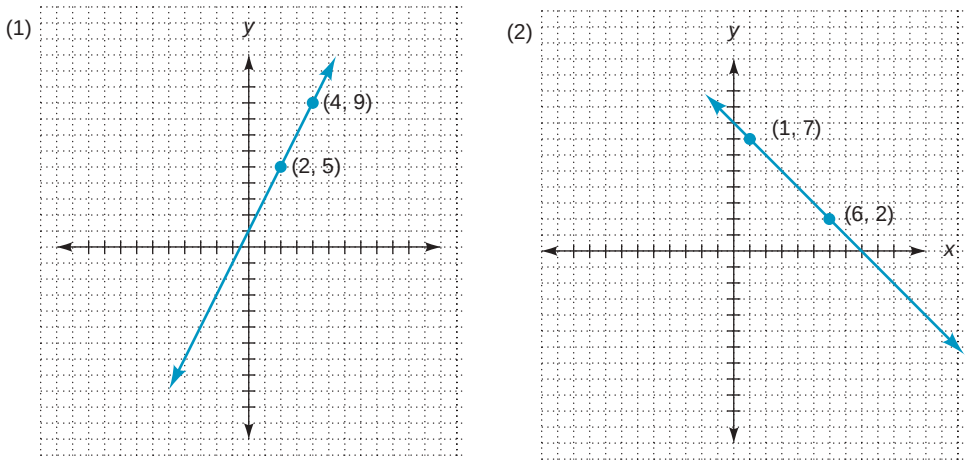
(a) (b)

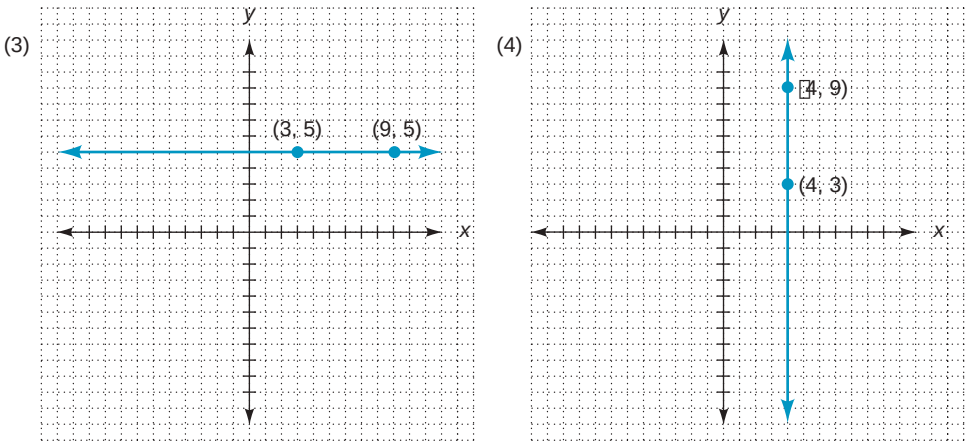
Figure 2.16

In part b, the slope is determined from the ordered pairs (2, 9) and (5, 3).

$$\text{Rise: } 3 - 9 = {}^{-}6 \qquad \text{Run: } 5 - 2 = 3 \qquad \text{Slope: } \frac{{}^{-}6}{3} = {}^{-}2$$

EXAMPLE D Find the rise and run for each pair of points, and determine the slope of the line, if it exists.





Solution **1.** Rise: $9 - 5 = 4$; run: $4 - 2 = 2$; slope: $\frac{4}{2} = 2$ **2.** Rise: $2 - 7 = {}^{-}5$; run: $6 - 1 = 5$; slope: $\frac{-5}{5} = {}^{-}1$ **3.** Rise: $5 - 5 = 0$; run: $9 - 3 = 6$; slope: $\frac{0}{6} = 0$ **4.** Rise: $9 - 3 = 6$; run: $4 - 4 = 0$; slope: undefined.

Notice that the line in graph 3 in Example D is parallel to the horizontal axis. All lines that are parallel to the horizontal axis will have a rise of zero and, therefore, a slope of zero. The line in graph 4 in Example D is parallel to the vertical axis. All lines that are parallel to the vertical axis will have a run of zero, and since division by zero is undefined, the slope for such lines is undefined. That is, lines parallel to the vertical axis *do not have a slope.* Graphs 1 and 2 in Example D show lines with positive and negative slopes. In general, lines that extend from lower left to upper right have a **positive slope,** and lines that extend from upper left to lower right have a **negative slope.**

Next, consider the three lines and their equations in Figure 2.17. Notice that for pairs of points on these lines, each $\frac{\text{rise}}{\text{run}}$ equals 2; and this slope can also be seen from the equations of the lines (below the graphs). In general, two lines with the same slope are **parallel.** Furthermore, the y coordinate of the point at which each line crosses the vertical axis, the ***y* intercept,** can be seen from the equation. It is zero for the line in part a, 1 for the line in b,

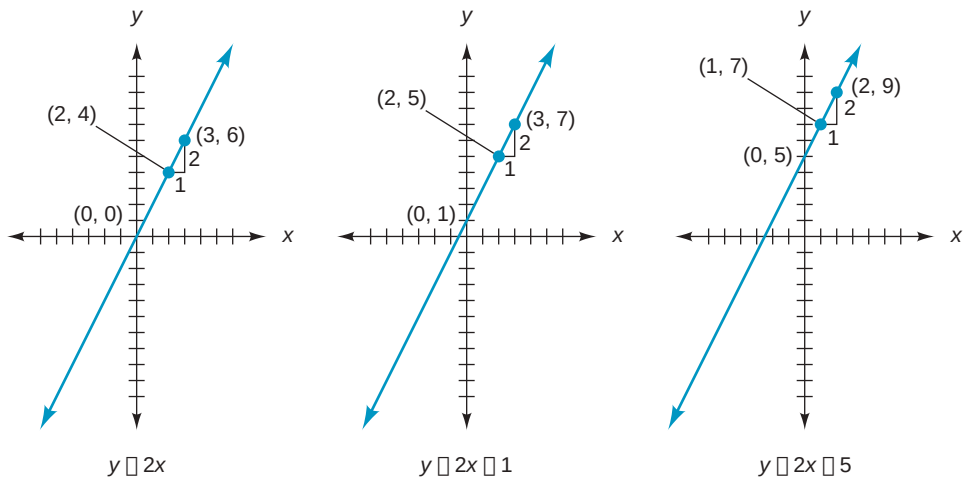


Figure 2.17

NCTM Standards

With strong middle-grades focus on linearity, students should learn about the idea that slope represents the constant rate of change in linear functions and be ready to learn in high school about classes of functions that have nonconstant rates of change. p. 40

and 5 for the line in c. The y intercept also can be easily obtained from these equations by setting $x = 0$.

In general, every line (except those parallel to the vertical axis) has an equation of the form

$$y = mx + b$$

where m and b are constants: and conversely, the graphs of such equations are lines. In this equation m is the slope of the line, and b is the y intercept. If a line is parallel to the vertical axis, its equation has the form $x = k$. For example, $x = 6$ is the line passing through (6, 0) and parallel to the vertical axis. Functions whose graphs are lines that are not parallel to the vertical axis are called **linear functions.** When the equation of a line is written in the form $y = mx + b$, it is said to be in **slope-intercept form,** because the slope m of the line and the y intercept b can be read from the equation.

When the slope and the y intercept of a line are known, the equation can be written immediately. This information is often given in applications. Consider **rates,** such as *miles per hour* or *cost per unit.* These are examples of linear functions. Suppose it costs \$8 per hour to rent a lawn mower. It will cost \$16 for 2 hours, \$24 for 3 hours, etc. If x denotes the number of hours and $f(x)$ the total cost, this information is described by the equation

$$f(x) = 8x$$

Now, if there is an initial fee of \$5 in addition to the hourly rate, the equation becomes

$$f(x) = 8x + 5$$

In general, the *rate* is the slope of a line, and the *initial cost* is the y intercept.

EXAMPLE E

A taxi meter starts at \$1.60 and increases at the rate of \$1.20 for every minute. Let x represent the number of minutes and $f(x)$ represent the total cost. Write an equation for the total cost as a function of the number of minutes.

Solution The initial fee is \$1.60, and each minute costs \$1.20. The total cost in dollars is $f(x) = 1.2x + 1.6$.

Many of the examples of linear functions from everyday life occur in rates we pay for services, such as electrical rates, phone rates, cable rates, etc.

EXAMPLE F

Research Statement

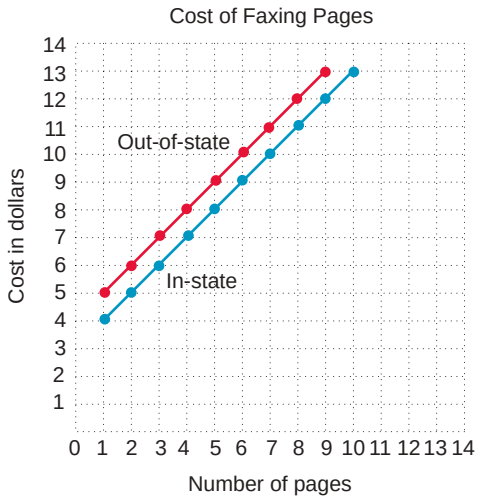
Teachers need to provide all students with experiences in which they identify the underlying rules for a variety of patterns that embody both constant and nonconstant rates of change.

Blume and Heckman

A copy center has the following rates for sending a fax of pages: in-state, \$4 for the first page and \$1 for each additional page; out-of-state, \$5 for the first page and \$1 for each additional page. The graphs of the functions defined by these rates are shown on the next page.

1. What is the domain of these functions?
2. What is the range of the in-state function? the out-of-state function?
3. Find the points on the graph that show each cost for faxing eight pages. What are the coordinates of these points? What is each cost?
4. What patterns do you see for these graphs?

5. If x represents a number from the domain of these functions and $f(x)$ and $g(x)$ are the corresponding numbers from the range for the in-state and the out-of-state function, respectively, write formulas for $f(x)$ and $g(x)$.



Solution **1.** The set of whole numbers: 1, 2, 3, 4, ... **2.** Range of in-state function: 4, 5, 6, ... Range of out-of-state function: 5, 6, 7, ... **3.** These are the two points above the number 8 on the horizontal axis. The coordinates are (8, 11) and (8, 12). The corresponding costs are \$11 and \$12. **4.** Possible patterns: The points on each graph lie on a straight line; to move from one point on the graph to the next point to the right, move one space right and up one space; the two graphs lie on lines that are parallel; the vertical distance between the graphs is 1. **5.** $f(x) = x + 3$ and $g(x) = x + 4$.

The graph of each line in Example F is called a **discrete graph** because the points of the graph are separate. The faxing rate is for a whole number of sheets; so, for example, there are no numbers in the domain between 1 and 2. Often however, as in Example F, the points of a discrete graph are connected to help us visualize changes in the graph and to distinguish between two or more graphs.

NONLINEAR FUNCTIONS

The graphs of the functions up to this point have been lines, or separate points on a line, because they have involved constant rates. The function in the next example illustrates a rate of change that is not constant, and its graph is not a straight line. This is an example of a nonlinear function.

EXAMPLE G

Fold a sheet of paper in half, then fold the resulting sheet in half again, and continue this process of folding in half. The number of regions formed is a function of the number of folds.

1. What are the numbers of regions for the following numbers of folds?

Number of folds	0	1	2	3	4
Number of regions	1				

2. Find a pattern in the numbers of regions, and predict the number of regions for five folds. Then determine the number of regions by folding paper.

NCTM Standards

Students should learn to distinguish linear relationships from nonlinear ones. In the middle grades, students should also learn to recognize and generate equivalent expressions, solve linear equations, and use simple formulas. p. 223

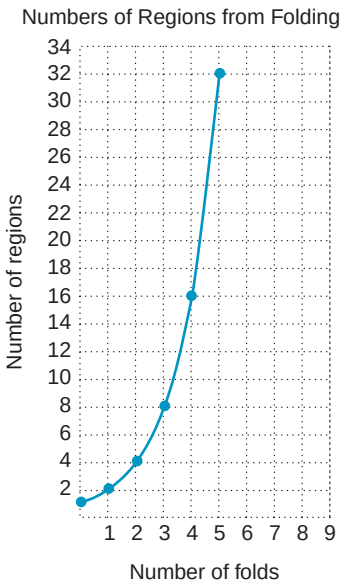
3. Graph this function for the first five folds. (Copy Grid Paper from the website.)
4. If x is an arbitrary number of folds and $f(x)$ is the corresponding number of regions, what is the algebraic rule for the function?
5. The graph shows that the number of regions appears to be doubling for each new fold. How can the paper folding activity be used to explain why the number of regions will continue to double?

Solution 1.

Number of folds	0	1	2	3	4
Number of regions	1	2	4	8	16

2. The numbers of regions for the first few folds are doubling; five folds, 32 regions.

3.



4. $f(x) = 2^x$ or $f(x) = \underbrace{2 \times 2 \times 2 \times \ldots \times 2}_{x \text{ times}}$ 5. After any given number of folds, the next fold will fold each of the existing regions in half. Notice that the points of the graph are connected to help show the rapid increase in the numbers of folds, but the graph is discrete.

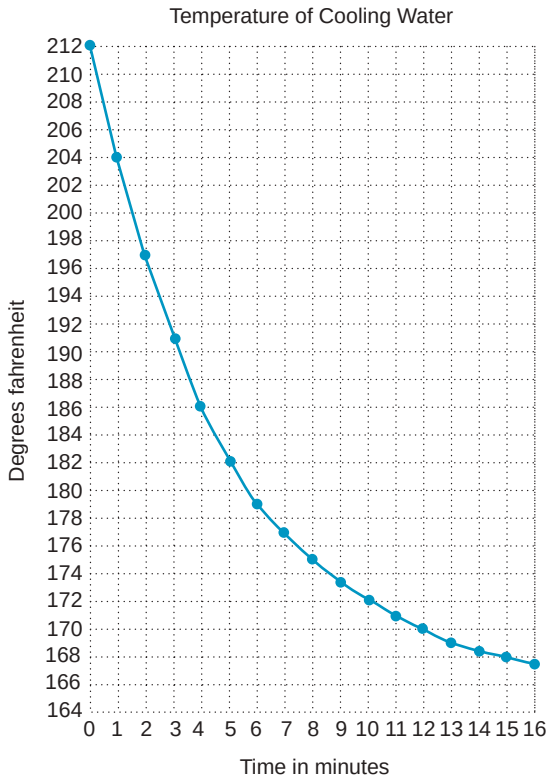
The graph in Example G curves upward because for each unit increase along the horizontal axis, the increase in the vertical direction is greater. A similar but opposite effect is seen in Example H.

EXAMPLE H

The fourth-graders at King Elementary School conducted an experiment to observe the rate at which water cools. They placed a thermometer in a beaker of water and heated the water to boiling (212°F). They recorded the water temperature every minute until the temperature dropped to just below 168°F. Then they plotted the results on a grid like the one shown on the next page. Notice that this is not a discrete graph because there is a temperature for each instant of time. The points of the graph can now be connected to form what is called a **continuous graph.**

1. How many degrees did the temperature drop during the first 2 minutes?
2. Did the temperature drop more in the first 2 minutes or in the second 2 minutes?

3. Often the variable t is used to represent time. If t represents the time in minutes and $f(t)$ is the corresponding temperature, use the graph to determine $f(0)$, $f(1.5)$, $f(4)$, $f(4.5)$, $f(8)$, $f(10)$, $f(12)$, $f(14)$, and $f(16)$.



4. Approximately how much did the temperature drop during the first 8 minutes compared to the last 8 minutes?

5. What conclusion does the graph suggest about the rate of cooling during this 16-minute period?

Solution **1.** 15°F **2.** First 2 minutes **3.** $f(0) = 212$°F, $f(1.5) = 200$°F, $f(4) = 186$°F, $f(4.5) = 184$°F, $f(8) = 175$°F, $f(10) = 172$°F, $f(12) = 170$°F, $f(14) = 168.5$°F, $f(16) = 167.5$°F **4.** 37°F for the first 8 minutes and 7.5°F for the second 8 minutes. The temperature decrease for the first 8 minutes was about 5 times the decrease for the second 8 minutes. **5.** The rate of cooling is more rapid at first and then slows down.

INTERPRETING GRAPHS

NCTM Standards

The NCTM *Curriculum and Evaluation Standards for School Mathematics* (p. 155) note that students frequently have experience graphing functions expressed in symbolic form, but that it is equally important that they be given opportunities to interpret graphs and translate from a graphical representation of a function to the symbolic form.

Examples I and J focus on interpreting graphs.

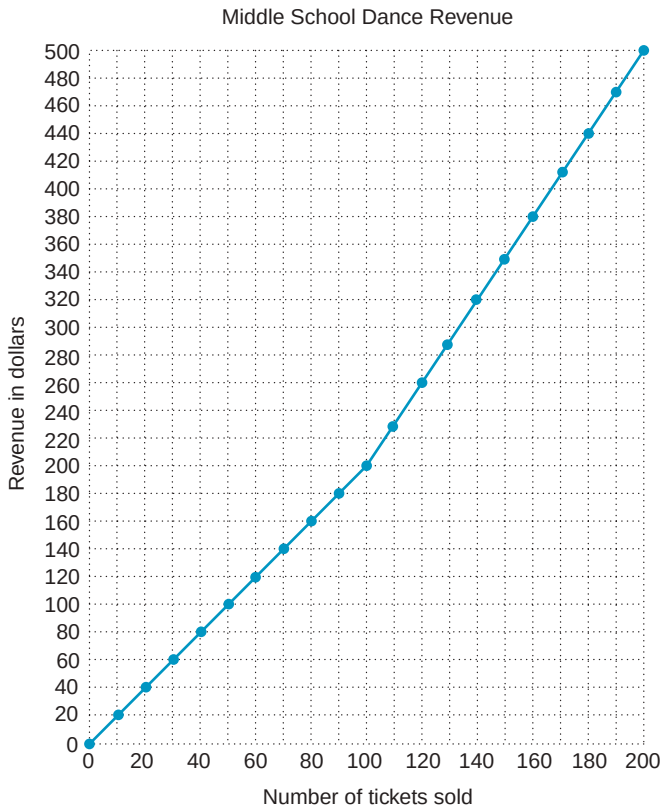
EXAMPLE I

The middle school sponsored a dance, and the graph of their revenue as a function of the number of tickets sold is shown below. After the sale of the first 100 tickets, the cost of the tickets increased, as shown by the steeper portion of the graph.

Research Statement

To provide effective instruction, teachers need to increase their knowledge of graphs and how to teach graphs.

Friel, Curcio, Bright



1. What was the revenue for the sale of the first 100 tickets?
2. What was the cost of each ticket for the first 100 tickets?
3. What was the revenue for the sale of the second 100 tickets?
4. What was the cost of each ticket for the second 100 tickets?
5. If $f(x)$ represents the revenue for the sale of x tickets, find $f(50)$ and $f(150)$.
6. Is this graph continuous or discrete?

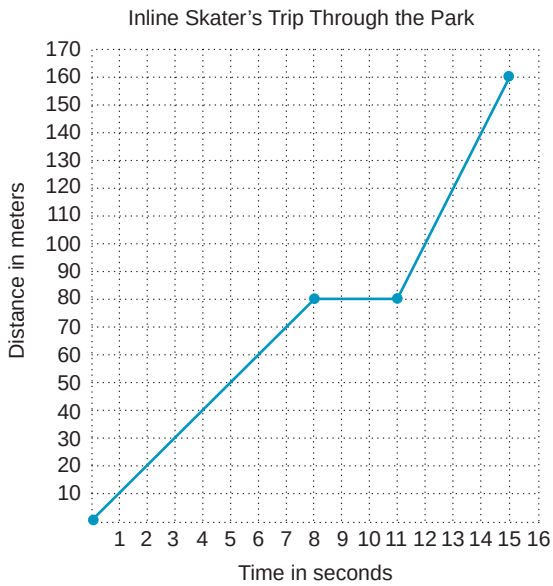
Solution **1.** \$200 **2.** \$2 per ticket **3.** \$500 − \$200 = \$300 **4.** \$3 per ticket **5.** $f(50) = 100$, $f(150) = 350$ **6.** Discrete

EXAMPLE J

The graph on the next page shows distance in meters as a function of time over a 15-second period for an inline skater's trip through the park.

1. What distance did the skater travel during the first 8 seconds?
2. What distance did the skater travel from the fifth to the eighth second?

3. What information does the graph indicate from the 8th to the 11th second?
4. What can be said about the skater's speed from the 11th to the 15th second compared to the speed for the first 8 seconds?
5. Is this graph continuous or discrete?



Solution **1.** 80 meters **2.** 30 meters **3.** There was no increase in distance during this time. **4.** It's faster—twice as fast. **5.** This graph is continuous because for each time, whether a whole number or not, there is a corresponding distance.

Technology Connection

Graphs of Functions

What happens to the graphs of $y = kx^2$ and $y = x^2 + k$ as k takes on different positive and negative values? In this investigation, you will look for patterns and make conjectures by using a graphing calculator to obtain the graphs of different forms of equations. This and similar questions are explored using Geometer's Sketchpad® student modules available at the companion website.

Mathematics Investigation
Chapter 2, Section 2
www.mhhe.com/bbn

PROBLEM-SOLVING APPLICATION

The introduction to functions and graphs provides a new problem-solving strategy, **drawing a graph.**

Problem

Students in the City Center School who live in the direction of Dolan Heights commute from school to home by taking bus 17 or the Dolan Heights subway. The bus leaves when school is over, and every 3 minutes it travels 1 mile. The subway leaves 7 minutes later, and every 3 minutes it travels 2 miles. What advice would you give to students who live in the direction of Dolan Heights and who want to use the method of travel that gets them home more quickly at the end of the school day? Is there a distance from the school that is reached at the same time by bus 17 as by the Dolan Heights subway?

NCTM Standards

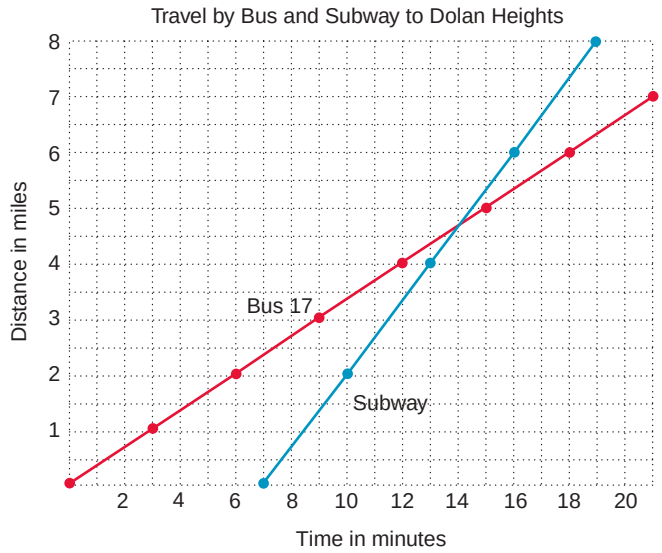
Even before formal schooling, children develop beginning concepts related to patterns, functions, and algebra. The recognition, comparison, and analysis of patterns are important components of a student's intellectual development. p. 91

Understanding the Problem Try a few numbers of minutes to find the distances that students could travel by taking bus 17 or the Dolan Heights subway. For example, in the first 6 minutes after school is out, students would travel 2 miles on the bus and 0 miles on

the subway. **Question 1:** What are the distances covered by bus and by subway at the end of 10 minutes?

Devising a Plan One method of solving the problem is to *form a table* for different numbers of times. Another method is to *draw a graph* of the times and distances for travel by bus and by subway. **Question 2:** If there is a time for which the distances covered by bus and by subway are equal, how will this be shown by the two graphs?

Carrying Out the Plan The graph of the distances traveled by bus can be plotted by repeatedly moving horizontally 3 spaces (3 minutes) and vertically 2 spaces (1 mile). Similarly, the graph of the distances traveled by subway is plotted by repeatedly moving horizontally 3 spaces (3 minutes) and vertically 4 spaces (2 miles). **Question 3:** What is the amount of time for which the distances traveled by bus and by subway are equal? What is this distance? How might students be advised in selecting the method of travel that gets them home more quickly?

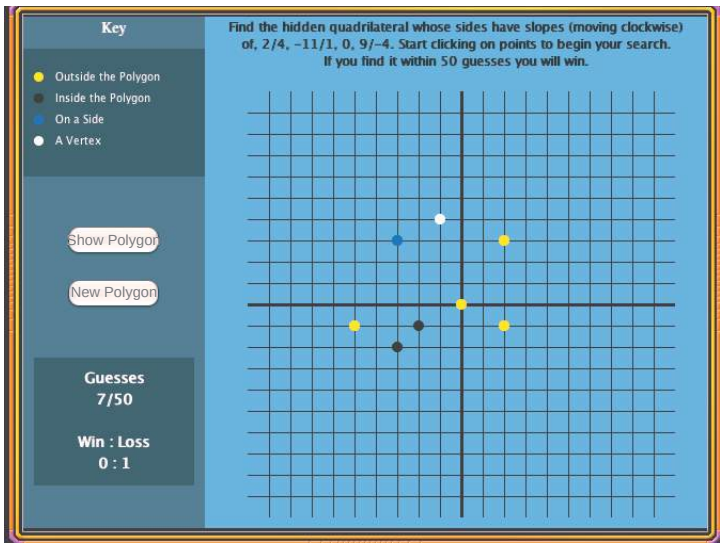


Looking Back The graphs show that the bus traveled the greater distance for the first 14 minutes, and after 14 minutes the subway traveled the greater distance. The vertical distances between the graphs show how much greater the distance by one method of travel is than the other. For example, after 11 minutes the bus has traveled approximately 1 mile farther than the subway. **Question 4:** What is the time at which the subway will have traveled approximately 1 mile farther than the bus?

Answers to Questions 1–4 **1.** Bus, $3\frac{1}{3}$ miles; subway, 2 miles. **2.** The time for which the distances are equal is shown by the intersection of the two graphs. **3.** 14 minutes; $4\frac{2}{3}$ miles. Students traveling less than $4\frac{2}{3}$ miles could be advised to take the bus, and those traveling greater distances could be advised to take the subway. **4.** 17 minutes.

Technology Connection

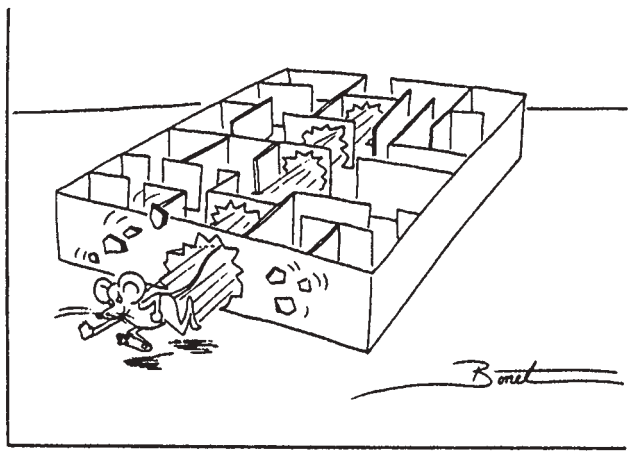
Try to find a hidden polygon on a coordinate system. Given the number of sides and the slopes of the sides, you will be told if the points you select are inside (black), outside (yellow), on the boundary (blue), or at a vertex of the polygon (white). Where would you select your next point or points for the information shown here?



Hunting for Hidden Polygons Applet, Chapter 2
www.mhhe.com/bbn

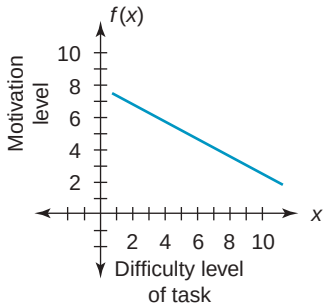
Exercises and Problems 2.2

Copies of the rectangular grid from the website can be used for these exercises.

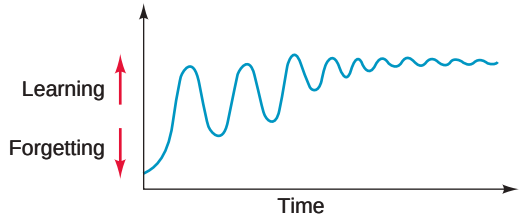


1. Experiments with rats at the University of London tested the conjecture that the motivation level for learning a task is a function of the difficulty of the task.*

What does this graph show about the motivation level as the difficulty level of the task increases? decreases?

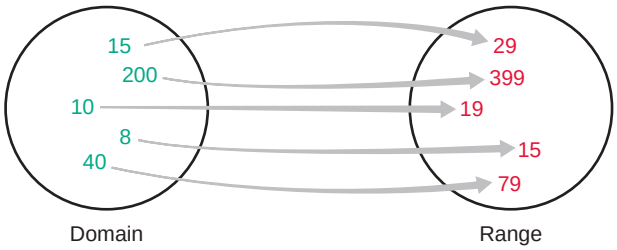


2. This graph shows the relationship between repeated exposure to learning and retention. Explain what this graph shows when the same topic is repeatedly reviewed and used over a period of time.



*P. L. Broadhurst, "Emotionality and the Yerkes-Dodson Law," *Journal of Experimental Psychology* 54, pp. 345–352.

3. Describe in words a rule for assigning each element of the domain to an element of the range for the arrow diagram.



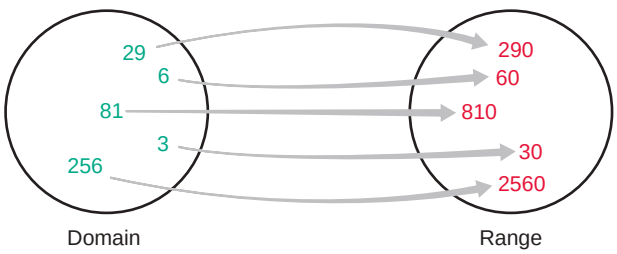
a. Write an algebraic rule for $f(x)$ that describes what each x in the domain corresponds to in the range.

b. Complete this table.

x	1	2	3	4	5	6	7	8	9	10
$f(x)$										

c. Using a rectangular grid, plot the points whose coordinates are given in the table in part b.

4. a. Describe in words a rule for assigning each element of the domain to an element of the range for the arrow diagram.



b. Write an algebraic rule for $g(x)$ that describes what each x in the domain corresponds to in the range.

c. Complete this table.

x	1	2	3	4	5	6	7	8	9	10
$g(x)$										

d. Using a rectangular grid, plot the points whose coordinates are given in the table in part c.

In exercises 5 and 6, determine which rules are functions. If the rule is not a function, explain why.

5. a. People assigned to their birthdays

b. Numbers assigned to numbers that are 10 times greater

c. People assigned to their telephone numbers

6. a. Circles assigned to their areas

b. Numbers assigned to any numbers that are greater

c. Pairs of numbers assigned to their products

In exercises 7 and 8, write an algebraic rule for each function.

7. a. Each whole number is assigned to the whole number that is 17 greater.

b. Each whole number is assigned to the number that is 2 less than 3 times the number.

8. a. Each whole number is assigned to the number that is 3 more than 4 times the number.

b. Each whole number greater than 10 is assigned to the number that is 6 less than the number.

9. Consider the function that relates the length x of the side of a square to the area $f(x)$ of the square.

a. Determine the range value $f(x)$ for each of the following domain values: 1, 2, 3, 4, and 5.

b. Use the coordinates in part a or a graphing calculator to sketch the graph of this function. (Copy the coordinate system from the website.)

c. What is the equation of this function?

d. Is this function linear or nonlinear?

10. Consider the function that relates the length x of the side of a square to the perimeter $f(x)$ of the square.

a. Determine the range value $f(x)$ for each of the following domain values: 1, 2, 3, 4, and 5.

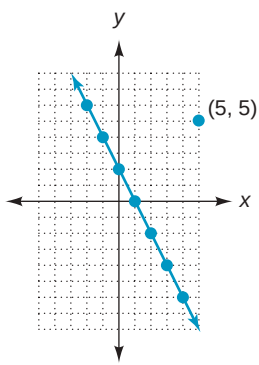
b. Use the coordinates in part a or a graphing calculator to sketch the graph of this function.

c. What is the equation of this function?

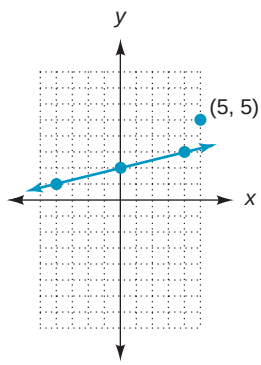
d. Is this function linear or nonlinear?

In exercises 11 and 12, determine the slope and equation of each line. (*Hint:* The slope and y intercept are all that are needed to write the equation of a line.)

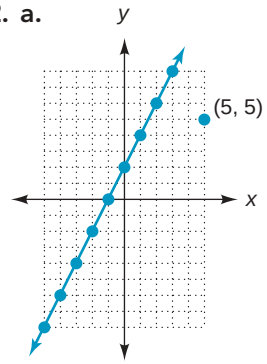
11. a.



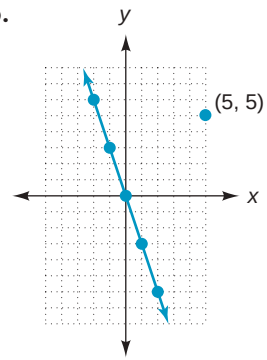
b.



12. **a.**

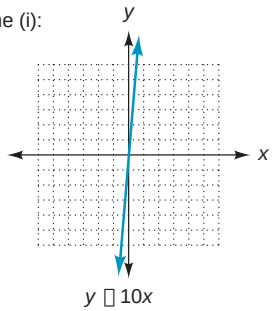


b.

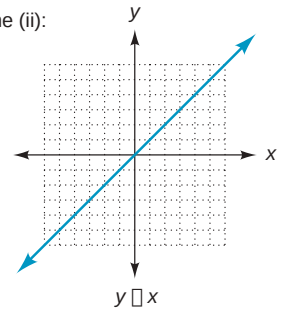


13. The equations of two linear functions and their graphs are shown below.

Line (i):



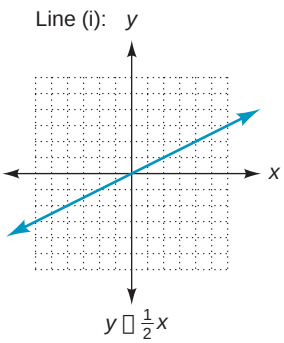
Line (ii):



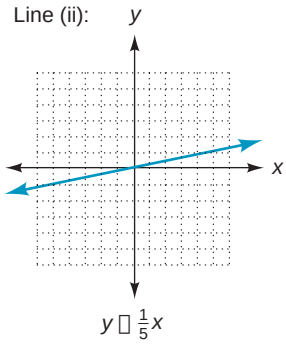
a. What is the slope of each line?
b. Can a line be drawn whose slope is greater than the slope of line i? If so, write the equation of such a line.
c. Is there any limit to how large the slope of a line can become? Explain your reasoning.
d. If a line contains the points $(^{-}2, 3)$ and is parallel to line ii, what is the equation of the line?

14. The equations of two linear functions and their graphs are shown below.

Line (i):



Line (ii):



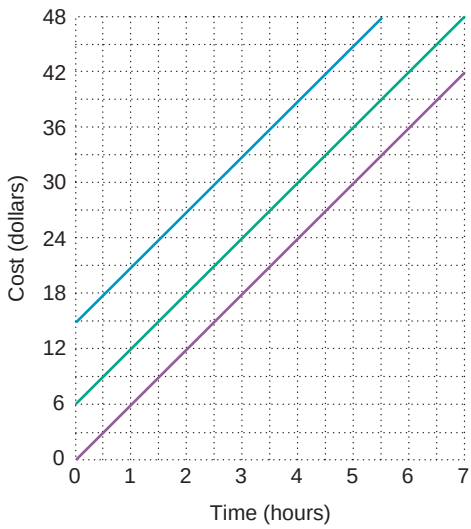
a. What is the slope of each line?
b. Can another line be drawn whose slope is positive and less than the slope of line ii? If so, write the equation of such a line.
c. For any line with a positive slope, is it possible to have another line with a smaller positive slope?
d. If a line intercepts the y axis at $(0, ^{-}4)$ and is parallel to line i, what is the equation of the line?

Reasoning and Problem Solving

15. During the past few years Great Britain's pound has been worth between \$1 and \$2 in U.S. currency. Suppose the rate of exchange is \$1.50 for each pound.
a. What is the value in dollars for 5 pounds?
b. What is the value in pounds of \$25.50?
c. The value in dollars is a function of the number of pounds. Letting x represent the number of pounds and $c(x)$ represent the value in dollars, write the equation for this function.
d. Sketch a graph of the function on a rectangular grid.

16. In recent years the Mexican peso has been worth between 5 cents and 20 cents in U.S. currency. Suppose the rate of exchange is 15 cents for each peso.
a. What is the value in dollars of 200 pesos?
b. What is the value in pesos of \$300?
c. The value in dollars is a function of the number of pesos. Letting x represent the number of pesos and $d(x)$ represent the value in dollars, write the equation for this function.
d. Sketch a graph of the function on a rectangular grid.

17. Leaky Boat Club charges \$6 per hour to rent a canoe. If you are a member of the club, there is no initial fee. Nonmembers who are state residents pay an initial fee of \$6, and out-of-state people pay an initial fee of \$15. The graphs of these rates are shown below.
a. Which line (upper, middle, or lower) represents the cost for state residents who are nonmembers?
b. How much more will it cost an out-of-state resident than a club member to rent a canoe for 3 hours? for 5 hours?



c. Using a rectangular grid, sketch these graphs and indicate the portions that correspond to the differences in part b.

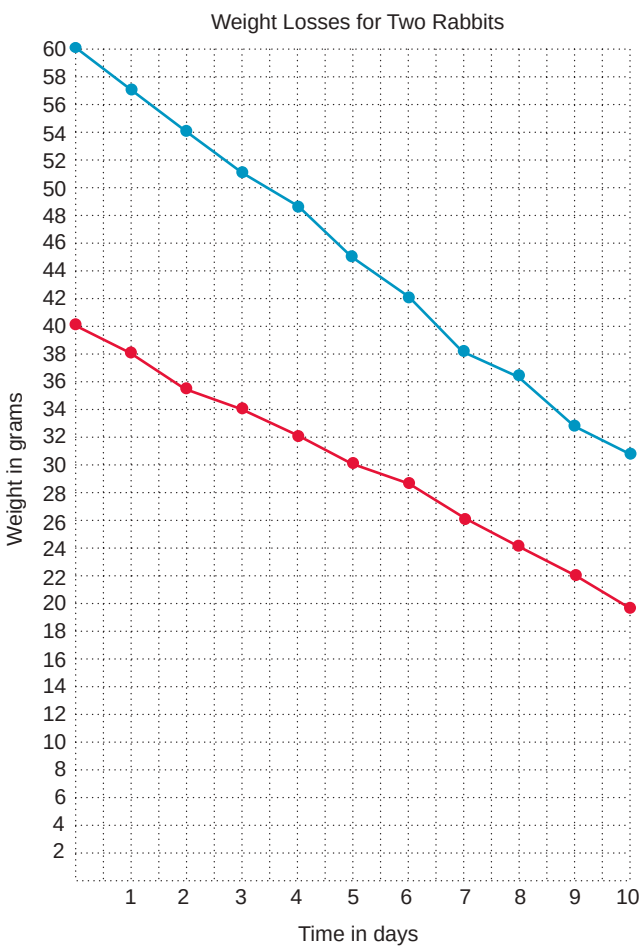
d. How much more will it cost a nonmember state resident than a club member to rent a canoe for 6 hours? Mark the portion of the copied graphs that indicates this difference.

18. One rabbit at the Morse Research Labs will be given a diet to lose approximately 2 grams each day, and another rabbit will be put on a diet to lose approximately 3 grams each day. The graphs of their weights (given here) for a 10-day period show the weights at the end of each day.

a. What is the weight of each rabbit at the beginning of the experiment?

b. How much weight will each rabbit lose after 5 days? Use a rectangular grid to sketch these graphs, and indicate the portion of the graphs that corresponds to the difference in their weights at the end of 5 days.

c. What is happening to the differences between the weights of the rabbits over the 10-day period?



19. In one location the telephone company charges an initial fee of \$60 for visiting a house plus \$20 for each telephone jack installed. Let x be the number of jacks and $c(x)$ the total cost.

a. Write an equation for the cost of having the telephone company visit a house and install a total of x jacks.

b. What is the value of $c(5)$?

c. Graph this function.

20. A bus transportation company charges a \$200 flat rate for a one-day trip (500 miles or less) plus \$25 for each person. Let x be the number of people taking a bus trip and $f(x)$ the total cost.

a. Write an equation for the cost of a bus trip for x people.

b. What is the value of $f(23)$?

c. Graph this function.

21. A racquetball club charges \$15 per month plus \$6 for each hour of court time, and only whole numbers of hours of court time can be purchased. Let x be the number of hours of playing racquetball in a given month, and $g(x)$ the total cost.

a. Write an equation for the cost per month of playing racquetball.

b. What is the value of $g(14)$?

c. Graph this function.

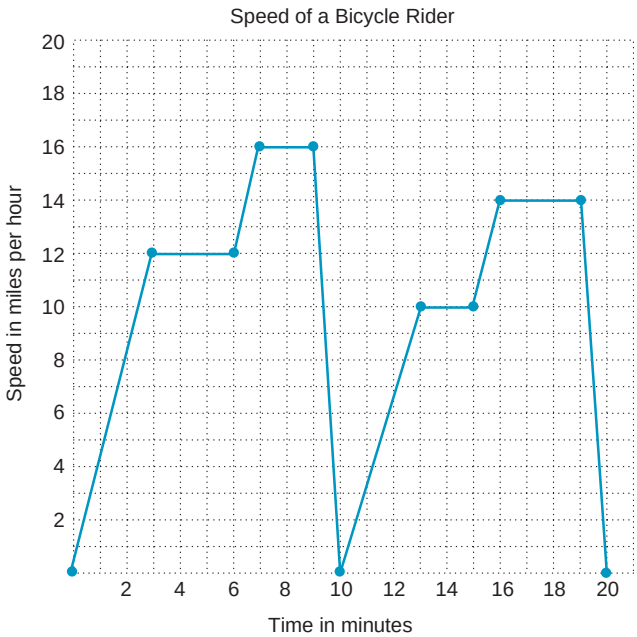
22. A cable television company charges \$60 per month plus \$5 for each additional channel. Let x be the number of additional channels and $p(x)$ the total cost per month.

a. Write an equation for the total monthly cost of the television cable service with x additional channels provided.

b. What is the value of $p(4)$?

c. Graph this function.

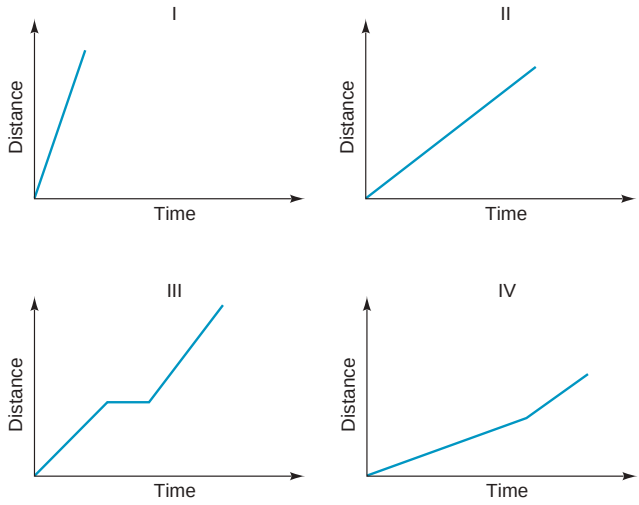
Use the following graph in problems 23 and 24. This graph shows a student's speed during a 20-minute bike ride.



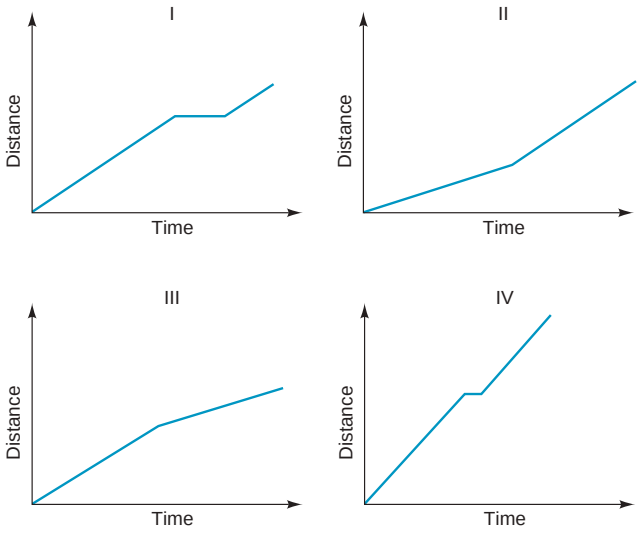
23. **a.** What was the student's speed from the third to the fifth minute?
b. At what times during the bike ride did the student come to a stop?
c. During what time intervals was the student's speed increasing?

24. **a.** What was the student's speed from the 16th to the 19th minute?
b. During what time intervals was the student's speed constant, that is, not changing?
c. During what time intervals was the student's speed decreasing?

25. Four children go to school along the same road. Joel walked half the distance and then jogged the rest of the way. Joan jogged all the way to school. Mary rode her bicycle but stopped to talk to a friend. Bob's father drove him to school in the family car.
a. The following graphs show distance as a function of time for each of these students. Match each student with a graph.
b. Which student took the longest to get to school?
c. Which student lives the farthest from school?
d. Which student took the least time to get to school?
e. Which student lives closest to school?

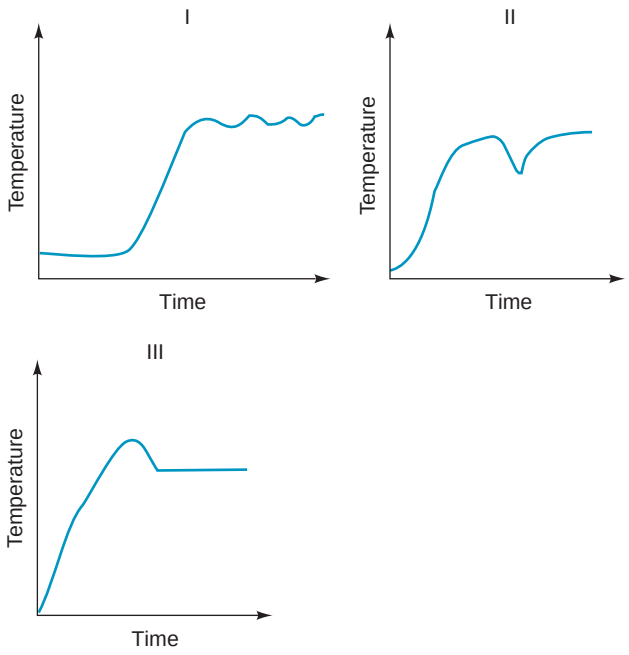


26. Sally, Tom, Bette, and Howard have jobs on weekends at the supermarket. The graphs on the next page show their distances traveled from home to work on a given day as functions of time. Tom walked halfway and then jogged the rest of the way. Sally jogged halfway and then walked the remaining distance. Bette skateboarded all the way but stopped to enjoy the view for 3 minutes. Howard rode his bike, but had to stop 1 minute at a stoplight.
a. Determine the graph that corresponds to each person's distance as a function of time.
b. Which student took the longest to get to work?
c. Which student lives the farthest from the supermarket?
d. Which student took the least time to get to work?
e. Which student lives closest to the supermarket?



27. Each of the following graphs represents the temperature of an oven as a function of time.

a. Which graph indicates that the oven door was opened once for a brief time during the cooking period? Mark the portion of the graph that indicates the open door.



b. Which graph indicates the oven was initially heated to a higher temperature than needed for the cooking?

c. Which graph shows the oven maintaining a more or less constant temperature during the cooking period with slight variations due to cooling and reheating? Mark this portion of the graph.

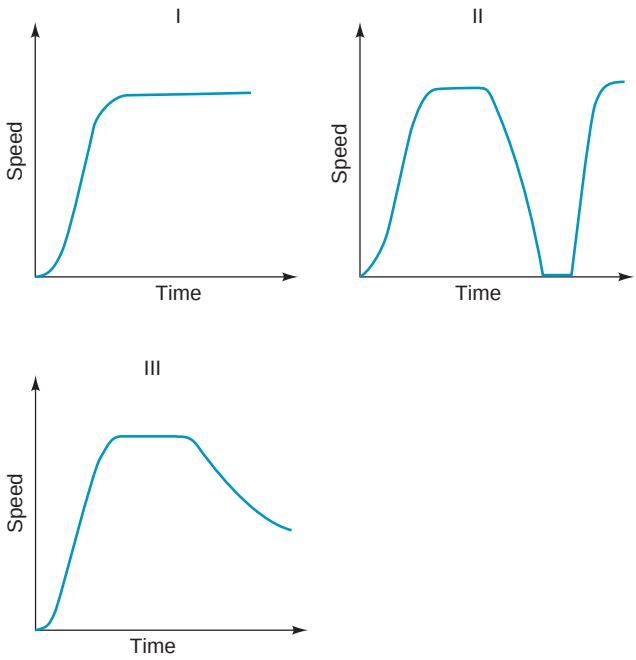
d. Which graph shows that the oven was on at a low temperature for a time before the heat was increased? Mark this portion of the graph.

28. Each of the following graphs represents the speed of a biker as a function of time.

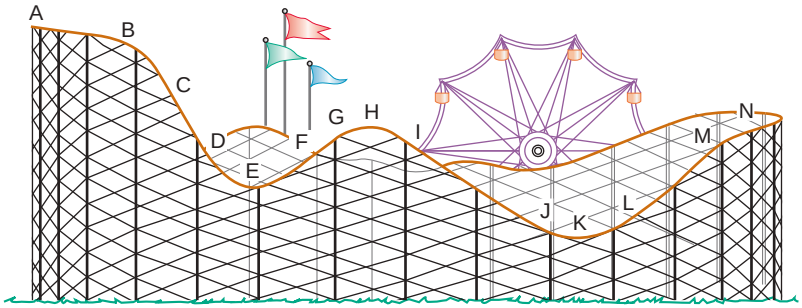
a. Which graph shows that the biker pedaled up a hill? Mark the portion of the graph that indicates the biker was pedaling up a hill.

b. Which graph shows the biker's constant speed was not interrupted? Mark the portion of the graph that indicates the speed was constant.

c. The biker stopped a few minutes for a repair. Which graph shows this? Mark the portion of the graph that indicates the time period for the repair.



Use the following sketch of a roller coaster in problems 29 and 30 to draw the graphs.*



29. Draw a graph of the speeds of the roller coaster from *A* to *H* as a function of the lettered locations on the roller coaster, without using numbers.

30. Draw a graph of the speeds of the roller coaster from *H* to *N* as a function of the lettered locations on the roller coaster, without using numbers.

31. Pat ran a 200-meter race from the swing set to the soccer goal net with her younger brother Hal. Pat runs 4 meters per second, and Hal runs 3 meters per second, so Pat gives her brother a 40-meter head start from the swing set.

a. Use a rectangular grid to form a graph for each person's distance as a function of time. (Copy the rectangular grid from the website.)

b. Mark the points on the graphs that show each person's distance from the swing set after 30 seconds.

c. How much time will have elapsed when they are both the same distance from the swing set?

d. Who will win the race?

e. When the winner wins the race, how far will the other person be from the soccer goal net?

32. At First National Bank the consumer pays a monthly checking account fee of $2 plus 10 cents for each check. At State Savings Bank there is a $1.50 monthly fee for checking accounts with a charge of 15 cents apiece for the first 15 checks and 10 cents apiece for each additional check beyond the 15th.

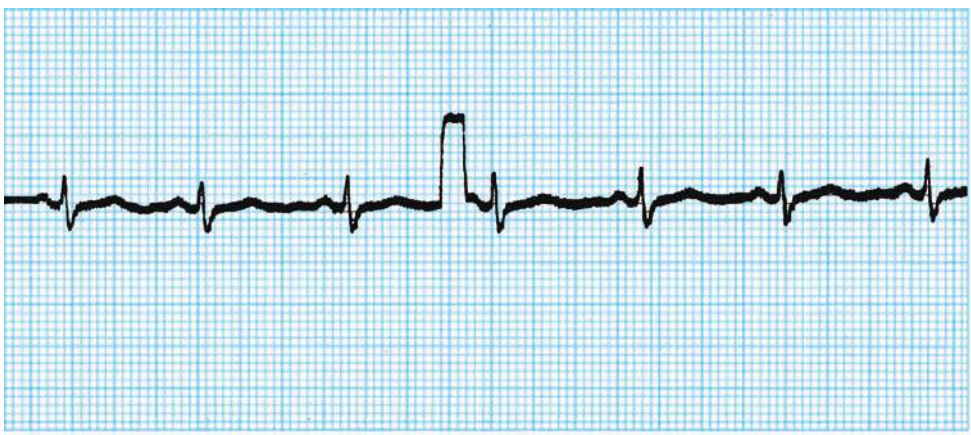
a. Use a rectangular grid to form a graph for both types of checking accounts. Label the costs on the grid for the first 15 checks.

b. Which bank charges more for 15 checks?

c. For what number of checks will the cost to the consumer be the same?

d. Determine which bank charges more for 30 checks, and then determine how much greater this cost is than that charged by the other bank.

33. Electrical impulses that accompany the beat of the heart are recorded on an electrocardiogram (EKG). The electrocardiograph measures electrical changes in millivolts (1 millivolt is $\frac{1}{1000}$ volt). The graph here shows the changes in millivolts as a function of time for a normal heartbeat.



Curriculum and Evaluation Standards for School Mathematics, p. 83.

a. How much time is represented on this graph if each small space on the horizontal axis represents .04 second?

b. The tall rectangular part of the graph was caused by a 10-millivolt signal from the EKG machine. Such a signal is called a *calibration pulse*. How long did this signal last?

c. This graph shows 7 heartbeats, or pulses. Approximately how much time is there between each pulse (from the end of one pulse to the end of the next pulse)? At this rate how many pulses will there be per minute?

34. Featured Strategy: Drawing a Graph. A household uses 4800 kWh of electricity each year. If purchased, the electricity is 11 cents per kWh, for a total cost of $528 per year. Installing solar voltaic panels that produce 3000 kWh per year costs $3960 after local and national energy incentives are applied. Find the number of years before the total cost of electricity for a household with solar panels will equal the total cost of electricity without solar panels.



a. Understanding the Problem. Let's look at the total cost of electricity at the end of the first year with solar panels. The household will buy 1800 kWh (4800 − 3000) of electricity at a cost of $198. So, including installation costs, they have paid $198 + $3960 = $4158 for the first year. Determine the total cost for electricity over a two-year period with solar panels and the total cost for electricity over a two-year period without solar panels.

b. Devising a Plan. One approach to solving this problem is to write and graph equations for the total cost of each system as a function of time. The equation for the total cost of electricity using solar panels at the end of x years is $s(x) = 3960 + 198x$ dollars. In terms of the variables x and $c(x)$, what is the equation for the total cost of electricity without solar panels at the end of x years?

c. Carrying Out the Plan. Draw a graph of the solar cost function, $s(x)$, and a graph of the cost function without solar panels, $c(x)$, together. Use the graphs to determine the number of years when the total costs are equal. That is, how many years will it be before the cost of installing solar panels pays for itself (the *break-even point*)?

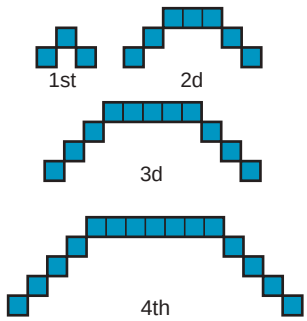
d. Looking Back. The vertical distance between the total cost graphs, $c(x)$ and $s(x)$, represents the difference in costs at any time x. Mark and label your graphs for the cost at the end of year 16 and compute the difference in costs for 16 years.

35. Following are the first four figures in a sequence that uses square tiles.

a. Let $f(n)$ represent the number of tiles in the nth figure. Complete the table for this function.

Figure number	1	2	3	4	5	6	7	8
$f(n)$	3							

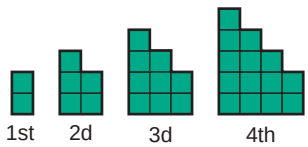
b. Graph the points of the function whose coordinates are in the table. What patterns do you see in the graph?



c. Find a pattern in the numbers of tiles, and determine the value of $f(20)$.

d. Write an algebraic rule for $f(n)$. Use this rule to find $f(350)$.

36. Here are the first four figures in a sequence that uses square tiles.



a. Let $g(n)$ represent the number of tiles in the nth figure. Complete the table for this function.

Figure number	1	2	3	4	5	6	7	8
$g(n)$	2							

b. Graph the points of the function whose coordinates are in the table. What patterns do you see in the graph?

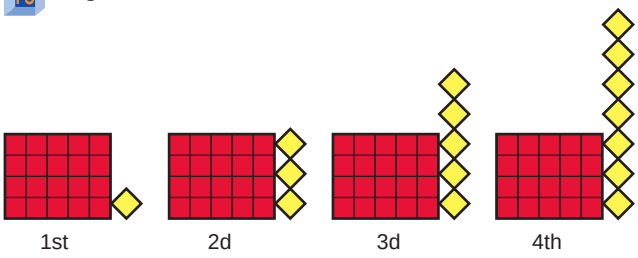
c. Find a pattern in the numbers of tiles, and determine the value of $g(18)$.

d. Write an algebraic rule for $g(n)$. Use this rule to find $g(475)$.

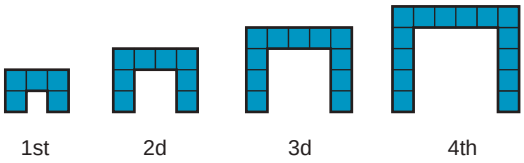
37. Consider the following two sequences of figures.

PS

Sequence 1:



Sequence 2:

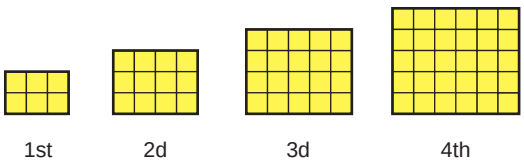


a. Find patterns in sequence 1 and sequence 2, and write an algebraic rule for the number of tiles in the *n*th figure of each sequence.
b. Use a rectangular grid to graph both functions defined by these rules.
c. Find the value of *n* for which the graphs intersect.
d. What information about these sequences is obtained by knowing the point of intersection?

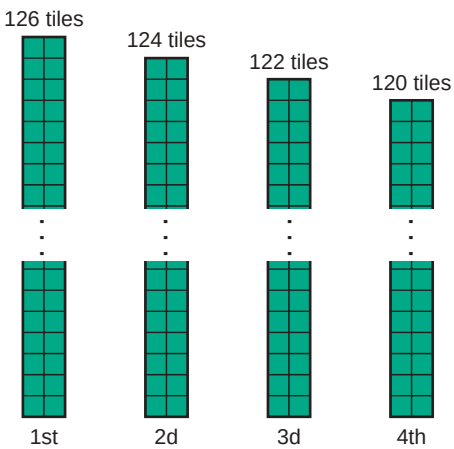
38. Consider the following two sequences of figures.

PS

Sequence 1:



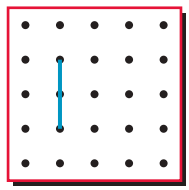
Sequence 2:



a. Find patterns in sequence 1 and sequence 2, and write an algebraic rule for the number of tiles in the *n*th figure of each sequence.
b. Use a rectangular grid to graph both functions defined by these rules.
c. Find the values of *n* for which the graphs intersect.
d. What information about these sequences is obtained by knowing the point of intersection?

Teaching Questions

1. A class of middle school students was forming line segments and slopes on geoboards. The slope was intuitively explained by the teacher by referring to the slope of a roof or the slope of a road. One student asked if the line segment on this geoboard had the greatest possible slope. Explain how you would answer this question.



2. Suppose that you have discussed slopes with your class. Then, after graphing temperatures of cooling water (Example H in this section), a student asks: "Does the temperature graph have a slope?". Research this question and form a response you could give to your student.

3. The Standards quote on page 87 discusses the need for students to learn the relationship of slopes of lines to rates of change. Write an example that would make sense to a middle school student to explain what is meant by *rate of change* and why the slope of a line is a constant rate of change.

4. The Standards quote on page 83 notes that from preschool on, students should encounter functions. If you were introducing this topic by playing "What's My Rule?" (see page 78) with your students, how would you use this game to explain to them the idea of a function?

Classroom Connections

1. On page 82 the example from the **Elementary School Text** asks a variety of questions about the small town. Here are two questions for you to solve, but others may occur to you. (a) Describe the location of the elementary school in relation to the fire house. If Violeta walks

from the library to the high school, and the length of the side of each block is .1 mile, what is the shortest path she can take along the coordinate grid? (b) Which two buildings are the furthest distance apart if you are walking along the coordinate grid blocks?

2. One of the most important connections in mathematics is the discovery by René Descartes that geometric figures can be represented on a coordinate system by equations. Using the footnote reference for the **Historical Highlight** on page 83, or other sources, find more information about this important link between geometry and algebra.

3. In the **Grades 3–5 Standards—Algebra** (see inside front cover) under *Use mathematical models . . .* , read the expectation and describe an example from Section 2.2 that satisfies all aspects of this expectation.

4. Even before formal schooling, children develop beginning concepts related to patterns, functions, and algebra (see **Standards** quote on page 92). Explain how patterns and functions are present in a child's prediction of the next element in the following sequence of squares, circles, and triangles.



MATH ACTIVITY 2.3

Deductive Reasoning Game

Purpose: Use deductive reasoning to discover winning strategies for a logic game.

Pica-Centro is a game for two players (or groups) in which one player uses **deductive reasoning** (obtaining conclusions from given information) to determine the digits of a number that is selected by the other player.† The game begins with player A choosing a three-digit number that contains no zeros and recording it on a slip of paper without showing it to player B. Player B then tries to determine this number by asking questions. Player B records all guesses and player A's responses in the table. Player B's first attempt is a guess, but after this, deductive reasoning is used based on player A's replies. Player A's replies are based on the following rules for each digit:

> **Pica:** A digit that is correct but not in the correct position.
>
> **Centro:** A digit that is both correct and in the correct position.

NCTM Standards

Beginning in the elementary grades, children can learn to disprove conjectures by finding counterexamples. At all levels, children will learn to reason inductively from patterns and specific cases. Increasingly over the grades, they should also learn to make effective deductive arguments. p. 59

***1.** Suppose player A chooses 574 and player B's first guess is 123 (see table below). Then player A responds by saying 0 pica and 0 centro, and player B records this response in the first row of the table. What does player B know from the 0 pica and 0 centro response?

***2.** Player B's second guess is 456 (see table) and player A responds with 2 pica and 0 centro, which player B records in the second row of the table. What does player B know from this response?

***3.** A logical third guess for player B is 654 or 546 or 465. Explain why.

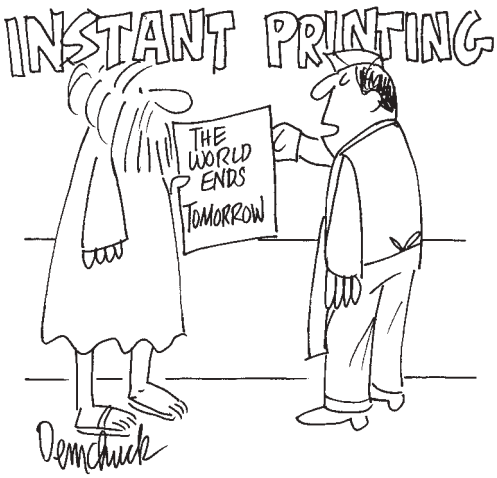
***4.** Suppose player B chooses 654 for the third guess (see table). Then player A would say 1 pica and 1 centro. What can player B deduce from this information? Explain your reasoning. Can player B conclude that the digit 4 is in the correct position? Can player B conclude that 5 is one of the digits in player A's number?

Guesses (Three Digits)			Responses	
			Pica	Centro
1	2	3	0	0
4	5	6	2	0
6	5	4	1	1

5. Make a Pica-Centro table and play this game with another person. Then reverse roles so that both players have a chance to use deductive reasoning. The player who requires the fewest guesses to determine the number is the winner.

†D. B. Aichele, "Pica-Centro, A Game of Logic," *The Arithmetic Teacher*.

Section 2.3 INTRODUCTION TO DEDUCTIVE REASONING



"When do you want it?"

PROBLEM OPENER

Mike won't take part in the school play if Sue is in it. Tim says that in order for him to participate in the play, Sue must be in it. If Mike is in the play, then Rhonda refuses to be part of it. The director insists that only one of the two girls and only one of the two boys will be in the play. Who will be chosen?

Lewis Carroll, well-known author of *Alice's Adventures in Wonderland,* also wrote books on logic. At the beginning of his *Symbolic Logic,** he states that logic will give you

> . . . the power to detect fallacies, and to tear to pieces flimsy illogical arguments which you will so continually encounter in books, in newspapers, in speeches, and even in sermons, and which so easily delude those who have never taken the trouble to master this fascinating art. Try it. That is all I ask of you!

As Lewis Carroll noted, examples of illogical reasoning are common. Consider the following statement:

If the world ends tomorrow, then you will not have to pay for the printing.

Suppose this statement is true. Does this mean that if the world does not end tomorrow, there will be a charge for the printing? This question will be answered in this section.

DEDUCTIVE REASONING

There are two main types of reasoning, *inductive* and *deductive,* and both are common in forming conclusions in our everyday activities. In Chapter 1 we saw that inductive reasoning is the process of forming conclusions on the basis of patterns and observations.

*Lewis Carroll, *Symbolic Logic and the Game of Logic* (New York: Dover Publications, Inc.).

EXAMPLE A

Observation An office clerk notices that a patient has never been on time for an appointment.

Conclusion This person will be late for his or her next appointment.
This conclusion may be true, but we cannot be sure from the given information.

Perhaps you can see why a conclusion based on inductive reasoning, as in Example A, is sometimes called an *informed guess.* **Deductive reasoning,** however, is the process of forming conclusions from one or more given statements as shown in Example B.

EXAMPLE B

Given statements

1. The sum of two numbers is 243.
2. One of the numbers is 56.

Conclusion The other number is 187.
This conclusion follows from the given information.

Examples A and B illustrate the difference between inductive and deductive reasoning: With *inductive reasoning* we form a conclusion that is probable or likely, and with *deductive reasoning* we form a conclusion based on given statements.

NCTM Standards

The *Curriculum and Evaluation Standards for School Mathematics* (p. 81) discusses the importance of both types of reasoning:

> Both inductive and deductive reasoning come into play as students make conjectures and seek to explain why they are valid. Whether encouraged by technology or by challenging mathematical situations posed in the classroom, this freedom to explore, conjecture, validate, and to convince others is critical to the development of mathematical reasoning in the middle grades.

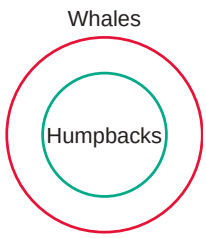
VENN DIAGRAMS

Circles, rectangles, etc. to represent sets were used by John Venn (1834–1923) as a visual aid in deductive reasoning. The following examples illustrate the convenience of *Venn diagrams* in representing information and drawing conclusions. Each given statement is called a **premise.**

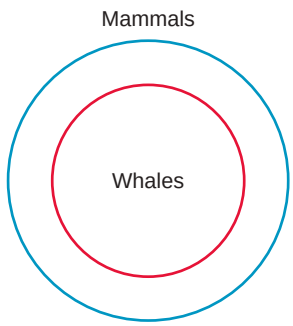
EXAMPLE C

Premises

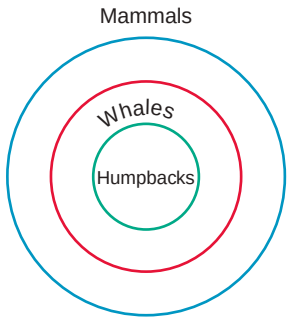
1. All humpbacks are whales.



2. All whales are mammals.



Conclusion All humpbacks are mammals.



NCTM Standards

Students should discuss their reasoning on a regular basis with the teacher and with one another, explaining the basis for their conjectures and the rationale for their mathematical assertions. Through these experiences, students should become more proficient in using inductive and deductive reasoning appropriately. p. 262

The diagram in Example C illustrates that the humpbacks are a subset of the whales and the whales are a subset of the mammals. So the humpbacks are a subset of the mammals. Example C also shows something important about deductive reasoning; namely, it is possible to obtain conclusions from given statements without necessarily having an understanding of the subject matter.

One of the conventions in using Venn diagrams is that if a region inside a circle represents a given set, then the region outside the circle represents all elements that are not in the set. In the next example, the region outside the amphibian circle represents all nonamphibians.

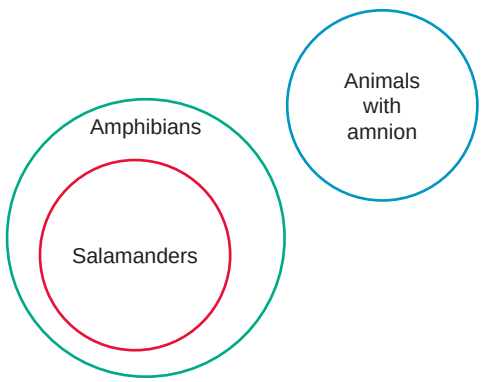
EXAMPLE D

Premises

1. All salamanders are amphibians.
2. Animals that develop an amnion are not amphibians.

NCTM Standards

Being able to reason is essential to understanding mathematics. Building on the considerable reasoning skills that children bring to school, teachers can help students learn what mathematical reasoning entails. p. 56



Conclusion Salamanders do not develop an amnion.

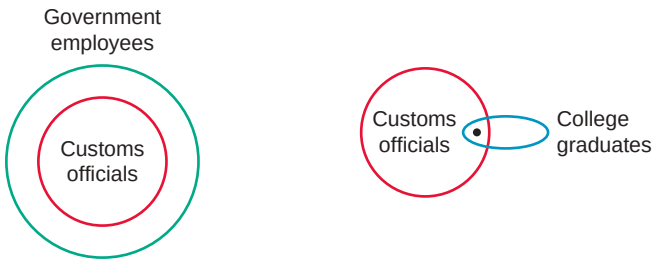
In Example D, premise 2 tells us that animals with an amnion are outside of the amphibian circle, and since the salamanders are inside the amphibian circle, we can conclude that salamanders do not develop an amnion.

Example E involves the word *some,* which means *at least one.* To illustrate with Venn diagrams that at least one element is in two different sets, we place a dot in the overlapping region of the two sets.

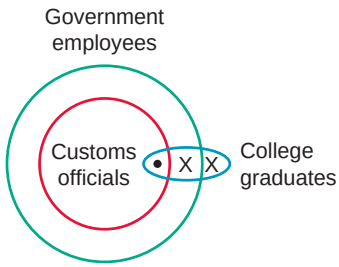
EXAMPLE E

Premises

1. All customs officials are government employees.
2. Some college graduates are customs officials.



Conclusion Some college graduates are government employees.



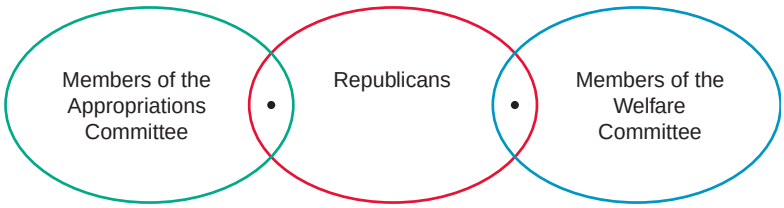
Notice that the diagram for the conclusion in Example E shows that some of the region for the college graduates overlaps the region of the government employees' circle and some of this region is outside the government circle. However, from the given information we cannot conclude that there are college graduates who are outside the government employees' circle, or that there are college graduates inside the government employees' circle but outside the customs officials' circle (see regions marked X).

In Examples B, C, D, and E, we have illustrated deductive reasoning. When a conclusion follows from the given information, as in these examples, we say that the conclusion is **valid** and that we have used **valid reasoning.** When a conclusion does not follow from the given information, we say that the conclusion is **invalid** (or **not valid**) and that we have used **invalid reasoning.** Invalid reasoning is illustrated in Example F. Notice the use of dots to indicate there is at least one person in the intersection of two pairs of these sets.

EXAMPLE F

Premises

1. Some members of the Appropriations Committee are Republicans.
2. Some Republicans are on the Welfare Committee.



Conclusion Some members of the Appropriations Committee are members of the Welfare Committee. (invalid)

It is possible that there are members of the Appropriations Committee who are also members of the Welfare Committee, but from the given information we are not forced to accept this conclusion. Therefore, the stated conclusion is invalid.

CONDITIONAL STATEMENTS

The *given information* in deductive reasoning is often a statement of the form "if . . . , then. . . ." A statement of this form has two parts: the "if" part is called the **hypothesis,** and the "then" part is called the **conclusion.** For example,

Hypothesis Conclusion

If a number is less than 3, then it is less than 8.

A statement in this form is called a **conditional statement.** Many statements that are not in if-then form can be rewritten as conditional statements.

EXAMPLE G

Write the following statements in if-then form.

1. All courses completed with a grade of C will not count for graduate credit.
2. Every public beach must have a lifeguard.
3. You will stay in good condition when you exercise every day.
4. Students will not be admitted after 5 p.m.
5. The patient will have a chance of recovering if he goes through the treatment.

Solution **1.** If a course is completed with a grade of C, then it will not count for graduate credit. **2.** If a beach is public, then it must have a lifeguard. **3.** If you exercise every day, then you will stay in good condition. **4.** If you are a student, then you will not be admitted after 5 p.m. **5.** If he goes through the treatment, then the patient will have a chance of recovering.

NCTM Standards

Students need to explain and justify their thinking and learn how to detect fallacies and critique others' thinking. They need to have ample opportunity to apply their reasoning skills and justify their thinking in mathematics discussions. p. 188

Every conditional statement "if *p,* then *q*" has three related conditional statements that can be obtained by negating and/or interchanging the *if part* and the *then part.* The new statements each have special names that show their relationship to the original statement.

Statement	If *p,* then *q*.
Converse	If *q,* then *p*.
Inverse	If not *p,* then not *q*.
Contrapositive	If not *q,* then not *p*.

EXAMPLE H

Write the converse, inverse, and contrapositive of the following conditional statement.

Statement: If a person lives in Maine, then the person lives in the United States.

Solution **Converse:** If a person lives in the United States, then the person lives in Maine. **Inverse:** If a person does not live in Maine, then the person does not live in the United States. **Contrapositive:** If a person does not live in the United States, then the person does not live in Maine.

The statement in Example H, "If a person lives in Maine, then the person lives in the United States," is diagrammed in Figure 2.18. Notice that the *if part* (hypothesis) is the inner of the two circles. The people who do not live in the United States are outside the larger circle. The same diagram also illustrates the information in the contrapositive of the statement in Example H: If a person does not live in the United States, then the person does not live in Maine. So the diagram in Figure 2.18 illustrates an important fact about if-then statements: A conditional statement and its contrapositive are **logically equivalent.** That is, if one is true, so is the other, and if one is false, so is the other.

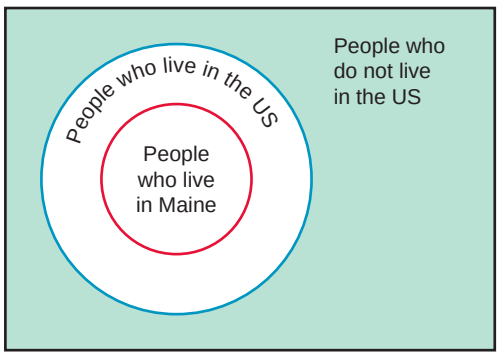


Figure 2.18

Figure 2.18 also illustrates another important fact: If a conditional statement is true, its *converse* and *inverse* are not necessarily true. Consider the converse in Example H: If a person lives in the United States, then the person lives in Maine. The region outside the small circle and inside the large circle shows that it is possible for a person to live in the United States and not live in Maine. So, a conditional statement is not logically equivalent to its converse. Consider the inverse in Example H: If a person does not live in Maine, then the person does not live in the United States. If a person does not live in Maine, this person is outside the small circle, but not necessarily outside the large circle. So we cannot conclude that the person does not live in the United States. This shows that a conditional statement is not logically equivalent to its inverse.

Example H illustrated that a conditional statement and its converse are not logically equivalent. That is, a conditional statement may be true, and its converse may be false. However, when it does happen that a conditional statement and its converse are both true, the two statements are often combined into a single statement by using the words *if and only if.* When this is done, the new statement is called a **biconditional** statement.

EXAMPLE I

The following statement and its converse are both true. Combine them into one statement by using the words if and only if.

Statement: If one of two numbers is zero, then the product of the two numbers is zero.

Converse: If the product of two numbers is zero, then one of the two numbers is zero.

Solution Biconditional: The product of two numbers is zero if and only if one of the numbers is zero. Or, it can be written: One of two given numbers is zero if and only if the product of the two numbers is zero.

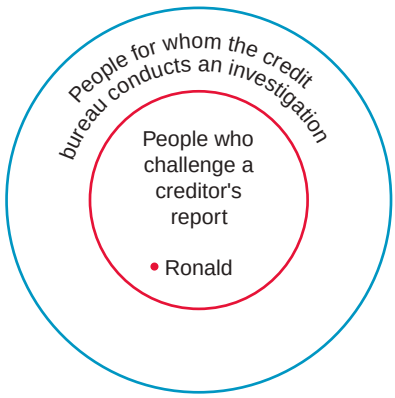
REASONING WITH CONDITIONAL STATEMENTS

The given information (premises) in Examples J and K contains conditional statements. Venn diagrams will be used to show whether the stated conclusions are valid or invalid.

EXAMPLE J

Premises

1. If a person challenges a creditor's report, then the credit bureau will conduct an investigation for that person.
2. Ronald C. Whitney challenged a creditor's report.



Conclusion The credit bureau will conduct an investigation for Ronald C. Whitney. (valid)

NCTM Standards

During grades 3–5, students should be involved in an important transition in their mathematical reasoning . . . formulating conjectures and assessing them on the basis of evidence should become the norm. p. 188

The diagram shows why the conclusion is valid. The two circles represent the information in statement 1. Since statement 2 says that Ronald C. Whitney challenged a creditor's report, he is represented by a point inside the small circle, which means that he is also in the large circle. So the credit bureau will conduct an investigation for Ronald C. Whitney.

Example J illustrates a characteristic of conditional statements: When a conditional statement is given (see premise 1) and the *if part* is satisfied (see premise 2), the *then part* will logically follow. This principle is known as the **law of detachment.** This law can be stated symbolically as follows:

Law of Detachment

Premises

1. If p, then q
2. p

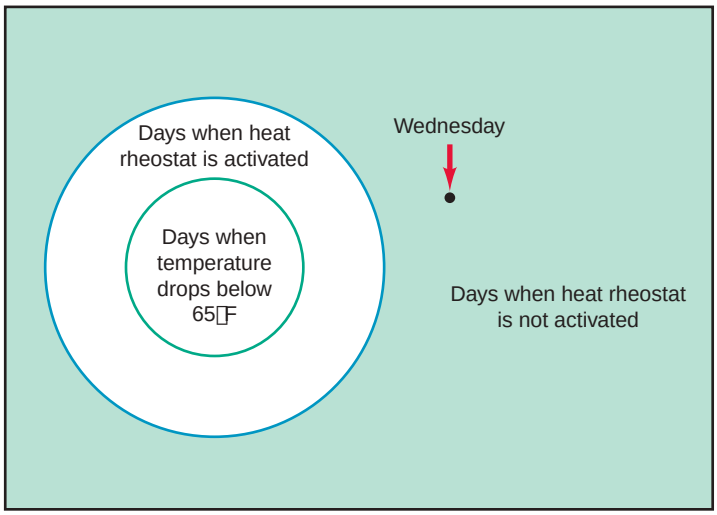
Conclusion q (valid)

Example K illustrates a different situation involving a conditional statement and valid reasoning.

EXAMPLE K

Premises

1. If the temperature drops below 65°F, then the heat rheostat is activated.
2. The heat rheostat was not activated on Wednesday.



Conclusion The temperature did not drop below 65°F on Wednesday. (valid)

The entire region outside the inner circle represents the days when the temperature did not drop below 65°F. Since the dot representing a Wednesday when the heat rheostat was not activated is outside the inner circle, the conclusion is valid.

Example K illustrates another law of reasoning: When a conditional statement is given (see premise 1) and the negation of the *then part* is given (see premise 2), the negation of the *if part* will logically follow. This principle is known as the **law of contraposition.**

Law of Contraposition

Premises
1. If p, then q
2. Not q

Conclusion Not p (valid)

Notice that the law of contraposition follows by replacing the premise "if p, then q" by its contrapositive, "If not q, then not p," and then using the law of detachment.

Premises	
1. If p, then q	1. If not q, then not p
2. Not q	2. Not q
Conclusion Not p (valid)	

Technology Connection

Differences of Squares

The number 65 can be written as the difference of two squares: $9^2 - 4^2 = 65$. What whole numbers can be written as the difference of two squares? Use the online 2.3 Mathematics Investigation to gather data and make conjectures.

Mathematics Investigation
Chapter 2, Section 3
www.mhhe.com/bbn

EXAMPLE L

Determine which of the two laws, *the law of detachment* or *the law of contraposition,* is used in each of the following examples of deductive reasoning.

1. **Premises**
   1. If the trip is over 300 miles, the campers will run out of fuel.
   2. The campers will not run out of fuel.

   Conclusion The trip is not over 300 miles. (valid)
2. **Premises**
   1. If Jan applies for the job, she will be hired.
   2. Jan applies for the job.

   Conclusion Jan will be hired. (valid)
3. **Premises**
   1. If Jones becomes mayor, the town will buy the Walker property.
   2. The town will not buy the Walker property.

   Conclusion Jones will not become mayor. (valid)

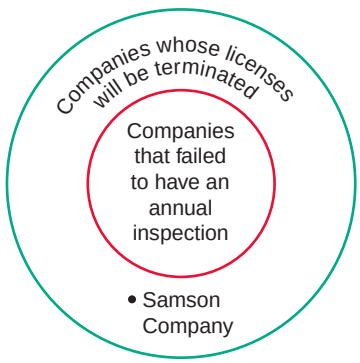
Solution The law of contraposition is used for 1 and 3 and the law of detachment for 2.

Examples M and N illustrate types of invalid reasoning that commonly occur when using conditional statements.

EXAMPLE M

Premises

1. If a company fails to have an annual inspection, then its license will be terminated.
2. The Samson Company's license was terminated.



Conclusion The Samson Company failed to have an annual inspection. (invalid)

The diagram shows that it is possible for the Samson Company to be inside the large circle (satisfying premise 2) but outside the small circle. Since we are not forced to accept the conclusion, it is invalid.

Symbolically, the type of invalid reasoning in Example M is written as

Premises

1. If p, then q
2. q

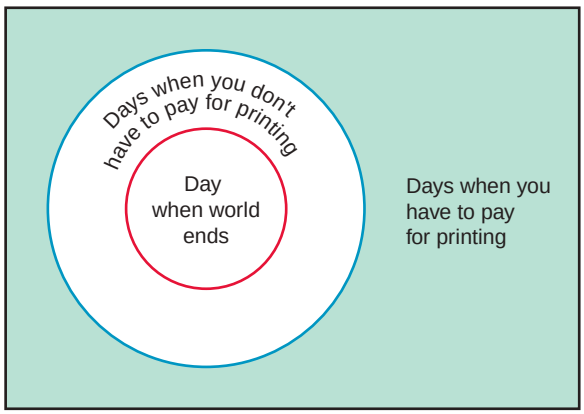
Conclusion p (invalid)

Example N answers the question posed on page 105.

EXAMPLE N

Premises

1. If the world ends tomorrow, then you will not have to pay for the printing.
2. The world does not end tomorrow.



Conclusion You will have to pay for the printing. (invalid)

The two circles in the diagram represent the information in premise 1. Notice that the days when you have to pay for the printing are all outside the large circle. The days when the world does not end tomorrow are outside the small circle, but these days may be inside the large circle. Thus, we are not forced to conclude that you will have to pay for the printing.

The type of invalid reasoning in Example N is written symbolically as

Premises

1. If p, then q
2. Not p

Conclusion Not q (invalid)

EXAMPLE O

Determine whether the following conclusions are valid or invalid. If valid, state whether the law of detachment or the law of contraposition is being used.

1. **Premises**

   a. If a person is a Florida resident, he or she will qualify for the supplemental food plan.

   b. Mallory is not a Florida resident.

   Conclusion Mallory will not qualify for the supplemental food plan.

2. Premises

a. If Jansen does not retire, the piano will be tuned.

b. The piano will not be tuned.

Conclusion Jansen will retire.

3. Premises

a. If the grant is approved, then the Antarctica expedition will be carried out.

b. The Antarctica expedition will be carried out.

Conclusion The grant will be approved.

4. Premises

a. If the new canoes arrive on time, the canoe races will be held.

b. The new canoes will arrive on time.

Conclusion The canoe races will be held.

NCTM Standards

Systematic reasoning is a defining feature of mathematics. It is found in all content areas and, with different requirements of rigor, at all grade levels. p. 57

Solution **1.** Invalid **2.** Valid, law of contraposition **3.** Invalid **4.** Valid, law of detachment

PROBLEM-SOLVING APPLICATION

What Is the Name of This Book? by Raymond M. Smullyan has many original and challenging problems in recreational logic.* The following problem from his book is solved using the problem-solving strategies of *drawing Venn diagrams* and *guessing and checking.*

Problem

An enormous amount of loot has been stolen from a store. The criminal (or criminals) took the loot away in a car. Three well-known criminals A, B, and C were brought to Scotland Yard for questioning. The following facts were ascertained.

1. No one other than A, B, or C was involved in the robbery.
2. C never pulls a job without using A (and possibly others) as an accomplice.
3. B does not know how to drive.

Is A innocent or guilty?

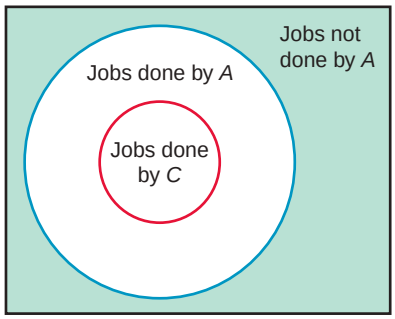
Understanding the Problem Statement 1 says that no one other than A, B, or C was involved in the robbery, but it does not say that all three were involved.

Devising a Plan One approach is to draw a Venn diagram of the given information to see what conclusions can be reached.

Carrying Out the Plan Statement 2 can be diagrammed by placing the jobs done by C inside the circle representing jobs done by A (see the following figure) to show that any time C pulls a job, A is also involved. The jobs not done by A are represented by points outside the large circle. Since we are trying to determine whether A is guilty, let's guess and

*Raymond M. Smullyan, *What Is the Name of This Book?*

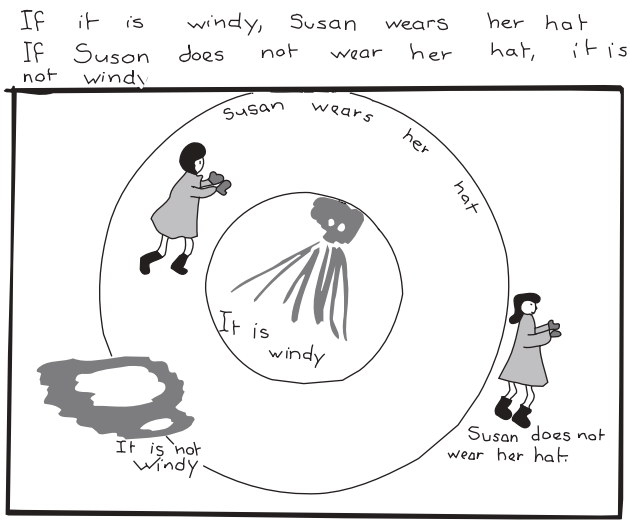
check the results from the diagram. If we guess that *A* is not guilty and select a point outside the large circle, then we know that *C* was not involved. **Question 1:** What does this line of reasoning show?



Looking Back Sometimes it is helpful to write a given statement in if-then form and then write its contrapositive. Statement 2 can be written as "If *C* pulls a job, then *A* pulls a job." **Question 2:** What is the contrapositive of this statement, and how does it help to solve the problem?

Answers to Questions 1–2 **1.** If *A* and *C* are not involved, this leaves only *B*, but *B* does not drive and could not have done the job alone. Therefore, *A* must be guilty. **2.** If *A* does not pull the job, then *C* does not pull the job. The contrapositive tells us that if *A* is not involved in the job, then *C* is not involved, which leaves only *B*. But *B* cannot do the job alone. Therefore, *A* must be guilty.

Exercises and Problems 2.3



This diagram and the statements above it were made by an elementary school student.* Use these statements in exercises 1 and 2.

1. If the first statement is true, is the second statement necessarily true? Explain how the diagram supports your conclusion.
2. If the second statement is true, is the first statement necessarily true? Use the diagram to explain your reasoning.

Rewrite each of the statements in exercises 3 and 4 in if-then form.

3. **a.** Taking a hard line with a bill collector may lead to a lawsuit.
   b. All employees in the Tripak Company must retire by age 65.
   c. There must be 2-hour class sessions if the class is to meet only twice a week.
   d. Every pilot must have a physical examination every 6 months.

*Nuffield Mathematics Project, *Logic*.

4. **a.** The parade will be on Thursday if Flag Day is on Thursday.
b. People under 13 years of age cannot obtain a driver's license.
c. A person who files a written application within 31 days of a termination notification will be issued a new policy.
d. All students in the Moreland district will be bused to the Horn Street School.

In exercises 5 and 6, draw a Venn diagram to illustrate each statement.

5. **a.** All truck drivers are strong people.
b. Some vegetables are green.
c. If an animal is a duck, then it has two legs.
d. If a person was born before 2000, then the person is more than 10 years old.

6. **a.** Every member of the Hillsville 500 Club is an alumnus of Hillsville High School.
b. A tree over the Forest Service's size limit will not be cut.
c. Some phones have a map application.
d. An animal is a mammal if it is a bat.

Write the converse, inverse, and contrapositive of each statement in exercises 7 through 10.

7. If you take a deduction for your home office, then you must itemize your deductions.

8. If the Democrats take California, they will win the election.

9. If switch *B* is pressed, the camera focus is on manual.

10. If the weather is fair, the opera will be sold out.

11. Consider the statement "There will be economic sanctions if they do not agree to U.N. inspections." Which of the following statements is logically equivalent to this statement?
a. If there are economic sanctions, then they will not agree to U.N. inspections.
b. If they do agree to U.N. inspections, then there will not be economic sanctions.
c. If there are no economic sanctions, then they will agree to U.N. inspections.

12. Consider the statement "If a number is less than 15, then it is less than 20." Which of the following statements is logically equivalent to this statement?
a. If a number is not less than 15, then it is not less than 20.
b. If a number is not less than 20, then it is not less than 15.
c. If a number is less than 20, then it is less than 15.

Write the contrapositive of each of the statements in exercises 13 and 14.

13. **a.** If you subtract \$750 for each dependent, then the computer will reject your income tax return.
b. The cards should be dealt again if there is no opening bid.
c. If you are not delighted, return the books at the end of the week's free sing-along.

14. **a.** If this door is opened after 10 p.m., an alarm will sound.
b. This crate of oranges came from the Johnson farm if it is not marked with JJ.
c. The common cold, flu, and other viral diseases occur when the immune system is weak.

Combine each statement and its converse in exercises 15 and 16 into a biconditional statement.

15. If you pay the Durham poll tax, then you are 18 years or older. If you are 18 years or older, then you pay the Durham poll tax.

16. If Smith is guilty, then Jones is innocent. If Jones is innocent, then Smith is guilty.

Write each biconditional statement in exercises 17 and 18 as two separate statements—a conditional statement and its converse.

17. Robinson will be hired if and only if she meets the conditions set by the board.

18. There will be negotiations if and only if the damaged equipment is repaired.

In exercises 19 through 22, sketch Venn diagrams to determine whether each conclusion follows logically from the premises. Explain your reasoning.

19. **Premises:** All flowers are beautiful. All roses are flowers. **Conclusion:** All roses are beautiful.

20. **Premises:** All teachers are smart. All nice people are smart. **Conclusion:** Some nice people are teachers.

21. **Premises:** Some truck drivers are rich. All musicians are rich. **Conclusion:** Some musicians are truck drivers.

22. **Premises:** All good students are good readers. Some math students are good students. **Conclusion:** Some math students are good readers.

Use the law of detachment or the law of contraposition to form a valid conclusion from each set of premises in exercises 23 through 26. Draw a Venn diagram to support your conclusion.

23. **Premises:** If anemia occurs, then something has interfered with the production of red blood cells. The production of red blood cells in this patient is normal.

24. **Premises:** If poison is present in the bone marrow, then production of red blood cells will be slowed down. This patient has poison in her bone marrow.

25. **Premises:** If a person has insufficient vitamin K, there will be a prothrombin deficiency. Mr. Keene does not have a prothrombin deficiency.

26. **Premises:** You are not eligible for a prize if you did not sign up for the steamboat trip. The boating club members are eligible for a prize.

Advertisements are often misleading and tempt people to draw conclusions that are favorable to a certain product. Determine which of the ads in exercises 27 through 31 present valid conclusions based on the first statements. Draw a diagram and explain your reasoning.

27. Great tennis players use Hexrackets. Therefore, if you use a Hexracket, you are a great tennis player.

28. If you follow our program, you will lose weight. You are not following our program if you do not lose weight.

29. It has been proved that the new double-shaft clubs result in longer drives. So if your drives are longer, then you are using these clubs.

30. People who use our aluminum siding are satisfied. Therefore, if you don't use our aluminum siding, you won't be satisfied.

31. If you take Sleepwell, you will have extra energy. Therefore, if you don't have extra energy, you are not taking Sleepwell.

Reasoning and Problem Solving

32. **Featured Strategy: Making a Table.** Janet Davis, Sally Adams, Collette Eaton, and Jeff Clark have occupations of architect, carpenter, diver, and engineer, but not necessarily in that order. You are told:
(1) The first letters of a person's last name and occupation are different.
(2) Jeff and the engineer go sailing together.
(3) Janet lives in the same neighborhood as the carpenter and the engineer. Determine each person's occupation.

a. **Understanding the Problem.** Each person has a different occupation. Janet can't be the diver. Why can't Sally be the architect?

b. **Devising a Plan.** One approach to this type of problem is to make a table with the names along one side and the occupations along another. Then *yes* or *no* can be written in the boxes of the table to record the given information. Explain why *no* can be written four times as shown in the following table.

	Architect	Carpenter	Diver	Engineer
Janet Davis			No	
Sally Adams	No			
Collette Eaton				No
Jeff Clark		No		

c. **Carrying Out the Plan.** Each row and column of the table should have exactly one *yes*. Continue filling out the table to solve this problem.

d. **Looking Back.** One advantage of using such a table is that once *yes* is written in a box, *no* can be written in several other boxes. Each *yes* provides a maximum of how many *no* boxes?

33. Dow, Eliot, Finley, Grant, and Hanley have the following occupations: appraiser, broker, cook, painter, and singer.
(1) The broker and the appraiser attended a father-and-son banquet.
(2) The singer, the appraiser, and Grant all belong to the same-gender club.
(3) Dow and Hanley both belong to a Women's Club.
(4) The singer told Finley that he liked science fiction.
(5) The cook owes Hanley $25.

If three of these people are men, determine each person's gender and occupation.

34. Lee has the flu, and he is concerned that he will not pass the mathematics exam. If the following three statements are true, will Lee pass the exam?
(1) Lee will not fail the mathematics exam if he finishes his computer program.
(2) If he goes to the theater, he does not have the flu.
(3) If he does not go to the theater, he will finish his computer program.

35. Huiru reads the proof of a theorem and finds it easy. Does it follow from the following four statements that the proof is not arranged in logical order?

(1) Huiru can't understand a proof if it is not arranged in logical order.
(2) If Huiru has trouble with a proof, it is not easy.
(3) If Huiru studies a proof without getting dizzy, it is one she understands.
(4) A proof gives Huiru trouble if she gets dizzy while studying it.

36. The morning after the big football game, the school guard found the goal posts missing! With better-than-average luck, the guard had three red-hot suspects by midmorning. The suspects—Andy, Dandy, and Sandy—were questioned, and they made the following statements.

Andy: (1) I didn't do it.
(2) I never saw Dandy before.
(3) Sure I know the football coach.

Sandy: (1) I didn't do it.
(2) Andy lied when he said he never saw Dandy before.
(3) I don't know who did it.

Dandy: (1) I didn't do it.
(2) Andy and Sandy are both pals of mine.
(3) Andy never stole anything.

One and only one of the three suspects is the prankster. One and only one of each person's three statements is false. Who lifted the posts?*

Teaching Questions

1. Two students were discussing the school announcement: "If it rains tomorrow, the tennis match will be cancelled." One student said this meant that if it was sunny, the match would not be cancelled, but the other student disagreed. Which student was correct? Explain how you would resolve their disagreement?

2. If one of your students drew the sketch at the beginning of Exercises and Problems 2.3 to illustrate the first statement above the sketch, and then used the sketch and the fact that the first statement was true to conclude that the second statement was true, would this be an example of inductive or deductive reasoning? Explain.

3. Suppose you told your students that the following two statements were true and asked if they could be sure of the conclusion.

(1) Everyone who is over 36 inches tall is allowed to ride on the roller coaster.
(2) Jay is allowed to ride on the roller coaster.
Conclusion: Jay is over 36 inches tall.

If most of your students thought the conclusion followed from the given information, explain how you would help these students to understand that even though statements (1) and (2) are true, Jay may not be over 36 inches tall.

Classroom Connections

1. In the **Process Standard—Reasoning and Proof** (see inside front cover), read the third and fourth expectations and then locate examples from Section 2.3 that satisfy these expectations.

2. The **Spotlight on Teaching** at the beginning of this chapter discusses the importance of reasoning in mathematics and poses the problem of forming a box of maximum volume by cutting out the corners along grid lines from a sheet of grid paper. (a) Of the nine boxes in this problem, which box has the greatest volume? (b) Is it possible to form a box of greater volume without cutting along the grid lines? (c) Is this method of determining the greatest volume an example of inductive or deductive reasoning?

3. The **Standards** quote on page 104 notes the need for students to use both inductive and deductive reasoning. Classify each of the following student conclusions as using inductive or deductive reasoning. Explain.
a. The box in $7 + \square = 15$ is equal to 8 because 7 plus 3 more is 10 and 10 plus 5 more is 15.
b. I know 35 is greater than 27 because when counting, 35 comes after 27.
c. An odd number plus an even number is odd because that was true for all the examples I tried.
d. The box in $4 \times \square = 32$ is a number less than 10 because 4×10 is 40.

4. The **Standards** quote on page 109 cites the need for students to learn to detect fallacies. One common source of fallacies is "reasoning from the converse," which happens when people conclude that because a statement is true, its converse is also true. Use examples and diagrams to show why "reasoning from the converse" is not valid reasoning.

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CHAPTER 2 REVIEW

1. Sets and Venn Diagrams

a. A **set** is described as a collection of objects called **elements.**

b. The elements of a set may be described with **words,** or they may be **listed.**

c. An **empty set,** or **null set,** is a set with no elements.

d. To show that k is **an element of** set S, we write $k \in S$.

e. **Venn diagrams** use circles, rectangles, or other shapes to illustrate sets.

2. Set Relations

a. Two sets are **disjoint** if they have no elements in common.

b. If every element of A is an element of B, then A is called a **subset** of B, written $A \subseteq B$.

c. If A is a subset of B and B has elements not contained in A, then A is called a **proper subset** of B.

d. Two sets are **equal** if they are subsets of each other.

e. Two sets can be put into **one-to-one correspondence** if it is possible to match each element in one set to exactly one element in the other set and conversely.

f. Two sets have the **same number** of elements if they can be put into one-to-one correspondence.

3. Set Operations

a. The **intersection** of sets A and B is the set of elements that are in both A and B, written as $A \cap B$.

b. The **union** of sets A and B is the set of elements that are in A or in B or in both A and B, written as $A \cup B$.

c. If A and B are disjoint subsets of a given set U, where $A \cup B = U$, then A and B are **complements** of each other, written as $A' = B$ and $B' = A$.

d. A **universal set** is the set that contains all the elements being considered in a given situation.

e. **Venn diagrams** are used to illustrate set relations and operations.

4. Finite and Infinite Sets

a. A set is **finite** if it is empty or can be put into one-to-one correspondence with the set $\{1, 2, 3, \ldots, n\}$, where n is a whole number.

b. A set is **infinite** if it is not finite.

5. Functions and Graphs

a. A **function** is two sets and a rule that assigns each element of the first set to exactly one element of the second set.

b. The first set of a function is called the **domain,** and the second set is called the **range.**

c. A function is a **linear function** if its graph has an equation that can be written in the form $y = mx + b$, where $\boldsymbol{m}$ and $\boldsymbol{b}$ are real numbers.

d. A function is **nonlinear** if its graph is not a line, that is, its graph does not have an equation of the form $y = mx + b$.

e. If the graph of a function consists of only separate points, it is called a **discrete graph.**

f. If two lines have the same slope, they are parallel.

6. Rectangular Coordinate System

a. The **rectangular coordinate system** is a method for locating the points on a plane by reference to a pair of perpendicular lines called the ***x* axis** and the ***y* axis.**

b. The first number in the ordered pair (x, y) is called the ***x* coordinate** and the second number is called the ***y* coordinate.**

c. The slope of a line containing points whose coordinates are (a, b) and (c, d) is the rise $(b - d)$ divided by the run $(a - c)$, for $a \neq c$.

d. An equation in the form $y = mx + b$ is called the **slope-intercept form** because the slope $\boldsymbol{m}$ and the y intercept $\boldsymbol{b}$ can be seen from the equation.

7. Deductive Reasoning

a. **Deductive reasoning** is the process of obtaining conclusions from one or more given statements, called **premises.**

b. When a conclusion follows from the given information, it is said to be **valid,** and the process of deriving the conclusion is called **valid reasoning.**

c. When a conclusion does not follow from the given information, the conclusion and the reasoning process are said to be **invalid.**

8. Conditional Statements

a. A statement in if-then form is called a **conditional statement.**

b. Every conditional statement "if p, then q" has three related statements:
Converse: If q, then p.
Inverse: If not p, then not q.
Contrapositive: If not q, then not p.

c. Two statements are said to be **logically equivalent** if when the first statement is true, the second statement is true and when the first statement is false, the second statement is false.

d. When a conditional statement is given (premise 1) and the *if part* is satisfied (premise 2), the *then part* will always follow. This principle is known as the **law of detachment.**

e. When a conditional statement is given (premise 1) and the negation of the *then part* is given, the negation of the *if part* will always follow. This principle is called the **law of contraposition.**

f. When a conditional statement and its converse are combined into one statement by using *if and only if,* the new statement is called a **biconditional statement.**

CHAPTER 2 TEST

1. Use the attribute pieces shown here to determine the sets satisfying the given conditions.



a. Hexagonal and yellow
b. Triangular or blue
c. Not (yellow or square)

2. Given the universal set $U = \{0, 1, 2, 3, 4, 5, 6\}$ and the sets $A = \{2, 4, 6\}$ and $B = \{1, 2, 3, 4\}$, determine the following sets.

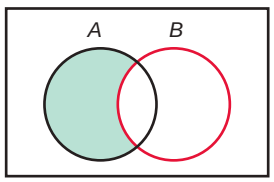
a. $A \cap B$ **b.** $A \cup B$
c. $A' \cap B$ **d.** $A \cup B'$

3. Sketch a Venn diagram to illustrate each of the following conditions.

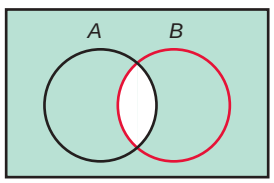
a. $E \cap F \neq \varnothing$
b. $E \subseteq G$
c. $F \cap G = \varnothing$ and $E \subseteq F$
d. $E \cap G \neq \varnothing$ and $(E \cup G) \cap F = \varnothing$
e. $E \subseteq G$ and $F \subseteq E$
f. $E \subseteq F$, $G \subseteq F$, and $E \cap G = \varnothing$

4. Use set notation to name the shaded regions shown here.

a.



b.



5. Answer each question, and draw a Venn diagram to support your conclusion.

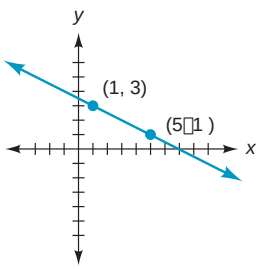
a. If $k \in R \cup S$, is $k \in R \cap S$?
b. If $x \in T \cap W$, is $x \in T \cup W$?
c. If $y \in R \cap S$, is $y \in S'$?

6. Consider the function that relates the length x of each line segment to $f(x)$, which is half of the length of the segment.

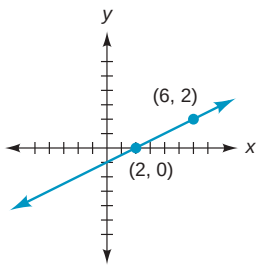
a. Determine the range value $f(x)$ for each of the following domain values: 1, 2, 3, 4, and 5.
b. Use the coordinates in part a or a graphing calculator to sketch the graph of this function. (Copy the coordinate system from the website.)
c. What is the equation of this function?
d. Is this function linear or nonlinear?

7. Determine the slope of each line.

a.



b.



c. If a line is parallel to the line in part a and contains the point (2, 3), what is the equation of the new line?

8. Write the equation of the line through the points (0, 4) and (3, 13) in the slope-intercept form.

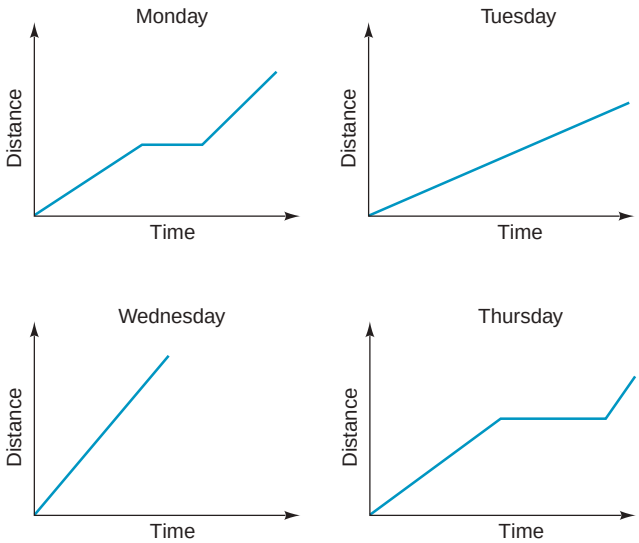
9. A travel company charges \$120 for insurance and \$55 a day to rent a trailer. Let x represent the number of days and y the total cost of renting a trailer.

a. Write an equation for the total cost of renting the trailer.
b. What is the total cost of renting the trailer for 10 days?
c. If the travel budget allows \$850 for trailer rental and a trailer can only be rented for a whole number of days, for how many days could you afford to rent the trailer?

10. An electrician normally charges \$75 per hour but on holidays the charge is \$95 per hour. During a certain period the electrician worked 60 hours and received \$4860. Let x represent the number of holiday hours worked and write an algebraic expression for the items in parts a to c.

a. The total amount of money received for the holiday hours worked.
b. The number of hours worked on nonholidays.
c. The total amount of money received for working on nonholidays.
d. How many holiday hours did the electrician work?

11. Alanna used the following types of transportation to go to school on each of 4 days: biking, walking, skateboarding, and jogging. On the day she jogged, she stopped for 3 minutes at a construction site, and on the day she skateboarded, she stopped for 6 minutes to talk with a friend. The graphs of her distances as functions of time are shown here. Determine which type of transportation she used each day if she only used each type of transportation once.



12. Of 75 cars that were inspected, 12 needed brake repair and 18 needed exhaust system repair. If the brakes or exhaust systems on 50 of the cars did not need repair, how many cars needed both brake and exhaust system repairs?

13. 150 men live in a certain town: 85 are married, 70 have a telephone, 75 own a car, 55 are married and have a telephone, 35 have a telephone and a car, 40 are married and have a car, and 30 are married, have a car, and have a telephone. How many men are single and do not have either a car or a telephone?

14. Rewrite each statement in if-then form.
a. People who are denied credit have a right to protest to the credit bureau.
b. All the children who were absent yesterday were absent again today.
c. Everybody at the party received a gift.

15. Write the converse, inverse, and contrapositive of each of the following statements.
a. If Mary goes fishing, then her husband goes with her.
b. If you join the book club, you will receive five free books.

16. "If the temperature drops below 10ºF, the culture dies." This statement is logically equivalent to which of the following statements?
(1) If the culture dies, the temperature drops below 10ºF.
(2) If the temperature does not drop below 10ºF, the culture does not die.
(3) If the culture does not die, the temperature does not drop below 10ºF.

17. Combine the following statement and its converse into a biconditional statement: If there are peace talks, then the prisoners will be set free. If the prisoners are set free, then there will be peace talks.

18. Determine whether each conclusion below is valid or invalid.
a. **Premises:** All mallards are aggressive birds. Some black ducks are aggressive birds. **Conclusion:** Some black ducks are mallards.
b. **Premises:** All geometry classes are interesting. Some math classes are geometry classes. **Conclusion:** Some math classes are interesting.
c. **Premises:** If people are happy, then they have enough to eat. All rich people have enough to eat. **Conclusion:** Some rich people are happy.

19. Use each set of premises to form a valid conclusion.
a. **Premises:** If a person is healthy, then the person has about 10 times as much lung tissue as necessary. The people in ward *B* have less lung tissue than necessary.
b. **Premises:** If an illegal move is made, the game pieces should be set up as they were before the move. An illegal move was made.

20. Determine whether each conclusion is valid or invalid.
a. **Premises:** You may keep the books if you like everything about them. John kept the books. **Conclusion:** John liked everything about the books.
b. **Premises:** If this year's tests are successful, the United States will be using laser communications by 2012. This year's tests were successful. **Conclusion:** The United States will be using laser communications by 2012.

CHAPTER

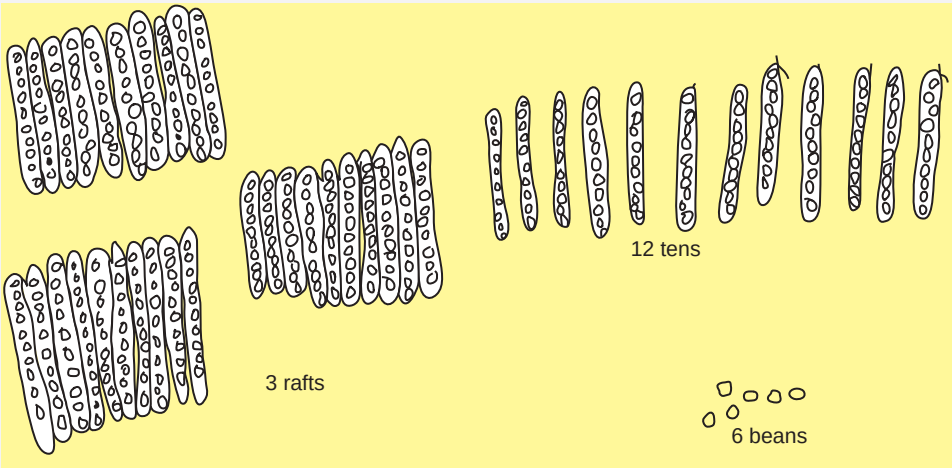
3 Whole Numbers

Spotlight on Teaching

Excerpts from NCTM's Standards for School Mathematics Grades Pre-K through 2*

Teachers have a very important role to play in helping students develop facility with computation. By allowing students to work in ways that have meaning for them, teachers can gain insight into students' developing understanding and give them guidance. . . . Consider the following hypothetical story, in which a teacher poses this problem to a class of second graders:

> We have 153 students at our school. There are 273 students at the school down the street. How many students are in both schools?
>
> As would be expected in most classrooms, the students give a variety of responses that illustrate a range of understandings. For example, Randy models the problem with bean sticks that the class has made earlier in the year, using hundreds rafts, tens sticks, and loose beans.

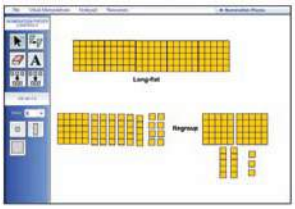


* *Principles and Standards for School Mathematics*, p. 86.

MATH ACTIVITY 3.1

Numeration and Place Value with Base-Five Pieces

Virtual Manipulatives



www.mhhe.com/bbn

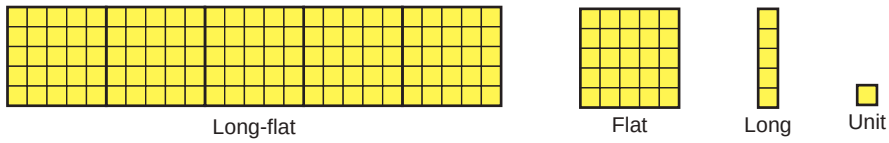
NCTM Standards

Concrete models can help students represent numbers and develop number sense, . . . but using materials, especially in a rote manner, does not ensure understanding. p. 80

Purpose: Explore whole number numeration concepts with base-five pieces. Base-five pieces are used to provide a fresh look at numeration concepts and to help develop a deeper understanding of numeration systems.

Materials: Base-Five Pieces in the Manipulative Kit or Virtual Manipulatives.

1. The four base-five pieces shown here are called **unit, long, flat,** and **long-flat.** Examine the numerical and geometric patterns of these pieces as they increase in size, and describe how you would design the next larger piece to continue your pattern.



***2.** The two collections shown here both contain 36 units, but collection 2 is called the **minimal collection** because it contains the smallest possible number of base-five pieces. Use your base-five pieces to determine the minimal collection for each of the following numbers of units.



a. 84 units
b. 147 units
c. 267 units

Long-Flats	Flats	Longs	Units

3. In base-five numeration, 3 flats, 2 longs, and 4 units are recorded by the numeral 324_{five}. Sketch the base pieces for each of the following numerals, and determine the total number of units in each collection.

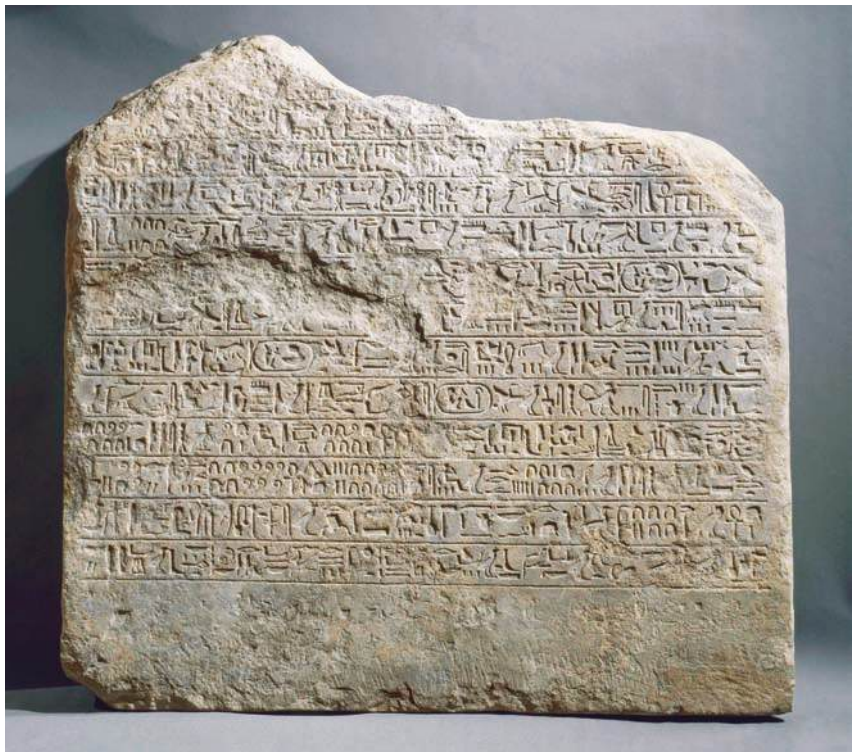
a. 1304_{five} **b.** 221_{five} **c.** 213_{five} **d.** 1023_{five}

***4.** Starting with the unit, sketch the first four base-three pieces.

a. What is the minimal collection of base-three pieces for 16 units?

b. What is the total number of units represented by 2112_{three} (2 long-flats, 1 flat, 1 long, 2 units)?

Section 3.1 NUMERATION SYSTEMS



Egyptian stone giving an account of the expedition of Amenhotep III in 1450 B.C.E.

PROBLEM OPENER

A 7 is written at the right end of a two-digit number, thereby increasing the value of the number by 700. Find the original two-digit number.

There are no historical records of the first uses of numbers, their names, and their symbols. A number is an idea or abstraction that represents a quantity. Written symbols for numbers are called **numerals** and probably were developed before number words, since it is easier to cut notches in a stick than to establish phrases to identify a number.

A logically organized collection of numerals is called a **numeration system.** Early numeration systems appear to have grown from tallying. In many of these systems, 1, 2, and 3 were represented by |, ||, and |||. By 3400 B.C.E. the Egyptians had an advanced system of numeration for numbers up to and exceeding 1 million. Their first few number symbols show the influence of the simple tally strokes (Figure 3.1).



Figure 3.1

Their symbol for 3 can be seen in the third row from the bottom of the stone inscriptions shown above. What other symbols for single-digit numerals can you see on this stone?

GROUPING AND NUMBER BASES

As soon as it became necessary to count large numbers of objects, the counting process was extended by grouping. Since the fingers furnished a convenient counting device, grouping by 5s was used in some of the oldest methods of counting. The left hand was generally used to keep a record of the number of objects being counted, while the right index finger pointed to the objects. When all 5 fingers had been used, the same hand would be used again to continue counting. In certain parts of South America and Africa, it is still customary to "count by hands": 1, 2, 3, 4, hand, hand and 1, hand and 2, hand and 3, etc.

EXAMPLE A

Use the "count by hands" system to determine the names of the numbers for each of the following sets of dots.

1. 
2. 
3. 

Solution **1.** 2 hands and 2. **2.** 3 hands and 4. **3.** 4 hands and 3.

NCTM Standards

Young children's earliest mathematical reasoning is likely to be about number situations, and their first mathematical representations will probably be of numbers. p. 32

The number of objects used in the grouping process is called the **base.** In Example A the base is five. By using the numerals 1, 2, 3, and 4 for the first four whole numbers and *hand* for the name of the base, it is possible to name numbers up to and including 24 (4 hands and 4).

Base Ten As soon as people grew accustomed to counting by the fingers on one hand, it became natural to use the fingers on both hands to group by 10s. In most numeration systems today, grouping is done by 10s. The names of our numbers reflect this grouping process. *Eleven* derives from the medieval German phrase *ein lifon,* meaning one *left over,* and *twelve* is from *twe lif,* meaning *two over ten.* The number names from 13 to 19 have similar derivations. *Twenty* is from *twe-tig,* meaning *two tens,* and *hundred* means *ten times ten.** When grouping is done by 10s, the system is called a **base-ten numeration system.**

ANCIENT NUMERATION SYSTEMS

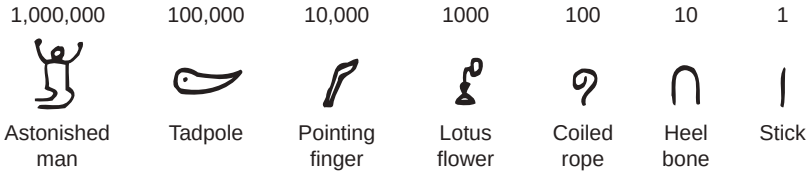
Egyptian Numeration The ancient Egyptian numeration system used picture symbols called **hieroglyphics** (Figure 3.2). This is a base-ten system in which each symbol represents a power of ten.

HISTORICAL HIGHLIGHT

There are many traces of base twenty from different cultures (see Example F). The Mayas of Yucatán and the Aztecs of Mexico had elaborate number systems based on 20. Greenlanders used the expression *one man* for twenty, *two men* for 40, etc. A similar system was used in New Guinea.

Evidence of grouping by 20 among the ancient Celtics can be seen in the French use of *quatre-vingt* (four-twenty) for 80. In our language the use of *score* suggests past tendencies to count by 20s. Lincoln's familiar Gettysburg Address begins, "Four score and seven years ago." Another example occurs in a childhood nursery rhyme: "Four and twenty blackbirds baked in a pie."

*H. W. Eves, *An Introduction to the History of Mathematics,* 3d ed. (New York: Holt, Rinehart and Winston, 1969), pp. 8–9.



Egyptian Symbols

Figure 3.2

EXAMPLE B

Write the following numbers, using Egyptian numerals.

1. 2342 **2.** 14,026

Solution 1.  2. 

The Egyptian numeration system is an example of an *additive numeration system* because each power of the base is repeated as many times as needed.

Additive Numeration System In an **additive numeration system,** some number b is selected for a base and symbols representing 1, b, b^2, b^3, etc., for powers of the base. Numbers are written by repeating these powers of the base the necessary number of times.

In the Egyptian numeration system, $b = 10$ and the powers of the base are 1, 10, 10^2, 10^3, etc. In an additive numeration system, the symbols can be written in any order. In Example B the powers of the base are descending from left to right, but the Egyptian custom was to write them in ascending powers from left to right, as shown in the stone inscriptions on page 125.

EXAMPLE C

Notice the numeral for 743 near the left end of the third row from the bottom of the Egyptian stone on page 125. The symbols for 3 ones, 4 tens, and 7 hundreds are written from left to right. What other Egyptian numerals can you find on this stone?

Solution It appears that the center of the third row up contains 75 and the left end of the fourth row up has 250. Parts of many other numerals can be seen.

Roman Numeration Roman numerals can be found on clock faces, buildings, gravestones, and the preface pages of books. Like the Egyptians, the Romans used base ten. They had a modified additive numeration system, because in addition to the symbols for powers of the base, there are symbols for 5, 50, and 500. The seven common symbols are

I	V	X	L	C	D	M
1	5	10	50	100	500	1000

Roman Numerals

Historical evidence indicates that C is from *centum,* meaning *hundred,* and M is from *milli,* meaning *thousand.* The origin of the other symbols is uncertain. The Romans wrote their numerals so that the numbers they represented were in decreasing order from left to right.

EXAMPLE D Write the following numbers, using Roman numerals.

1. 2342 **2.** 1996

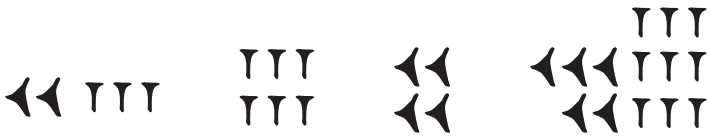
Solution **1.** MMCCCXXXXII **2.** MDCCCCLXXXXVI

When a Roman numeral is placed to the left of a numeral for a larger number, its position indicates subtraction, as in IX for 9, XL for 40, XC for 90, CD for 400, or CM for 900. The subtractive principle was recognized by the Romans, but they did not make much use of it.* (In fact, the subtractive principle has only been in common use for about the past 200 years.) Compare the preceding Roman numeral for 1996 with the following numeral written using the subtractive principle:

MCMXCVI

The Romans had relatively little need for large numbers, so they developed no general system for writing them. In the inscription on a monument commemorating the victory over the Carthaginians in 260 B.C.E., the symbol  for 100,000 is repeated 23 times to represent 2,300,000.

Babylonian Numeration The Babylonians developed a base-sixty numeration system. Their basic symbols for 1 through 59 were additively formed by repeating  for 1 and  for 10. Four such numerals are shown here.



23 6 40 59

To write numbers greater than 59, the Babylonians used their basic symbols for 1–59 and the concept of *place value*. **Place value** is a power of the base, and the Babylonian place values were 1, 60, 60^2, 60^3, etc. Their basic symbols had different values depending on the position or location of the symbol. For example, $135 = 2(60) + 15(1)$, so the Babylonians wrote their numeral for 2 to represent 2×60 and their numeral for 15 for the number of units, as shown next. Generally, the first position from right to left represented the number of units, the second position the number of 60s, the third position the number of 60^2s, etc.



$2(60) + 15(1) = 135$ $22(60^2) + 3(60) + 30(1) + 79{,}410$

EXAMPLE E Write the following numbers using Babylonian numeration.

1. 47 **2.** 2473 **3.** 10,821

Solution **1.**  **2.**  **3.** 

$47(1)$ $41(60) + 13(1)$ $3(60^2) + 21(1)$

*D. E. Smith, *History of Mathematics,* 2d ed. (Lexington, MA: Ginn 1925), p. 60.

Technology Connection

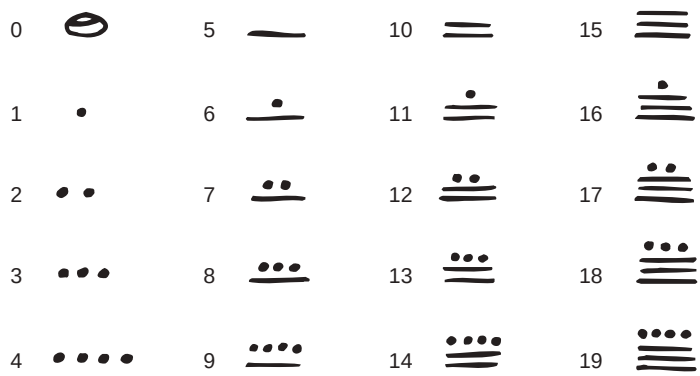
The Number 6174

Select any four-digit number and form a new number by placing its digits in decreasing order from left to right. Then form the reverse of this number and subtract the smaller from the larger. Continue this process and you will find a surprising result. Explore this and related questions in this investigation.

Mathematics Investigation
Chapter 3, Section 1
www.mhhe.com/bbn

The solution to the third part of Example E illustrates a weakness in the Babylonian system. The number 10,821 is equal to $3(60^2) + 0(60) + 21(1)$, but because there was no symbol for zero in the Babylonian system, there was no way to indicate 0(60), that is, the missing power of 60. Babylonians who saw the symbols in the third part of Example E might have thought it represented $3(60) + 21(1)$. A larger gap between symbols was sometimes used to indicate that a power of the base was missing, and later, symbols were used to indicate a missing power of the base.

Mayan Numeration The Mayas used a modified base-twenty numeration system that included a symbol for zero. Their basic symbols for 0 through 19 are shown in Figure 3.3. Notice that there is grouping by 5s within the first 20 numbers.



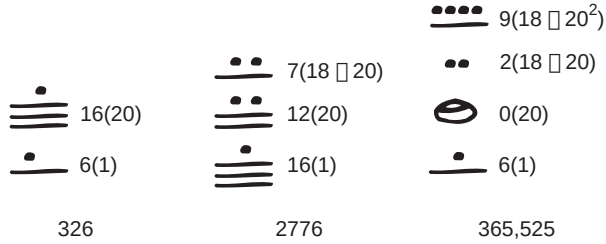
Mayan Symbols

Figure 3.3

To write numbers greater than 19, the Mayas used their basic symbols from 0 to 19 and place value. They wrote their numerals vertically with one numeral above another, as shown in Example F, with the powers of the base increasing from bottom to top. The numeral in the bottom position represented the number of units. The numeral in the second position represented the number of 20s. Because the Mayan calendar had 18 months of 20 days each, the place value of the third position was 18×20 rather than 20^2. Above this position, the next place values were 18×20^2, 18×20^3, etc.

EXAMPLE F

These three Mayan numerals represent the following numbers: $16(20) + 6(1) = 326$; $7(18 \times 20) + 12(20) + 16(1) = 2776$; and $9(18 \times 20^2) + 2(18 \times 20) + 0(20) + 6(1) = 65{,}526$.

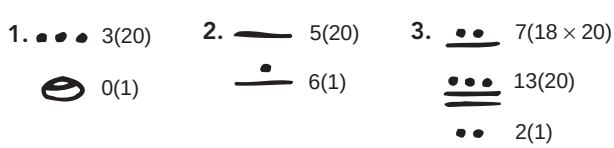


EXAMPLE G

Write the following numbers using Mayan numerals.

1. 60 **2.** 106 **3.** 2782

Solution



Notice the necessity in the Mayan system for a symbol that has the same purpose as our numeral zero. In part 1 of Example G, their symbol for zero occupies the lower place and tells us that the three dots have a value of 3×20 and there are zero 1s.

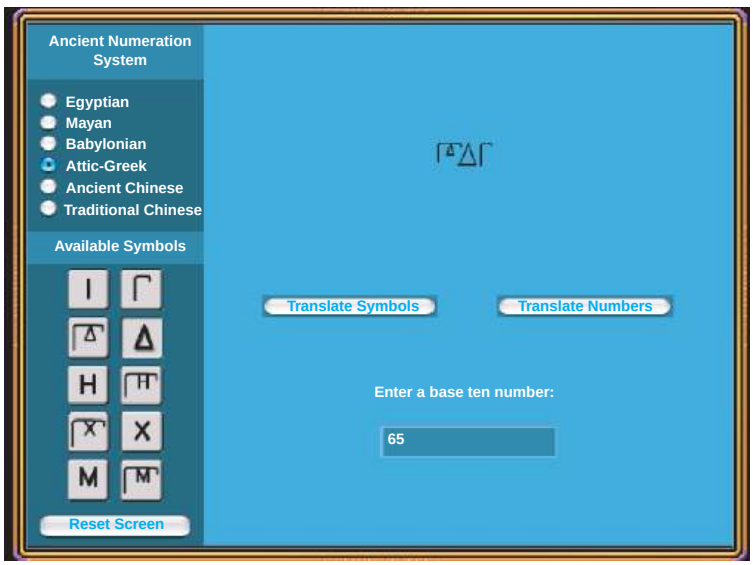
HISTORICAL HIGHLIGHT



There is archaeological evidence that the Mayas were in Central America before 1000 B.C.E. During the Classical Period (300 to 900), they had a highly developed knowledge of astronomy and a 365-day calendar with a cycle going back to 3114 B.C.E. The pyramid of Kukulkan at Chichén Itzá, pictured at the left, was used as a calendar: four stairways, each with 91 steps and a platform at the top, made a total of 365. Their year was divided into 18 months of 20 days each with 5 extra days for holidays. Because their numeration system was developed mainly for calendar calculations, they used 18×20 for the place value in the third position, rather than 20×20.

Technology Connection

Can you decipher this Attic-Greek ancient numeration system? To decipher this system, drag the symbols onto the workspace and click "Translate Symbols" to determine their value. There are five other ancient numeration systems for you to decipher in this applet.



Deciphering Ancient Numeration Systems Applet, Chapter 3
www.mhhe.com/bbn

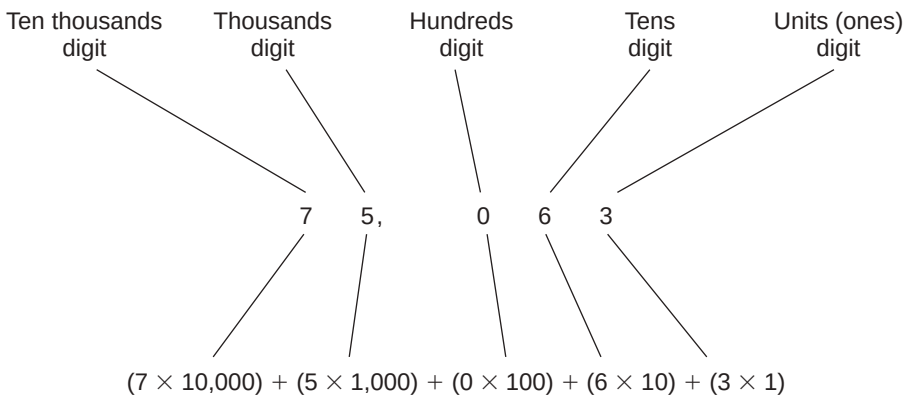
Hindu-Arabic Numeration Much of the world now uses the Hindu-Arabic numeration system. This positional numeration system was named for the Hindus, who invented it, and the Arabs, who transmitted it to Europe. It is a base-ten numeration system in which place value is determined by the position of the **digits** 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Each digit in a numeral has a name that indicates its position.

EXAMPLE H

Here are the names and values of the digits in 75,063.

NCTM Standards

It is absolutely essential that students develop a solid understanding of the base-ten numeration system and place-value concepts by the end of grade 2. p. 81



When we write a number as the sum of the numbers represented by each digit in its numeral (see Example H), we are writing the number in **expanded form.** Another common method of writing a number in expanded form is to write the powers of the base using exponents. For example, $7(10^4) + 5(10^3) + 0(10^2) + 6(10^1) + 3(1)$.

The Hindu-Arabic numeration system is an example of a positional numeration system. In general,

Positional Numeration System In a **positional numeration system,** a number is selected for a base and basic symbols are adopted for 0, 1, 2, . . . up to one less than the base. (In our numeration system these basic symbols are the 10 digits 0, 1, 2, . . . , 9.) Whole numbers are represented in a positional numeration system by writing one or more basic symbols side by side with their positions indicating increasing powers of the base.

EXAMPLE I

Determine the value of each underlined digit and its place value.

1. 7<u>0</u>24 **2.** 3<u>7</u>0,189 **3.** 49,<u>2</u>38

Solution **1.** The value is 0, and the place value is hundreds. **2.** The value is 70,000, and the place value is ten thousands. **3.** The value is 200, and the place value is hundreds.

READING AND WRITING NUMBERS

In English the number names for the whole numbers from 1 to 20 are all single words. The names for the numbers from 21 to 99, with the exceptions of 30, 40, 50, etc., are compound number names that are hyphenated. These names are hyphenated even when they occur as parts of other names. For example, we write *three hundred forty-seven* for 347.

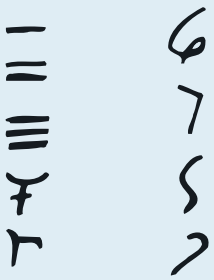
Numbers with more than three digits are read by naming each group of three digits (the **period** of the digits). Within each period, the digits are read as we would read any number from 1 to 999, and then the name of the period is recited. The names for the first few periods are shown in the following example.

EXAMPLE J Read the following number.

$$\underbrace{2\ 3,}_{\text{Trillion}}\quad\underbrace{4\ 7\ 8,}_{\text{Billion}}\quad\underbrace{5\ 0\ 6,}_{\text{Million}}\quad\underbrace{0\ 4\ 2,}_{\text{Thousand}}\quad 3\ 1\ 9$$

Solution This number is read as twenty-three trillion, four hundred seventy-eight billion, five hundred six million, forty-two thousand, three hundred nineteen. *Note:* The word *and* is not used in reading a whole number.

HISTORICAL HIGHLIGHT



There are various theories about the origin of our digits. It is widely accepted, however, that they originated in India. Notice the resemblance of the Brahmi numerals for 6, 7, 8, and 9 to our numerals. The Brahmi numerals for 1, 2, 4, 6, 7, and 9 were found on stone columns in a cave in Bombay dating from the second or third century B.C.E.* The oldest dated European manuscript that contains our numerals was written in Spain in 976. In 1299, merchants in Florence were forbidden to use these numerals. Gradually, over a period of centuries, the Hindu-Arabic numeration system replaced the more cumbersome Roman numeration system.

*J. R. Newman, *The World of Mathematics* (New York: Simon and Schuster, 1956), pp. 452–454.

ROUNDING NUMBERS

If you were to ask a question such as "How many people voted in the 2008 presidential election?" you might be told that in "round numbers" it was about 131 million. Approximations are often as helpful as the exact number, which in this example is 131,257,328.

Research Statement

Research on students' number sense shows that students continue to have difficulty representing and thinking about large numbers.

Sowder and Kelin

One method of **rounding** a number to the nearest million is to write the nearest million greater than the number and the nearest million less than the number and then choose the closer number. Of the following numbers, 131,257,328 is closer to 131,000,000.

132,000,000
131,257,328 rounds to 131,000,000
$\longrightarrow$
131,000,000

The more familiar approach to rounding a number uses place value and is stated in the following rule.

Rule for Rounding Whole Numbers

1. Locate the digit with the place value to which the number is to be rounded, and check the digit to its right.
2. If the digit to the right is 5 or greater, then each digit to the right is replaced by 0 and the digit with the given place value is increased by 1.
3. If the digit to the right is 4 or less, each digit to the right of the digit with the given place value is replaced by 0.

EXAMPLE K

Round 131,257,328 to the following place values.

1. Ten thousands **2.** Thousands **3.** Hundreds

Solution

1. Ten thousands place

↓

131,257,328 rounds to 131,260,000

⟶

2. Thousands place

↓

131,257,328 rounds to 131,257,000

⟶

3. Hundreds place

↓

131,257,328 rounds to 131,257,300

⟶

MODELS FOR NUMERATION

NCTM Standards

NCTM's K–4 Standard, *Number Sense and Numeration* in the *Curriculum and Evaluation Standards for School Mathematics* (p. 39), says that place value is a critical step in the development of children's understanding of number concepts:

> Since place-value meanings grow out of grouping experiences, counting knowledge should be integrated with meanings based on grouping. Children are then able to use and make sense of procedures for comparing, ordering, rounding, and operating with larger numbers.

There are many models for illustrating positional numeration and place value. The bundles-of-sticks model and base-ten number pieces will be introduced in Examples L, M, and N and then used to model operations on whole numbers in the remainder of this chapter.

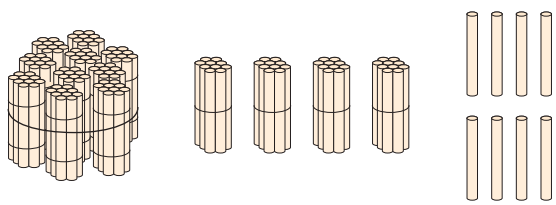
Bundles-of-Sticks (or Straws) Model In this model, units and tens are represented by single sticks and bundles of 10 sticks, respectively. One hundred is represented by a bundle of 10 bundles.

EXAMPLE L

The following figure shows the bundle-of-sticks model for representing 148.

NCTM Standards

Using concrete materials can help students learn to group and ungroup by tens. For example, . . . to express "23" as 23 ones (units), 1 ten and 13 ones, or 2 tens and 3 ones. p. 81



100 40 8

Base-Ten Pieces In this model, the powers of 10 are represented by objects called **units, longs,** and **flats:** 10 units form a long, and 10 longs form a flat (Figure 3.4). Higher powers of the base can be represented by sets of flats. For example, 10 flats placed in a row are called a **long-flat** and represent 1000.

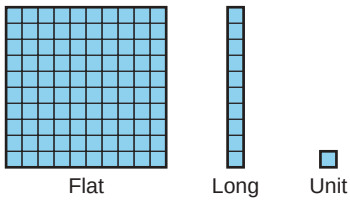
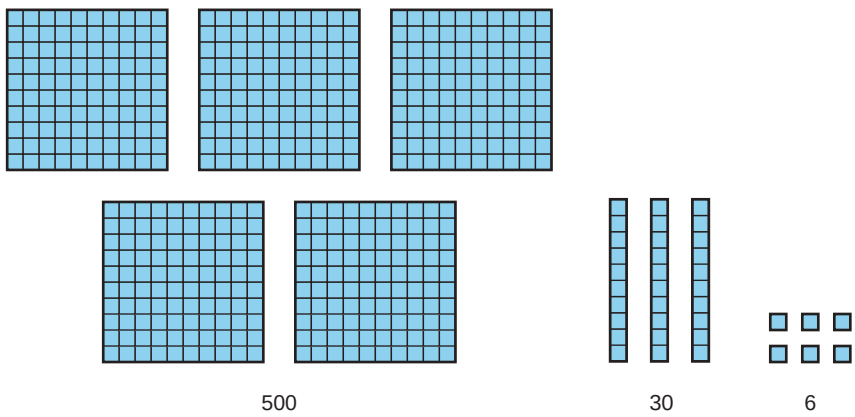


Figure 3.4

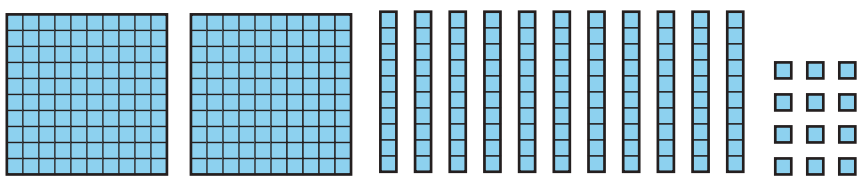
EXAMPLE M Sketch base-ten pieces to represent 536.

Solution



Bundles of sticks and base-ten pieces can be used to illustrate the concept of **regrouping:** changing one collection to another that represents the same number.

EXAMPLE N Sketch the minimum number of base-ten pieces needed to replace the following collection. Then determine the base-ten number represented by the collection.



Solution The new collection will have 3 flats, 2 longs, and 2 units. This collection of base-ten pieces represents 322.

Base-Five Numeration The base-five pieces are models for powers of 5, and as in the case of base ten, there are pieces called **units, longs,** and **flats:** 5 units form a long, and 5 longs form a flat. The next higher power of 5 is represented by placing 5 flats end to end to form a **long-flat.**

EXAMPLE O

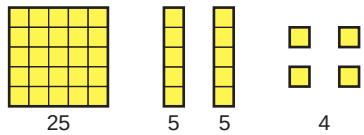
Sketch the minimum number of base-five pieces to represent the following number of units.

1. 39 units **2.** 115 units **3.** 327 units

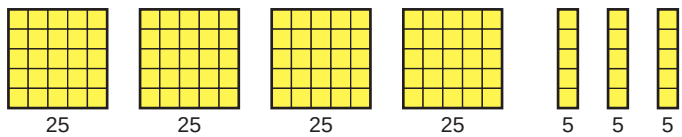
NCTM Standards

Representing numbers with various physical materials should be a major part of mathematics instruction in the elementary school grades. p. 33

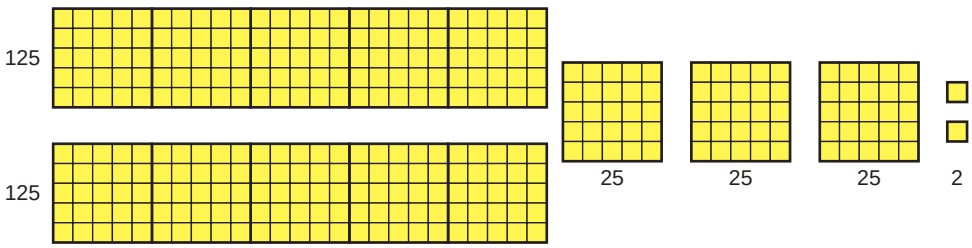
Solution **1.** A collection with 1 flat, 2 longs, and 4 units



2. A collection with 4 flats, 3 longs, and 0 units



3. A collection with 2 long-flats, 3 flats, 0 longs, and 2 units



Research Statement

No more than half of the 4th and 5th grade students interviewed demonstrated an understanding that the 5 in 25 represents five of the objects and the 2, the remaining 20.

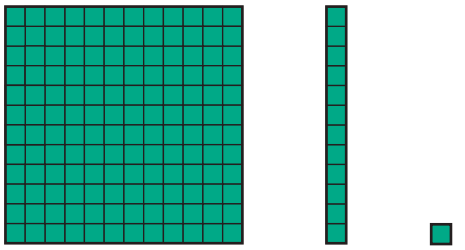
Ross

Positional numeration is used to write numbers in various bases by writing the numbers of long-flats, flats, longs, and units from left to right, just as we do in base ten. From Example O, 39 in base-ten numeration is written as 124_{five} in base-five numeration. Similarly, 115 is written as 430_{five}, and 327 is written as 2302_{five}. Since base ten is the standard base, we do not write the subscript to show that a number is being written in base-ten numeration.

PROBLEM-SOLVING APPLICATION

The next problem introduces the strategy of **reasoning by analogy,** which involves forming conclusions based on similar situations. For example, we know that when we add two numbers, *the greater the numbers, the greater the sum.* Reasoning by analogy, we might conclude that *the greater the numbers, the greater the product.* In this case the conclusion is true. This type of reasoning, however, is not always reliable; the conclusion *the greater the numbers, the greater the difference* would be false. The problem-solving strategies of *reasoning by analogy* and *using a model* are used on the next page to solve a problem involving base-twelve positional numeration.

Problem



How can numbers be written in base-twelve positional numeration?

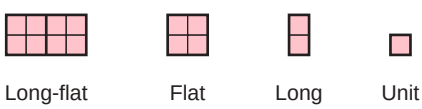
Understanding the Problem The count-by-hands method of counting, which was introduced in the opening pages of this section, is a base-five system. **Question 1:** In that system, what digits are needed to name any number from 1 to 24?

Devising a Plan Consider a similar problem: Why are 0, 1, 2, 3, . . . , 9 the only digits needed in base ten? Referring to the base-ten pieces, we know that if there are more than nine of one type of base-ten piece, we can replace each group of 10 pieces by a piece representing the next higher power of 10. This suggests using similar pieces for base twelve. The first three base-twelve pieces are shown above. **Question 2:** How can this model be extended?

Carrying Out the Plan To count in base twelve, we can say 1, 2, 3, 4, 5, 6, 7, 8, 9, but then we need new symbols for ten and eleven because 10 in base twelve represents 1 long and 0 units, which equals twelve units; and 11 in base twelve represents 1 long and 1 unit, which equals thirteen units. One solution is to let T represent the number 10 and E represent the number eleven. Then the first few numerals in base twelve are 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E, 10, 11, 12, 13, . . . , where 12 represents 1 long and 2 units (fourteen units), etc. In base-twelve positional numeration, 3 flats, 2 longs, and 8 units are written as 328_{twelve}. **Question 3:** Why does any base-twelve numeral require only the twelve symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T, and E?

Looking Back These models suggest ways to visualize other number bases, such as base two, base seven, or base sixteen. **Question 4:** What digits are needed in base two, and what would the base-two pieces look like?

Answers to Questions 1–4 **1.** 0, 1, 2, 3, and 4. For example, 2 hands and 1, 3 hands and 4, etc. **2.** The next base-twelve piece has a row of 12 flats. **3.** Whenever there are 12 of any base-twelve pieces, they can be replaced by the next larger base-twelve piece. If there are no pieces of a given type, the 0 is needed in the numeral to indicate this. **4.** The only digits needed in base two are 0 and 1. The first four base-two pieces are shown here.



Technology Connection

One skill in helping children acquire number sense and familiarity with place value is counting up by 10s from a given number. This can be practiced with a calculator that has a constant function key or is programmed to achieve this function. On such calculators, which are designed for elementary school students, the following key strokes will produce a sequence of numbers, with each number being 10 more than the preceding number.

Keystrokes	View Screen
26	26
[ ] 10 [ ]	36
[ ]	46
[ ]	56

In a similar manner, but using [−], a calculator with a constant function enables students to practice counting down by 10s. If a calculator does not have a constant function, the preceding sequence 26, 36, 46, . . . can be generated on most calculators by entering 26 and repeatedly pressing [+] 10 [=]. Similarly, the keystrokes below will produce the decreasing sequence 54, 44, 34, ...

Keystrokes	View Screen
64 [−] 10 [ ]	54
[−] 10 [ ]	44
[−] 10 [ ]	34
[−] 10 [ ]	24

HISTORICAL HIGHLIGHT

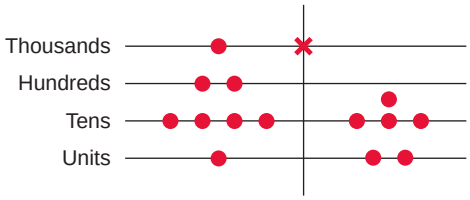


In the fifteenth and sixteenth centuries, there were two opposing opinions on the best numeration system and methods of computing. The *abacists* used Roman numerals and computed on the abacus and the *algorists* used the Hindu-Arabic numerals and place value. The sixteenth-century print at the left shows an abacist competing against an algorist. The abacist is seated at a reckoning table with four horizontal lines and a vertical line down the middle. Counters, or chips, placed on lines represented powers of 10. The thousands line was marked with a cross to aid the eye in reading numbers. If more lines were needed, every third line was marked with a cross. This practice gave rise to our modern custom of separating groups of three digits in a numeral by a comma.*

*D. E. Smith, *History of Mathematics,* 2d ed. (Lexington, MA: Ginn, 1925), pp. 183–185.

Exercises and Problems 3.1

The chips on the lines of this reckoning table each represent one of the indicated powers of 10. Use this table to answer 1 and 2.



1. What number is represented on the left side of the reckoning table?

2. Each chip in a space between the horizontal lines represents half as much as it would on the line above. What number is represented on the right side of this reckoning table?

Use the following counting system of a twentieth-century Australian tribe to answer exercises 3 and 4.

Neecha	Boolla	Boolla Neecha	Boolla Boolla
1	2	3	4

3. If this system were continued, what would be the names for 5 and 6?

4. How would even numbers differ from odd numbers in a continuation of this system?

5. In the base-five system of counting by fingers and grouping by hands, the name for 7 is *1 hand and 2*.
a. What is the name for 22 in this system?
b. If 25 is called a *hand of hands,* what is the name for 37 in this system?

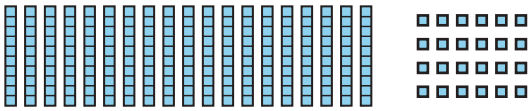
6. The following number names are literal translations of number words taken from primitive languages in various parts of the world.* Follow this pattern, and write in the missing names.

5	whole hand
6	one on the other hand
8	
10	
11	1 on the foot
15	
16	
20	person
21	1 on the hands of the next person
25	
30	
40	

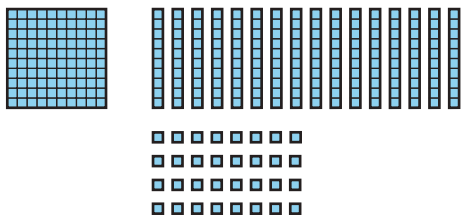
*D. Smeltzer, *Man and Number,* pp. 14–15.

Sketch the minimum number of base-ten pieces needed to replace each set in 7 and 8.

7.



8.



Sketch base pieces for the unit, long, and flat for each of the bases in exercises 9 and 10.

9. **a.** Base seven
b. Base three

10. **a.** Base five
b. Base twelve

Sketch the minimum number of base pieces for the bases given in exercises 11 and 12 to represent the following set of units. Then write the number of units in positional notation for the given base.



11. **a.** Base seven
b. Base five

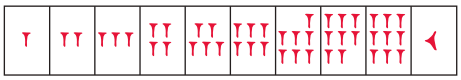
12. **a.** Base twelve
b. Base three

In exercises 13 and 14, determine the total number of units to which the base pieces are equivalent for the given base.

13. **a.** Base five: 3 flats, 4 longs, 3 units
b. Base eight: 6 flats, 0 longs, 5 units

14. **a.** Base twelve: 8 flats, 7 longs, 8 units
b. Base three: 2 long-flats, 1 flat, 2 longs, 1 unit

15. The numeration systems on the next page were used at different times in different geographic locations. Compare these sets of numerals for the numbers 1 through 10. What similarities can you find? For each system, what evidence is there of grouping by 5s?



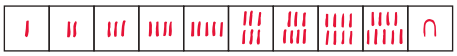
Babylonian numerals 1–10



Roman numerals 1–10



Mayan numerals 1–10



Egyptian numerals 1–10

In exercises 16 and 17, write each number in the given system.

16. **a.** Egyptian numeration: 3275
b. Roman numeration: 406
c. Babylonian numeration: 8063
d. Mayan numeration: 48

17. **a.** Egyptian numeration: 40,208
b. Roman numeration: 1776
c. Babylonian numeration: 4635
d. Mayan numeration: 172

18. The Greek numerals shown below date from about 1200 B.C.E. Use these symbols and the additive numeration system to write 2483.

1	10	100	1000
			

19. The Attic-Greek numerals were developed sometime prior to the third century B.C.E. and came from the first letters of the Greek names for numbers. Use the clues in the following table to find the missing numerals. What base is used in this system?

1	4	8		26
				

32	52	57	206	
				

Write each number in exercises 20 and 21 in expanded form.

20. **a.** 256,049 **b.** 4033_{five}

21. **a.** 7,082,555 **b.** 14321_{five}

Determine the value of each underlined digit and its place value in exercises 22 and 23.

22. **a.** 3$\underline{7}$2,089
b. $\underline{1}11{,}111_{\text{five}}$
c. $\underline{9}2{,}441{,}000$

23. **a.** $43\underline{1}2_{\text{five}}$
b. 70$\underline{0}$,000
c. $\underline{2}{,}947{,}831$

Write the names of the numbers in exercises 24 and 25.

24. **a.** 5,438,146
b. 31,409
c. 816,447,210,361
d. 62,340,782,000,000

25. **a.** 4040
b. 793,428,511
c. 30,197,733
d. 5,210,999,617

Round the numbers in exercises 26 and 27 to the nearest given place value.

26. 375,296,588
a. Million
b. Hundred thousand
c. Ten million
d. Thousand

27. 43,668,926
a. Hundred thousand
b. Ten thousand
c. Thousand
d. Hundred

Make a sketch of the given model for each number of units in exercises 28 and 29.

28. **a.** 136, using base-ten pieces
b. 47, using the bundle-of-sticks model
c. 108, using base-five pieces
d. 35, using base-three pieces

29. **a.** 108, using the bundle-of-sticks model
b. 570, using base-ten pieces
c. 93, using base-five pieces
d. 70, using base-twelve pieces

In exercises 30 and 31, enter the number in the top view screen into your calculator. What numbers and operations can be entered to change the screen to the one under it without changing the digits that are the same in both screens?

30. a.

1034692.
1834692.

b.

938647.
908047.

c.

36859.
3859.

31. a.

72913086.
78913086.

b.

3270521.
3470821.

c.

7496146.
749146.

In exercises 32 and 33, assume that the calculator view screen displays nine digits and that numbers are entered into the calculator using only the keys 1, 2, 3, 4, 5, 6, 7, 8, and 9.

32. a. What is the greatest whole number that can be formed in the calculator view screen if each of these keys is used exactly once?

b. What is the greatest whole number that can be formed in the calculator view screen if a key can be used more than once?

33. a. What is the smallest whole number that can be formed to fill the calculator view screen if each of these keys is used exactly once?

b. What is the smallest whole number that can be formed to fill the calculator view screen if a key can be used more than once?

If each number and operation in exercises 34 and 35 is entered into a calculator in the order in which it occurs from left to right, what number will appear in the calculator view screen?

34. a. $3 \times 1000 + 4 \times 100 + 0 \times 10 + 7$

b. $8 \times 10{,}000 + 3 \times 10 + 1$

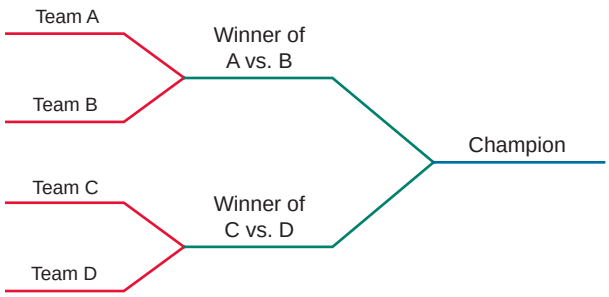
35. a. $7 \times 100{,}000 + 7 \times 1000 + 7$

b. $12 \times 1000 + 8 \times 100 + 3 \times 10 + 2$

Reasoning and Problem Solving

36. Featured Strategies: Making a Drawing, Making a Table, and Finding a Pattern. A single-elimination basketball tournament has 247 teams competing for the championship. If the tournament sponsors must pay $20 to have each game refereed, what is the total cost of referees for the tournament?

a. Understanding the Problem. For every two teams that play each other, there is a winner and a loser. The loser is eliminated from the tournament, and the winner plays another team. The following brackets for a four-team tournament show that three games are needed to determine a champion.



If the number of teams entered in the tournament is not a power of 2, byes are necessary. That is, some teams will be unopposed in the first round so that the number of teams for the second round will be a power of 2. Draw a set of brackets for a seven-team tournament. How many teams will be unopposed in the first round?

b. Devising a Plan. One approach to solving this problem is to use small numbers and to look for a pattern. Complete the following table.

No. of teams	2	3	4	5	6	7	8
Total no. of games	1		3				

c. Carrying Out the Plan. To solve this problem, use the approach suggested in part b and inductive reasoning, or use your own plan. What is the total cost of referees for the tournament?

d. Looking Back. The brackets in part a directed our attention to the winning teams. The total number of games played can be more easily determined by thinking about the losing teams. Each game that is played determines one loser. How many losing teams will there be in the tournament?

37. Powers of 2 are used in base-two numeration systems (**binary numbers**). Here are the first seven powers of 2.

1	2	4	8	16	32	64
2^0	2^1	2^2	2^3	2^4	2^5	2^6

Amy and Joel found that the first 25 whole numbers either are powers of two or can be written as a sum of powers of two so that each power of two is used only once or not at all. For example, $25 = 16 + 8 + 1$. Is this true for whole numbers greater than 25? Try a few such numbers and form a conjecture.

38. What four-digit whole number satisfies the following conditions? The sum of the digits is 6; the number is less than 1200; none of the four digits are equal; and the tens digit is an odd number.

39. A three-digit number satisfies the following conditions: The digits are consecutive whole numbers; the sum of each pair of digits is greater than 4 and less than 10; and the tens digit is an even number. What is the number?

40. What is Jared's favorite six-digit number if the tens digit is his favorite digit, the sum of the hundred thousands digit and the thousands digit is his favorite digit, and the digits in his number from the largest place value to the smallest place value are consecutive numbers?

41. The third, fourth, and fifth floors of a business building are being remodeled. The rooms will be numbered using all the whole numbers from 300 to 599. The front door of each room will be numbered with bronze digits. How many bronze numerals for the digit 3 will be needed to number these rooms?

42. What is the two-digit number that satisfies the following conditions? The tens digit is larger than the units digit; the sum of the digits is 11; and if the digits are reversed and the resulting two-digit number is subtracted from the original number, the difference is 27.

Teaching Questions

1. How would you answer the student who asks: "If we use base ten because we have ten fingers, what would our numeration look like if we used base twenty since we have twenty fingers and toes?"
2. One mathematics educator expressed the opinion that we should not teach ancient numeration systems by saying, "leave ancient numeration systems to the ancients." Make a case for or against teaching ancient numeration systems in elementary schools.
3. Explain what you think is meant by "number sense" in the following quote and give an example of how students might use a concrete model ineffectively as noted in this statement. "Concrete models can help students represent numbers and develop number sense, . . . , but using materials, especially in a rote manner, does not ensure understanding." Standards 2000, p. 80.

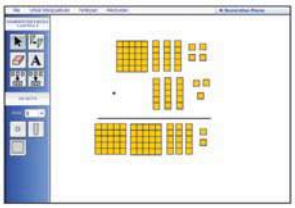
Classroom Connections

1. In the **PreK–2 Standards—Number and Operations** (see inside front cover) under *Understand Numbers . . .*, read the expectation that involves the use of multiple models. Name three models from Section 3.1 and explain how they satisfy this expectation.
2. Compare the **Standards** quote on page 131 with the **Research** statement on page 135. What conclusions can you draw from these two statements? Explain why you think that more than half the fifth and sixth grade students have this deficiency.
3. The **Standards** quote on page 133 notes that "concrete materials can help students learn to group and ungroup by tens." Use one of the models from this section to illustrate and explain how this can be done.
4. The one-page **Math Activity** at the beginning of this section introduces base-five pieces. Explain some of the advantages of using base five to help understand our base-ten numeration system.
5. The **Historical Highlight** on page 126 gives examples of number bases from different cultures. Check the Internet for at least two other examples of ancient numeration systems, other than those in this section, that used different number bases. If possible, offer a guess as to why each number base might have been selected.
6. The **Spotlight on Teaching** at the beginning of Chapter 3 shows an elementary school student's use of the bean sticks model. Find the mistake in the student's sketch. What questions could you ask this student to determine if the mistake was or was not a conceptual error?

MATH ACTIVITY 3.2

Addition and Subtraction with Base-Five Pieces

Virtual Manipulatives

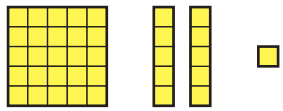


www.mhhe.com/bbn

Purpose: Play base-five games to explore grouping in whole number addition and subtraction.

Materials: Two dice and Base-Five Pieces in the Manipulative Kit or Virtual Manipulatives.

***1. Trading-Up Game** (two to four players) Use your base-five pieces to play. On a player's turn two dice are rolled. The product (or the game can be played with sums) of the two numbers on the dice is the number of units the player wins. At the end of a player's turn, the pieces should be traded (regrouped) so that the total winnings are represented by the *minimal collection* (smallest possible number of base-five pieces). The first player to get 1 long-flat or more wins the game. Suppose on a player's first turn, two 6s were rolled on the dice. The product of 36 is represented by the base-five pieces shown here. On the player's second turn, a 4 and 5 were rolled on the dice. What is the player's minimal collection after the second turn?



2. The following tables show the numbers of base-five pieces that were won in the Trading-Up Game by three players after each player had seven turns. (LF, F, L, and U denote long-flat, flat, long, and unit, respectively.) Use your base-five pieces to determine the minimal collection each player had at the end of the game. Who won the game?

Player 1

LF	F	L	U
		3	3
		4	4
		3	0
		2	2
		4	4
		4	0
		2	2
Total			

Player 2

LF	F	L	U
		2	0
		2	2
		4	0
		3	3
		4	0
	1	1	0
		2	0
Total			

Player 3

LF	F	L	U
		1	0
		2	2
	1	0	0
		3	1
	1	1	0
		3	3
		1	1
Total			

***3.** In the first three turns of a Trading-Up Game, a player won 124_{five} units and in the next three turns won a total of 134_{five} units. Use your base-five pieces to represent each number, and determine the minimum collection of flats, longs, and units the player had after six turns. In the remaining turns, what is the minimal collection of base-five pieces the player will need to obtain one long-flat? Explain how you arrived at your answer.

4. Trading-Down Game Use your base-five pieces to play. Each player begins this game with one long-flat and removes the number of units determined by the product of the numbers on the dice. The object is to be the first player to get rid of all base-five pieces. Suppose that after four turns in this game a player has 3 flats, 1 long, and 0 units. If the player rolls double 6s on the fifth turn, what is the minimal collection of base-five pieces the player will have after the fifth turn?

Section 3.2 ADDITION AND SUBTRACTION



PROBLEM OPENER

Use each of the digits 0 through 9 exactly once to obtain the smallest whole-number difference.

$$\square\,\square\,\square\,\square\,\square$$
$$-\ \square\,\square\,\square\,\square\,\square$$

Children learn addition at an early age by using objects. If 2 clams are *put together* with 3 clams, the total number of clams is the **sum** $2 + 3$. The idea of *putting sets together,* or *taking their union,* is often used to define addition.

Addition of Whole Numbers If set R has r elements and set S has s elements, and R and S are disjoint, then the **sum** of r plus s, written $r + s$, is the number of elements in the union of R and S. The numbers r and s are called **addends.**

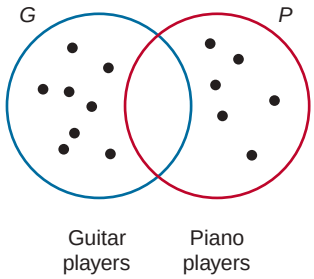
In the definition of addition, R and S must be disjoint sets. Otherwise, you could not determine the total number of elements in two sets by adding the number of elements in one set to the number of elements in the other set.

EXAMPLE A

There are eight people in a group who play the guitar and six who play the piano. These are the only people in the group.

1. What is the minimum number of people in this group?
2. What is the maximum number of people in this group?
3. In which case (question 1 or question 2) can the answer be found by adding the number of people who play guitar to the number of people who play piano?

Solution **1.** 8 if the 6 piano players also play the guitar. **2.** 14 if the sets of piano players and guitar players are disjoint. **3.** Question 2, as illustrated on the next page.



NCTM Standards

Calculators should be available at appropriate times as computational tools, particularly when many or cumbersome computations are needed to solve problems. However, when teachers are working with students on developing computational algorithms, the calculator should be set aside to allow this focus. p. 32

MODELS FOR ADDITION ALGORITHMS

An **algorithm** is a step-by-step procedure for computing. Algorithms for addition involve two separate procedures: (1) adding digits and (2) regrouping, or "carrying" (when necessary), so that the sum is written in positional numeration. The term *carrying* probably originated back at a time when a counter, or chip, was actually carried to the next column on a counting board. Traditionally, a substantial portion of the school mathematics curriculum has involved practice with pencil-and-paper algorithms. Since calculators and computers are readily available, there can be less emphasis on written algorithms. It will always be important, however, to understand algorithms—especially in mental mathematics and estimation.

There are many models for providing an understanding of addition algorithms. Example B shows how to illustrate the sum of two numbers by using the bundle-of-sticks model. The sticks representing these numbers can be placed below each other, just as the numerals are in the addition algorithm. The sum is the total number of sticks in the bundles plus the total number of individual sticks.

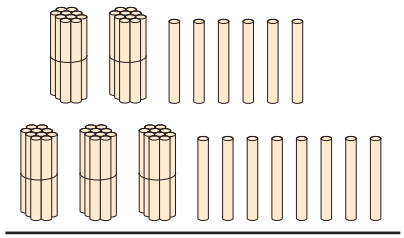
EXAMPLE B

Research Statement

Elementary school students often incorrectly employ a "when in doubt, add" strategy. This is attributed to an aspect of poorly developed conceptual knowledge.

Kroll and Miller

The numbers 26 and 38 are represented in the following figure. To compute 26 + 38, we must determine the total number of sticks. There is a total of 5 bundles of sticks (5 tens) and 14 sticks (14 ones). Since there are 14 single sticks, they can be regrouped into 1 bundle of 10 sticks and 4 more. Thus, there are a total of 6 bundles and 4 sticks. In the addition algorithm, a 4 is recorded in the units column and the extra 10 is recorded by writing a 1 in the tens column.



Tens	Ones
$^{1}2$	6
+ 3	8
6	4

NCTM Standards

The use of concrete materials such as the base-ten pieces or the bundle-of-sticks model provides opportunities for students to develop their own methods of computing. The *Curriculum and Evaluation Standards for School Mathematics* (p. 95) recognizes the value of such activities:

As they begin to understand the meaning of operations and develop a concrete basis for validating symbolic processes and situations, students should design their own algorithms and discuss, compare, and evaluate them with their peers and teacher.

Students using the model in Example B might find it natural to combine all the single sticks first, next combine the bundles of 10, and then do the regrouping. This can lead to an algorithm called **partial sums.** In this method, the digits for each place value are added, and the partial sums are recorded before there is any regrouping.

Two methods of writing partial sums are shown in Example C. In part 1 there is seldom a need for regrouping, because if there is more than one digit in the partial sum, the digits are placed in different columns. In part 2 the regrouping can be done beginning with any partial sum with more than one digit.

EXAMPLE C

1.

$$\begin{array}{r} 345 \\ +\ 278 \\ \hline 13 \\ 11 \\ 5 \\ \hline 623 \end{array}$$

2.

$$\begin{array}{rl} 345 = & 3 \text{ hundreds} + 4 \text{ tens} + 5 \\ +278 = & 2 \text{ hundreds} + 7 \text{ tens} + 8 \\ \hline & 5 \text{ hundreds} + 11 \text{ tens} + 13 \\ \text{Regrouping:} & 6 \text{ hundreds} + 2 \text{ tens} + 3 \\ = 623 & \end{array}$$

Left-to-Right Addition Some students might begin the process of combining the sticks in Example B by first combining the bundles of 10. Since children learn to read from left to right, some may find it natural to add in this direction. The next example illustrates this process in computing the sum of two three-digit numbers.

EXAMPLE D

To compute 897 + 537 from left to right, we first add 8 and 5 in the hundreds column (see below). In the second step, 9 and 3 are added in the tens column, and because regrouping (carrying) is necessary, 3 in the hundreds column is scratched out and replaced by 4. In the third step, we add the units digits. Again regrouping is necessary, so 2 in the tens column is scratched out and replaced by 3.

First step	Second step	Third step
897	897	897
+537	+537	+537
13	1 ~~3~~ 2	1 ~~3~~ ~~2~~ 4
	4	4 3

The early Hindus and later the Europeans added from left to right. The Europeans called this algorithm the **scratch method.**

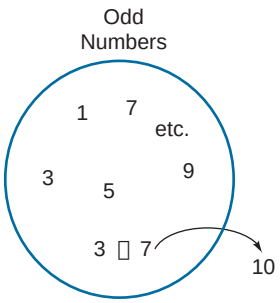
NUMBER PROPERTIES

A few fundamental properties for operations on whole numbers are so important that they are given special names. Four properties for addition are introduced here, and the corresponding properties for multiplication are given in Section 3.3.

NCTM Standards

Research has shown that learning about number and operations is a complex process for children (e.g., Fuson). p. 32

Closure Property for Addition If you were to select any two whole numbers, their sum would be another whole number. This fact is expressed by saying that the whole numbers are **closed for the operation of addition.** In general, the word *closed* indicates that when an operation is performed on any two numbers from a given set, the result is also in the set, rather than outside the set. For example, the set of whole numbers is not closed for subtraction, because sometimes the difference between two whole numbers is a negative number. Consider another example. If we select any two numbers from the set of odd numbers {1, 3, 5, 7, . . . }, the sum is not another odd number. So the set of odd numbers is not closed for addition. To test for closure, students sometimes find it helpful to draw a circle and write the numbers from a given set inside. Then if the set is *closed*, the results of the operation will be inside the circle. If the given operation produces *at least one* result that is outside the circle, the set is *not closed* for the given operation.



Closure Property For every pair of numbers in a given set, if an operation is performed, and the result is also a number in the set, the set is said to be **closed for the operation.** If one example can be found where the operation does not produce an element of the given set, then the set is **not closed for the operation.**

EXAMPLE E

Determine whether the set is closed or not closed for the given operation.

1. The set of odd numbers for subtraction.
2. The set of odd numbers for multiplication.
3. The set of whole numbers for division.

Solution **1.** The set of odd numbers is not closed for subtraction. For example, $23 - 3$ is not an odd number. **2.** The set of odd numbers is closed for multiplication; the product of any two odd numbers is another odd number. **3.** The set of whole numbers is not closed for division. For example, $\frac{2}{3}$ is not a whole number.

Identity Property for Addition Included among the whole numbers is a very special number, zero. Zero is called the **identity for addition** because when it is added to another number, there is *no change.* That is, adding 0 to any number leaves the identity of the number unchanged. For example,

$$0 + 5 = 5 \qquad 17 + 0 = 17 \qquad 0 + 0 = 0$$

Zero is unique in that it is the only number that is an identity for addition.

Identity Property for Addition For any whole number b,

$$0 + b = b + 0 = b$$

and 0 is the unique identity for addition.

Associative Property for Addition In any sum of three numbers, the middle number may be added to (associated with) either of the two end numbers. This property is called the **associative property for addition.**

EXAMPLE F

$$\underbrace{147 + (20 + 6)} = \underbrace{(147 + 20) + 6}$$

Associative property for addition

Associative Property for Addition For any whole numbers a, b, and c,

$$a + (b + c) = (a + b) + c$$

When elementary school students compute by breaking a number into a convenient sum, as in Example G, the *associative property of addition* plays a role. Arranging numbers to produce sums of 10 is called *making 10s.*

EXAMPLE G

$$8 + 7 = \underbrace{8 + (2 + 5)} = \underbrace{(8 + 2) + 5} = 10 + 5 = 15$$

Associative property for addition

Commutative Property for Addition When two numbers are added, the numbers may be interchanged (commuted) without affecting the sum. This property is called the **commutative property for addition.**

EXAMPLE H

$$\underbrace{257 + 498} = \underbrace{498 + 257}$$

Commutative property for addition

Commutative Property for Addition For any whole numbers a and b,

$$a + b = b + a$$

As the addition table in Figure 3.5 on the next page shows, the commutative property for addition roughly cuts in half the number of basic addition facts that elementary school students must memorize. Each sum in the shaded part of the table has a corresponding equal sum in the unshaded part of the table.

Technology Connection

Palindromic Differences

Begin with any three-digit number, reverse its digits, and subtract the smaller from the larger. If this process is continued, will the result eventually be a palindromic number? The online 3.2 Mathematics Investigation will carry out this reversing and subtracting process and quickly supply you with data for making conjectures.

Mathematics Investigation
Chapter 3, Section 2
www.mhhe.com/bbn

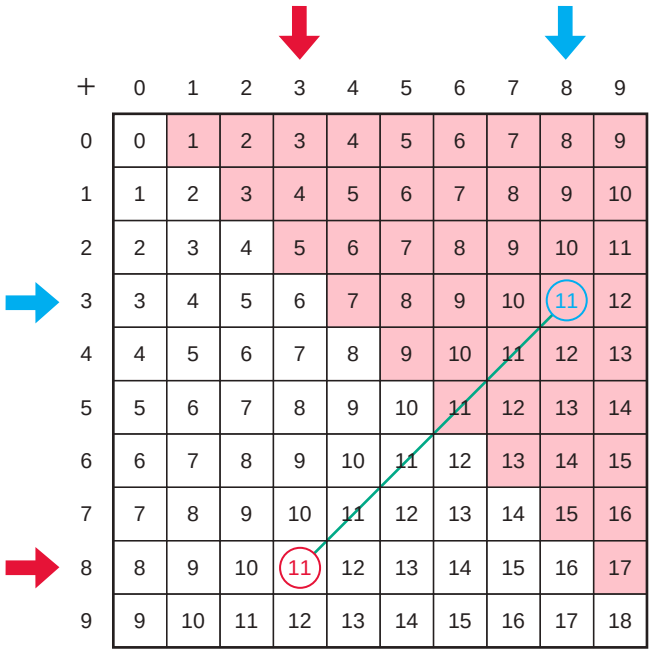


Figure 3.5

EXAMPLE I

If we know that $3 + 8 = 11$, then, by the commutative property for addition, $8 + 3 = 11$. What do you notice about the locations of these sums in the addition table?

Solution The sums of $3 + 8$ and $8 + 3$ are in opposite parts of the table. If the shaded part of the table is folded onto the unshaded part of the table, these sums will coincide. That is, the table is symmetric about the diagonal from upper left to lower right.

The commutative property also enables us to select convenient combinations of numbers when we are adding.

EXAMPLE J

The numbers 26, 37, and 4 are arranged more conveniently on the right side of the following equation than on the left, because $26 + 4 = 30$ and it is easy to compute $30 + 37$.

$$26 + \underbrace{37 + 4} = 26 + \underbrace{4 + 37} = (26 + 4) + 37 = 30 + 37$$

Commutative property for addition

INEQUALITY OF WHOLE NUMBERS

The inequality of whole numbers can be understood intuitively in terms of the locations of numbers as they occur in the counting process. For example, 3 is less than 5 because it is named before 5 in the counting sequence. This ordering of numbers can be illustrated with a number line. A **number line** is formed by beginning with any line and marking off two points, one labeled 0 and the other labeled 1, as shown in Figure 3.6 on the next page. This **unit segment** is then used to mark off equally spaced points for consecutive whole numbers. For any two numbers, the one that occurs on the left is less than the one that occurs on the right.

One method of marking off unit lengths to form a number line is to use the edges of base-ten pieces, such as the *long,* for marking off 10 units (see Figure 3.6). This use of base-ten pieces provides a link between the region model and the linear model for illustrating numbers.

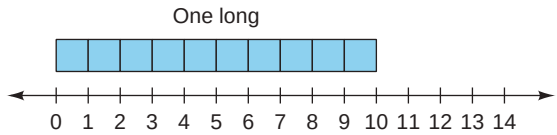


Figure 3.6

The inequality of whole numbers is defined in terms of addition.

Inequality of Whole Numbers For any two whole numbers m and n, m is less than n (written $m < n$) if and only if there is a nonzero whole number k such that $m + k = n$.

An inequality can be written with the inequality symbol opening to the right or to the left. For example, $4 < 9$ means that 4 is **less than** 9; and $9 > 4$ means that 9 is **greater than** 4. Sometimes the inequality symbol is combined with the equality symbol: $\leq$ means **less than or equal to,** and $\geq$ means **greater than or equal to.**

HISTORICAL HIGHLIGHT

The symbols $<$ and $>$ were first used by English surveyor Thomas Harriot in 1631. There is no record of why Harriot chose these symbols, but the following conjecture is logical and will help you to remember their meanings. The distances between the ends of the bars in the equality symbol are equal, and in an equation (for example, $3 = 1 + 2$) the number on the left of the equality symbol equals the number on the right. Similarly, $3 < 4$ indicates that 3 is less than 4, because the distance between the bars on the left is less than the distance between the bars on the right. The reasoning is the same whether we write $3 < 4$ or $4 > 3$. These symbols could easily have evolved into our present notation, $<$ and $>$, in which the bars completely converge to prevent any misjudgment of the distances.*

*This is one of two conjectures on the origin of the inequality symbols, described by H. W. Eves in *Mathematical Circles* (Boston: Prindle, Weber, and Schmidt, 1969), pp. 111–113.

Research Statement

For students in grades K–2, learning to see the part to whole relations in addition and subtraction situations is one of their most important accomplishments in arithmetic.

Resnick

MODELS FOR SUBTRACTION ALGORITHMS

Subtraction is often explained as the *taking away* of a subset of objects from a given set. The word *subtract* literally means to *draw away from under.*

The process of taking away, or subtraction, may be thought of as the opposite of the process of putting together, or addition. Because of this dual relationship, subtraction and addition are called **inverse operations.** This relationship is used to define subtraction in terms of addition.

Subtraction of Whole Numbers For any whole numbers r and s, with $r \geq s$, the **difference** of r **minus** s, written $r - s$, is the whole number c such that $r = s + c$. The number c is called the **missing addend.**

NCTM Standards

By the end of grade 2, children should know the basic addition and subtraction combinations, should be fluent in adding two-digit numbers, and should have methods for subtracting two-digit numbers. p. 33

The definition of subtraction says that we can compute the difference $17 - 5$ by determining the **missing addend,** that is, finding the number that must be added to 5 to give 17. Store clerks use this approach when making change. Rather than subtracting 83 cents from \$1.00 to determine the difference, they pay back the change by counting up from 83 to 100.

After negative numbers are introduced, there is no need to require r to be greater than or equal to s in the definition of subtraction. In the early school grades, however, before negative numbers appear, most examples involve subtracting a smaller number from a larger one.

Three concepts of subtraction occur in problems: the **take-away concept,** the **comparison concept,** and the **missing addend concept.**

Take-Away Concept Suppose that you have 12 stamps and give away 7. How many stamps will you have left? Figure 3.7 illustrates $12 - 7$ by showing 7 objects being taken away from 12 objects.



Figure 3.7

Take-away concept showing $12 - 7 = 5$

Comparison Concept Suppose that you have 12 stamps and someone else has 7 stamps. How many more stamps do you have than the other person? In this case we compare one collection to another to determine the difference. Figure 3.8 shows that there are 5 more stamps in one collection than in the other

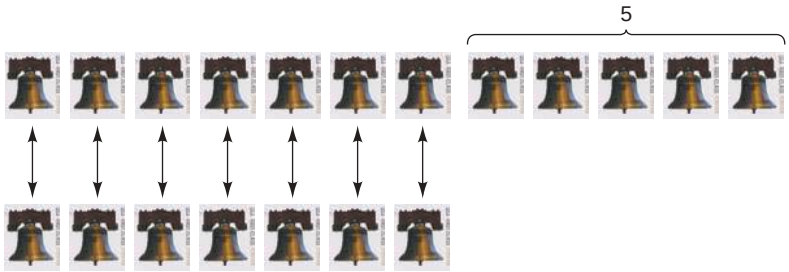


Figure 3.8

Comparison concept showing $12 - 7 = 5$

Missing Addend Concept Suppose that you have 7 stamps and you need to mail 12 letters. How many more stamps are needed? In this case we can count up from 7 to 12 to determine the missing addend. Figure 3.9 shows that 5 stamps should be added to 7 stamps to form a collection of 12 stamps.

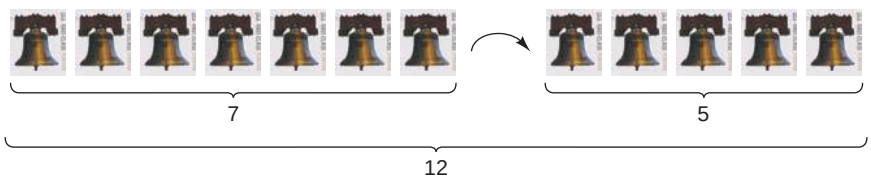


Figure 3.9

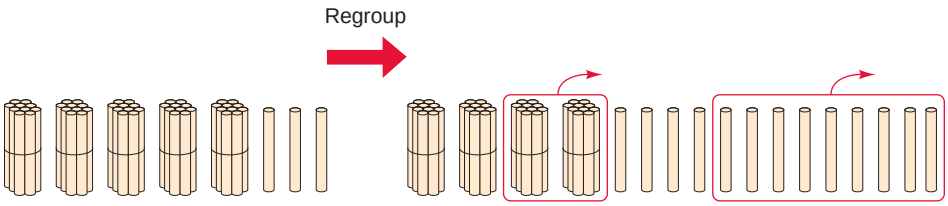
Missing addend concept showing $12 - 7 = 5$

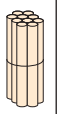
There are two types of examples to consider in explaining the steps in finding the difference between two multidigit numbers: examples in which regrouping (borrowing) is not needed and those in which regrouping (borrowing) is needed.

The bundle-of-sticks model and the take-away concept of subtraction are used in Example K to illustrate the subtraction algorithm with regrouping.

EXAMPLE K

To illustrate $53 - 29$, we begin with 5 bundles of sticks (5 tens) and 3 sticks (3 ones), as shown. To take away 9 sticks, we must regroup one bundle, to form 13 single sticks. Once this has been done, we can take away 2 bundles of sticks and 9 sticks, leaving 2 bundles of sticks and 4 single sticks. In the algorithm, the regrouping is recorded by crossing out 5 and writing 4 above it.



Tens	Ones
	
$\cancel{5}^{4}$	3
$-$ 2	9
2	4

Technology Connection

Sums and differences can be computed on calculators with algebraic logic by entering the numbers from left to right as they occur in equation form. For instance, $475 + 381 - 209$ is computed by the following key strokes.

Keystrokes	View Screen
475	475
+	475
381	381
−	856
209	209
=	647

When numbers and operations are entered into some calculators, such as the one in Figure 3.10, they are displayed on the view screen from left to right as illustrated. If more numbers and operations are entered than can be displayed on the view screen of this calculator, previous entries are pushed off the left end of the screen but are retained internally in the calculator's memory.



Figure 3.10

14-8

1 Introduce

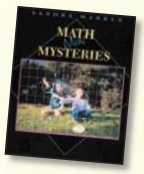
Activity Choice 1 • Hands-On

Write the number 642 on the board. Give each student eleven 10 × 10 squares of grid paper.

- **How many small squares are there on each large square?** 100
- Have students cut single squares to represent ones and columns of squares to represent tens.
- Have them model the number 642.
- Ask students to suggest other numbers to make with their squares.

Activity Choice 2 • Literature

Introduce the lesson with "You and the Statue" by Sandra Markle from *Math Mini Mysteries*. For a related math activity, see p. TR65.



2 Teach

Use place-value workmats and base-ten blocks.

- **Can you subtract 2 from 4?** yes
- Have students put 4 units in the ones column on their place-value chart. **Can you subtract 5 from 3?** no **What will you have to do?** regroup hundreds
- Have students trade a flat for 10 rods. **What number have you made?** 434
- Have students move one yellow counter into the tens column. **What number is in the tens column?** 13 (130)
- **Can you subtract 5 rods from 13 rods? (50 from 130)** yes Point out that you have regrouped hundreds to solve the problem.

Get Ready Use the section at the top of p. 457 to reinforce the lesson concept. Guide students in regrouping hundreds.

Check Observe students as you work through Exercises 1–4 as a class.

Exercise 5 Assess student comprehension before assigning practice exercises.

COMMON ERROR!

If students have trouble adding the regrouped hundred to the number that is already in the tens place, have them write out the addition problem in the margin. For example: 4 tens plus 10 tens, or 40 plus 100, equals 140.

Name _____

Hands-On Activity

Regroup Hundreds

Get Ready

Main Idea

I will regroup hundreds to subtract three-digit numbers.

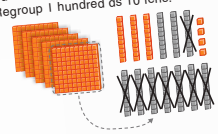
Find 539 − 285.

Step 1: Subtract the ones.



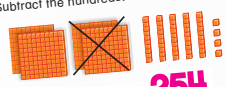
hundreds	tens	ones
5	3	9
− 2	8	5
		4

Step 2: Subtract the tens. You cannot subtract 8 from 3. Regroup 1 hundred as 10 tens.



hundreds	tens	ones
4	13	
5	3	9
− 2	8	5
	5	4

Step 3: Subtract the hundreds.



hundreds	tens	ones
4	13	
5	3	9
− 2	8	5
2	5	4

539 − 285 = **254**

Check

5. Sample answer: Regroup when there are not enough ones or tens to subtract.

Use base-ten blocks and WorkMat 7. Subtract.

1. 568 − 197 = **371**
2. 628 − 442 = 186
3. 759 − 5 = 754
4. 341 − 22 = 319
5. Talk About It How do you know when to regroup?

four hundred fifty-seven 457

Chapter 14 Lesson 8

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Reteach (p. 41) BL

Name _____

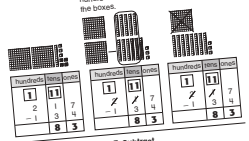
14-8 Reteach

Regroup Hundreds

Step 1 Subtract the ones. Write how many ones are left.

Step 2 Subtract the tens. Regroup 1 hundred as 10 tens. Write the new number of hundreds and tens in the boxes.

Step 3 Subtract the hundreds. Write how many hundreds are left.



Use models and WorkMat 7. Subtract.

1. 827 − 433 = 394
2. 245 − 153 = 92
3. 597 − 489 = 108
4. 762 − 234 = 528
5. 624 − 325 = 299
6. 943 − 144 = 799

41

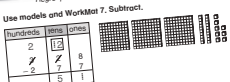
Skills Practice (p. 42) OL

Name _____

14-8 Skills Practice

Regroup Hundreds

Use models and WorkMat 7. Subtract.



1. 567 − 295 = 272
2. 912 − 562 = 350
3. 727 − 382 = 345
4. 838 − 445 = 395
5. 478 − 416 = 62
6. 648 − 377 = 271
7. 946 − 268 = 678
8. 256 − 131 = 125
9. 871 − 596 = 275
10. 158 − 98 = 60

Solve. Show your work.

11. Penny had 347 pumpkins for sale. She sold 255 pumpkins. How many pumpkins did Penny have left? 92 pumpkins

92

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Calculators can be used to strengthen students' understanding of place value and algorithms for computing. Earlier in this section we discussed partial sums and left-to-right addition. The next keystrokes illustrate these methods for computing 792 + 485 + 876. Notice that the first view screen shows the sum of the hundreds; the second screen shows the sum of the hundreds and tens; and the last screen shows the sum of the original three numbers.

Keystrokes	View Screen
700 ☐ 400 ☐ 800 ☐	1900
90 ☐ 80 ☐ 70 ☐	2140
2 ☐ 5 ☐ 6 ☐	2153

MENTAL CALCULATIONS

Mental calculations are important because they often prove the quickest and most convenient method of obtaining an answer. Performing mental computations requires us to combine a variety of skills: the abilities to use various algorithms, to understand place value and base-ten numeration, and to use number properties. Mental calculations are useful in obtaining exact answers, and they are a prerequisite to estimating. Let's consider a few techniques for performing mental calculations.

Compatible Numbers for Mental Calculation One mental calculating technique is to look for pairs of numbers whose sum or difference is easy to compute. For example, it is convenient to combine 17 and 43 in the following computation.

$$17 - 12 + 43 = 17 + 43 - 12 = 60 - 12 = 48$$

Using pairs of numbers that are especially easy to compute with is the calculating technique called **compatible numbers for mental calculations.**

EXAMPLE L

Do the following computations using compatible numbers for mental calculations.

1. 17 + 12 + 23 + 45
2. 12 − 15 + 82 − 61 + 55

Solution **1.** One possibility is to notice that 17 + 23 = 40; then 40 + 45 = 85, and adding 12 produces 97. Another possibility is to notice that 12 + 23 = 35. Then 35 + 45 = 80, and adding 17 produces 97. **2.** Here is one possibility: 55 − 15 = 40 and 82 − 61 = 21. Then 40 + 21 = 61, and adding 12 produces 73.

Substitutions for Mental Calculation Using the method of **substitutions,** a number is broken down into a convenient sum or difference of numbers. You can easily compute the sum 127 + 38 in your head in many ways. Here are three possibilities:

$$127 + (3 + 35) = (127 + 3) + 35 = 130 + 35 = 165$$
$$127 + (30 + 8) = (127 + 30) + 8 = 157 + 8 = 165$$
$$(125 + 2) + 38 = 125 + (2 + 38) = 125 + 40 = 165$$

EXAMPLE M

Do each computation mentally by substituting a convenient sum or difference for one of the given numbers.

1. $57 + 24$

2. $163 - 46$

Solution Here is one possibility for each computation: **1.** $57 + 20 + 4 = 77 + 4 = 81$. **2.** $163 - 40 - 6 = 123 - 6 = 117$.

Equal Differences for Mental Calculation The method of **equal differences** uses the fact that the difference between two numbers is unchanged when both numbers are increased or decreased by the same amount. Figure 3.11 illustrates why this is true when both numbers are increased. No matter how many tiles (see blue tiles) are adjoined to the two rows in this figure, the difference between the numbers of tiles in the two rows is $11 - 7 = 4$.

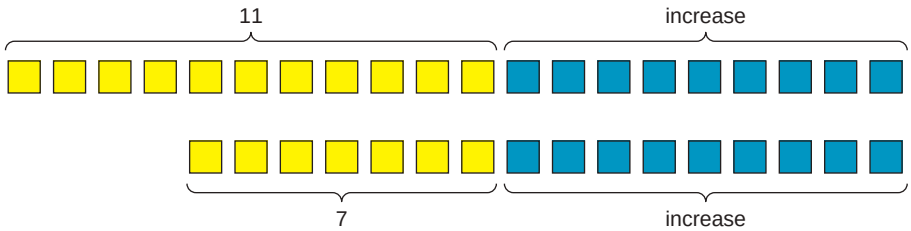


Figure 3.11

Replacing a difference by an equal but more convenient difference can be very useful.

EXAMPLE N

To compute $47 - 18$, first find a more convenient but equal difference by increasing or decreasing both numbers by the same amount.

Solution Here are several differences that are more convenient for computing $47 - 18$.

$49 - 20$	(both numbers were increased by 2)
$50 - 21$	(both numbers were increased by 3)
$30 - 1$	(both numbers were decreased by 17)
$40 - 11$	(both numbers were decreased by 7)

The difference, 29, is easy to compute in any of these forms.

Add-Up Method for Mental Calculation A convenient mental method for subtracting is to **add up** from the smaller to the larger number, using several easy steps. For example, to compute $54 - 19$, first add 1 to 19 to obtain 20 and then 34 to 20 to obtain 54. The difference is the sum of the "add ups": $54 - 19 = 1 + 34 = 35$.

EXAMPLE O

Compute each difference by adding up from the smaller to the larger number.

1. $53 - 17$

2. $135 - 86$

Solution **1.** From 17 to 20 is 3, and from 20 to 53 is 33. So the difference is $3 + 33 = 36$. **2.** From 86 to 100 is 14, and from 100 to 135 is 35. So the difference is $14 + 35 = 49$.

ESTIMATION OF SUMS AND DIFFERENCES

In recent years, the teaching of estimation has become a top priority in school mathematics programs. Often in everyday applications we need to make a quick calculation that does not have to be exact to serve the purpose at hand. For example, when shopping, we may want to estimate the total cost of the items selected in order to avoid an unpleasant surprise at the checkout counter. Estimation is especially important for developing "number sense" and predicting the reasonableness of answers. With the increased use of calculators, estimation helps students to determine if the correct keys have been pressed.

There are some difficulties in teaching estimation. First, the best estimating technique to use often depends on the numbers involved and the context of the problem. Second, there is no correct answer. An estimate is a "ballpark" figure, and for a given problem there will often be several different estimates.

There are many techniques for estimating. Three common ones—*rounding, using compatible numbers for estimation,* and *front-end estimation*—are explained below. After obtaining an estimation, we sometimes need to know if it is less than or greater than the actual answer. This can often be determined from the method of estimation used.

Rounding If an approximate sum or difference is all that is needed, we can round the numbers before computing. The type of problem will often determine to what place value the numbers will be rounded. The following estimates are obtained by rounding to the nearest hundreds or thousands. The symbol $\approx$ means **approximately equal to.**

EXAMPLE P

Obtain an estimation by rounding each number to the place value of the leading digit.

1. $624 - 289 - 132$
2. $4723 + 419 + 1040$
3. $812 - 245$

Solution 1. $\approx 600 - 300 - 100 = 200$ 2. $\approx 5000 + 400 + 1000 = 6400$ 3. $\approx 800 - 200 = 600$

Some people prefer rounding each number to the same place value. If each number in part 2 of Example P were rounded to the nearest thousand, 419 would be rounded to 0 and the approximate sum would become $5000 + 0 + 1000 = 6000$. Even when numbers have the same number of digits, they do not have to be rounded to the same place value. A different estimation could be obtained in part 3 of Example P by rounding 245 to 250 (the nearest ten). We could then use the add-up method to obtain a difference of 550.

$$812 - 245 \approx 800 - 250 = 550$$

Compatible Numbers for Estimation Sometimes a computation can be simplified by replacing one or more numbers by approximations in order to obtain **compatible numbers.** For example, to approximate $342 + 250$, we might replace 342 by 350.

$$342 + 250 \approx 350 + 250 = 600$$

Using compatible numbers is a common estimating technique.

EXAMPLE Q

Use compatible numbers for estimation to obtain each sum or difference. Without computing the actual answer, predict whether your estimate is too small or too big.

1. $88 + 37 + 66 + 24$
2. $142 - 119$
3. $127 + 416 - 288$

Solution Here are some estimations. Others may occur to you. **1.** $90 + 40 + 70 + 20 = 220$, which is greater than the actual answer. **2.** $140 - 120 = 20$, which is less than the actual answer. **3.** $130 + 400 - 300 = 230$, which is less than the actual answer.

Front-End Estimation The method of **front-end estimation** is similar to left-to-right addition, but involves only the leading digit of each number.

Suppose you have written checks for \$433, \$684, and \$228 and wish to quickly estimate the total. Using front-end estimation, we see that the sum of the leading digits is 12, so the estimated sum is 1200.

$$433 + 684 + 228 \approx 400 + 600 + 200 = 1200$$

This method of estimation is different from rounding to the highest place value. For example, in the preceding sum, 684 is replaced by 600, rather than the rounded value of 700.

The next example shows how front-end estimation is used when the leading digit of each number in a sum does not have the same place value.

$$3827 + 458 + 5031 + 311 \approx 3000 + 400 + 5000 + 300 = 8700$$

Notice that in estimating the sums in these two examples, the digits beyond the leading digit of each number are not used. Thus, when front-end estimation is used for sums, the estimation is always less than or equal to the exact sum.

Front-end estimation is used for estimating both sums and differences in Example R.

EXAMPLE R Use front-end estimation to estimate each sum or difference.

1. $1306 + 7247 + 3418$

2. $4718 - 1335$

3. $527 + 4215 + 718$

4. $7316 - 547$

Solution **1.** $1306 + 7247 + 3418 \approx 1000 + 7000 + 3000 = 11{,}000$ **2.** $4718 - 1335 \approx 4000 - 1000 = 3000$ **3.** $527 + 4215 + 718 \approx 500 + 4000 + 700 = 5200$ **4.** $7316 - 547 \approx 7000 - 500 = 6500$

Large errors from computing on a calculator, such as those produced by pressing an incorrect key, can sometimes be discovered by techniques for estimating. Suppose, for example, that you wanted to add 417, 683, and 228, but you entered 2228 on the calculator rather than 228. The sum of the three numbers you intended to add when rounded to the nearest hundred is 1300, but the erroneous calculator sum will be 3328. The difference of more than 2000 between the estimation and the calculator sum indicates that the computation should be redone.

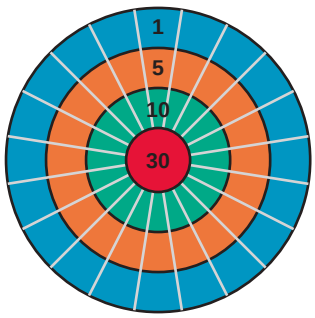
Sum		Estimation
	(rounding)	
417	⟶	400
683	⟶	700
+ 228	⟶	+ 200
		1300

PROBLEM-SOLVING APPLICATION

The following problem introduces the strategy of **making an organized list.** This problem-solving strategy is closely associated with another strategy called *eliminating possibilities.* Next to guessing and checking, one of the most common approaches to solving problems is to systematically search for or eliminate possibilities.

Problem

Karen and Angela are playing darts on the board shown below. Each player throws three darts on her turn and adds the numbers on the regions that are hit. The darts always hit the dartboard, and when a dart lands on a line, the score is the larger of the two numbers. After four turns Karen and Angela notice that their sums for each turn are all different. How many different sums are possible?



Understanding the Problem **Question 1:** What are the largest and smallest possible sums?

Devising a Plan Here are two approaches to finding all the sums. Since the lowest sum is 3 and the highest sum is 90, we can list the numbers from 3 through 90 and determine which can be obtained. Or we can *make an organized list* showing the different regions the three darts can strike. **Question 2:** For example, if the first two darts land in regions 1 and 5, what are the possible scores after the third dart is thrown?

Carrying Out the Plan Use one of the above approaches or one of your own to find the different sums and determine how each can be obtained from the dartboard. **Question 3:** How many different sums are possible?

Looking Back Instead of four regions, suppose the dartboard had three regions. **Question 4:** How many different sums would be possible on a dartboard with three regions numbered 1, 5, and 10?

Answers to Questions 1–4 **1.** The largest sum is 90, and the smallest is 3. **2.** The possible sums are 7, 11, 16, and 36. **3.** 20 different sums. **4.** 10 different sums.

HISTORICAL HIGHLIGHT



This adding machine was developed by Blaise Pascal in 1642 for computing sums. The machine is operated by dialing a series of wheels with digits from 0 to 9. To carry a number to the next column when a sum is greater than 9, Pascal devised a ratchet mechanism that would advance a wheel 1 digit when the wheel to its right made a complete revolution. The wheels from right to left represent units, tens, hundreds, etc.

Exercises and Problems 3.2

Use the following information in exercises 1 and 2. To compute 854 + 629, using the adding machine described on the previous page, we first turn the hundreds, tens, and units wheels 8, 5, and 4 notches, respectively. We then dial these same wheels 6, 2, and 9 more notches. The sum will appear on indicators at the top of the machine.

1. a. Which of these wheels will make more than one revolution for this sum?
b. Can this sum be computed by left-to-right addition, that is, by turning the hundreds wheel for both hundreds digits, 8 and 6; turning the tens wheel for 5 and 2; and turning the units wheel for 4 and 9?

2. a. Which two wheels will be advanced one digit because of carrying?
b. Can this sum be computed by turning the wheels in different orders, such as the tens wheel 5, the units wheel 9, the hundreds wheel 8, the units wheel 4, the hundreds wheel 6, and the tens wheel 2?

Determine the minimum number of long-flats, flats, longs, and units for the bases in exercises 3 and 4 if the pieces in set *A* are combined with the pieces in set *B*. (*Reminder:* In some cases regrouping will be needed.) Then write numerals in positional numeration for sets *A* and *B* and the numerals for their sum in the given base.

3. a. Base five
A: 2 flats, 3 longs, 2 units
B: 1 flat, 2 longs, 3 units
b. Base twelve
A: 8 flats, 5 longs, 2 units
B: 2 flats, 9 longs, 5 units

4. a. Base three
A: 2 flats, 2 longs, 2 units
B: 2 flats, 1 long, 2 units
b. Base ten
A: 5 flats, 7 longs, 7 units
B: 2 flats, 6 longs, 5 units

Determine the minimum number of pieces in exercises 5 and 6 that need to be combined with set *B* to obtain set *A* for the given base. Then write numerals in positional numeration for sets *A* and *B* and the numerals for their difference in the given base.

5. a. Base eight
A: 5 flats, 2 longs, 3 units
B: 2 flats, 6 longs, 5 units
b. Base five
A: 3 flats, 4 longs, 2 units
B: 1 flat, 3 longs, 4 units

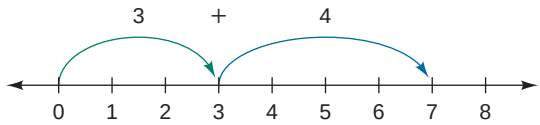
6. a. Base twelve
A: 7 flats, 9 longs, 6 units
B: 5 flats, 8 longs, 9 units
b. Base ten
A: 6 flats, 6 longs, 2 units
B: 2 flats, 9 longs, 3 units

Sketch base pieces for exercises 7 and 8 to illustrate each computation. Show regrouping.

7. a. 106 + 38
b. $41_{five} - 23_{five}$, using the take-away concept
c. 161 − 127, using the comparison concept
d. $142_{five} + 34_{five}$
e. 157 − 123, using the missing addend concept

8. a. 46 + 27
b. 52 − 36, using the take-away concept
c. 35 − 18, using the comparison concept
d. $33_{five} + 43_{five}$
e. $434_{five} - 312_{five}$, using the missing addend concept

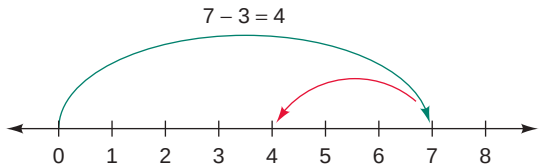
Addition is illustrated on a number line by a series of arrows, as shown here. Use a number line to illustrate the equalities in exercises 9 and 10.



9. a. 2 + 5 = 5 + 2
b. (2 + 4) + 1 = 1 + (2 + 4)

10. a. (3 + 4) + 1 = (4 + 1) + 3
b. (2 + 3) + 4 = 2 + (4 + 3)

Subtraction is illustrated on a number line by arrows that represent numbers. The number being subtracted is represented by an arrow from right to left, as shown here. Use a number line to illustrate the equations in exercises 11 and 12.



11. a. $(6 - 3) - 2 = 1$
b. $6 - 6 = 0$

12. a. $(4 + 5) - 7 = 2$
b. $(9 - 2) - 6 = 1$

Compute the sums in exercises 13 and 14, using the given method. Describe an advantage of each method.

13. a. Left-to-right addition

$$\begin{array}{r} 726 \\ +508 \\ \hline \end{array}$$

b. Partial sums

$$\begin{array}{r} 974 \\ +382 \\ \hline \end{array}$$

14. a. Left-to-right addition

$$\begin{array}{r} 4763 \\ +9607 \\ \hline \end{array}$$

b. Partial sums

$$\begin{array}{r} 476 \\ +947 \\ \hline \end{array}$$

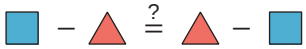
Which number property shows that the two sides of each equation in 15 and 16 are equal?

15. a. $(38 + 13) + 17 = 38 + (13 + 17)$
b. $(47 + 62) + 12 = (62 + 47) + 12$

16. a. $2 \times (341 + 19) = 2 \times (19 + 341)$
b. $13 + (107 + 42) = (13 + 107) + 42$

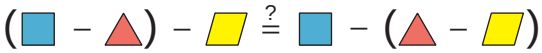
Try some whole numbers in exercises 17 and 18 to determine whether the properties hold.

17. a. Is subtraction commutative?



b. Is the set of even numbers closed for addition?

18. a. Is subtraction associative?



b. Is the set of odd numbers closed for addition?

Error analysis: Some types of student errors and misuses of addition are very common. Describe the types of errors illustrated in exercises 19 and 20.

19. a.

$$\begin{array}{r} 47 \\ +\ 86 \\ \hline 123 \end{array}$$

b.

$$\begin{array}{r} 16 \\ +48 \\ \hline 91 \end{array}$$

20. a.

$$\begin{array}{r} 56 \\ +\ 78 \\ \hline 1214 \end{array}$$

b.

$$\begin{array}{r} 35 \\ +46 \\ \hline 171 \end{array}$$

Error analysis: One common source of elementary school students' errors in subtraction is adding rather than subtracting. When addition is taught first, the students' responses become so automatic that later on they write 8 for the difference $5 - 3$. Try to detect the reason for the error in each computation in 21 and 22.

21. a.

$$\begin{array}{r} 84 \\ -36 \\ \hline 52 \end{array}$$

b.

$$\begin{array}{r} 52 \\ -38 \\ \hline 24 \end{array}$$

22. a.

$$\begin{array}{r} 46 \\ -27 \\ \hline 73 \end{array}$$

b.

$$\begin{array}{r} 94 \\ -37 \\ \hline 12 \end{array}$$

In exercises 23 and 24, compute exact answers mentally by using *compatible numbers* or *substitutions for mental calculations.* Show your method.

23. a. $23 + 25 + 28$
b. $128 - 15 + 27 - 50$
c. $83 + 50 - 13 + 24$

24. a. $208 + 554$
b. $1398 + 583$
c. $130 + 25 + 70 + 10$

In exercises 25 and 26, use the *equal-differences mental calculation method* to find a difference which is more convenient for mental computation. Show your work.

25. a. $6502 - 152$
b. $894 - 199$
c. $14{,}200 - 2700$

26. a. $435 - 198$
b. $622 - 115$
c. $245 - 85$

In exercises 27 and 28, use the *add-up mental calculation method* to compute exact differences. Record the numbers you use in the add-up process.

27. a. $400 - 185$
b. $535 - 250$
c. $135 - 47$

28. a. $92 - 56$
b. $842 - 793$
c. $2310 - 2105$

In exercises 29 and 30, *round* each number in the table to the place value of its leading digit and then compute an estimate of the sum of each row of numbers, as shown in the example at the top of the table.

				≈ Sum
	83 (Think 80)	47 (Think 50)	112 (Think 100)	230
29. a.	102	38	21	
b.	26	43	59	
30. a.	25	212	81	
b.	27	68	18	

In exercises 31 and 32, estimate each sum or difference by replacing one or both numbers by *compatible numbers.* Show your replacements.

31. a. $359 - 192 \approx$ **b.** $712 + 293 \approx$
c. $882 + 245 \approx$ **d.** $1522 - 486 \approx$

32. a. $3906 + 1200 \approx$ **b.** $684 - 317 \approx$
c. $918 - 366 \approx$ **d.** $2243 - 1589 \approx$

In exercises 33 and 34, use front-end estimation to estimate each sum.

33. a. $362 + 408 + 978$
b. $16 + 49 + 87 + 33$
c. $7215 + 5102 + 8736$

34. a. $472 + 821 + 306 + 512$
b. $4721 + 2015 + 3681$
c. $62 + 85 + 31 + 24 + 88$

A homeowner has the following bills to pay for the month of March. Use this information in exercises 35 and 36.

Electricity	$86	Food	$541
Heat	$128	Doctor bills	$477
Water and sewage	$94	Gas and oil	$73
Property taxes	$163	Car payments	$148
Life insurance	$230	Home mortgage	$570
Car insurance	$65	Dentist bills	$109
House insurance	$58	Recreation	$14

35. a. Obtain an estimation of how much she owes by rounding each bill to the nearest hundred.
b. Can the homeowner pay these bills with a monthly salary of $1800?

36. a. Estimate the amount owed by rounding each bill to the nearest $10.
b. What is the difference between the actual amount of the bills and the estimation obtained in part a?

In exercises 37 and 38 a sequence of numbers is generated by beginning with the first number entered into the calculator and repeatedly carrying out the given keystrokes. Beginning with the first number entered, write each sequence that is produced by the given keystrokes.

37. a. Enter 8723 and repeat the keystrokes [+] 100 [=] seven times.
b. Enter 906 and repeat the keystrokes [−] 10 [=] six times.

38. a. Enter 4337 and repeat the keystrokes [+] 1000 [=] seven times.
b. Enter 8004 and repeat the keystrokes [−] 100 [=] six times.

The constant function on some calculators will repeatedly carry out addition or subtraction of the second number entered by repeated pressing of the [=] key. For example, [1] [7] [3] [+] [8] [2] [=] [=] [=] will produce the sequence 255, 337, and 419. Assume that such a calculator is used in exercises 39 and 40.

39. a. If the keys [2] [7] [1] [4] [+] [1] [4] [5] are pressed and then [=] is pressed six times, what are the next six numbers in the sequence after 2714?
b. What is the 10th number after 2714 in the sequence from part a?
c. If [+] is replaced by [−] in part a, what are the next six numbers of the sequence after 2714?

40. a. If the keys [7] [9] [3] [+] [2] [8] are pressed and then [=] is pressed five times, what are the next five numbers in the sequence after 793?
b. What is the eighth number after 793 in the sequence in part a?
c. If [+] is replaced by [−] in part a, what are the next five numbers of the sequence after 793?

Calculators with constant functions (see exercises 39 and 40) can be used by schoolchildren to practice counting forward or backward by various numbers. Assume that such a calculator is used in exercises 41 and 42.

41. List the keys to be pressed on a calculator with a constant function to obtain the first six numbers of the following sequences by counting:
a. Forward by 2s, beginning with 2
b. Backward by 3s, beginning with 30
c. Forward by 5s, beginning with 20

42. List the keys to be pressed on a calculator with a constant function to obtain the first six numbers of the following sequences by counting:
a. Forward by 10s, beginning with 10
b. Backward by 2s, beginning with 100
c. Forward by 3s, beginning with 6

Reasoning and Problem Solving

43. A dealer has 30 cars with air conditioning and 22 cars with standard transmissions. These are the only cars on the lot.

a. What is the minimum number of cars the dealer has on the lot?

b. What is the maximum number of cars?

c. What is the total number of cars if there are 17 cars with both air conditioning and standard transmissions?

d. In which case above can the answer be found by adding the number of cars with air conditioning to the number of cars with standard transmissions?

44. Featured Strategy: Working Backward. This is a two-person game called *Force Out*. An arbitrary number is selected, and from it the players take turns subtracting any single-digit number greater than zero. The player who is forced to obtain zero loses the game. Describe a strategy for winning this game.

a. Understanding the Problem. On each player's turn only one single-digit number (1, 2, 3, 4, 5, 6, 7, 8, or 9) may be subtracted. If you can get the remaining number to be 1, then you will win. Select a number and play the game to become familiar with the rules. If the number is 15 and it is your turn to play, what number should you subtract?

b. Devising a Plan. One approach to solving this problem is to play the game for small numbers to see if you can hit upon an idea for a winning strategy. Another approach is to *work backward* to find the numbers that will guarantee you a win. Explain how you can win if the remaining number is greater than 1 and less than 11, and it is your turn.

c. Carrying Out the Plan. Select an approach to look for a solution to this problem. Explain why you can win if you get the remaining number to be 11 and it's your opponent's turn to play. Describe a strategy for winning the game if it's your turn to play and the final digit in the number is not a 1 (1, 11, 21, 31, 41, etc.).

d. Looking Back. Let's revise the game so that the players subtract any number from 1 to 19. Suppose you are to take the first turn and the starting number is 76. Describe a strategy for winning the game.

45. A class survey found that 26 students watched the Olympics on television Saturday and 21 watched on Sunday. Of those who watched the Olympics on only one of these days, 11 chose Saturday and 6 chose Sunday. If every student watched at least one of these days, how many students are in the class?

46. Calculators with a constant function can be programmed to add a constant number to any number that is entered into the calculator. Let's assume that a certain stock has paid a bonus of \$164 to each stockholder and that this amount is to be added to their accounts.

Stockholder Accounts
\$4,728
\$13,491
\$6,045
\$21,437
\$10,418
\$9,366

The first step shown for one such calculator adds 164 to 4728. To add 164 to the remaining accounts, it is necessary only to enter the next number and press [=]. Find the new account balances for these view screens.

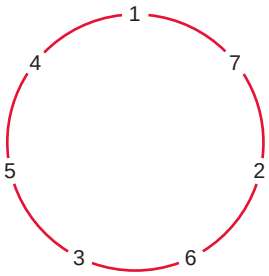
	Keystrokes	View Screen
a.	4728 [+] 164 [=]	
b.	13491 [=]	
c.	6045 [=]	
d.	21437 [=]	
e.	10418 [=]	
f.	9366 [=]	

47. Slate and brick are sold by weight. At one company the slate or brick is placed on a loading platform that weighs 83 kilograms. A forklift moves the slate or brick and the platform onto the scales. The weight of the platform is then subtracted from the total weight to obtain the weight of the slate or brick.

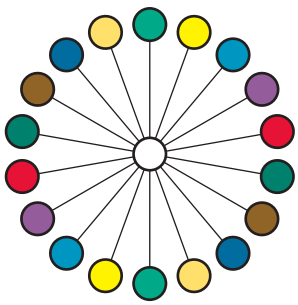
Assume that a calculator with a constant function that subtracts a constant from any number entered is used to determine the weight of the slate or brick below. Find the resulting weights for the following calculator view screens, if 83 is subtracted from the weights in lines b–f by entering a number and pressing [=].

	Keystrokes	View Screen
a.	748 [−] 83 [=]	
b.	807 [=]	
c.	1226 [=]	
d.	914 [=]	
e.	1372 [=]	
f.	655 [=]	

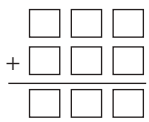
48. This circle contains the whole numbers from 1 to 7. By adding two or more *neighbor numbers* (numbers that are next to each other), we can get every number from 8 to 28. How can the numbers 1, 2, 3, 4, 5, and 6 be placed around a circle to obtain all sums from 7 to 21?



49. How can the whole numbers from 1 to 19 be placed into the 19 disks of the diagram below so that any three numbers on the same line through the center will give the same sum?



50. Use every digit from 1 to 9 exactly once to compute this sum.



51. The values of C, K, G, and F in this number puzzle are four different digits from 0 to 9. What are these digits?

$$\begin{array}{r} CCC \\ +\ \ K \\ \hline GFFG \end{array}$$

Teaching Questions

1. What explanation would you offer to a middle school student who asks: "Why do we use subtraction to determine how many more books Elayna has than Seth, if nothing is being taken away"?
2. Some teachers teach one method (algorithm) to their class for adding multidigit numbers. Others encourage their students to invent and use their own methods for adding. As a teacher, what position would you take on this issue? Give some reasons for your decision.
3. Three different concepts of subtraction are presented in this section. Do you think it is important for school students to be taught about these concepts and/or their names? Explain why or why not. Write a story problem that illustrates each concept.

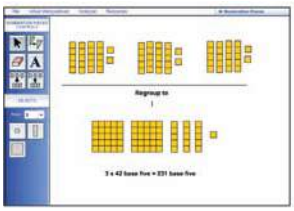
Classroom Connections

1. On page 152, the example from the **Elementary School Text** illustrates $539 - 285$ with base-ten blocks while recording the corresponding symbolic steps in a place-value table. (a) The regrouping in this example illustrates subtraction that begins with the units digit (right to left). Sketch the base-ten pieces for 539 and illustrate the steps for subtracting 285 by beginning with the hundreds digit (left to right). Show how you would record your steps in a place value table. (b) For computing $539 - 285$ with base-ten pieces, which method, right to left or left to right, do you prefer? Explain.
2. The **Research** statement on page 144 speaks of "poorly developed conceptual knowledge." (a) What is meant by this phrase and how can this be avoided when teaching addition and subtraction? (b) Explain how you, as a teacher, would design a class activity that encourages students to devise their own algorithm for addition.
3. Compare the **Research** statement on page 144 with the **Standards** quote on page 146. What are the Standards suggesting that can help with student difficulties in learning about operations?
4. It is common for school students to confuse the inequality symbols $>$ and $<$. One method for recalling their meaning is to think of the symbol as a "mouth" that always bites the "larger number." Compare this method with that suggested in the **Historical Highlight** on page 149. List some advantages of the method described in the highlight.
5. Read the expectation in the **PreK–2 Standards—Number and Operations** (see inside front cover), under *Understand Meanings of Operations . . .* that involves understanding meanings of subtraction and give three examples of how the presentation of subtraction in Section 3.2 satisfies this expectation.
6. Some teachers feel that calculators should not be used in the elementary grades until the basic addition facts are memorized. Compare and discuss the **Standards** quotes on pages 144 and 150. As a teacher, what would be your policy regarding the use of calculators in the classroom? Explain why you would have this policy.

MATH ACTIVITY 3.3

Multiplication with Base-Five Pieces

Virtual Manipulatives

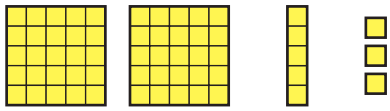


www.mhhe.com/bbn

Purpose: Model whole number multiplication ideas using base-five pieces.

Materials: Base-Five Pieces in the Manipulative Kit or Virtual Manipulatives.

1. Use your base-five pieces to represent 213_{five}. Then determine the minimal collection for a group of four of these sets to illustrate $4 \times 213_{five}$. This activity illustrates multiplication as *repeated addition.*



***2.** Use your base-five pieces to determine the minimal collection for each of the following products. Then write the base-five numeral for each product.

a. $2 \times 444_{five}$ **b.** $4 \times 234_{five}$ **c.** $3 \times 1042_{five}$

3. In base five, the numeric value *five* is written as 10_{five}. Thus, a product such as $10_{five} \times 13_{five}$ can be computed by forming five collections of 1 long and 3 units.

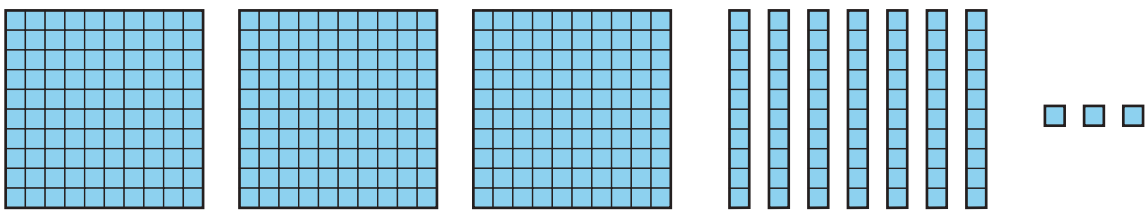
a. Use your base-five pieces to determine the minimal collection for $10_{five} \times 13_{five}$, and then write the base-five numeral for the product.

b. Repeat part a for the product $10_{five} \times 123_{five}$.

c. Explain, in terms of base-five pieces, why multiplication of a number by 10_{five} has the effect of affixing a zero onto the right of the numeral.

4. In base four, the numeric value *four* is written as 10_{four}. Draw a sketch of base-four pieces to illustrate the products $10_{four} \times 13_{four}$ and $10_{four} \times 322_{four}$, and write the base-four numeral for each product beneath its sketch.

***5.** What base is illustrated by the set of multibase pieces shown here? Determine the minimal collection for a group of six of these sets and write the product that is illustrated.



Section **3.3**

MULTIPLICATION



Corning Tower at the Empire State Plaza, Albany, New York

PROBLEM OPENER

Lee has written a two-digit number in which the units digit is her favorite digit. When she subtracts the tens digit from the units digit, she gets 3. When she multiplies the original two-digit number by 21, she gets a three-digit number whose hundreds digit is her favorite digit and whose tens and units digits are the same as those in her original two-digit number. What is her favorite digit?

The skyscraper in the photo here is called the Corning Tower. A window-washing machine mounted on top of the building lowers a cage on a vertical track so that each column of 40 windows can be washed. After one vertical column of windows has been washed, the machine moves to the next column. The rectangular face of the building visible in the photograph has 36 columns of windows. The total number of windows is $40 + 40 + 40 + \ldots + 40$, a sum in which 40 occurs 36 times. This sum equals the product 36×40, or 1440. We are led to different expressions for the sum and product by considering the rows of windows across the floors. There are 36 windows in each floor on this face of the building and 40 floors. Therefore, the number of windows is $36 + 36 + 36 + \ldots + 36$, a sum in which 36 occurs 40 times. This sum is equal to 40×36, which is also 1440. For sums such as these in which one number is repeated, multiplication is a convenient method for doing addition.

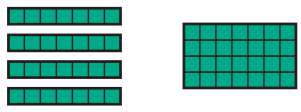
Historically, multiplication was developed to replace certain special cases of addition, namely, the cases of *several equal addends.* For this reason we usually see **multiplication** of whole numbers explained and defined as **repeated addition.**

Multiplication of Whole Numbers For any whole numbers r and s, the product of r and s is the sum with s occurring r times. This is written as

$$r \times s = \underbrace{s + s + s + \ldots + s}_{r \text{ times}}$$

If $r \neq 0$ and $s \neq 0$, r and s are called **factors.**

One way of representing multiplication of whole numbers is with a **rectangular array** of objects, such as the rows and columns of windows at the beginning of this section. Figure 3.12 shows the close relationship between the use of *repeated addition* and *rectangular arrays* for illustrating products. Part (a) of the figure shows squares in 4 groups of 7 to illustrate $7 + 7 + 7 + 7$, and part (b) shows the squares pushed together to form a 4×7 rectangle.



$7 + 7 + 7 + 7$ $\quad$ 4×7

(a) $\quad$ (b)

Figure 3.12

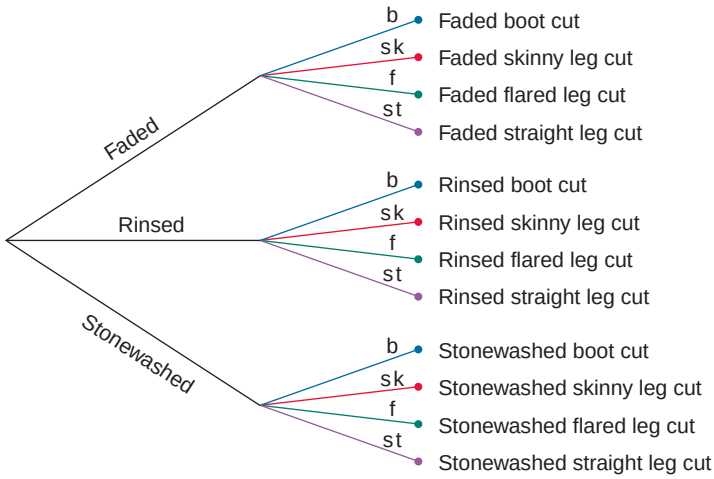
In general, $r \times s$ is the number of objects in an $r \times s$ rectangular array.

Another way of viewing multiplication is with a figure called a **tree diagram.** Constructing a tree diagram is a counting technique that is useful for certain types of multiplication problems.

EXAMPLE A

A catalog shows jeans available in faded, rinsed, or stonewashed fabric and in boot (b), skinny leg (sk), flared leg (f), or straight leg (st) cuts. How many types of jeans are available?

Solution A tree diagram for this problem is shown below. The tree begins with 3 branches, each labeled with one of the types of fabric. Each of these branches leads to 4 more branches, which correspond to the cuts. The tree has $3 \times 4 = 12$ endpoints, one for each of the 12 different types of jeans.



NCTM Standards

Research provides evidence that students will rely on their own computational strategies (Cobb et al.). Such inventions contribute to their mathematical development (Gravemeijer; Steffe). p. 86

MODELS FOR MULTIPLICATION ALGORITHMS

Physical models for multiplication can generate an understanding of multiplication and suggest or motivate procedures and rules for computing. There are many suitable models for illustrating multiplication. Base-ten pieces are used in the following examples.

Figure 3.13 illustrates 3×145, using base-ten pieces. First 145 is represented as shown in (a). Then the base-ten pieces for 145 are tripled. The result is 3 flats, 12 longs, and 15 units, as shown in (b). Finally, the pieces are regrouped: 10 units are replaced by 1 long, leaving 5 units; and 10 longs are replaced by 1 flat, leaving 3 longs. The result is 4 flats, 3 longs, and 5 units, as shown in (c).

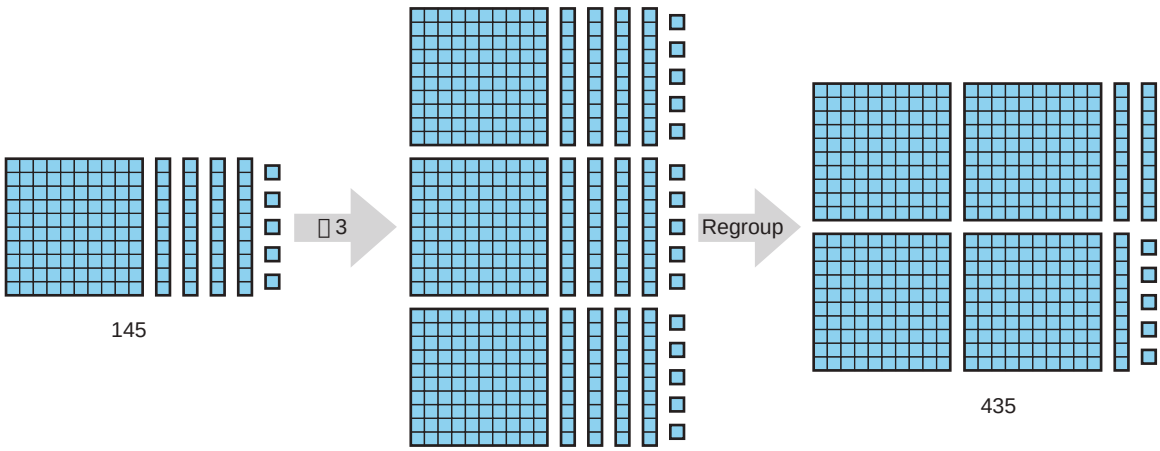


Figure 3.13

Base-ten pieces can be used to illustrate the pencil-and-paper algorithm for computing. Consider the product 3×145 shown in Figure 3.13. First a 5, indicating the remaining 5 units in part (c), is recorded in the units column, and the 10 units that have been regrouped are recorded by writing 1 in the tens column (see below). Then 3 is written in the tens column for the remaining 3 longs, and 1 is recorded in the hundreds column for the 10 longs that have been regrouped.

Flats	Longs	Units
1	1	
1	4	5
	×	3
4	3	5

Figure 3.14 on page 167 illustrates how multiplication by 10 can be carried out with base-ten pieces. Multiplying by 10 is especially convenient because 10 units can be placed together to form 1 long, 10 longs to form 1 flat, and 10 flats to form 1 long-flat (row of flats).

To multiply 34 and 10, we replace each base-ten piece for 34 by the base-ten piece for the next higher power of 10 (see Figure 3.14). We begin with 3 longs and 4 units and end with 3 flats, 4 longs, and 0 units. This illustrates the familiar fact that the product of any whole number and 10 can be computed by placing a zero at the right end of the numeral for the whole number.

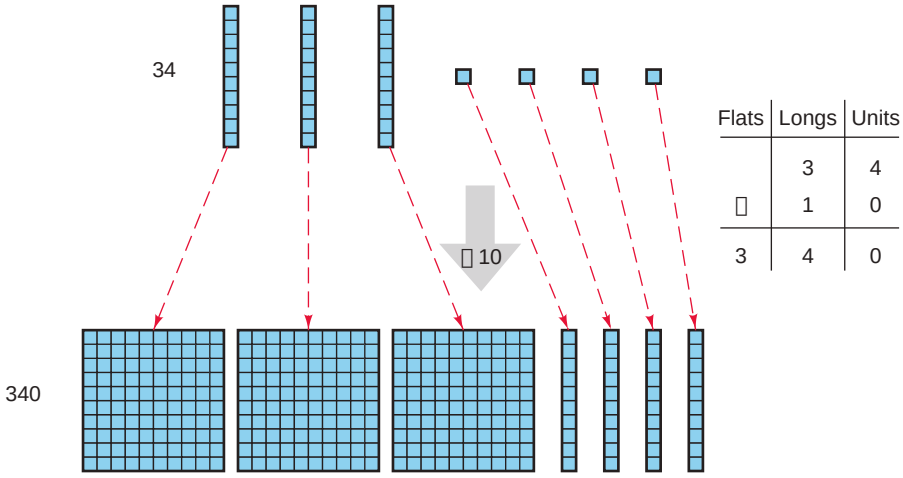


Figure 3.14

Computing the product of two numbers by repeated addition of base-ten pieces becomes impractical as the size of the numbers increases. For example, computing 18×23 requires representing 23 with base-ten pieces 18 times. For products involving two-digit numbers, rectangular arrays are more convenient.

To compute 18×23, we can draw a rectangle with dimensions 18 by 23 on grid paper (Figure 3.15). The product is the number of small squares in the rectangular array. This number can be easily determined by counting groups of 100 flats and strips of 10 longs. The total number of small squares is 414. Notice how the array in Figure 3.15 can be viewed as 18 horizontal rows of 23, once again showing the connection between the repeated-addition and rectangular-array views of multiplication.

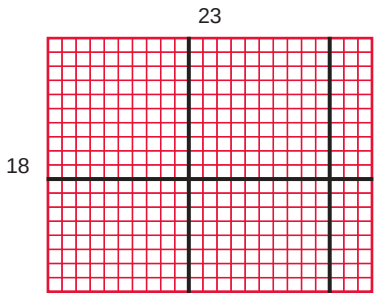


Figure 3.15

The pencil-and-paper algorithm for multiplication computes **partial products.** When a two-digit number is multiplied by a two-digit number, there are four partial products.

The product 13×17 is illustrated in Figure 3.16. The four regions of the grid formed by the heavy lines represent the four partial products. Sometimes it is instructive to draw arrows from each partial product to the corresponding region on the grid.

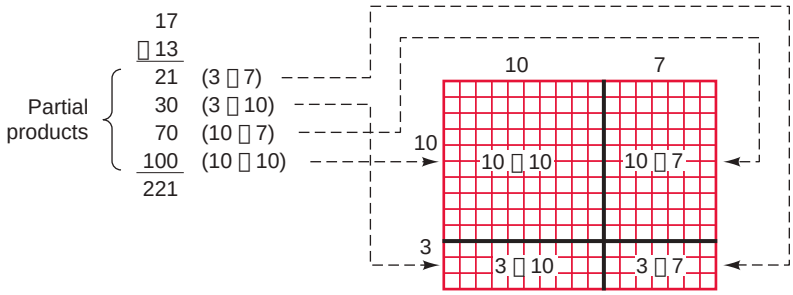


Figure 3.16

HISTORICAL HIGHLIGHT

$\rightarrow 1 \times 52 = 52$
$\rightarrow 2 \times 52 = 104$
$4 \times 52 = 208$
$\rightarrow 8 \times 52 = 416$

$$\begin{array}{r} 52 \\ 104 \\ +416 \\ \hline 572 \end{array}$$

One of the earliest methods of multiplication is found in the Rhind Papyrus. This ancient scroll (ca. 1650 B.C.E.), more than 5 meters in length, was written to instruct Egyptian scribes in computing with whole numbers and fractions. Beginning with the words "Complete and thorough study of all things, insights into all that exists, knowledge of all secrets . . . ," it indicates the Egyptians' awe of mathematics. Although most of its 85 problems have a practical origin, there are some of a theoretical nature. The Egyptians' algorithm for multiplication was a succession of doubling operations, followed by addition as shown in the example at the left. To compute 11×52, they would repeatedly double 52, then add *one* 52, *two* 52s, and *eight* 52s to get *eleven* 52s.

NUMBER PROPERTIES

Four properties for addition of whole numbers were stated in Section 3.2. Four corresponding properties for multiplication of whole numbers are stated below, along with one additional property that relates the operations of addition and multiplication.

Closure Property for Multiplication This property states that the product of any two whole numbers is also a whole number.

> **Closure Property for Multiplication** For any whole numbers a and b,
>
> $a \times b$ is a unique whole number.

Identity Property for Multiplication The number 1 is called an **identity for multiplication** because when multiplied by another number, it leaves the identity of the number unchanged. For example,

$$1 \times 14 = 14 \qquad 34 \times 1 = 34 \qquad 1 \times 0 = 0$$

The number 1 is unique in that it is the only number that is an identity for multiplication.

> **Identity Property for Multiplication** For any whole number b,
>
> $1 \times b = b \times 1 = b$
>
> and 1 is the unique identity for multiplication.

Commutative Property for Multiplication This number property says in any product of numbers, two numbers may be interchanged (commuted) without affecting the product. This property is called the **commutative property for multiplication.** For example,

$$\underbrace{347 \times 26}_{\uparrow} = \underbrace{26 \times 347}_{\uparrow}$$

Commutative property for multiplication

Commutative Property for Multiplication For any whole numbers a and b,

$$a \times b = b \times a$$

NCTM Standards

Using area models, properties of operations such as commutativity of multiplication become more apparent. p. 152

The commutative property is illustrated in Figure 3.17, which shows two different views of the same rectangular array. Part (a) represents 7×5, and part (b) represents 5×7. Since part (b) is obtained by rotating part (a), both figures have the same number of small squares, so 7×5 is equal to 5×7.

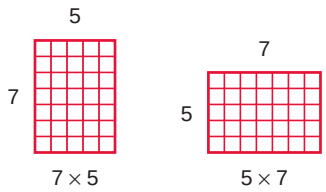


Figure 3.17

As the multiplication table in Figure 3.18 shows, the commutative property for multiplication approximately cuts in half the number of basic multiplication facts that elementary school students must memorize. Each product in the shaded part of the table corresponds to an equal product in the unshaded part of the table.

EXAMPLE B

Since $3 \times 7 = 21$, we know by the commutative property for multiplication that $7 \times 3 = 21$. What do you notice about the location of each product in the shaded part of the table relative to the location of the corresponding equal product in the unshaded part of the table?

Solution If the shaded part of the table is folded along the diagonal onto the unshaded part, each product in the shaded part will coincide with an equal product in the unshaded part. In other words, the table is symmetric about the diagonal from upper left to lower right.

×	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

Figure 3.18

Notice that the numbers in the rows of the multiplication table in Figure 3.18 form arithmetic sequences, for example, 2, 4, 6, 8, . . . and 3, 6, 9, 12. . . . One reason that children learn to count by 2s, 3s, and 5s is to acquire background for learning basic multiplication facts.

Associative Property for Multiplication In any product of three numbers, the middle number may be associated with and multiplied by either of the two end numbers. This property is called the **associative property for multiplication.** For example,

$$\underbrace{6 \times (7 \times 4)}_{\uparrow} = \underbrace{(6 \times 7) \times 4}_{\uparrow}$$

Associative property for multiplication

Associative Property for Multiplication For any whole numbers a, b, and c,

$$a \times (b \times c) = (a \times b) \times c$$

Figure 3.19 illustrates the associative property for multiplication. Part (a) represents 3×4, and (b) shows 5 of the 3×4 rectangles. The number of small squares in (b) is $5 \times (3 \times 4)$. Part (c) is obtained by subdividing the rectangle (b) into 4 copies of a 3×5 rectangle. The number of small squares in (c) is $4 \times (3 \times 5)$, which, by the commutative property for multiplication, equals $(5 \times 3) \times 4$. Since the numbers of small squares in (b) and (c) are equal, $5 \times (3 \times 4) = (5 \times 3) \times 4$.

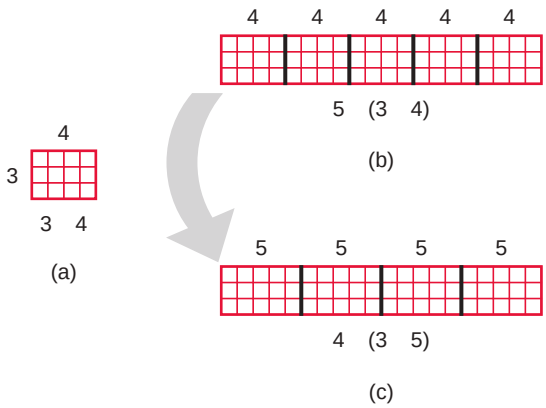


Figure 3.19

The commutative and associative properties are often used to obtain convenient combinations of numbers for mental calculations, as in Example C.

EXAMPLE C Try computing $25 \times 46 \times 4$ in your head before reading further.

Solution The easy way to do this is by rearranging the numbers so that 25×4 is computed first and then 46×100. The following equations show how the commutative and associative properties permit this rearrangement.

Associative property for multiplication

$$\underbrace{(25 \times 46)}_{\uparrow} \times 4 = \underbrace{(46 \times 25) \times 4}_{\uparrow} = 46 \times (25 \times 4)$$

Commutative property for multiplication

Distributive Property When multiplying a sum of two numbers by a third number, we can add the two numbers and then multiply by the third number, or we can multiply each number of the sum by the third number and then add the two products.

For example, to compute $35 \times (10 + 2)$, we can compute 35×12, or we can add 35×10 to 35×2. This property is called the **distributive property for multiplication over addition.**

$$35 \times 12 = \underbrace{35 \times (10 + 2)}_{\text{Distributive property}} = (35 \times 10) + (35 \times 2)$$

Distributive Property for Multiplication over Addition For any whole numbers a, b, and c,

$$a \times (b + c) = a \times b + a \times c$$

One use of the distributive property is in learning the basic multiplication facts. Elementary schoolchildren are often taught the "doubles" ($2 + 2 = 4$, $3 + 3 = 6$, $4 + 4 = 8$, etc.) because these number facts together with the distributive property can be used to obtain other multiplication facts.

EXAMPLE D How can $7 \times 7 = 49$ and the distributive property be used to compute 7×8?

Solution

$$7 \times 8 = \underbrace{7 \times (7 + 1)}_{\text{Distributive property}} = 49 + 7 = 56$$

The distributive property can be illustrated by using rectangular arrays, as in Figure 3.20. The dimensions of the array in (a) are 6 by $(3 + 4)$, and the array contains 42 small squares. Part (b) shows the same squares separated into two rectangular arrays with dimensions 6 by 3 and 6 by 4. Since the number of squares in both figures is the same, $6 \times (3 + 4) = (6 \times 3) + (6 \times 4)$.

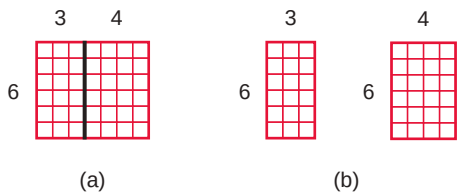


Figure 3.20

The distributive property also holds for multiplication over subtraction.

Explore

Math Activity for 3-2
Multiply Mentally

MAIN IDEA

I will mentally multiply a one-digit factor by a two-digit factor.

Math Online

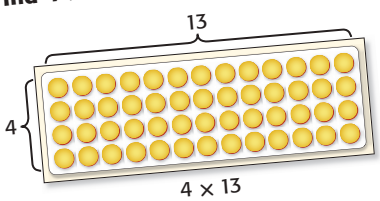
macmillanmh.com

• Concepts in Motion

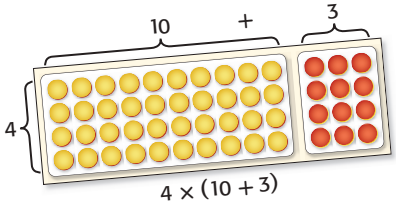
It may be hard to find a product like 4×13 mentally, even if you use counters. If you separate the counters into smaller groups called *partial products*, it is easier to multiply.

ACTIVITY

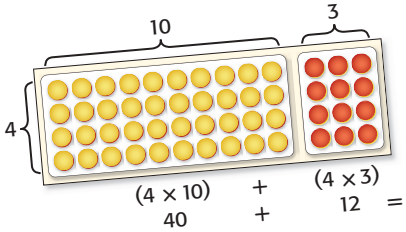
1 **Find 4×13 mentally using partial products.**



Model 4×13 by arranging counters in 4 rows and 13 columns.



Separate 13 into two numbers that are each easily multiplied by 4.



Multiply to find the number of counters in each group. Then add.

Rewrite 4×13 as $(4 \times 10) + (4 \times 3)$. This is useful since it is easier to find $(4 \times 10) + (4 \times 3)$ mentally than to find 4×13. So, 4×13 is 52.

Think About It

1. To find 4×13, you can also find $4 \times (9 + 4)$. Why is it easier to find $4 \times (10 + 3)$ mentally than it is to find $4 \times (9 + 4)$?

2. Which expression would you use to find 7×19 mentally: $7 \times (13 + 6)$ or $7 \times (10 + 9)$? Explain.

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EXAMPLE E Show that the two sides of the following equation are equal.

$$6 \times (20 - 8) = (6 \times 20) - (6 \times 8)$$

Solution $6 \times (20 - 8) = 6 \times 12 = 72$ and $(6 \times 20) - (6 \times 8) = 120 - 48 = 72$.

MENTAL CALCULATIONS

In the following paragraphs, three methods are discussed for performing mental calculations of products. These methods parallel those used for performing mental calculations of sums and differences.

Compatible Numbers for Mental Calculation We saw in Example C that the commutative and associative properties permit the rearrangement of numbers in products. Such rearrangements can often enable computations with compatible numbers.

EXAMPLE F Find a more convenient arrangement that will yield compatible numbers, and compute the following products mentally.

1. $5 \times 346 \times 2$
2. $2 \times 25 \times 79 \times 2$

Solution **1.** $5 \times 2 \times 346 = 10 \times 346 = 3460$. **2.** $2 \times 2 \times 25 \times 79 = 100 \times 79 = 7900$.

Substitutions for Mental Calculation In certain situations the distributive property is useful for facilitating mental calculations. For example, to compute 21×103, first replace 103 by $100 + 3$ and then compute 21×100 and 21×3 in your head. Try it.

$$21 \times 103 = \underbrace{21 \times (100 + 3)}_{\uparrow} = \underbrace{2100 + 63}_{\uparrow} = 2163$$

Distributive property

Occasionally it is convenient to replace a number by the difference of two numbers and to use the fact that multiplication distributes over subtraction. Rather than compute 45×98, we can compute 45×100 and subtract 45×2.

$$45 \times 98 = \underbrace{45 \times (100 - 2)}_{\uparrow} = \underbrace{4500 - 90}_{\uparrow} = 4410$$

Distributive property

EXAMPLE G Find a convenient substitution, and compute the following products mentally.

1. 25×99
2. 42×11
3. 34×102

Solution **1.** $25 \times (100 - 1) = 2500 - 25 = 2475$. **2.** $42 \times (10 + 1) = 420 + 42 = 462$. **3.** $34 \times (100 + 2) = 3400 + 68 = 3468$.

NCTM Standards

Other relationships can be seen by decomposing and composing area models. For example, a model for 20×6 can be split in half and the halves rearranged to form a 10×12 rectangle, showing the equivalence of 10×12 and 20×6. p. 152

Equal Products for Mental Calculation This method of performing mental calculations is similar to the *equal differences* method used for subtraction. It is based on the fact that the product of two numbers is unchanged when one of the numbers is divided by a given number and the other number is multiplied by the same number. For example, the product 12×52 can be replaced by 6×104 by dividing 12 by 2 and multiplying 52 by 2. At this point we can mentally calculate 6×104 to be 624. Or we can continue the process of dividing and multiplying by 2, replacing 6×104 by 3×208, which can also be mentally calculated.

Figure 3.21 illustrates why one number in a product can be halved and the other doubled without changing the product. The rectangular array in part (a) of the figure represents 22×16. If this rectangle is cut in half, the two pieces can be used to form an 11×32 rectangle, as in (b). Notice that 11 is half of 22 and 32 is twice 16. Since the rearrangement has not changed the number of small squares in the two rectangles, the products 22×16 and 11×32 are equal.

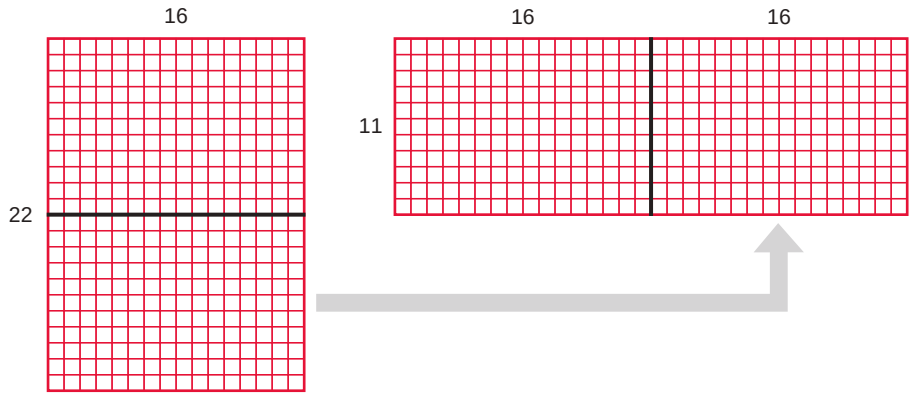


Figure 3.21

The equal-products method can also be justified by using number properties. The following equations show that $22 \times 16 = 11 \times 32$. Notice that multiplying by $\frac{1}{2}$ and 2 is the same as multiplying by 1. This is a special case of the inverse property for multiplication, which is discussed in Section 5.3.

$$\begin{aligned} 22 \times 16 &= 22 \times 1 \times 16 && \text{identity property for multiplication} \\ &= 22 \times \left(\frac{1}{2} \times 2\right) \times 16 && \text{inverse property for multiplication} \\ &= \left(22 \times \frac{1}{2}\right) \times (2 \times 16) && \text{associative property for multiplication} \\ &= 11 \times 32 \end{aligned}$$

EXAMPLE H

Use the method of equal products to perform the following calculations mentally.

1. 14×4
2. 28×25
3. 15×35

Solution **1.** $14 \times 4 = 7 \times 8 = 56$. **2.** $28 \times 25 = 14 \times 50 = 7 \times 100 = 700$. **3.** $15 \times 35 = 5 \times 105 = 525$.

NCTM Standards

Instruction should emphasize the development of an estimation mindset. Children should come to know what is meant by an estimate, when it is appropriate to estimate, and how close an estimate is required in a given situation. If children are encouraged to estimate, they will accept estimation as a legitimate part of mathematics. p. 115

ESTIMATION OF PRODUCTS

The importance of estimation is noted in NCTM's K–4 Standard, *Estimation* in the *Curriculum and Evaluation Standards for School Mathematics*. The techniques of *rounding,* using *compatible numbers for estimation,* and *front-end estimation* are used in the following examples.

Rounding Products can be estimated by rounding one or both numbers. Computing products by rounding is somewhat more risky than computing sums by rounding, because any error due to rounding becomes multiplied. For example, if we compute 47×28 by rounding 47 to 50 and 28 to 30, the estimated product $50 \times 30 = 1500$ is greater than the actual product. This may be acceptable if we want an estimate greater than the actual product. For a closer estimate, we can round 47 to 45 and 28 to 30. In this case the estimate is $45 \times 30 = 1350$.

EXAMPLE 1

NCTM Standards

When students leave grade 5, . . . they should be able to solve many problems mentally, to estimate a reasonable result for a problem, . . . and to compute fluently with multidigit whole numbers. p. 149

Use rounding to estimate these products. Make any adjustments you feel might be needed.

1. 28×63
2. 81×57
3. 194×26

Solution Following is one estimate for each product. You may find others. **1.** $28 \times 63 \approx 30 \times 60 = 1800$. Notice that since 63 is greater than 28, increasing 28 by 2 has more of an effect on the estimate than decreasing 63 to 60 (see Figure 3.22). So the estimate of 1800 is greater than the actual answer. **2.** $81 \times 57 \approx 80 \times 60 = 4800$. **3.** $194 \times 26 \approx 200 \times 25 = 5000$.

Figure 3.22 shows the effect of estimating 28×63 by rounding to 30×60. Rectangular arrays for both 28×63 and 30×60 can be seen on the grid. The green region shows the increase from rounding 28 to 30, and the red region shows the decrease from rounding 63 to 60. Since the green region ($2 \times 60 = 120$) is larger than the red region ($3 \times 28 = 84$), we are adding more than we are removing, and so the estimate is greater than the actual product.

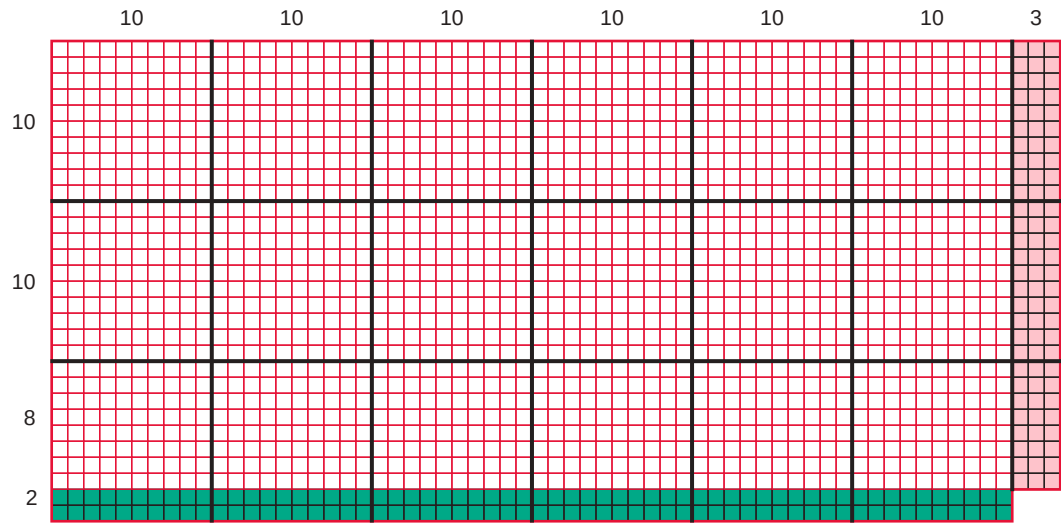


Figure 3.22

Compatible Numbers for Estimation Using compatible numbers becomes a powerful tool for estimating products when it is combined with techniques for performing mental calculations. For example, to estimate $4 \times 237 \times 26$, we might replace 26 by 25 and use a different ordering of the numbers.

$$4 \times 237 \times 26 \approx 4 \times 25 \times 237 = 100 \times 237 = 23{,}700$$

EXAMPLE J

Use compatible numbers to estimate these products.

1. $2 \times 117 \times 49$
2. $34 \times 46 \times 3$

Solution **1.** $2 \times 117 \times 49 \approx 2 \times 117 \times 50 = 100 \times 117 = 11{,}700$. **2.** $34 \times 46 \times 3 = (3 \times 34) \times 46 \approx 100 \times 46 = 4600$.

Front-End Estimation This technique is similar to that used for computing sums. The leading digit of each number is used to obtain an estimated product. To estimate 43×72, the product of the leading digits of the numbers is $4 \times 7 = 28$, so the estimated product is 2800.

$$43 \times 72 \approx 40 \times 70 = 2800$$

Similarly, front-end estimation can be used for estimating the products of numbers whose leading digits have different place values.

$$61 \times 874 \approx 60 \times 800 = 48{,}000$$

EXAMPLE K

Use front-end estimation to estimate these products.

1. 64×23 **2.** 68×87 **3.** 237×76 **4.** $30{,}328 \times 419$

Solution **1.** $64 \times 23 \approx 60 \times 20 = 1200$ **2.** $68 \times 87 \approx 60 \times 80 = 4800$ **3.** $237 \times 76 \approx 200 \times 70 = 14{,}000$ **4.** $30{,}328 \times 419 \approx 30{,}000 \times 400 = 12{,}000{,}000$



Order of Operations

Special care must be taken on some calculators when multiplication is combined with addition or subtraction. The numbers and operations will not always produce the correct answer if they are entered into the calculator in the order in which they appear.

EXAMPLE L

Compute $3 + 4 \times 5$ by entering the numbers into your calculator as they appear from left to right.

Solution Some calculators will display 35, and others will display 23. The correct answer is 23 because multiplication should be performed before addition:

$$3 + 4 \times 5 = 3 + 20 = 23$$

Mathematicians have developed the convention that when multiplication occurs with addition and/or subtraction, the multiplication should be performed first. This rule is called the **order of operations.**



Some calculators are programmed to follow the order of operations. On this type of calculator, any combination of products with sums and differences and without parentheses can be computed by entering the numbers and operations in the order in which they occur from left to right and then pressing [=]. If a calculator does not follow the order of operations, the products can be computed separately and recorded by hand or saved in the calculator's memory.

EXAMPLE M Use your calculator to evaluate $34 \times 19 + 82 \times 43$. Then check the reasonableness of your answer by using estimation and mental calculations.

Solution The exact answer is 4172. An estimate can be obtained as follows:

$$34 \times 19 + 82 \times 43 \approx 30 \times 20 + 80 \times 40 = 600 + 3200 = 3800$$

Notice that the estimation in Example M is 372 less than the actual product. However, it is useful in judging the reasonableness of the number obtained from the calculator: It indicates that the calculator answer is most likely correct. If $34 \times 19 + 82 \times 43$ is entered into a calculator as it appears from left to right and if the calculator is not programmed to follow the order of operations, then the incorrect result of 31,304 will be obtained, which is too large by approximately 27,000.

PROBLEM-SOLVING APPLICATION

There is an easy method for mentally computing the products of certain two-digit numbers. A few of these products are shown here.

$$25 \times 25 = 625 \qquad 24 \times 26 = 624 \qquad 71 \times 79 = 5609$$
$$37 \times 33 = 1221 \qquad 35 \times 35 = 1225 \qquad 75 \times 75 = 5625$$

The solution to the following problem reveals the method of mental computation and uses *rectangular grids* to show why the method works.

Problem

What is the method of mental calculation for computing the products of the two-digit numbers shown above, and why does this method work?

Understanding the Problem There are patterns in the digits in these products. One pattern is that the two numbers in each pair have the same first digit. Find another pattern. **Question 1:** What types of two-digit numbers are being used?

Devising a Plan Looking for patterns may help you find the types of numbers and the method of computing. Another approach is to represent some of these products on a grid. The following grid illustrates 24×26; the product is the number of small squares in the rectangle. To determine this number, we begin by counting large groups of squares. There are 6 hundreds.

Technology Connection

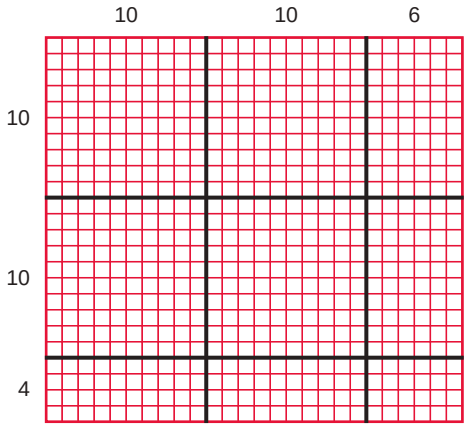
Number Chains

Start with any two-digit number, double its units digit, and add the result to its tens digit to obtain a new number. Repeat this process with each new number, as shown by the example here. What eventually happens? Use the online 3.3 Mathematics Investigation to generate number chains beginning with different numbers and to look for patterns and form conjectures.

$14 \rightarrow 9 \rightarrow 18 \rightarrow 17 \rightarrow 15$

Mathematics Investigation
Chapter 3, Section 3
www.mhhe.com/bbn

Question 2: Why is this grid especially convenient for counting the number of hundreds?



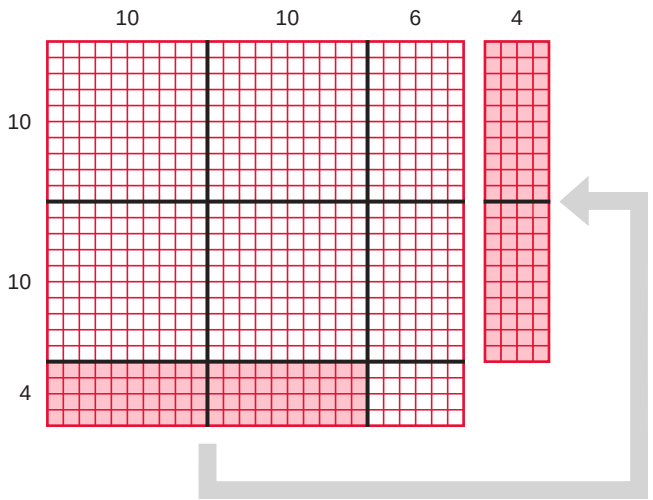
Carrying Out the Plan Sketch grids for one or more of the products being considered in this problem. For each grid it is easy to determine the number of hundreds. This is the key to solving the problem. **Question 3:** What is the solution to the original problem?

Looking Back Consider the following products of three-digit numbers:

$$103 \times 107 = 11{,}021 \qquad 124 \times 126 = 15{,}624$$

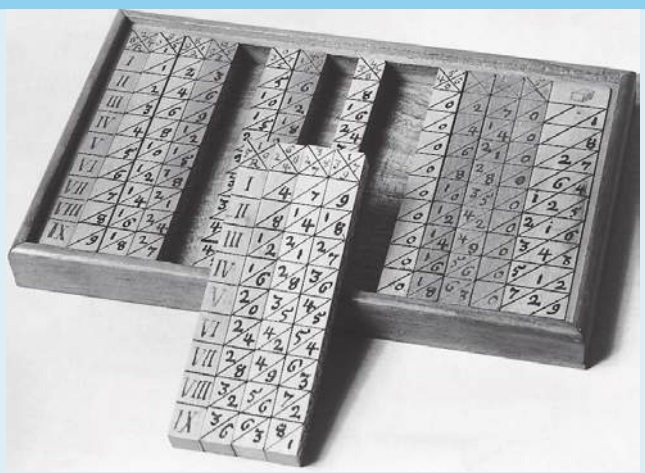
Question 4: Is there a similar method for mentally calculating the products of certain three-digit numbers?

Answers to Questions 1–4 **1.** In each pair of two-digit numbers, the tens digits are equal and the sum of the units digits is 10. **2.** The two blocks of 40 squares at the bottom of the grid can be paired with two blocks of 60 squares on the right side of the grid to form two more blocks of 100, as shown below. Then the large 20×30 grid represents 6 hundreds. The 4×6 grid in the lower right corner represents 4×6.



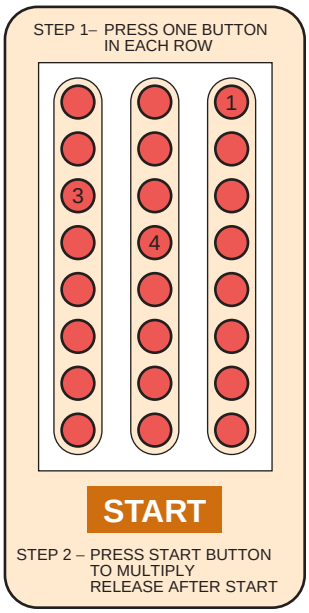
3. The first two digits of the product are formed by multiplying the tens digit by the tens digit plus 1. The remaining digits of the product are obtained by multiplying the two units digits. **4.** Yes. For 124×126: $12 \times 13 = 156$ and $4 \times 6 = 24$, so $124 \times 126 = 15{,}624$.

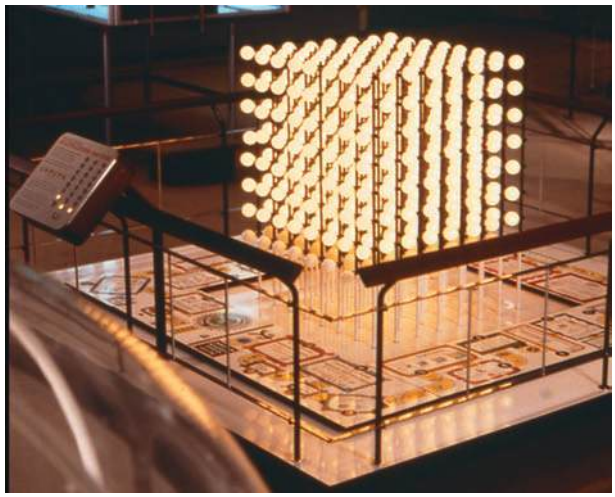
HISTORICAL HIGHLIGHT



As late as the seventeenth century, multiplication of large numbers was a difficult task for all but professional clerks. To help people "do away with the difficulty and tediousness for calculations," Scottish mathematician John Napier (1550–1617) invented a method of using rods for performing multiplication. Napier's rods—or *bones* as they are sometimes called—contain multiplication facts for each digit. For example, the rod for the 4s has 4, 8, 12, 16, 20, 24, 28, 32, and 36. This photograph of a wooden set shows the fourth, seventh, and ninth rods placed together for computing products that have a factor of 479. For example, to compute 6×479 look at row VI of the three rods for 479. Adding the numbers along the diagonals of row VI results in the product of 2874.

Exercises and Problems 3.3

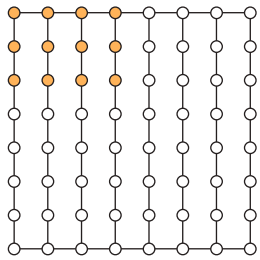




An exhibit illustrating multiplication at the California Museum of Science and Industry.

©2005 Eames Office LLC (www.eamesoffice.com)

This cube of lights illustrates products of three numbers from 1 through 8. Each time three buttons are pressed on the switch box, the product is illustrated by lighted bulbs in the $8 \times 8 \times 8$ cube of bulbs. Buttons 3, 4, and 1 are for the product $3 \times 4 \times 1$. The 12 bulbs in the upper left corner of the cube will be lighted for this product, as shown in the following figure. Whenever the third number of the product is 1, the first two numbers determine a rectangular array of lighted bulbs on the front face of the cube (facing switch box).



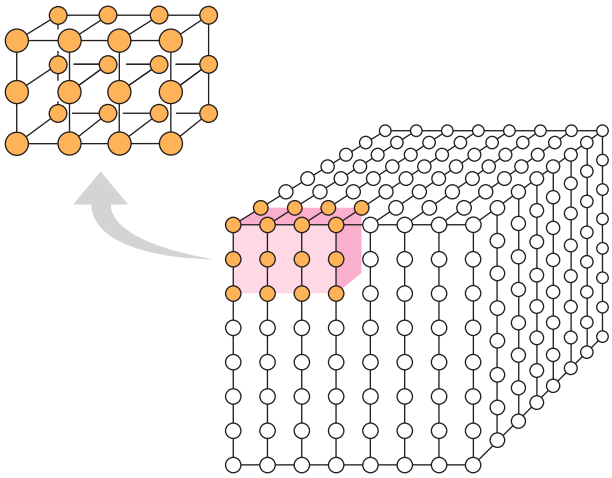
$3 \times 4 \times 1$

Describe the bulbs that will be lighted for the products in exercises 1 and 2.

1. a. $7 \times 3 \times 1$ **b.** $2 \times 8 \times 1$

2. a. $5 \times 4 \times 1$ **b.** $8 \times 8 \times 1$

The third number in a product illustrated by the cube of bulbs determines the number of times the array on the front face is repeated in the cube. The 24 bulbs in the upper left corner of the next figure will be lighted for $3 \times 4 \times 2$.



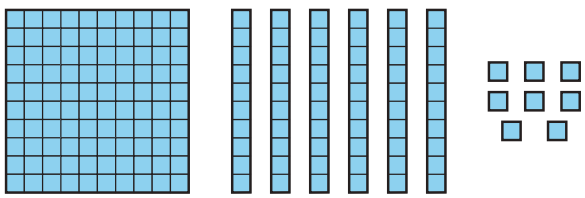
Describe the bulbs that will be lighted for the products in exercises 3 and 4.

3. a. $6 \times 4 \times 3$ **b.** $1 \times 8 \times 8$

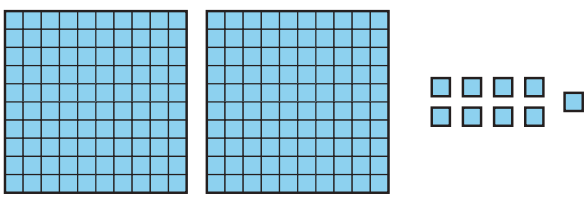
4. a. $2 \times 2 \times 2$ **b.** $8 \times 1 \times 8$

Sketch a new set of base pieces for each product in exercises 5 and 6, and then show regrouping.

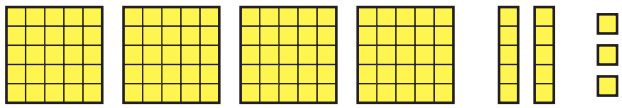
5. a. Multiply 168 by 3.



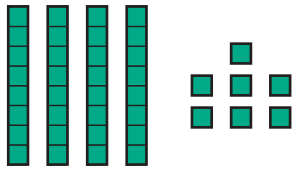
b. Multiply 209 by 4.



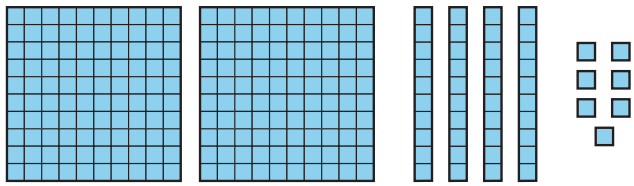
c. Multiply 423_{five} by 3.



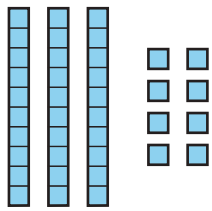
d. Multiply 47_{eight} by 5.



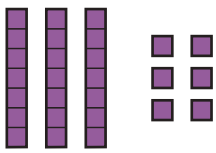
6. a. Multiply 247 by 2.



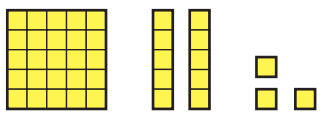
b. Multiply 38 by 5.



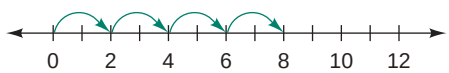
c. Multiply 36_{seven} by 5.



d. Multiply 123_{five} by 4.



Multiplication of whole numbers can be illustrated on the number line by a series of arrows beginning at 0. This number line shows 4×2.



Draw arrow diagrams for the products in exercises 7 and 8.

7. a. 3×4 **b.** 2×5

c. Use the number line to show that $3 \times 4 = 4 \times 3$.

8. a. 2×6 **b.** 6×2
c. Use the number line to show that $2 \times (3 + 2) = 2 \times 3 + 2 \times 2$.

Error analysis. Students who know their basic multiplication facts may still have trouble with the steps in the pencil-and-paper multiplication algorithm. Try to detect each type of error in exercises 9 and 10, and write an explanation.

9. a.
$$\begin{array}{r} {\scriptstyle 2}\\ 27\\ \times\ 4\\ \hline 48 \end{array}$$
b.
$$\begin{array}{r} {\scriptstyle 2}\\ 18\\ \times\ 3\\ \hline 34 \end{array}$$

10. a.
$$\begin{array}{r} {\scriptstyle 4}\\ 54\\ \times\ 6\\ \hline 342 \end{array}$$
b.
$$\begin{array}{r} {\scriptstyle 1}\\ 34\\ \times\ 24\\ \hline 76 \end{array}$$

In exercises 11 and 12, use base-ten grids to illustrate the partial products that occur when these products are computed with pencil and paper. Draw arrows from each partial product to its corresponding region on the grid. (Copy the Base-Ten Grid from the website or use the Virtual Manipulatives.)

11. a.
$$\begin{array}{r} 24\\ \times\ 7\\ \hline \end{array}$$
b.
$$\begin{array}{r} 56\\ \times\ 43\\ \hline \end{array}$$

12. a.
$$\begin{array}{r} 34\\ \times\ 26\\ \hline \end{array}$$
b.
$$\begin{array}{r} 39\\ \times\ 47\\ \hline \end{array}$$

Which number property is being used in each of the equalities in exercises 13 and 14?

13. a. $3 \times (2 \times 7 + 1) = 3 \times (7 \times 2 + 1)$
b. $18 + (43 \times 7) \times 9 = 18 + 43 \times (7 \times 9)$
c. $(12 + 17) \times (16 + 5)$
$= (12 + 17) \times 16 + (12 + 17) \times 5$

14. a. $(13 + 22) \times (7 + 5) = (13 + 22) \times (5 + 7)$
b. $(15 \times 2 + 9) + 3 = 15 \times 2 + (9 + 3)$
c. $59 + 41 \times 8 + 41 \times 26 = 59 + 41 \times (8 + 26)$

Determine whether each set in exercises 15 and 16 is closed for the given operation. Explain your answer. If not closed give an example to show why not.

15. a. The set of even whole numbers for multiplication.
b. The set of whole numbers less than 100 for addition.
c. The set of all whole numbers whose units digits are 6 for multiplication.

16. a. The set of all whole numbers whose units digits are 0 for multiplication.
b. The set of whole numbers less than 1000 for multiplication.
c. The set of whole numbers greater than 1000 for multiplication.

In exercises 17 and 18, compute the exact products mentally, using *compatible numbers.* Explain your method.

17. a. $2 \times 83 \times 50$ **b.** $5 \times 3 \times 2 \times 7$

18. a. $4 \times 2 \times 25 \times 5$ **b.** $5 \times 17 \times 20$

In exercises 19 and 20, compute the exact products mentally, using substitution and the fact that multiplication distributes over addition. Show your use of the distributive property.

19. a. 25×12 **b.** 15×106

20. a. 18×11 **b.** 14×102

In exercises 21 and 22, compute the exact products mentally, using the fact that multiplication distributes over subtraction. Show your use of the distributive property.

21. a. 35×19 **b.** 30×99

22. a. 51×9 **b.** 40×98

In exercises 23 and 24, use the method of *equal products* to find numbers that are more convenient for making exact mental calculations. Show the new products that replace the original products.

23. a. 24×25 **b.** 35×60

24. a. 16×6 **b.** 36×5

In exercises 25 and 26, *round* the numbers and mentally estimate the products. Show the rounded numbers, and predict whether the estimated products are greater than or less than the actual products. Explain any adjustment you make to improve the estimates.

25. a. 22×17 **b.** 83×31

26. a. 71×56 **b.** 205×29

In exercises 27 and 28, use *compatible numbers* to estimate the products. Show your compatible-number replacements, and predict whether the estimated products are greater than or less than the actual products.

27. a. $4 \times 76 \times 24$ **b.** $3 \times 34 \times 162$

28. a. $5 \times 19 \times 74$ **b.** $2 \times 63 \times 2 \times 26$

In exercises 29 and 30, estimate the products, using *front-end estimation*. Show two estimates for each product, one using only the tens digits and one using combinations of the tens and units digits.

29. a. 36×58 **b.** 42×27

30. a. 62×83 **b.** 14×62

In exercises 31 and 32, *round* the given numbers and estimate each product. Then sketch a rectangular array for the actual product, and on the same figure sketch the rectangular array for the product of the rounded numbers. Shade the regions that show increases and/or decreases due to rounding. (Copy the Base-Ten Grid from the website or use the Virtual Manipulatives.)

31. **a.** 18×62 **b.** 43×29

32. **a.** 17×28 **b.** 53×31

In exercises 33 and 34, circle the operations in each expression that should be performed first. Estimate each expression mentally, and show your method of estimating. Use a calculator to obtain an exact answer, and compare this answer to your estimate.

33. **a.** $62 \times 45 + 14 \times 29$ **b.** $36 + 18 \times 40 + 15$

34. **a.** $114 \times 238 - 19 \times 605$ **b.** $73 - 50 + 17 \times 62$

In exercises 35 and 36, a number sequence is generated by beginning with the first number entered into the calculator and repeatedly carrying out the given keystrokes. Beginning with the first number entered, write each sequence that is produced by the given keystrokes.

35. **a.** Enter 5 and repeat the keystrokes [×] [3] [=] eight times.
b. Enter 20 and repeat the keystrokes [+] [5] [=] nine times.

36. **a.** Enter 91 and repeat the keystrokes [−] [2] [=] fourteen times.
b. Enter 3 and repeat the keystrokes [×] [2] [=] six times.

An elementary school calculator with a constant function is convenient for generating a geometric sequence. For example, the sequence 2, 4, 8, 16, 32, . . . , will be produced by entering 1 [×] 2 and repeatedly pressing [=]. Write the first four terms of each sequence in exercises 37 and 38.

37.

	Keystrokes	View Screen
a.	81 □ 5 □ □ □ □ □	
b.	119 □ 4 □ □ □ □ □	

38.

	Keystrokes	View Screen
a.	17 □ 3 □ □ □ □ □	
b.	142 □ 6 □ □ □ □ □	

In exercises 39 and 40, estimate the second factor so that the product will fall within the given range. Check your answer with a calculator. Count the number of tries it takes you to land in the range.

	Product	Range
Example	$22 \times$ _____	(900, 1000)
	$22 \times 40 = 880$	Too small
	$22 \times 43 = 946$	In the range in two tries

39. **a.** $32 \times$ _____ (800, 850)
b. $95 \times$ _____ (1650, 1750)

40. **a.** $103 \times$ _____ (2800, 2900)
b. $6 \times$ _____ (3500, 3600)

41. There are many patterns in the multiplication table (page 169) that can be useful in memorizing the basic multiplication facts.
a. What patterns can you see?
b. There are several patterns for products involving 9 as one of the numbers being multiplied. Find two of these patterns.

Reasoning and Problem Solving

42. A student opened her math book and computed the sum of the numbers on two facing pages. Then she turned to the next page and computed the sum of the numbers on these two facing pages. Finally, she computed the product of the two sums, and her calculator displayed the number 62,997. What were the four page numbers?

43. Harry has $2500 in cash to pay for a secondhand car, or he can pay $500 down and $155 per month for 2 years. If he doesn't pay the full amount in cash, he knows he can make $150 by investing his money. How much will he lose if he uses the more expensive method of payment?

44. Kathy read 288 pages of her 603-page novel in 9 days. How many pages per day must she now read in order to complete the book and return it within the library's 14-day deadline?

45. A store carries five styles of backpacks in four different sizes. The customer also has a choice of two different kinds of material for three of the styles. If Vanessa is only interested in the two largest backpacks, how many different backpacks would she have to choose from?

46. Featured Strategy: Making an Organized List. The five tags shown below are placed in a box and mixed. Three tags are then selected at a time. If a player's score is the product of the numbers, how many different scores are possible?



a. Understanding the Problem. The problem asks for the number of different scores, so each score can be counted only once. The tags 6, 5, and 1 produce a score of 30. Find three other tags that produce a score of 30.

b. Devising a Plan. One method of solving the problem is to form an organized list. If we begin the list with the number 3, there are six different possibilities for sets of three tags. List these six possibilities.

c. Carrying Out the Plan. Continue to list the different sets of three tags and compute their products. How many different scores are there?

d. Looking Back. A different type of organized list can be formed by considering the scores between 6 (the smallest score) and 90 (the greatest score). For example, 7, 8, and 9 can be quickly thrown out. Why?

Find patterns in problems 47 and 48 and determine if they continue to hold for the next few equations. If so, will they continue to hold for more equations? Show examples to support your conclusions.

47.
$$1 \times 9 + 2 = 11$$
$$12 \times 9 + 3 = 111$$
$$123 \times 9 + 4 = 1111$$

48.
$$1 \times 99 = 99$$
$$2 \times 99 = 198$$
$$3 \times 99 = 297$$

49. a. Select some two-digit numbers and multiply them by 99 and 999. Describe a few patterns and form some conjectures.

b. Test your conjectures on some other two-digit numbers. Predict whether your conjectures will continue to hold, and support your conclusions with examples.

c. Do your conjectures continue to hold for three-digit numbers times 99 and 999?

50. a. Select some two-digit numbers and multiply them by 11. Describe some patterns and form a conjecture.

b. Test your conjecture on some other two-digit numbers. Predict whether your conjectures will continue to hold, and support your conclusions with examples.

c. Does your conjecture hold for three-digit numbers times 11?

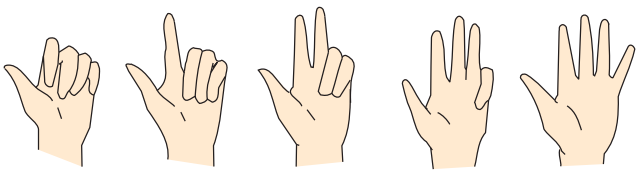
51. Samir has a combination lock with numbers from 1 to 25. This is the type of lock that requires three numbers to be opened: turn right for the first number, left for the second number, and right for the third number. Samir remembers the first two numbers, and they are not equal; but he can't remember which one is first and which is second. Also, he has forgotten the third number. What is the greatest number of tries he must make to open the lock?

52. When 6-year-old Melanie arrived home from school, she was the first to eat cookies from a freshly baked batch. When 8-year-old Felipe arrived home, he ate twice as many cookies as Melanie had eaten. When 9-year-old Hillary arrived home, she ate 3 fewer cookies than Felipe. When 12-year-old Nicholas arrived, he ate 3 times as many cookies as Hillary. Nicholas left 2 cookies, one for each of his parents. If Nicholas had eaten only 5 cookies, there would have been 3 cookies for each of his parents. How many cookies were in the original batch?

53. A mathematics education researcher is studying problem solving in small groups. One phase of the study involves pairing a third-grade girl with a third-grade boy. If the researcher wants between 70 and 80 different boy-girl pairings and there are 9 girls available for the study, how many boys are needed?

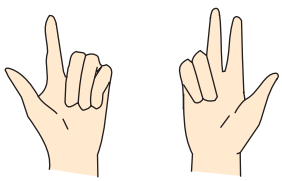
54. A restaurant owner has a luncheon special that consists of a cup of soup, half of a sandwich, and a beverage. She wants to advertise that a different luncheon of three items can be purchased 365 days of the year for $4.99 apiece. If she has 7 different kinds of soup and 6 different kinds of sandwiches, how many different kinds of beverages are needed to provide at least 365 different luncheons?

In problems 55 and 56, use the following system of finger positions to compute the products of numbers from 6 to 10. Here are the positions for the digits from 6 to 10.



The two numbers that are to be multiplied are each represented on a different hand. The sum of the raised fingers is the number of 10s, and the product of the closed fingers is the number of 1s.

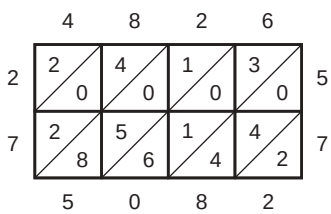
55. Explain how the position illustrated below shows that $7 \times 8 = 56$.



56. Describe the positions of the fingers for 7×6. Does the method work for this product?

One of the popular schemes used for multiplying in the fifteenth century was called the **lattice method.** The two numbers to be multiplied, 4826 and 57 in the example shown here, are written above and to the right of the lattice. The partial products are written in the cells. The sums of numbers along the diagonal cells, beginning at the lower right with 2, 4 + 4 + 0, etc., form the product 275,082.

Show how the lattice method can be used to compute the products in exercises 57 and 58.



57. 34×78

58. 306×923

Teaching Questions

1. A student who was experimenting with different methods of multiplying two whole numbers noticed that when she increased one number and decreased the other by the same amount and then multiplied, she did not get the correct answer. Explain how you would help this student.
2. Two students were discussing their lesson on different number bases for positional numeration systems. One student said there is an advantage in using our base-ten system over other systems because to multiply by ten, just put a zero at the end of the number. The other student said the same thing was true when multiplying by five in base five or eight in base eight. Which student is correct? What would you say to the students in this situation? If it works the same in different base systems explain why. If not, explain why not.
3. The use of calculators is becoming more common in schools, but the NCTM Standards recommends that students know the basic addition and multiplication facts for single-digit numbers. Discuss some reasons why it is important for students to know these facts and to be able to do these computations quickly without the use of a calculator.

Classroom Connections

1. On page 172 the example from the **Elementary School Text** illustrates the distributive property. Unlike the other number properties, the distributive property involves two operations. Explain how multiplication is modeled and how addition is modeled in this example.
2. Read the expectation in the **Grades 3–5 Standards—Number and Operations** (see inside front cover) under *Understand Meanings of Operations . . .* , that involves the distributive property of multiplication over addition and cite an example from the mental calculation techniques in Section 3.3 that shows the usefulness of this property.
3. The example in the **Standards** quote on page 174 says that an area model for 20×6 can be split in half to show that $20 \times 6 = 10 \times 12$. Use sketches involving an area model to show that these two products are equal. Use sketches of the area model and the process of splitting into thirds to show that $15 \times 8 = 5 \times 24$.

4. The **Standards** quote on page 166 notes that when students develop computational strategies, it helps contribute to their mathematical development. Explain how students might compute 23×41 using their own strategies, and how using such strategies, as opposed to only using a standard algorithm, might contribute to their mathematical development.

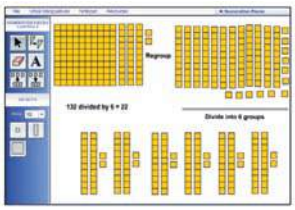
5. Activities 3 and 4 in the one-page **Math Activity** at the beginning of this section examine the meaning of 10 in base five and base four. What is the meaning of 10 in base three and in base twelve? Explain why examining the meaning of 10 in various bases can be helpful to understanding the meaning of 10 in base ten.

6. On page 175, the **Standards** quotes both refer to estimation. Explain how you would help your students to know when an estimate is reasonable.

MATH ACTIVITY 3.4

Division with Base-Five Pieces

Virtual Manipulatives

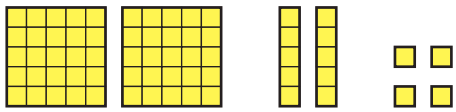


www.mhhe.com/bbn

Purpose: Explore whole number division models using base-five and base-ten pieces.

Materials: Base-Five and Base-Ten Pieces in the Manipulative Kit or Virtual Manipulatives.

***1.** Form the set of base-five pieces shown here. If these pieces are divided equally into four sets, regrouping as necessary, what is the minimal collection for each set? This is an example of the **sharing concept** of division.



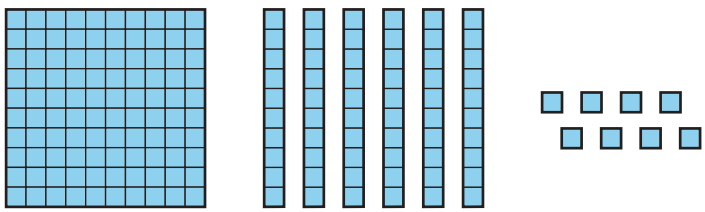
2. Use the *sharing concept* of division and base-five pieces to compute the following quotients. Write the answer in base-five notation.

***a.** $33_{five} \div 2$ **b.** $344_{five} \div 3$ **c.** $34_{five} \div 4$

3. Compute the following quotients by using the *sharing concept* of division and the base-five pieces. For example, the numeral 11_{five} represents 6, so to compute $220_{five} \div 11_{five}$, the base-five pieces for 2 flats and 2 longs should be divided into six equal collections.

***a.** $220_{five} \div 11_{five}$ **b.** $330_{five} \div 20_{five}$ **c.** $434_{five} \div 12_{five}$ **d.** $430_{five} \div 10_{five}$

4. a. Form the collection shown below with your base-ten pieces. Determine the minimal collection each person would receive if these pieces were divided equally among twelve people. Show a sketch of the solution.



b. Using the preceding collection of pieces, and regrouping as necessary, determine the number of people who could receive 12 units each, if the total number of units are divided into equal collections of 12 units. Show a sketch of the solution.

c. Notice in 4a and 4b that the sketches are different, but both illustrate division. Do they both provide an answer to $168 \div 12$? Explain why or why not.

Section 3.4 DIVISION AND EXPONENTS



General Motors Terex Titan and Chevrolet Luv pickup

PROBLEM OPENER

Using exactly four 4s and only addition, subtraction, multiplication, and division, write an expression that equals each of the numbers from 1 to 10. You do not have to use all the operations, and numbers such as 44 are permitted.*

One common use of division is to compare two quantities. In the photograph above, consider the relative sizes of the Terex Titan dump truck and the Luv pickup, which is on the Titan's dump body. The Terex Titan can carry 317,250 kilograms; the Luv pickup has a limit of 450 kilograms. We can determine the number of Luv loads it requires to equal one Titan load by dividing 317,250 by 450. The answer is 705, which means the Luv pickup will have to haul 705 loads to fill the Titan just once! Sitting in the back of the Luv pickup is a child holding a toy truck. If the toy truck holds 3 kilograms of sand, how many of its loads will be required to fill the Titan?

The division operation used in comparing the sizes of the Terex Titan and the Luv pickup can be checked by multiplication. The load weight of the smaller truck times 705 should equal the load weight of the larger truck. The close relationship between division and multiplication can be used to define division in terms of multiplication.

Division of Whole Numbers For any whole numbers r and s, with $s \neq 0$, the quotient of r divided by s, written $r \div s$, is the whole number k, if it exists, such that $r = s \times k$.

*Similar equations exist for five 5s, six 6s, etc. See R. Crouse and J. Shuttleworth, "Playing with Numerals," *Arithmetic Teacher* 1, no. 5, pp. 417–419.

EXAMPLE A

Mentally calculate each quotient.

1. $18 \div 3$ **2.** $24 \div 6$ **3.** $35 \div 5$

Solution **1.** $18 \div 3 = 6$ since $18 = 3 \times 6$. **2.** $24 \div 6 = 4$ since $24 = 6 \times 4$. **3.** $35 \div 5 = 7$ since $35 = 5 \times 7$.

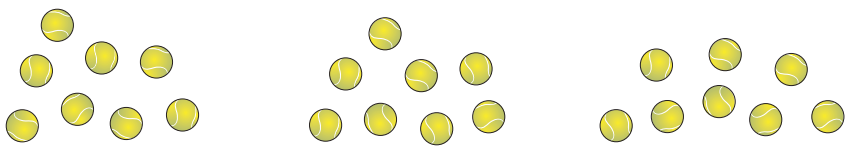
The definition of division, along with Example A, shows why multiplication and division are called **inverse operations.** We arrive at division facts by knowing multiplication facts.

There are three basic terms used in describing the division process: *dividend, divisor,* and *quotient.* In problem 1 of Example A, 18 is the **dividend,** 3 is the **divisor,** and 6 is the **quotient.** Over the centuries, division has acquired two meanings or uses. David Eugene Smith, in *History of Mathematics,* speaks of the twofold nature of division and refers to the sixteenth-century authors who first clarified the differences between its two meanings.* These two meanings of division, known as *sharing* (*partitive*) and *measurement* (*subtractive*), are illustrated in Examples B and C.

EXAMPLE B

Suppose you have 24 tennis balls, which you want to divide equally among 3 people. How many tennis balls would each person receive?

Solution The answer can be determined by separating (partitioning) the tennis balls into 3 equivalent sets. The following figure shows 24 balls divided into 3 groups and illustrates $24 \div 3 = 8$. The divisor 3 indicates the number of groups and the quotient 8 indicates the number of tennis balls in each group. This problem illustrates the **sharing (partitive) concept** of division.



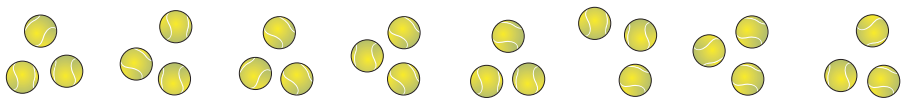
EXAMPLE C

Suppose you have 24 tennis balls and want to give 3 tennis balls to as many people as possible. How many people would receive tennis balls?

NCTM Standards

In prekindergarten through grade 2, students should begin to develop an understanding of the concepts of multiplication and division. . . . They can investigate division with real objects and through story problems, usually ones involving the distribution of equal shares. p. 84

Solution The answer can be determined by subtracting, or measuring off, as many sets of 3 as possible. The following figure of 24 tennis balls shows the result of this measuring process and illustrates $24 \div 3 = 8$. The divisor 3 is the number of balls in each group, and the quotient 8 is the number of groups. This problem illustrates the **measurement (subtractive) concept** of division.



MODELS FOR DIVISION ALGORITHMS

Of the four basic pencil-and-paper algorithms, the algorithm for division, called **long division,** is the most difficult and has traditionally required the most classroom time to master. As the use of calculators in schools increases, long division, especially for three- and

* D. E. Smith, *History of Mathematics,* 2d ed. (Lexington, MA: Ginn, 1925), p. 130.

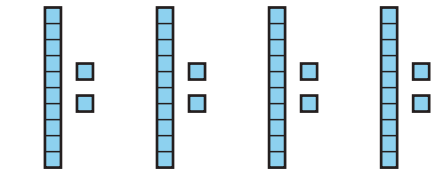
four-digit numbers, will be deemphasized. However, an understanding of division and of algorithms for determining quotients will remain important for mental calculations, estimation, and problem solving.

There are several physical models for illustrating division. Base-ten pieces are used in Examples D and E.

EXAMPLE D

Compute 48 ÷ 4 by sketching base-ten pieces.

Solution **1.** One possibility is to use the sharing (partitive) concept of division, placing 1 long in each of four groups and then 2 units in each group, as shown in the following figure. The *size* of each group, or 12, is the quotient of 48 ÷ 4.

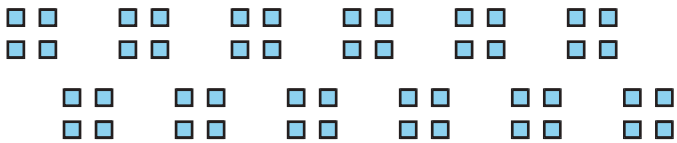


48 ÷ 4 = 12 using the sharing concept of division

NCTM Standards

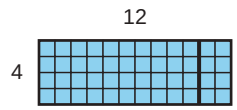
By creating and working with representations (such as diagrams or concrete objects) of multiplication and division situations, students can gain a sense of the relationships among the operations. p. 33

2. Another approach is to use the measurement (subtractive) concept of division to form as many groups of 4 units as possible. In this case there are 12 groups of 4 units each, as shown in the next figure. The *number* of groups, namely 12, is the quotient of 48 ÷ 4.



48 ÷ 4 = 12 using the measurement concept of division

3. A third possibility is to use 4 longs and 8 units to form a rectangular array with one dimension of 4, as shown next. The other dimension is 12, the quotient of 48 ÷ 4.



48 ÷ 4 = 12 using the array method of division

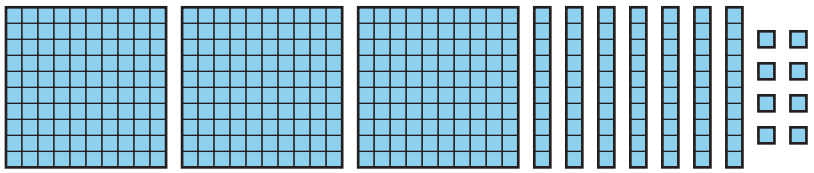
Notice in solution 3 of Example D that by viewing the rectangular array as 4 rows of 12 units each, we are making use of the sharing (partitive) concept of division for computing 48 ÷ 4, and by viewing the array as 12 columns of 4 units each, we are making use of the measurement (subtractive) concept of division for computing 48 ÷ 4.

Example E shows how base-ten pieces can be used to illustrate the steps in the long-division algorithm.

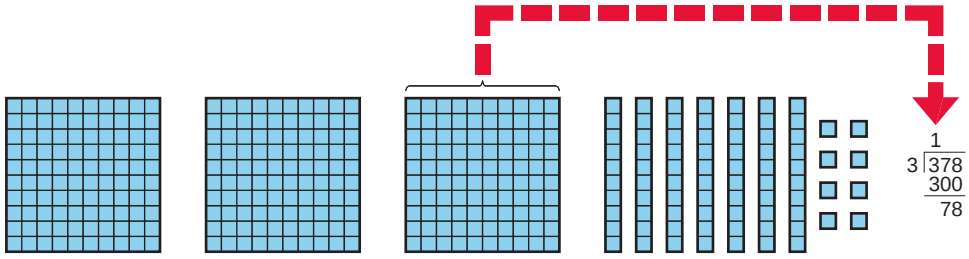
EXAMPLE E

This example illustrates 378 ÷ 3 by using the sharing (partitive) concept of division. Four steps are described. In each step, as the base-ten pieces are divided into groups, the groups are matched to the quotient of the long-division algorithm.

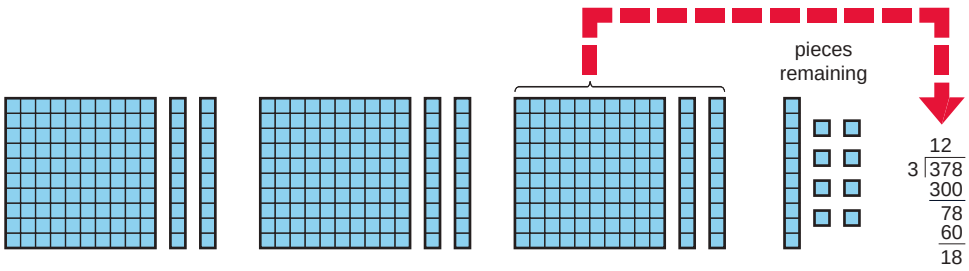
Step 1. Begin with 3 flats, 7 longs, and 8 units to represent 378.



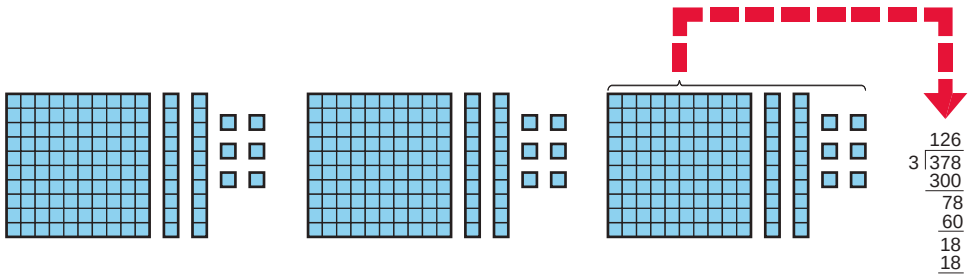
Step 2. Share the flats by placing 1 flat in each of 3 groups. This leaves 7 longs and 8 units.



Step 3. Share the longs by placing 2 longs in each of the 3 groups, leaving 1 long and 8 units.



Step 4. Regroup the remaining long into 10 units and share the 18 units by placing 6 units in each of the 3 groups.



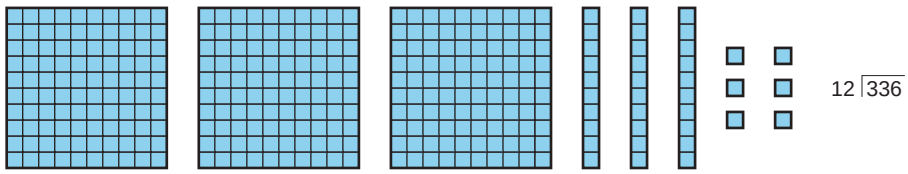
Notice that each of the final groups of base-ten pieces represents the quotient 126.

For small divisors, as in Example E, the sharing concept of division is practical because the number of groups is small. For larger divisors, as in Example F, rectangular arrays are convenient. In recent years, the rectangular-array approach to illustrating division has become more common.

EXAMPLE F

This example illustrates 336 ÷ 12 by using a rectangular array. Three steps are described, and each step is related to the quotient of the long-division algorithm.

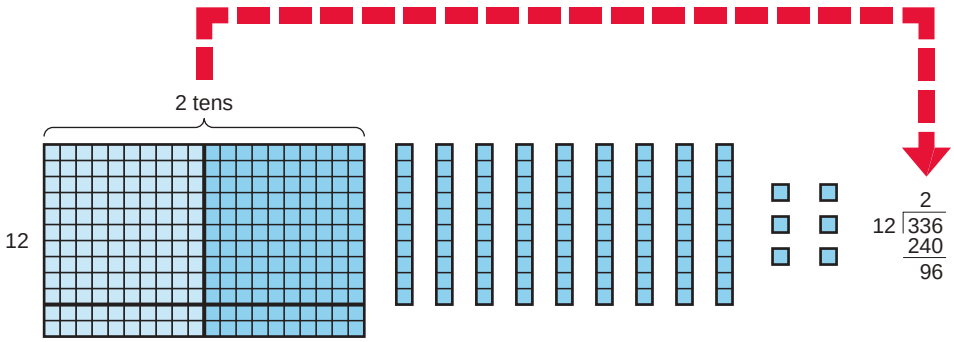
Step 1. Begin with 3 flats, 3 longs, and 6 units to represent 336.



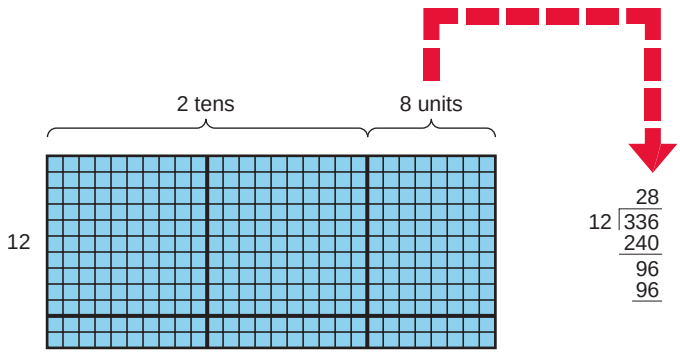
NCTM Standards

As students move from third to fifth grade, they should consolidate and practice a small number of computational algorithms for addition, subtraction, multiplication, and division that they understand well and can use routinely. p. 155

Step 2. Start building a rectangle with one dimension of 12. This can be done by beginning with 1 flat and 2 longs (see light blue region). Then a second flat and 2 more longs can be added by regrouping the third flat into 10 longs. This leaves 9 longs and 6 units.



Step 3. Continue building the rectangle by extending it with the remaining 9 longs and 6 units. To accomplish this, regroup one of the longs into 10 units.



The final dimension of the rectangle in Example F is 28, the quotient of 336 ÷ 12. Notice that the rectangular-array illustration of division is a visual reminder of the close relationship between division and multiplication: The product of the two dimensions, or 12 × 28, is 336, the number represented by the original set of base-ten pieces.

Explore

Math Activity for 8-1
Model Division

In division, the **dividend** is the number that is being divided. The **divisor** is the number that divides the dividend. The **quotient** is the result.

$$\text{divisor})\overline{\text{dividend}} \qquad \text{quotient}$$

MAIN IDEA

I will explore dividing by one-digit numbers.

You Will Need

base-ten blocks

New Vocabulary

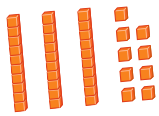
- **dividend**
- **divisor**
- **quotient**
- **remainder**

ACTIVITY

1 **Find 39 ÷ 3.**

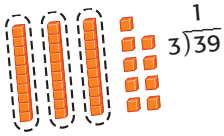
Step 1 **Model the dividend, 39.**

Use 3 tens and 9 ones to show 39.



Step 2 **Divide the tens.**

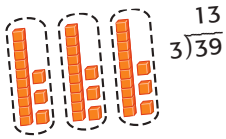
The divisor is 3. So, divide the tens into 3 equal groups. There is a ten in each group.



Step 3 **Divide the ones.**

Divide the ones into 3 equal groups. There are 1 ten and 3 ones in each group.

So, 39 ÷ 3 = 13.





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DIVISION ALGORITHM THEOREM

We have seen that the sum or product of two whole numbers is always another whole number and that this fact is called the *closure* property. Subtraction and division of whole numbers, on the other hand, are not closed. That is, the difference or quotient of two whole numbers is not always another whole number.

EXAMPLE G

1. $12 - 15$ is not a whole number because there is no whole number c such that $12 = 15 + c$.
2. $38 \div 7$ is not a whole number because there is no whole number k such that $38 = 7 \times k$.

At times we want to solve problems involving division of whole numbers even though the quotient is not a whole number. In the case of $38 \div 7$, we can determine that the greatest whole number quotient is 5 and the remainder is 3.

$$38 = 7 \times 5 + 3$$

Technology Connection

Calculators intended for schoolchildren are sometimes designed to display the whole number quotient and remainder when one whole number is divided by another. The view screens for two such calculators are shown in Figure 3.23. These screens show the quotient and remainder for $66{,}315 \div 7$. Notice that these calculators have a special key to indicate division with a whole number remainder.



Figure 3.23

If you do not have such a calculator, whole number quotients and remainders can be obtained from any calculator. The following view screen shows the quotient in terms of a whole number and a decimal.

66315 ÷ 7 = 9473.571429

The whole number quotient is 9473, and the whole number remainder can be determined as follows: $66{,}315 - 7 \times 9473 = 4$.

Technology Connection

Another approach to obtaining whole number quotients and remainders is to use repeated subtraction. This approach has the added advantage of reinforcing the measurement (subtractive) concept of division, and it can be used on most calculators. The following steps and displays show that 489 ÷ 134 has a quotient of 3 (because 134 has been subtracted 3 times) and a remainder of 87. The process of subtracting 134 continues until the view screen shows a number that is less than 134 and greater than or equal to 0.

Keystrokes	View Screen
489	489
− 134 □	355
− 134 □	221
− 134 □	87

Calculators that display whole number quotients and remainders help children to see that whenever one whole number is divided by another nonzero whole number, the quotient is always greater than or equal to 0; and the remainder is always less than the divisor.

The fact that such quotients q and remainders r always exist is guaranteed by the following theorem.

Division Algorithm Theorem For any whole numbers a and b, with **divisor** $b \neq 0$, there are whole numbers q **(quotient)** and r **(remainder)** such that

$$a = bq + r$$

and $0 \leq r < b$.

This theorem says that the remainder r is always less than the divisor b. If $r = 0$, then the quotient $a \div b$ is the whole number q.

EXAMPLE H

Use a calculator and one of the preceding approaches to determine the whole number quotient and remainder.

1. 81,483 ÷ 26 **2.** 37,641 ÷ 227

3. 707,381 ÷ 112 **4.** 51,349 ÷ 57

Solution **1.** Quotient 3133 and remainder 25. **2.** Quotient 165 and remainder 186. **3.** Quotient 6315 and remainder 101. **4.** Quotient 900 and remainder 49.

MENTAL CALCULATIONS

NCTM Standards

The teacher plays an important role in helping students develop and select an appropriate computational tool (calculator, paper-and-pencil algorithm, or mental strategy). p. 156

A major strategy in performing mental calculations is replacing a problem by one that can be solved more easily. This approach was used in Sections 3.2 and 3.3 for mentally calculating sums, differences, and products; it is described here for division.

Equal Quotients for Mental Calculation In calculating a quotient mentally, sometimes it is helpful to use the method of **equal quotients,** in which we divide or multiply both the divisor and the dividend by the same number.

EXAMPLE I

1. The quotient $144 \div 18$ can be replaced by $72 \div 9$ by dividing both 144 and 18 by 2. We know from our basic multiplication facts that $9 \times 8 = 72$, so

$$144 \div 18 = 72 \div 9 = 8$$

2. The quotient $1700 \div 50$ can be replaced by multiplying both 1700 and 50 by 2. Then dividing by 100 can be done mentally to obtain a quotient of 34.

$$1700 \div 50 = 3400 \div 100 = 34$$

The *equal quotients* calculating technique can be illustrated by the array model. Recall that this model illustrates the close connection between multiplication and division. For example, a 6 by 7 array of 42 tiles not only illustrates the product $6 \times 7 = 42$, but also the quotient $42 \div 6 = 7$. Figure 3.24a shows that when both numbers in the quotient $42 \div 6$ are divided by a number, in this case 2, the quotient remains unchanged. The first array illustrates $42 \div 6$ and that the quotient is 7, the top side of the array. The second array is obtained by dividing the first array in half. Notice that both the left side of the array (6) and the total number of tiles (42) in the first array are cut in half to obtain the array that illustrates $21 \div 3$. However, the top side of the second array, which is 7 (the quotient of $21 \div 3$), has remained unchanged.



Figure 3.24a

Similarly, Figure 3.24b illustrates why both numbers in the quotient $42 \div 6$ can be multiplied by a number, in this case 3, and the quotient remains unchanged. The first array illustrates $42 \div 6$, and shows that the quotient is 7, as in Figure 3.24a. The second array is obtained by tripling the first array. Notice that both the left side of the array (6) and the total number of tiles (42) in the first array are tripled to obtain the array that illustrates $126 \div 18$. Also notice that the top side of the second array, which is 7 (the quotient of $126 \div 18$), has remained unchanged.

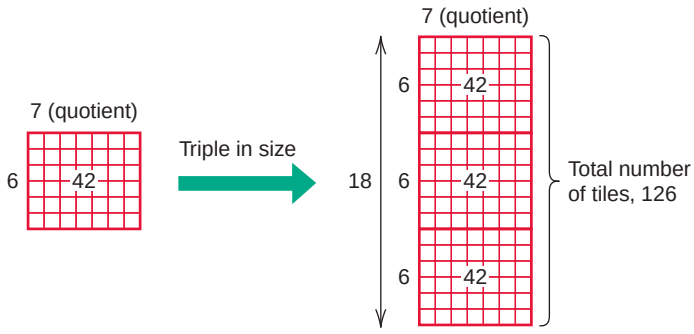


Figure 3.24b

Usually when mentally computing the quotient of two whole numbers, we want to divide both numbers by the same number to obtain smaller numbers. However, when we use the *equal-quotients* technique for mentally computing quotients of fractions and decimals, it is often more convenient to multiply both numbers in the quotient to obtain compatible numbers. (See the use of *equal quotients* in Sections 5.3 and 6.2.)

EXAMPLE J Replace each quotient by equal quotients until you can calculate the answer mentally.

1. $180 \div 12$

2. $900 \div 36$

3. $336 \div 48$

Solution Here are three solutions. Others are possible. **1.** $180 \div 12 = 60 \div 4 = 15$ (Divide by 3). **2.** $900 \div 36 = 300 \div 12 = 100 \div 4 = 25$ (Divide by 3 twice). **3.** $336 \div 48 = 112 \div 16 = 56 \div 8 = 7$ (Divide by 3, then by 2).

ESTIMATION OF QUOTIENTS

Rounding Often we wish to obtain a rough comparison of two quantities in order to determine how many times bigger (or smaller) one is than the other. This may require finding an estimation for a quotient. *Rounding* numbers is one method of estimating a quotient.

EXAMPLE K Mentally estimate each quotient by rounding one or both numbers.

1. $472 \div 46$

2. $145 \div 23$

3. $8145 \div 195$

Solution Here are some possibilities. **1.** $472 \div 46 \approx 460 \div 46 = 10$; or $472 \div 46 \approx 500 \div 50 = 10$. **2.** $145 \div 23 \approx 150 \div 25 = 6$. **3.** $8145 \div 195 \approx 8000 \div 200 = 40$.

HISTORICAL HIGHLIGHT



Emilie de Breteuil, 1706–1749

France, during the post-Renaissance period, offered little opportunity for the education of women. Emilie de Breteuil's precocity showed itself in many ways, but her true love was mathematics. One of her first scientific works was an investigation regarding the nature of fire, which was submitted to the French Academy of Sciences in 1738. It anticipated the results of subsequent research by arguing that both light and heat have the same cause or are both modes of motion. She also discovered that different-color rays do not give out an equal degree of heat. Her book *Institutions de physique* was originally intended as an essay on physics for her son. She produced instead a comprehensive textbook, not unlike a modern text, which traced the growth of physics, summarizing the thinking of the philosopher-scientists of her century. The work established Breteuil's competence among her contemporaries in mathematics and science.*

* L. M. Osen, *Women in Mathematics* (Cambridge, MA: The MIT Press, 1974), pp. 49–69.

Rounding to obtain an approximate quotient can be combined with the process of finding equal quotients (dividing or multiplying both the divisor and the dividend by the same number).

EXAMPLE L

Estimate each of the following by first *rounding* and then using *equal quotients.*

1. 427 ÷ 72

2. 139 ÷ 18

Solution **1.** 427 ÷ 72 ≈ 430 ÷ 70 = 43 ÷ 7 ≈ 6. **2.** 139 ÷ 18 ≈ 140 ÷ 18 = 70 ÷ 9 ≈ 8; or 139 ÷ 18 ≈ 140 ÷ 20 = 14 ÷ 2 = 7.

Compatible Numbers for Estimation Replacing numbers with *compatible numbers* is a useful technique for mentally estimating quotients.

EXAMPLE M

Find one or two compatible numbers to replace the given numbers, and mentally estimate the quotient.

1. 92 ÷ 9

2. 59 ÷ 16

3. 485 ÷ 24

Solution Here is one possibility for each quotient: **1.** 92 ÷ 9 ≈ 90 ÷ 9 = 10. **2.** 59 ÷ 16 ≈ 60 ÷ 15 = 4. **3.** 485 ÷ 24 ≈ 500 ÷ 25 = 20.

Front-End Estimation This technique can be used to obtain an estimated quotient of two numbers by using the leading digit of each number. Consider the following example, where both *compatible numbers* and *front-end estimation* are used.

$$783 \div 244 \approx 700 \div 200 = 7 \div 2 = 3\tfrac{1}{2}$$

In the example just shown, the two numbers being divided have the same number of digits. A front-end estimation also can be obtained for the quotient of two numbers when the leading digits have different place values, as in 8326 ÷ 476.

$$8326 \div 476 \approx 8000 \div 400 = 80 \div 4 = 20$$

EXAMPLE N

Use front-end estimation to estimate each quotient.

1. 828 ÷ 210

2. 7218 ÷ 2036

3. 4128 ÷ 216

Solution **1.** 828 ÷ 210 ≈ 800 ÷ 200 = 8 ÷ 2 = 4. **2.** 7218 ÷ 2036 ≈ 7000 ÷ 2000 = 7 ÷ 2 = $3\frac{1}{2}$. **3.** 4128 ÷ 216 ≈ 4000 ÷ 200 = 40 ÷ 2 = 20.



Technology Connection

Estimation techniques can help check the reasonableness of results from calculator computations as noted in the *Curriculum and Evaluation Standards for School Mathematics* (p. 37).

> Estimation is especially important when children use calculators. If they need to compute 4783 ÷ 13, for example, a quick estimate can be found by using "compatible

numbers." In this case 4783 is about 4800 and 13 is about 12, so 4783 ÷ 13 is about 4800 ÷ 12. The dividing can be done mentally, since 48 and 12 are "compatible numbers" for division. Thus, 4783 ÷ 13 is about 400. This rough estimate provides children with enough information to decide whether the correct keys were pressed and whether the calculator result is reasonable.

EXPONENTS

The large numbers used today were rarely needed a few centuries ago. The word *billion,* which is now commonplace, was not adopted until the seventeenth century. Even now, billion means different things to different people. In the United States it represents 1,000,000,000 (one thousand million), and in England it is 1,000,000,000,000 (one million million).

Our numbers are named according to powers of 10. The first, second, and third powers of 10 are the familiar ten, hundred, and thousand. After this, only every third power of 10 has a new or special name: million, billion, trillion, etc.

$10^0 = 1$	one
$10^1 = 10$	ten
$10^2 = 100$	one hundred
$10^3 = 1000$	**one thousand**
$10^4 = 10{,}000$	ten thousand
$10^5 = 100{,}000$	one hundred thousand
$10^6 = 1{,}000{,}000$	**one million**
$10^7 = 10{,}000{,}000$	ten million
$10^8 = 100{,}000{,}000$	one hundred million
$10^9 = 1{,}000{,}000{,}000$	**one billion**
$10^{10} = 10{,}000{,}000{,}000$	ten billion
$10^{11} = 100{,}000{,}000{,}000$	one hundred billion
$10^{12} = 1{,}000{,}000{,}000{,}000$	**one trillion**

The operation of raising numbers to a power is called **exponentiation.**

Technology Connection

Sums and Differences

Find a pattern in the equations shown here. Does this pattern continue to hold for other two-digit numbers? Analyze this pattern and form conjectures regarding similar questions in this investigation.

$20^2 - 3^2 = 23 \times 17$

$50^2 - 7^2 = 57 \times 43$

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Exponentiation For any number b and any whole number n, with b and n not both zero,

$$b^n = \underbrace{b \times b \times b \times b \times \ldots \times b}_{b \text{ occurs } n \text{ times}}$$

where b is called the **base** and n is called the **exponent.** In case $n = 0$ or $n = 1$, $b^0 = 1$ and $b^1 = b$.

EXAMPLE O

Evaluate each expression.

1. 3^4 **2.** 2^5

3. 5^0 **4.** 3^1

Solution **1.** 81 **2.** 32 **3.** 1 **4.** 3

A number written in the form b^n is said to be in **exponential form.** In general, the number b^n is called the ***n*th power** of b. The second and third powers of b, b^2 and b^3, are usually called ***b* squared** and ***b* cubed.** This terminology was inherited from the ancient Greeks, who pictured numbers as geometric arrays of dots. Figure 3.25 illustrates 2^2, a 2×2 array of dots in the form of a square, and 2^3, a $2 \times 2 \times 2$ array of dots in the form of a cube.

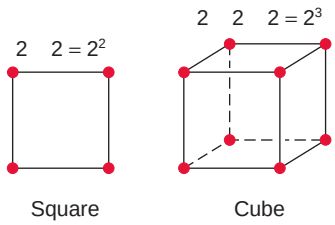


Figure 3.25

A number greater than or equal to 1 that can be written as a whole number to the second power is called a **square number** or a **perfect square** (1, 4, 9, 16, 25, . . .), and a number greater than or equal to 1 that can be written as a whole number to the third power is called a **perfect cube** (1, 8, 27, 64, 125, . . .).

Laws of Exponents Multiplication and division can be performed easily with numbers that are written as powers of the same base. To multiply, we add the exponents; to divide, we subtract the exponents.

EXAMPLE P

Evaluate each product or quotient. Write the answer in both exponential form and positional numeration.

1. $2^4 \times 2^3$ **2.** $2^8 \div 2^3$

Solution **1.** $2^4 \times 2^3 = (2 \times 2 \times 2 \times 2) \times (2 \times 2 \times 2) = 2^7 = 128.$

2. $2^8 \div 2^3 = \dfrac{2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2} = 2^5 = 32.$

The equations in Example P are special cases of the following rules for computing with exponents.

Laws of Exponents For any number a and all whole numbers m and n, except for the case where the base and exponents are both zero,

$$a^n \times a^m = a^{n+m}$$

$$a^n \div a^m = a^{n-m} \quad \text{for } a \neq 0$$

The primary advantage of exponents is their compactness, which makes them convenient for computing with very large numbers and (as we shall see in Section 6.3) very small numbers.

EXAMPLE Q

1. In our galaxy there are 10^{11} (100 billion) stars, and in the observable universe there are 10^9 (1 billion) galaxies. If every galaxy had as many stars as ours, there would be $10^9 \times 10^{11}$ stars. Write this product in exponential form.
2. If 1 out of every 1000 stars had a planetary system, there would be $10^{20} \div 10^3$ stars with planetary systems. Write this quotient in exponential form.
3. If 1 out of every 1000 stars with a planetary system had a planet with conditions suitable for life, there would be $10^{17} \div 10^3$ such stars. Write this quotient in exponential form.

Solution **1.** 10^{20} **2.** 10^{17} **3.** 10^{14}

Technology Connection

Numbers raised to a whole-number power can be computed on a calculator by repeated multiplication, provided the products do not exceed the capacity of the calculator's view screen. On most calculators the steps shown in Figure 3.26 will produce the number represented by 4^{10}, if the process is carried out to step 10. Try this sequence of steps on your calculator.

Keystrokes	View Screen
4	4
× 4 =	16
× 4 =	64
× 4 =	256

Figure 3.26

The number of steps in the preceding process can be decreased by applying the rule for adding exponents, namely $a^n \times a^m = a^{n+m}$. To compute 4^{10}, first compute 4^5 on the calculator and then multiply the result, 1024, by itself.

$$4^{10} = 4^5 \times 4^5 = 1024 \times 1024 = 1{,}048{,}576$$

Some calculators have exponential keys such as y^x, x^y, or $\wedge$ for evaluating numbers raised to a power. To compute a number y to some exponential power x, enter the base y into the calculator, press the exponential key, and enter the exponent x. The steps in evaluating 4^{10} are shown in Figure 3.27.

Keystrokes	View Screen
4 (base)	4
y^x	4
10 (exponent)	10
=	1048576

Figure 3.27

Since 10 is a common base when exponents are used, some calculators have a key such as 10^x or 10^n. Depending on the brand of calculator, the exponent may have to be

entered before pressing the exponential key or after, as shown by the following keystrokes.

5 [10^x] [100000] or [10^x] 5 [100000]

Numbers that are raised to powers frequently have more digits than the number of places in the calculator's view screen. If you try to compute 4^{15} on a calculator with only eight places in its view screen, there will not be room for the answer in positional numeration. Some calculators will automatically convert to scientific notation when numbers in positional numeration are too large for the view screen (see Section 6.3), and others will print an error message such as *Error* or *E.*

ORDER OF OPERATIONS

The rules for *order of operations,* discussed in Section 3.3, can now be extended to include division and raising numbers to powers. The order of operations requires that numbers raised to a power be evaluated first; then products and quotients are computed in the order in which they occur from left to right; finally, sums and differences are calculated in the order in which they occur from left to right. An exception to the rule occurs when numbers are written in parentheses. In this case, computations within parentheses are carried out first.

EXAMPLE R Evaluate the following expressions.

1. $4 \times 6 + 16 \div 2^3$
2. $4 \times (6 + 16) \div 2^3$
3. $220 - 12 \times 7 + 15 \div 3$
4. $24 \div 4 \times 2 + 15$

Solution **1.** 26 (First replace 2^3 by 8; then compute the product and quotient; then add). **2.** 11 (First replace $6 + 16$ by 22; then replace 2^3 by 8; then compute the product and quotient). **3.** 141 (First replace 12×7 by 84 and $15 \div 3$ by 5; then compute the difference and sum). **4.** 27 (First compute $24 \div 4$; then multiply by 2; then add 15).

Technology Connection

Calculators that are programmed to follow the order of operations are convenient for computing expressions involving several different operations. You may wish to try problem 3 in Example R on your calculator, entering in the numbers and operations as they appear from left to right and then pressing the equality key, to see if you obtain 141.

PROBLEM-SOLVING APPLICATION

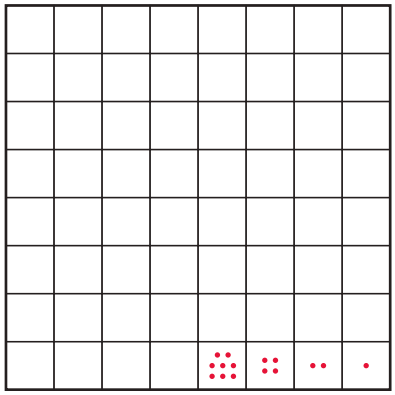
The following problem involves numbers in exponential form and is solved by using the strategies of *making a table* and *finding a pattern.*

Problem

There is a legend that chess was invented for the Indian king Shirham by the grand visier Sissa Ben Dahir. As a reward, Sissa asked to be given 1 grain of wheat for the first square of the chessboard, 2 grains for the second square, 4 grains for the third square, then 8 grains, 16 grains, etc., until each square of the board had been accounted for. The king was surprised

at such a meager request until Sissa informed him that this was more wheat than existed in the entire kingdom. What would be the sum of all the grains of wheat for the 64 squares of the chessboard?

Understanding the Problem The numbers of grains for the first few squares are shown in the following figure.



The numbers 1, 2, 4, 8, 16, 32, . . . form a geometric sequence whose common ratio is 2. Sometimes it is convenient to express these numbers as powers of 2.

$$1 \quad 2 \quad 2^2 \quad 2^3 \quad 2^4 \quad 2^5 \quad \ldots$$

Question 1: How would the number of grains for the 64th square be written as a power of 2?

Devising a Plan Computing the sum of all 64 binary numbers would be a difficult task. Let's form a table for the first few sums and look for a pattern. Compute the next three totals in the following table. **Question 2:** How is each total related to a power of 2?

Square	Number of Grains	Total
1	1	1
2	$1 + 2$	3
3	$1 + 2 + 2^2$	7
4	$1 + 2 + 2^2 + 2^3$	
5	$1 + 2 + 2^2 + 2^3 + 2^4$	
6	$1 + 2 + 2^2 + 2^3 + 2^4 + 2^5$	

Carrying Out the Plan Find a pattern in the preceding table, and use it to express the sum of the grains for all 64 squares. **Question 3:** What is this sum, written as a power of 2?

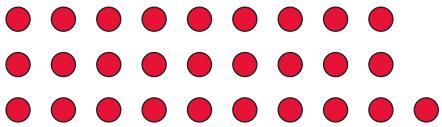
Looking Back King Shirham was surprised at the total amount of grain because the number of grains for the first few squares is so small. There is more grain for each additional square than for all the preceding squares combined. **Question 4:** Why is the number of grains for the 64th square greater than the total number of grains for the first 63 squares?

Answers to Questions 1–4 **1.** 2^{63}. **2.** The total in each row is 1 less than a power of 2. **3.** The total number of grains is $2^{64} - 1$. **4.** The total number of grains for the first 63 squares is $2^{63} - 1$, but there are 2^{63} grains for the 64th square.

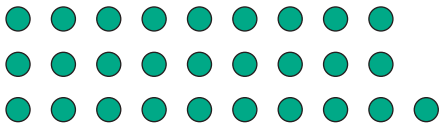
Exercises and Problems 3.4

In exercises 1 and 2, circle groups of chips to illustrate each concept of division.

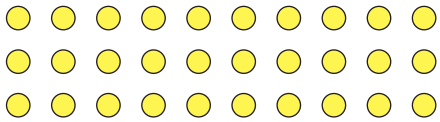
1. a. $28 \div 7$ using the *sharing (partitive) concept*



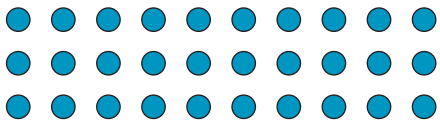
b. $28 \div 7$ using the *measurement (subtractive) concept*



2. a. $30 \div 6$ using the *sharing (partitive) concept*



b. $30 \div 6$ using the *measurement (subtractive) concept*



In exercises 3 and 4, write each division as a multiplication.

3. a. $68 \div 17 = 4$ **b.** $414 \div 23 = 18$

4. a. $288 \div 8 = 36$ **b.** $a \div b = c$

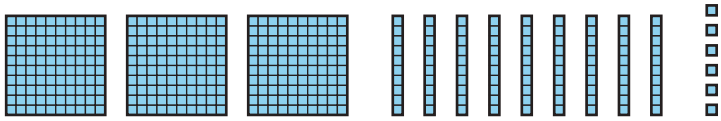
In exercises 5 and 6, write each multiplication as a division.

5. a. $14 \times 24 = 336$ **b.** $9 \times 8 = 72$

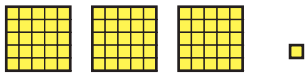
6. a. $360 \times 10 = 3600$ **b.** $r \times s = t$

Illustrate each quotient in exercises 7 and 8 by using the given concept of division. Sketch any new pieces that are necessary to show regrouping.

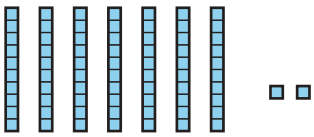
7. a. $396 \div 132$, using the *measurement (subtractive) concept*



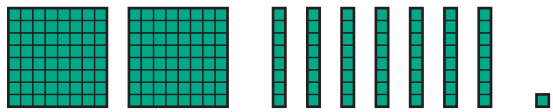
b. $301_{\text{five}} \div 4_{\text{five}}$ using the *sharing (partitive) concept*



8. a. $72 \div 18$ using the *measurement (subtractive) concept*



b. $271_{\text{eight}} \div 5_{\text{eight}}$ using the *sharing (partitive) concept*



In exercises 9 and 10, use base-ten pieces to illustrate the long-division algorithm for each quotient. In separate steps show which base-ten pieces correspond to each digit in the quotient.

9. a. $7\overline{)392}$ **b.** $5\overline{)320}$

10. a. $4\overline{)96}$ **b.** $3\overline{)426}$

In exercises 11 and 12, use a rectangular array of base-ten pieces to illustrate each quotient. (Copy the Base-Ten Grid from the website or use the Virtual Manipulatives.)

11. a. $72 \div 12$ **b.** $286 \div 26$

12. a. $238 \div 14$ **b.** $391 \div 23$

In exercises 13 and 14, show a rectangle that uses all the base-ten pieces and has the given dimension. Regrouping may be needed. Label both dimensions of the rectangle. Write a multiplication fact and a division fact illustrated by each rectangle. (Copy the Base-Ten Grid from the website or use the Virtual Manipulatives.)

13. a. 6 flats, 0 longs, and 8 units, with one dimension of 32
b. 2 flats, 2 longs, and 1 unit, with one dimension of 13
c. 2 flats, 9 longs, and 4 units, with one dimension of 21

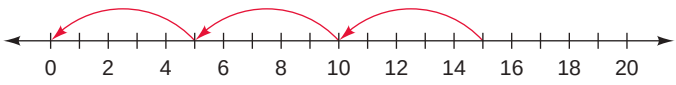
14. a. 1 flat, 1 long, and 7 units, with one dimension of 13
b. 3 flats, 3 longs, and 0 units, with one dimension of 15
c. 5 flats, 1 long, and 8 units, with one dimension of 14

In exercises 15 and 16, determine which quotients can be computed and what they equal. Try these quotients on a calculator.

15. a. $0 \div 4$ **b.** $4 \div 0$ **c.** $0 \div 0$

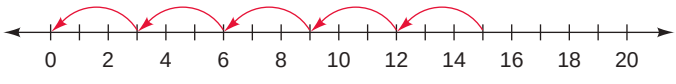
16. a. $39 \div 0$ **b.** $(14 - 14) \div (7 - 7)$ **c.** $0 \div 10$

17. a. What division fact is illustrated by the arrows on the number line?



b. Draw an arrow diagram for $18 \div 6$, using the *measurement (subtractive) concept* of division.

18. a. What division fact is illustrated by the arrows on the number line?



b. Draw an arrow diagram for $16 \div 8$, using the *measurement (subtractive) concept* of division.

The examples of long division in exercises 19 and 20 illustrate different types of errors. Locate and explain each type of error.

19. a.
$$\begin{array}{r} 56\ \text{R4} \\ 8\overline{)4052} \\ 40 \\ \hline 52 \\ 48 \\ \hline 4 \end{array}$$

b.
$$\begin{array}{r} 68 \\ 3\overline{)258} \\ 24 \\ \hline 18 \\ 18 \\ \hline \end{array}$$

20. a.
$$\begin{array}{r} 370 \\ 7\overline{)2149} \\ 21 \\ \hline 49 \\ 49 \\ \hline \end{array}$$

b.
$$\begin{array}{r} 29\ \text{R20} \\ 4\overline{)136} \\ 8 \\ \hline 56 \\ 36 \\ \hline 20 \end{array}$$

Try some numbers for the variables in each equation in exercises 21 and 22. Does the right side equal the left side? Can you find a case in which the equation does not hold? It takes only one counterexample to show that a property does not hold.

21. a. $(\square + \triangle) \div \parallelogram \stackrel{?}{=} (\square \div \parallelogram) + (\triangle \div \parallelogram)$

b. $\square \div \triangle \stackrel{?}{=} \triangle \div \square$

22. a. $\square \div (\triangle \div \parallelogram) \stackrel{?}{=} (\square \div \triangle) \div \parallelogram$

b. $(\square - \triangle) \div \parallelogram \stackrel{?}{=} (\square \div \parallelogram) - (\triangle \div \parallelogram)$

Determine whether each set in exercises 23 and 24 is closed or not closed for the given operation. Explain your answer. If not closed, give an example to show why not.

23. a. The set of odd whole numbers for division
b. {0, 1} for division
c. The set of multiples of 10 for division

24. a. The set of even whole numbers for division
b. The set of whole numbers less than 1000 for division
c. The set of whole numbers greater than or equal to 1000 for division

Find the greatest whole number quotient and the remainder in exercises 25 and 26.

25. a. $47{,}208 \div 674$
b. $2018 \div 17$
c. $1{,}121{,}496 \div 465$

26. a. $13{,}738 \div 24$
b. $107{,}253 \div 86$
c. $988{,}604 \div 236$

In exercises 27 and 28, use the method of *equal quotients* to replace the divisor and the dividend with smaller numbers. Show the new quotient that replaces the original quotient. Repeat this process, if necessary, until you can mentally calculate the exact quotient.

27. a. $90 \div 18$ **b.** $84 \div 14$

28. a. $400 \div 16$ **b.** $144 \div 16$

In exercises 29 and 30, round or use *compatible numbers* to mentally estimate the quotient. Show the new quotient, and predict whether it is greater than or less than the exact quotient.

29. a. $250 \div 46$ **b.** $82 \div 19$
c. $486 \div 53$

30. a. $203 \div 50$ **b.** $8145 \div 195$
c. $241 \div 31$

In exercises 31 and 32, use *front-end estimation* to mentally estimate the quotient.

31. a. $623 \div 209$ **b.** $7218 \div 1035$

32. a. $938 \div 31$ **b.** $5634 \div 713$

Compute the products and quotients in exercises 33 and 34. Leave your answers in exponential form.

33. **a.** $5^{14} \times 5^{20}$ **b.** $10^{12} \div 10^{10}$

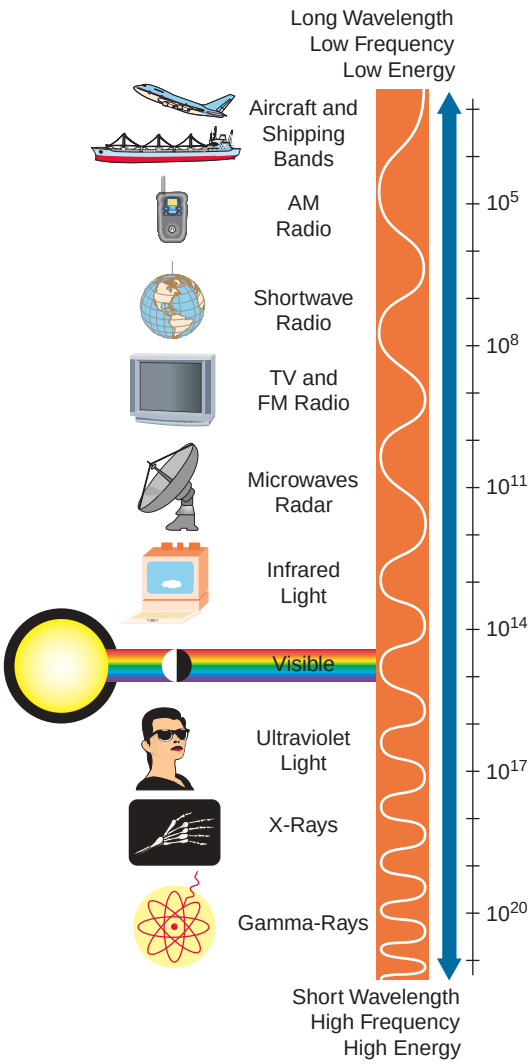
34. **a.** $10^{32} \div 10^{15}$ **b.** $3^{22} \times 3^{8}$

Evaluate the expressions in exercises 35 and 36.

35. **a.** $6 + 4 \times 8 - 3$ **b.** $5 \times 10 - 2 \times 6$
c. $5 \times (10 - 2) \times 6$ **d.** $45 \div 3 \times 5 - 2$

36. **a.** $8 - 5 + 2 + 9$ **b.** $15 \times (80 + 170) \div 5^2$
c. $160 \div 2^4 + 2^7 - 75$ **d.** $1440 \div 12 \times 10 + 18$

The chart here shows the electromagnetic spectrum with approximate frequencies of some common types of waves. Visible light waves, for example, have a frequency between 10^{14} and 10^{15} cycles per second. Exercises 37 and 38 involve the frequencies of waves from this chart.



37. **a.** If a radio frequency is 10^8 cycles per second and gamma rays have a frequency of 10^{20} cycles per second, the gamma-ray frequency is how many times the radio frequency?
b. The frequency of a gamma ray at 10^{20} cycles per second is how many times the frequency of a long wave at 10^5 cycles per second?

38. **a.** The frequency of television is 10^9 cycles per second. If a type of x-ray has a frequency that is 10^{11} times the television frequency, what is the x-ray frequency?
b. If the frequency of infrared waves is 10^{13} cycles per second and it is 100 times the frequency of microwaves, what is the microwave frequency?

Beneath each equation in exercises 39 through 41 is a sequence of calculator steps. Determine whether the sequence produces the correct answer. If not, revise the steps so that the correct answer is obtained from a calculator.

39. $8 \times (12 \div 3) = 32$
1. Enter 8
2. [×]
3. Enter 12
4. [÷]
5. Enter 3
6. [=]

40. $3 \times 4 + 7 = 19$
1. Enter 3
2. [×]
3. Enter 4
4. [+]
5. Enter 7
6. [=]

41. $17 - 3 \times 5 = 2$
1. Enter 17
2. [−]
3. Enter 3
4. [×]
5. Enter 5
6. [=]

42. Some calculators for students in grade school have a division key [INT÷] which determines the whole number quotient and remainder when dividing one whole number by another. Assume an elementary school class has such calculators and the students wish to determine the number of buses needed by six schools in the district, if each bus holds 30 students.

a. For each calculator view screen, determine the whole-number quotient (first bracket of the screen) and remainder (second bracket of the screen) for the given numbers of students.
b. Determine the total number of buses needed by the school district if buses do not pick up children at more than one school.

	Keystrokes	View Screen
1.	132 Int▯ 30 ▯	____ ____
2.	171 Int▯ 30 ▯	____ ____
3.	83 Int▯ 30 ▯	____ ____
4.	227 Int▯ 30 ▯	____ ____
5.	168 Int▯ 30 ▯	____ ____
6.	200 Int▯ 30 ▯	____ ____

43. Six classes in a middle school will go on a field trip in vans that each hold seven students. To determine the total number of vans needed, the students will use a calculator with a whole number division key [INT÷] as illustrated in the screens below.
a. For each calculator view screen, determine the whole-number quotient (first bracket of the screen) and the remainder (second bracket of the screen) for the given numbers of students.
b. If each class rides only in vans driven by parents of students in the class, what is the total number of vans that will be needed?

	Keystrokes	View Screen
1.	32 Int▯ 7 ▯	____ ____
2.	26 Int▯ 7 ▯	____ ____
3.	34 Int▯ 7 ▯	____ ____
4.	30 Int▯ 7 ▯	____ ____
5.	27 Int▯ 7 ▯	____ ____
6.	24 Int▯ 7 ▯	____ ____

Calculators with a constant function can be used to generate a geometric sequence by dividing by a common ratio. The second number entered in step 1 in exercises 44 and 45 is the *common ratio* for each sequence, and pressing [=] in each succeeding step divides the number in the view screen by the common ratio.
a. What are the numbers for the six view screens?
b. How many numbers are there in the sequence that begins with the first number in the view screen before obtaining a number less than 1?

44.

	Keystrokes	View Screen
1.	1062882 ▯ 3 ▯	____ ____
2.	▯	____ ____
3.	▯	____ ____
4.	▯	____ ____
5.	▯	____ ____
6.	▯	____ ____

45.

	Keystrokes	View Screen
1.	11529602 ▯ 7 ▯	
2.	▯	
3.	▯	
4.	▯	
5.	▯	
6.	▯	

The key [INT÷], which is used in exercises 46 and 47, displays a whole-number quotient (first bracket of the screen) and remainder (second bracket of the screen) in the calculator's view screen. Determine each quotient and remainder.

46. a. 25684 [INT÷] 58 = ____ ____
b. 6551 [INT÷] 112 = ____ ____

47. a. 8683 [INT÷] 17 = ____ ____
b. 7666 [INT÷] 605 = ____ ____

Assume that in exercises 48 and 49 a calculator is used that follows the *order of operations* and that pressing the keys [6] [10^x] produces 1000000 in the view screen and pressing [2] [y^x] [3] [=] produces 8. Determine the number that will appear in the view screen for each set of keystrokes.

48. a. 18 [+] 4 [10^x] [=]
b. 2 [×] 7 [y^x] 3 [=]

49. a. 14 [+] 6 [10^x] [÷] 2 [=]
b. 251 [+] 9 [y^x] 4 [=]

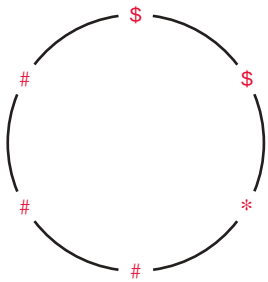
Reasoning and Problem Solving

50. Featured Strategy: Finding a Pattern. The chart below illustrates a repeating pattern. If this pattern continues, what symbol will be in the 538th square?

$	$	*	#	#	#	$	$	*	#	#	#	$	· · ·

a. Understanding the Problem. To become more familiar with the problem, extend the pattern a few more squares. What symbol will be in the 19th square?

b. Devising a Plan. Because the pattern repeats itself after six squares, it is suggestive of a clock with six symbols. What symbol occurs in squares 6, 12, 18, etc.? Alternatively, you could think of the pattern as pieces of tile 6 squares long. To make the length 32 squares, how many tiles and squares would you need?



c. Carrying Out the Plan. Choose a method for finding the symbol on the 538th square. Explain your method.

d. Looking Back. The lengths and symbols of repeating patterns vary. What will be the 345th digit in the following number if the pattern continues? 142,857,142,857 . . .

PS Find a pattern in each set of equations in exercises 51 and 52, and use inductive reasoning to predict the next equation. Evaluate both sides of your new equation to check the results. Use your pattern to determine the 12th equation.

51. $1^2 + 2^2 + 2^2 = 3^2$
$2^2 + 3^2 + 6^2 = 7^2$
$3^2 + 4^2 + 12^2 = 13^2$

52. $1^3 + 2^3 = 3^2$
$1^3 + 2^3 + 3^3 = 6^2$
$1^3 + 2^3 + 3^3 + 4^3 = 10^2$

PS Look for some patterns in the following triangle of numbers. Use these patterns in exercises 53 and 54.

$$\begin{aligned} 1 &= 1 \\ 3 + 5 &= 8 \\ 7 + 9 + 11 &= 27 \\ 13 + 15 + 17 + 19 &= 64 \\ 21 + 23 + 25 + 27 + 29 &= 125 \end{aligned}$$

53. What is the 12th row of this triangle, and what is its sum?

54. What is the 10th row of this triangle, and what is its sum?

PS If each number in exercises 55 and 56 is expanded and written in positional numeration, what is the digit in the units place?

55. a. 2^{103} **b.** 6^{52}

56. a. 17^{65} **b.** 14^{81}

57. PS Suppose you had a chance to work for 22 weeks and could choose one of two methods of payment. You could choose to be paid \$1 the first week, \$2 the second week, \$4 the third week, \$8 the fourth week, etc., with the amount doubling each week; or you could choose to receive \$2 million in one lump sum.
a. Which method would result in the greater payment?
b. What is the difference in the amounts between these two types of payments?

58. PS A school district receives a grant with the stipulation that the money be divided equally among the city's seven elementary schools. Each school is to divide its money equally among grades K through 5, and each grade is to divide its money equally for furniture, books, and science equipment. If each grade obtained \$1,345.20 for science equipment from the grant, what was the original amount of the grant?

59. PS Parker let his neighbor tap his 15 sugar maple trees to obtain sap for making maple syrup. In return, his neighbor agreed to divide the maple syrup equally between himself, Parker, and another person who would help to collect the sap. The sap was collected over a 30-day period during March and April. Parker was given 2 quarts of syrup, but he wondered if he had been cheated. He learned that each sugar maple tree gives approximately 1 gallon of sap every 5 days and that 40 gallons of sap must be boiled down to obtain 1 gallon of maple syrup. Given this information, how many quarts of maple syrup should have been given to Parker?

60. PS You have rented a car for one day at a cost of \$28 per day. You also buy the insurance which is \$12 per day. When you begin your trip, the tank is full with 12 gallons of gasoline. When you return the car, the gas gauge shows nearly empty. The clerk argues that you should pay for 12 gallons of gasoline, but you think this is unfair. You know that the odometer read 24,140 when you started the trip and 24,425 when you returned. So you ask the clerk how many miles this car gets for each gallon of gasoline. The clerk says 30 miles per gallon. How much gasoline should you pay for?

Teaching Questions

1. How would you respond to a middle school student who says: "My father showed me that 15 ÷ 3 is 5 by dividing 15 chips into 3 piles of 5 chips each, and he says my method of removing 3 chips as many times as possible from 15 chips is subtraction and not division."
2. One fifth-grade math book illustrated the *equal quotients* technique with an example: 60 ÷ 12 = 30 ÷ 6. If a student asked you why this method works, what would be your explanation? Show how you can illustrate this with diagrams.
3. Write two story problems for 28 ÷ 5: one that exemplifies the *sharing* meaning of division; and the other the *measurement* (subtractive) meaning of division. Illustrate each solution with a diagram of base-ten pieces. For each solution explain how you would deal with the remainder.
4. Suppose you were teaching an elementary school class and the parents of one of your students was opposed to the use of calculators in teaching mathematics. Explain what you would say to these parents to help convince them that your use of calculators was beneficial to learning mathematics.

Classroom Connections

1. In the **PreK–2 Standards—Number and Operations** (see inside front cover) under *Understand Meanings of Operations* . . . read the expectation that entails division and illustrate this expectation with examples.
2. On page 192 the example from the **Elementary School Text** uses base-ten blocks to model one of the two concepts of division. (a) Which concept is it? How do you know? (b) Sketch base-ten pieces to model the same problem using the other division concept and explain the process.
3. Illustrate 48 ÷ 6 using the recommendation in the **Standards** quote on page 189. Explain how your illustration shows the relationship between multiplication and division.
4. The **Historical Highlight** on page 196 features Emilie de Breteuil and her accomplishments in mathematics and science. Research history of mathematics books and/or the Internet and write a paper on some of Emilie de Breteuil's contributions in mathematics and science or write a paper about the contributions of another prominent woman scientist/mathematician from the eighteenth century.
5. Compare and discuss the **Standards** quotes on pages 166 and 191. As a teacher, how would you encourage students to develop their own methods of computing?

CHAPTER 3 REVIEW

1. **Numeration systems**
   - a. A logically organized collection of numerals is called a **numeration system.**
   - b. The number of objects used in the grouping process is called the **base.**
   - c. In an **additive numeration system,** each symbol is repeated as many times as needed.
   - d. The **Egyptian** and **Roman numeration systems** are additive numeration systems.
   - e. In a **positional numeration system,** the position of each digit indicates a power of the base.
   - f. In a base-ten positional numeration system, the power of 10 associated with each digit is called its **place value.**
   - g. The **Mayan** and **Hindu-Arabic numeration systems** are positional numeration systems.
2. **Reading and rounding numbers**
   - a. Numbers with more than three digits are read by naming **periods.** Each period has three digits.
   - b. The next three periods after the ones, tens, and hundreds digits are called **thousands, millions,** and **billions.**
   - c. To **round** a number to the nearest million means to choose the million that is closest to the number.
3. **Models for numeration**
   - a. The **bundle-of-sticks model** and **base-ten pieces** are two models for numeration systems.
   - b. **Regrouping** is replacing one collection of pieces in a model by another collection that represents the same number.
4. **Whole-number operations**
   - a. **Addition** is defined in terms of sets.
   - b. **Subtraction** is defined as the inverse operation of addition.
   - c. **Multiplication** is defined as repeated addition.
   - d. **Division** is defined as the inverse operation of multiplication.
   - e. There are three concepts of subtraction: the **take-away concept,** the **comparison concept,** and the **missing addend concept.**
   - f. There are two concepts of division: the **sharing (partitive) concept** and the **measurement (subtractive) concept.**
   - g. The **Division Algorithm Theorem** states for any whole numbers a and b, with **divisor** $b \neq 0$, there are whole numbers q **(quotient)** and r **(remainder)** such that $a = bq + r$, and $0 \leq r < b$.
   - h. The operation of raising numbers to a power is called **exponentiation.**
   - i. A number written in the form b^n is said to be in **exponential form;** b^n is called the ***n*th power** of b.
   - j. For any whole numbers a, n, and m, not allowing 0^0, $a^n \times a^m = a^{n+m}$ and $a^n \div a^m = a^{n-m}$, for $a \neq 0$.
5. **Algorithms for operations**
   - a. An **algorithm** is a step-by-step procedure for computing.
   - b. The algorithms for addition and multiplication involve computing **partial sums** and **partial products.**
   - c. **Left-to-right addition** is an algorithm that begins with the digits on the left (digits of greatest place value).
6. **Models for operations**
   - a. The **bundle-of-sticks model** and **base-ten pieces** are two models for the four basic operations on whole numbers.
   - b. A **rectangular array** is a visual method of illustrating the product of two whole numbers.
   - c. Constructing a tree diagram is a counting technique that involves products of whole numbers.
7. **Number properties**
   - a. **Closure property for addition:** For any whole numbers a and b, $a + b$ is a unique whole number.
   - b. **Closure property for multiplication:** For any whole numbers a and b, $a \times b$ is a unique whole number.
   - c. **Identity property for addition:** For any whole number b, $0 + b = b + 0 = b$, and 0 is a unique identity for addition.
   - d. **Identity property for multiplication:** For any whole number b, $1 \times b = b \times 1 = b$, and 1 is a unique identity for multiplication.
   - e. **Commutative property for addition:** For any whole numbers a and b, $a + b = b + a$.
   - f. **Commutative property for multiplication:** For any whole numbers a and b, $a \times b = b \times a$.
   - g. **Associative property for addition:** For any whole numbers a, b, and c, $a + (b + c) = (a + b) + c$.
   - h. **Associative property for multiplication:** For any whole numbers a, b, and c, $a \times (b \times c) = (a \times b) \times c$.
   - i. **Distributive property for multiplication over addition:** For any whole numbers a, b, and c, $a \times (b + c) = (a \times b) + (a \times c)$.

8. **Inequality of whole numbers**
   a. For any whole numbers m and n, m is **less than** n if and only if there is a nonzero whole number k such that $m + k = n$. This property is written as $m < n$ or $n > m$.
   b. The inequality symbol and the equality symbol can be combined as $\geq$, which means **greater than or equal to,** or as $\leq$, which means **less than or equal to.**

9. **Mental calculations**
   a. **Compatible numbers for mental calculation** is the technique of using pairs of numbers that are especially convenient for mental calculations.
   b. **Substitution for mental calculation** is the technique of breaking a number into a convenient sum, difference, or product.
   c. **Equal differences for mental calculation** is the technique of increasing or decreasing both numbers in a difference by the same amount. With such a change, the difference between the two numbers stays the same.
   d. **Adding up for mental calculation** is the technique of finding a difference by adding up from the smaller number to the larger number.
   e. **Equal products for mental calculation** is a type of substitution that uses the fact that the product of two numbers remains the same when one of the numbers is divided by a given number and the other number is multiplied by the given number.
   f. **Equal quotients for mental calculation** is a type of substitution that uses the fact that the quotient of two numbers remains the same when both numbers are divided by the same number.

10. **Estimation**
    a. **Rounding** is the technique of replacing one or both numbers in a sum, difference, product, or quotient by an approximate number to obtain an estimation.
    b. **Compatible numbers for estimation** is the technique of computing estimations by replacing one or more numbers with convenient approximate numbers.
    c. **Front-end estimation** involves computing with only the leading digit of each number and is used for obtaining estimations of sums, differences, products, and quotients.

11. **Order of operations**
    a. The **order of operations** requires that when combinations of operations occur in an expression, numbers raised to a power be evaluated first, next products and quotients in the order in which they occur from left to right, then sums and differences in the order in which they occur from left to right.
    b. When parentheses are used in an expression, computations within the parentheses should be performed first.

CHAPTER 3 TEST

1. How would 226 be written in each of the following numeration systems?
- **a.** Egyptian
- **b.** Roman
- **c.** Babylonian
- **d.** Mayan

2. Determine the value of each underlined digit and its place value.
- **a.** 1<u>4</u>,702,301
- **b.** 36,0<u>0</u>7,285

3. Round 6,281,497 to the nearest
- **a.** Hundred thousand
- **b.** Hundred
- **c.** Thousand

4. Represent 123, using the given model.
- **a.** Bundle-of-sticks model
- **b.** Base-ten pieces
- **c.** Base-five pieces
- **d.** Base-seven pieces

5. Sketch base-ten pieces to illustrate each computation.
- **a.** $245 + 182$
- **b.** $362 - 148$

6. Compute the sum, using the given method.

a. *Left-to-right addition*

$$\begin{array}{r} 483 \\ +274 \\ \hline \end{array}$$

b. *Partial sums*

$$\begin{array}{r} 864 \\ +759 \\ \hline \end{array}$$

7. Use *equal differences* to find a replacement for each number that is more convenient for mental calculation. Show the new numbers and determine the answer.
- **a.** $65 - 19$
- **b.** $843 - 97$

8. Use *front-end estimation* to estimate the value of each computation.
- **a.** $321 + 435 + 106$
- **b.** $7410 - 2563 + 4602$
- **c.** 32×56
- **d.** $3528 \div 713$

9. Use *equal products* to find a replacement for each number that is more convenient for mental computation. Show the new numbers and determine the answer.
- **a.** 18×5
- **b.** 25×28

10. Compute 43×28 by showing the partial products. Sketch a rectangular grid to illustrate the product, and draw arrows from each partial product to its corresponding region on the grid.

11. Evaluate the following expressions.
- **a.** $6 \times 4 \times 5 - 3$
- **b.** $48 \div 4 + 2 \times 10$
- **c.** $(8 + 3) \times 5 - 2$

12. Compute the product or quotient. Leave your answer in exponential form.
- **a.** $3^{12} \div 3^4$
- **b.** $7^4 \times 7^6$

13. Sketch base-ten pieces to show how to compute $52 \div 4$ by using (a) the sharing (*partitive*) concept of division, (b) the measurement (*subtractive*) concept of division, and (c) a rectangular array to illustrate the quotient.

14. *Round* or use *compatible numbers* to mentally estimate each of the following. Show your number replacements.
- **a.** $473 + 192$
- **b.** $534 - 203$
- **c.** 993×42
- **d.** $350 \div 49$

15. Determine whether each statement is true or false for operations on the set of whole numbers.
- **a.** If the differences involved are whole numbers, multiplication is distributive over subtraction.
- **b.** Addition is commutative.
- **c.** If the differences involved are whole numbers, subtraction is associative.
- **d.** For nonzero whole numbers, division is commutative.
- **e.** The set of whole numbers is closed for subtraction.

16. Find a pattern in the equations below, and use inductive reasoning to predict the right side of the third equation. Then predict the fourth equation. Evaluate both sides of the fourth equation and determine if the pattern holds for this equation.

$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 =$$

17. There were 61 athletes at the annual sports banquet who played on either the football team or the baseball team. If 49 were on the football team and 18 were on the baseball team, how many players were on both teams?

18. If pizza is sold in four different sizes and can be ordered plain or with any one of five different toppings, how many different types of pizza are there?

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CHAPTER

4 Number Theory

Spotlight on Teaching

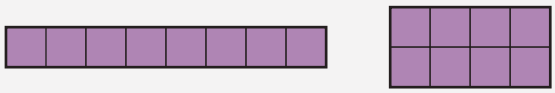
Excerpts from NCTM's Standard 6 for Grades 5–8*

Number theory offers many rich opportunities for explorations that are interesting, enjoyable, and useful. These explorations have payoffs in problem solving, in understanding and developing other mathematical concepts, in illustrating the beauty of mathematics, and in understanding the human aspects of the historical development of number.

Challenging but accessible problems from number theory can be easily formulated and explored by students. For example, building rectangular arrays with a set of tiles can stimulate questions about divisibility and prime, composite, square, even, and odd numbers. (See the figure below.)



Only 1 rectangle can be made with 7 tiles, so 7 is prime.



More than 1 rectangle can be made with 8 tiles, so 8 is composite.

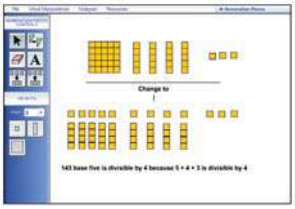
This activity and others can be extended to investigate other interesting topics, such as abundant, deficient, or perfect numbers; triangular and square numbers; cubes; palindromes; factorials; and Fibonacci numbers.

**Curriculum and Evaluation Standards for School Mathematics,* p. 91.

MATH ACTIVITY 4.1

Divisibility with Base-Ten Pieces

Virtual Manipulatives



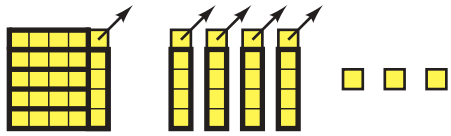
www.mhhe.com/bbn

Purpose: Use base-five and base-ten pieces to understand the divisibility tests for 3, 4, and 9.

Materials: Base-Five and Base-Ten Pieces in the Manipulative Kit or Virtual Manipulatives.

***1.** The following diagram shows that the base-five pieces representing 143_{five} can be divided evenly into groups of 4. Each flat and long is divided into groups of 4 with 1 unit remaining (marked by arrows). The 8 remaining units (1 + 4 + 3) can be evenly divided into two groups of 4.

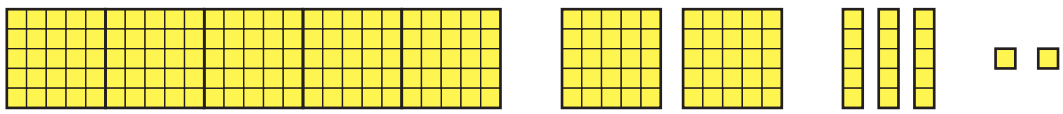
143_{five}



a. Explain how the diagram above illustrates the following divisibility rule: A three-digit base-five number is divisible by 4 if the sum of its digits is divisible by 4. Form an arbitrary collection of base-five pieces to illustrate your reasoning.

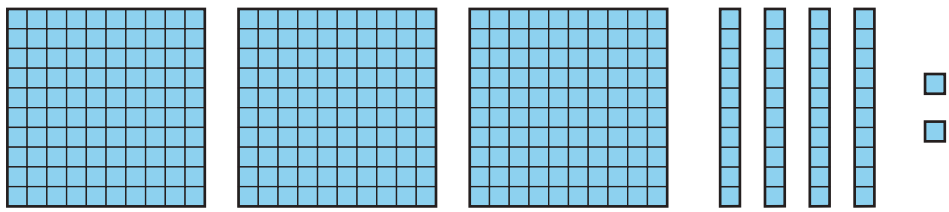
b. Use your base-five pieces to form the following collection. Show whether or not the divisibility rule from part a works for the four-digit number 1232_{five}. Write a general statement for determining when a base-five number is divisible by 4.

1232_{five}



2. Form some collections of base-ten pieces and show how divisibility by 3 and divisibility by 9 in base ten is similar to divisibility by 4 in base five.

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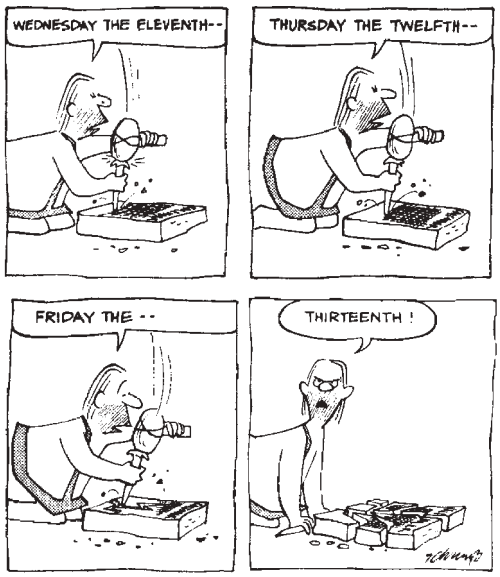


a. Based on your examples, state a rule for divisibility by 3 and a rule for divisibility by 9. Draw diagrams to support your conclusions.

b. Are all base-ten numbers that are divisible by 3 also divisible by 9, and vice versa? Explain.

3. Use your base-ten pieces to form the collection shown in activity 2. Look for an easy method of determining if these pieces can be evenly divided into four equal groups. Try your method on other collections of base-ten pieces. Show diagrams. Write a general statement describing a method for determining when a base-ten number can be divided by 4.

Section 4.1 FACTORS AND MULTIPLES



PROBLEM OPENER

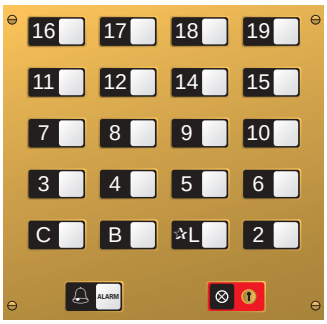
Fifty pennies are placed side by side. Each second penny is replaced by a nickel, each third coin is replaced by a dime, each fourth coin is replaced by a quarter, and each fifth coin is replaced by a half-dollar. What is the value of the 50 coins?

HISTORICAL HIGHLIGHT

The Pythagoreans (ca. 500 B.C.E.), a brotherhood of mathematicians and philosophers, believed that numbers had special meanings that could account for all aspects of life. For example, the number 1 represented reason, 2 stood for opinion, 4 was symbolic of justice, and 5 represented marriage. Even numbers were considered weak and earthly, and odd numbers were viewed as strong and heavenly. The numbers 1, 2, 3, and 4 also represented fire, water, air, and earth, and the fact that $1 + 2 + 3 + 4$ equals 10 had many meanings. When only 9 heavenly bodies could be found, including the earth, sun, moon, and the sphere of stars, the Pythagoreans imagined a tenth to "balance the earth."*

*M. Kline, *Mathematics in Western Culture* (New York: Oxford University Press, 1953), p. 77.

Number theory is the study of nonzero whole numbers and their relationships. Historically, certain numbers have had special attraction. Perhaps you have a favorite number. Seven is a common favorite number; 3 is also popular. There may be historical reasons for the preference for 3. For example, in the French phrase *très bien,* which means very good, *très* is derived from the word for 3. One of the oldest superstitions is that odd numbers (1, 3, 5, 7, 9, 11, . . .) are lucky. One exception is the common fear of 13, called



Elevator panel for a hotel with no 13th floor

triskaidekaphobia. Often hotels will not have a floor numbered 13, and motels will not have a room 13. Is this true where you live? Some people have a particular fear of Friday the 13th. Do you know someone with triskaidekaphobia?

MODELS FOR FACTORS AND MULTIPLES

One important type of relationship in number theory is that between a factor and a multiple. If one number is a **factor** of a second number or divides the second (as 3 is a factor of 12), then the second number is a **multiple** of the first (as 12 is a multiple of 3).

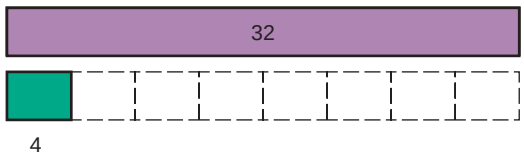
Factor and Multiple If a and b are whole numbers and $a \neq 0$, then

a is a factor of b if and only if there is a whole number c such that $ac = b$.

We can say that **a divides b** or that **b is a multiple of a.**

Let's look at two models for illustrating this relationship: the *linear model* and the *rectangular model.* Rods such as those shown in Figure 4.1 are one type of **linear model.** To determine whether one number is a factor of another, or divides another, we mark off the rod representing the second number, using a rod representing the proposed factor.

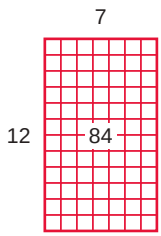
In Figure 4.1, the rod for 4 units can be marked off 8 times on the rod that represents 32 units. This shows that $8 \times 4 = 32$. So 8 and 4 are factors of 32, and 32 is a multiple of both 8 and 4.



8 and 4 are factors of 32 32 is a multiple of both 8 and 4

Figure 4.1

Figure 4.2 illustrates the **rectangular model.** In this model, one number is represented by a rectangular array of squares or tiles, and the two dimensions of the rectangle are factors of the number. One way to determine whether a whole number k is a factor of a whole number b is to try building a rectangular array of b tiles such that one dimension of the array is k.



12 and 7 are factors of 84 84 is a multiple of both 12 and 7

Figure 4.2

In Figure 4.2, the rectangular array of 84 tiles has 12 rows of tiles. That is, the 84 tiles are divided evenly into 12 rows, so 12 is a factor of 84. Since the 84 tiles are also divided evenly into 7 columns of tiles, 7 is also a factor of 84. Thus, the two dimensions of the rectangle, 7 and 12, are factors of 84, and 84 is a multiple of both 12 and 7.

Another way of indicating 12 divides 84 (is a factor of 84) is to write $12|84$, where the vertical line means *divides*.

> If a and b are whole numbers such that a **divides** b (a is a factor of b), we write $a|b$.
> If a does not divide b, we write $a \nmid b$.

EXAMPLE A To become more familiar with the *divides* relationship, classify the following statements as true or false.

1. $15|60$
2. $8|30$
3. $3 \nmid 19$
4. $18|18$
5. $2|0$

Solution 1. True 2. False 3. True 4. True 5. True

Notice that *divides* signifies a *relationship* between two numbers; it indicates that one number is divisible by another. It does not indicate the operation of division, that is, dividing one number by another. For example, $3|15$ tells us that *3 divides 15* and should not be confused with the fraction $\frac{3}{15}$, which means *3 divided by 15* and is equal to the fraction $\frac{1}{5}$.

PROBLEM-SOLVING APPLICATION

The following problem is solved by using factors and multiples and features the strategies of *guessing and checking* and *making an organized list.*

Problem

A factory uses machines to sort cards into piles. On one occasion a machine operator obtained the following curious result. When a box of cards was sorted into 7 equal groups, there were 6 cards left over; when the box of cards was sorted into 5 equal groups, there were 4 left over; and when it was sorted into 3 equal groups, there were 2 left. If the machine cannot sort more than 200 cards at a time, how many cards were in the box?

Understanding the Problem Sorting the cards into groups of 7 is like dividing by 7. Since there were 6 cards left, we know 7 is not a factor of the original number of cards. **Question 1:** How can we be sure that 5 and 3 are not factors of the number of cards in the box?

Devising a Plan One approach is to *guess and check* a few numbers to become more familiar with the problem. Even as you start to guess, you can throw out certain numbers. For example, there could not have been 100 cards, because 5 divides 100. Another approach is to find numbers satisfying one of the conditions. For example, since dividing by 7 leaves a remainder of 6, we can *make an organized list* of numbers satisfying this condition:

13, 20, 27, 34, 41, 48, 55, 62, 69, 76, 83, 90, 97, 104, 111, 118, . . .

Then we can find the numbers in this list that leave a remainder of 4 when divided by 5 and a remainder of 2 when divided by 3. **Question 2:** What is the smallest number in this list that leaves a remainder of 4 when divided by 5?

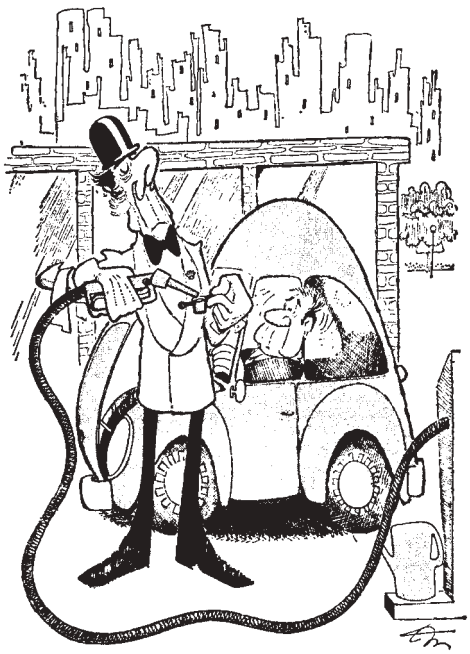
Carrying Out the Plan The numbers 34, 69, and 104 leave a remainder of 4 when divided by 5. **Question 3:** Which of these numbers leaves a remainder of 2 when divided by 3? The answer to this question is the solution to the original problem.

Looking Back Another approach is to change the original problem by noticing that if there had been 1 more card in the box, then 7, 5, and 3 would have been factors of the number of cards. Once this number has been found, the number 1 can be subtracted to obtain the original number of cards. **Question 4:** What is the smallest nonzero whole number that is divisible by 7, 5, and 3?

Answers to Questions 1–4 **1.** When the total number of cards was divided by 5, there was a remainder of 4; and when it was divided by 3, there was a remainder of 2. If these numbers were factors, there would be no remainders. **2.** 34 **3.** 104 **4.** 105

DIVISIBILITY TESTS

During a gasoline shortage, Oregon adopted an "odd and even" system of gas rationing. Drivers with odd-numbered license plates could get gasoline on the odd-numbered days of the month, and those with even-numbered plates could get gasoline on the even-numbered days. Some people whose license plate numbers ended in zero were confused as to whether their numbers were odd or even. The solution to this problem lies in the definition of odd and even numbers.



Odd and Even Numbers Any whole number that has 2 as a factor is called an **even number,** and any whole number that does not have 2 as a factor is called an **odd number.**

EXAMPLE B Would a person with the following license plate have purchased gasoline on the odd-numbered or even-numbered days?



Solution On even-numbered days, because 1080 has 2 as a factor ($2 \times 540 = 1080$).

In the following paragraphs we will examine a few simple tests for determining whether a number is divisible by 2, 3, 4, 5, 6, or 9 without carrying out the division. Before looking at these tests, we present three divisibility properties.

Divisibility Properties

1. If a number divides each of two other numbers, then it divides their sum. If $a|b$ and $a|c$, then $a|(b + c)$.
2. If a number divides one of two numbers but not the other, then it will not divide their sum. If $a|b$ and $a \nmid c$, then $a \nmid (b + c)$.
3. If one number divides another number, then it will divide the product of that number with any other whole number. If $a|b$, then $a|kb$.

Figures 4.3, 4.4, and 4.5 illustrate three divisibility properties. There are other divisibility properties in Exercises and Problems 4.1.

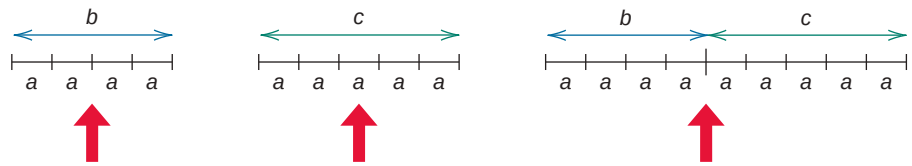


Figure 4.3

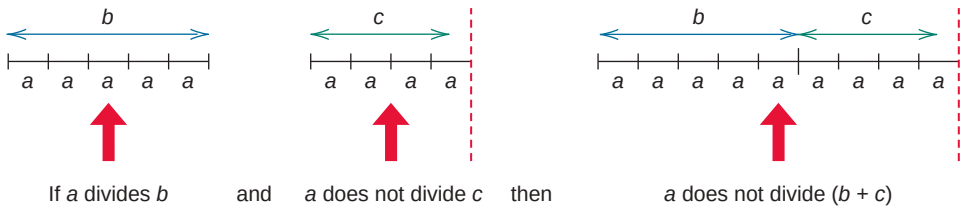


Figure 4.4

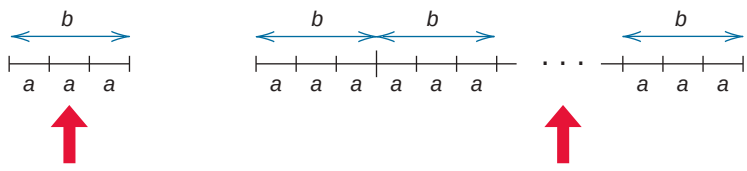


Figure 4.5

EXAMPLE C

Here are examples of the three divisibility properties.

1. Since $6|54$ and $6|48$, we can conclude that $6|(54 + 48)$. That is, $6|102$.
2. Since $7|42$ but $7 \nmid 50$, we can conclude that $7 \nmid (42 + 50)$. That is, $7 \nmid 92$.
3. Since $13|26$, we can conclude that 13 divides any multiple of 26. For example, $13|(9 \times 26)$, or $13|234$.

Test for Divisibility by 2 or 5

A number is **divisible by 2** if the number represented by the units digit is divisible by 2. Thus, a number is divisible by 2 if its units digit is 0, 2, 4, 6, or 8.

A number is **divisible by 5** if the number represented by the units digit is divisible by 5. Thus, a number is divisible by 5 if its units digit is 0 or 5.

EXAMPLE D

Classify each statement as true or false and explain why.

1. $2|13{,}776$ 2. $5|3135$ 3. $2|2461$

Solution 1. True. Since $2|6$, we know that $2|13{,}776$. 2. True. Since $5|5$, we know that $5|3135$. 3. False. Since $2 \nmid 1$, we know that $2 \nmid 2461$.

Let's look at a base-ten representation of 2573 to see why these tests work. Figure 4.6 on the next page shows that each long-flat, flat, and long can be divided into two equal parts (see dotted lines). So whether 2 divides 2573 depends on whether 2 divides 3. Since 3 units cannot be divided into two equal groups of units, 2 does not divide 2573.

Similarly, we can use Figure 4.6 (on the next page) to see that 2573 is not divisible by 5. Think about how you would divide each long-flat, flat, and long into 5 equal parts. Since 3 units cannot be divided into 5 equal groups of units, 5 does not divide 2573.

Now we will consider the same example, using the divisibility properties that were illustrated in Figures 4.3 to 4.5. The expanded form of 2573 is

$$2573 = \underbrace{2 \times 10^3 + 5 \times 10^2 + 7 \times 10}_{} + 3 \times 1$$

Since 2 divides 10^3, by divisibility property 3 it also divides any multiple of 10^3 and, in particular, it divides 2×10^3. Similarly, since 2 divides 10^2 and 10, it also divides 5×10^2 and 7×10. Then, by divisibility property 1, 2 divides the sum of these numbers, which is the portion of the preceding equation indicated by the brace. Therefore, by divisibility property 1, 2 will divide the right side of the equation if 2 divides 3; by divisibility property 2, if 2 does not divide 3, it will not divide the right side of the equation. Since 2 does not divide 3, it does not divide 2573. Similarly, since 5 divides 10^3, 10^2, and 10, it divides the portion of the equation indicated by the brace. However, since 5 does not divide 3, it does not divide 2573. A general proof of the divisibility tests for 2 or 5 for any arbitrary whole number follows the same reasoning.

Test for Divisibility by 3 or 9

A number is **divisible by 3** if the sum of its digits is divisible by 3.

A number is **divisible by 9** if the sum of its digits is divisible by 9.

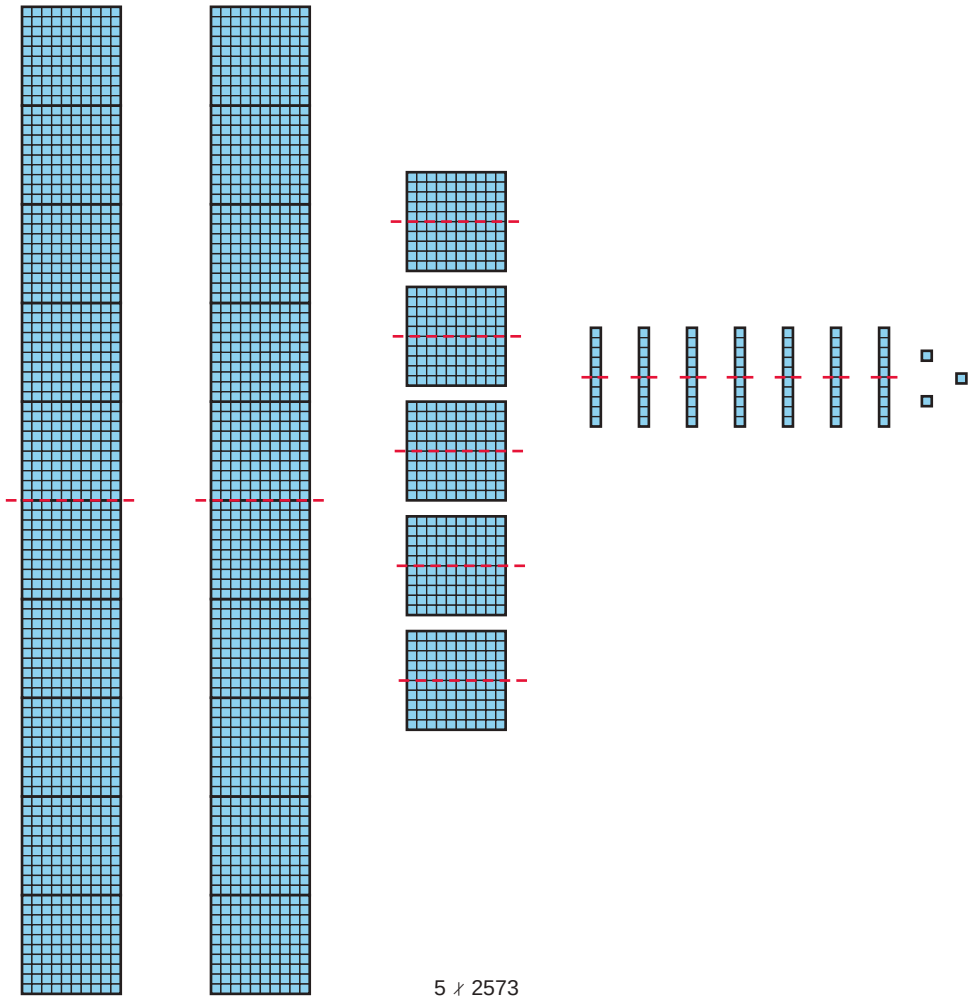


Figure 4.6

EXAMPLE E Classify each statement as true or false, and explain why.

1. $3|2847$ **2.** $9|147{,}389$ **3.** $3|270{,}415$

Solution **1.** True, because 3 divides $2 + 8 + 4 + 7 = 21$. **2.** False, because 9 does not divide $1 + 4 + 7 + 3 + 8 + 9 = 32$. **3.** False, because 3 does not divide $2 + 7 + 0 + 4 + 1 + 5 = 19$.

To visualize why the tests for divisibility by 3 and 9 work, consider the base-ten piece representation for 2847 in Figure 4.7 on the next page. If 1 unit is removed from a long-flat, the remaining 999 units can be divided into 3 equal groups. Similarly, if 1 unit is removed from each flat, the remaining 99 units can be divided into 3 equal groups; and if 1 unit is removed from each long, the remaining 9 units can be divided into 3 equal groups. Thus, whether 3 divides 2847 depends on whether 3 divides the remaining units, which consist of 2 units from the long-flats, 8 from the flats, 4 from the longs, and 7 from the units. Since 3 divides $2 + 8 + 4 + 7 = 21$, it also divides 2847.

NCTM Standards

Tasks, such as the following, involving factors, multiples, prime numbers, and divisibility can afford opportunities for problem solving and reasoning: (1) Explain why the sum of the digits of any multiple of 3 is itself divisible by 3; and (2) A number of the form *abcabc* always has several prime-number factors. Which prime numbers are always factors of a number of this form? Why? p. 217

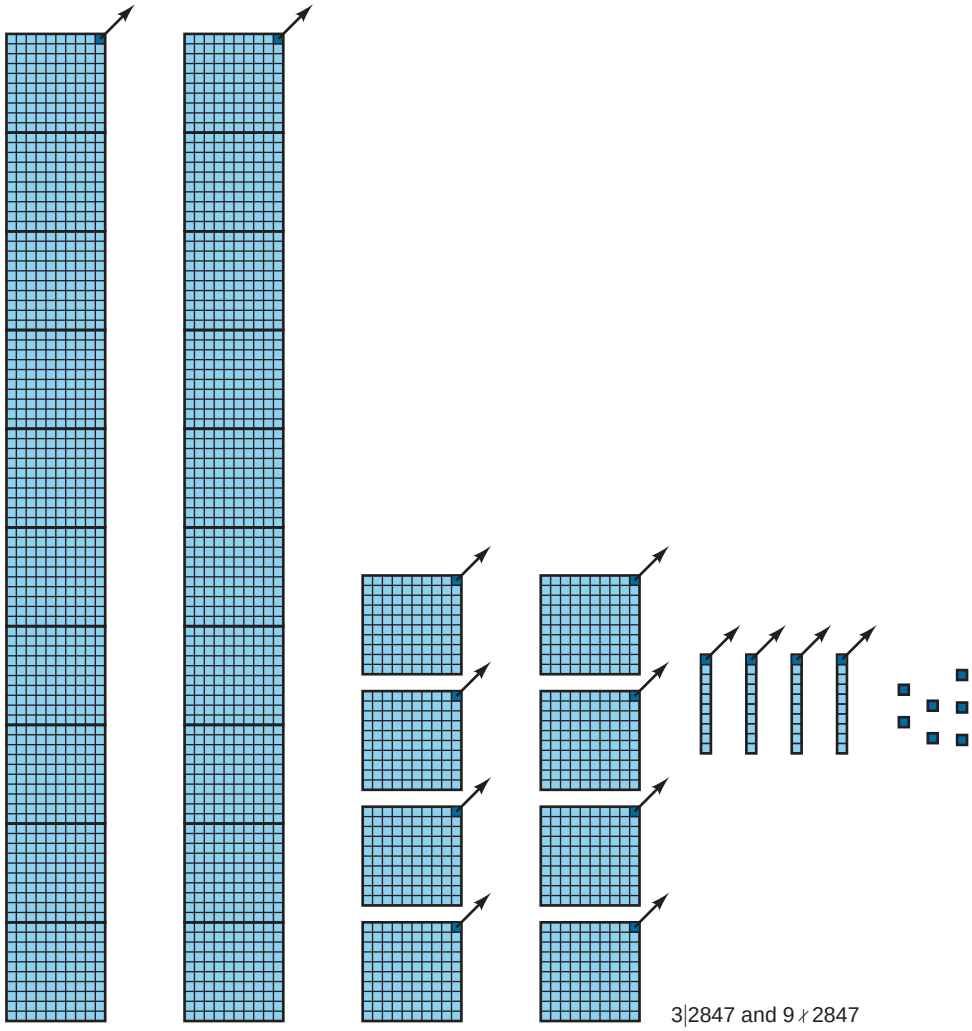


Figure 4.7

Figure 4.7 shows why 2847 is not divisible by 9. Once a unit piece is removed from each long-flat, flat, and long, 9 will divide the remaining number of units in each of these base-ten pieces. Thus, whether 9 divides 2847 depends on whether 9 divides $2 + 8 + 4 + 7$. Since 9 does not divide 21, we can conclude that 9 does not divide 2847.

Let's examine the divisibility tests for 3 and 9 on 2847 by using the divisibility properties from Figures 4.3 through 4.5. The expanded form of 2847 is shown below. Notice the use of the distributive property to obtain the third equation from the second equation. The commutative and associative properties for addition are also needed to obtain the arrangement of numbers in the third equation.

$$\begin{aligned} 2847 &= 2 \times 10^3 + 8 \times 10^2 + 4 \times 10 + 7 \times 1 \\ &= 2 \times (999 + 1) + 8 \times (99 + 1) + 4 \times (9 + 1) + 7 \times 1 \\ &= \underbrace{2 \times 999 + 8 \times 99 + 4 \times 9} + (2 + 8 + 4 + 7) \times 1 \end{aligned}$$

Since 3 divides 9, 99, and 999, by divisibility properties 3 and 1 it also divides the portion of the equation indicated by the brace. Therefore, because of divisibility properties 1 and 2, to determine if 2847 is divisible by 3, it is necessary only to determine if the remaining portion of the equation containing the sum $2 + 8 + 4 + 7$ is divisible by 3. Since 3 does divide this sum, we know that 3 divides 2847.

The expanded form of 2847 in the preceding equation also shows why a similar test works for divisibility by 9. Since 9 divides the portion of the equation indicated by the brace, it will divide 2847 if and only if it divides 2 + 8 + 4 + 7. In this case, 9 does not divide this sum, so it does not divide 2847. A general proof of the divisibility tests for 3 or 9 for any arbitrary whole number follows essentially the same reasoning.

Test for Divisibility by 6

A number is **divisible by 6** if and only if the number is divisible by both 2 and 3.

EXAMPLE F

Apply the preceding test to determine which of the following numbers are divisible by 6.

1. 561,781 **2.** 2,100,000,472 **3.** 123,090,534

Solution **1.** 561,781 is not divisible by 2 (because it is odd), so it is not divisible by 6. **2.** 2,100,000,472 is not divisible by 3 (because the sum of its digits is 16), so it is not divisible by 6. **3.** 123,090,534 is even and divisible by 3, so it is divisible by 6.

Test for Divisibility by 4

A number is **divisible by 4** if the number represented by the last two digits is divisible by 4.

EXAMPLE G

Use the preceding test to determine which of the following numbers are divisible by 4.

1. 65,932 **2.** 476,025,314 **3.** 113,775,920

Solution **1.** $4 \mid 65{,}932$ because $4 \mid 32$. **2.** $4 \nmid 476{,}025{,}314$ because $4 \nmid 14$. **3.** $4 \mid 113{,}775{,}920$ because $4 \mid 20$.

The following expanded form shows why the test for divisibility by 4 works for 65,932.

$$65{,}932 = \underbrace{6 \times 10^4 + 5 \times 10^3 + 9 \times 10^2}_{} + 3 \times 10 + 2 \times 1$$

Since 4 is a factor of 10^4, 10^3, and 10^2, by divisibility properties 3 and 1 it will divide the portion of the expanded form indicated by the brace. Thus, whether 4 divides 65,932 depends only on whether 4 divides 32. Similar reasoning can be used to prove the test for divisibility by 4 for any arbitrary whole number. (There are questions in Exercises and Problems 4.1 requiring the use of base-ten pieces to illustrate the divisibility-by-4 test.)

PRIME AND COMPOSITE NUMBERS

NCTM Standards

NCTM's K–4 standard on number operations in the *Curriculum and Evaluation Standards for School Mathematics* (p. 42) recommends that terms such as *factor* and *multiple* be introduced informally:

> The notions of factors and multiples can prompt interesting explorations. Children can find the factors of a number using tiles or graph paper. This can lead to an investigation of numbers that have only two factors (prime numbers) and numbers with two equal factors (square numbers).

Research Statement

Mathematics achievement is increased through the long-term use of concrete instructional materials and students' attitudes toward mathematics are improved when they have instruction with concrete materials provided by teachers knowledgeable about their use.

Sowell

One way to classify whole numbers is to examine their factors. It is possible to find all factors of a number by building rectangular arrays. Figure 4.8 shows rectangular arrays for 16 and 30 by increasing widths. What observations and conjectures can be made from these rectangles?

Here are a few observations: (1) The factors of the numbers occur in pairs (two for each rectangle) except when a number has a square array, as in the case of 16. (2) The numbers 1, 2, 4, 8, and 16 are factors of 16; and 1, 2, 3, 5, 6, 10, 15, and 30 are factors of 30. (3) There is a duplication of rectangles for the number 16 after a width of 4 is reached for the 4×4 square array. There is a duplication for the number 30 after a width of 5 is reached for the 5×6 array.

Figure 4.8 suggests that square arrays, or rectangles that are close to square arrays, are the turning point about which further rectangles have dimensions (or factors) that repeat. To find the factors of 16, we need to consider only arrays with widths up to 4 because 4 is the width of the square array. For the number 30, we know we have passed the turning point for finding its factors when we get to the width of 6 because the second factor must be less than 6 if the product of the two factors is to be 30.

Consider the preceding observations for finding the factors of 188. We first notice that 188 is not a square number because $13 \times 13 = 169$ and $14 \times 14 = 196$. Thus, all rectangles for 188 will have at least one dimension (factor) less than 14. Systematically checking arrays with widths less than 14, we find rectangular arrays with these dimensions: 1×188, 2×94, and 4×47. So the factors of 188 are 1, 2, 4, 47, 94, and 188.

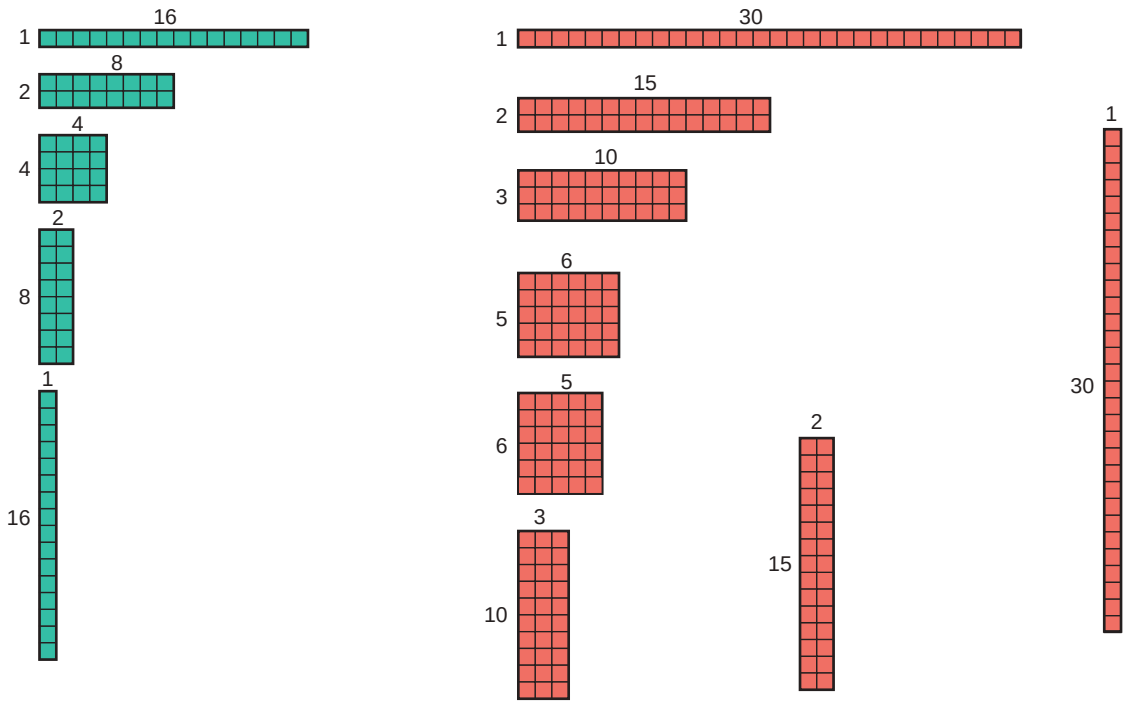


Figure 4.8

Figure 4.9 shows the arrays for 11 and 10. There is only one array for 11, and its factors are 1 and 11. There are two arrays for 10, and its factors are 2, 5, 1, and 10.



Figure 4.9

Arrays can also be sketched for the numbers from 1 to 9. The table in Figure 4.10 lists the factors for the numbers from 1 to 12. Notice that the numbers 2, 3, 5, 7, and 11 have

NCTM Standards

Students should recognize that different types of numbers have particular characteristics; for example square numbers have an odd number of factors and prime numbers have only two factors. p. 151

exactly two factors. These numbers are called **prime numbers.** In general, any nonzero whole number with exactly two factors is called a **prime number.** Nonzero whole numbers with more than two factors are called **composite numbers.** Since 1 has only one factor, it is classified as *neither prime nor composite.*

No.	Factors	No. of Factors (Divisors)
1	1	1
2	1, 2	2
3	1, 3	2
4	1, 2, 4	3
5	1, 5	2
6	1, 2, 3, 6	4
7	1, 7	2
8	1, 2, 4, 8	4
9	1, 3, 9	3
10	1, 2, 5, 10	4
11	1, 11	2
12	1, 2, 3, 4, 6, 12	6

Figure 4.10

EXAMPLE H

The table in Figure 4.10 suggests some questions.

1. Will there be numbers other than 1 with only one factor?
2. What kinds of numbers have an odd number of factors?
3. Are there numbers with more than six factors?

Solution **1.** No. **2.** Square numbers. For a square number, there will be a square array, which contributes only one factor. So for a square number, the total number of factors will always be odd. **3.** Yes. Any number of factors is possible.

EXAMPLE I

List the numbers from 13 to 20, and determine whether they are prime or composite.

Solution The numbers 13, 17, and 19 are prime (each can be represented by only one rectangular array), and 14, 15, 16, 18, and 20 are composite (each has two or more rectangular arrays).

There is no largest prime because there are an infinite number of prime numbers. Some very large primes have been discovered. From 1876 to 1951, this 39-digit number was the largest known prime:

170,141,183,460,469,231,731,687,303,715,884,105,727

Now computers make it possible to find a large prime every few months. In August 2008, for example, the Great Internet Mersenne Prime Search (GIMPS) collaborative project found the largest currently known prime number: $2^{43,112,609} - 1$. This number is 12,978,189 digits long.

Prime numbers are difficult to locate because they do not occur in predictable patterns. In fact, there are arbitrarily large stretches of consecutive whole numbers that include no primes. For example, between the numbers 396,733 and 396,833 there are 99 composite numbers.

HISTORICAL HIGHLIGHT

Knowing the factors of numbers has long been valuable in research involving prime numbers. In 1659 J. H. Rahn published a table listing the factors for all numbers up to 24,000, and in 1668 John Pell of England extended this table to 100,000. The greatest achievement of this sort is the table by J. P. Kulik (1773–1863) from the University of Prague. His table covers all numbers up to 100,000,000.* Finding the factors of numbers is currently of major interest to cryptographers and intelligence agencies, whose code solutions are often based on the prime factors of very large whole numbers.

*H. W. Eves, *An Introduction to the History of Mathematics,* 3d ed. (New York: Holt, Rinehart and Winston, 1969), p. 149.

PRIME NUMBER TEST

One method of determining if a number is prime is to check whether it has any factors other than itself and 1. This is where the divisibility tests can be useful.

EXAMPLE J

Which of the following numbers are prime?

1. 43,101 **2.** 24,638 **3.** 53

Solution **1.** Since $3|(4 + 3 + 1 + 0 + 1)$, we know that $3|43{,}101$, so this number is not prime. **2.** Since $2|8$, we know that $2|24{,}638$, so this number is not prime. **3.** 53 is prime.

Technology Connection

Frequency of Primes

Do the numbers of primes in the intervals 1 to 100, 100 to 200, 200 to 300, etc., tend to decrease? Does the number of pairs of *twin primes,* such as 5, 7 and 11, 13 that have a difference of 2, decrease for these intervals? Use the online 4.1 Mathematics Investigation to print out the primes in these intervals and form some conjectures.

Mathematics Investigation
Chapter 4, Section 1
www.mhhe.com/bbn

To determine if a number has factors other than itself and 1, we need only try dividing by prime numbers (2, 3, 5, 7, . . .). There is no need to divide by composite numbers (4, 6, 8, 9, . . .). For example, if 4 divides a number, then 2 divides the number. In other words, if 2 does not divide a number, then 4 will not divide the number.

Let's consider how we might determine whether 53 is a prime number. The divisibility tests show that 53 is not divisible by 2, 3, or 5, and we know from basic multiplication facts that 7 is not a factor of 53. This means that rectangles whose dimensions are 2, 3, 5, or 7 cannot be built with 53 tiles. Now by the observations from Figure 4.8 on page 224, we need to consider only arrays up to 8×8, since this is the first square array that has more than 53 tiles. Since there are no arrays for 53 with widths less than 8, 53 is a prime number. This example suggests the following theorem.

Prime Number Test Suppose n is a whole number and k is the smallest whole number such that $k \times k$ is greater than n. If there is no prime number less than k that is a factor of n, then n is a prime number.

This theorem tells us which primes we should try as divisors to determine if a number is prime or composite.

EXAMPLE K

1. Is 421 prime or composite? **2.** Is 667 a prime number?

Solution **1.** Since $23 \times 23 > 421$ and $19 \times 19 < 421$, we need only consider 2, 3, 5, 7, 11, 13, 17, and 19 as possible factors of 421. Since none of these primes are factors, 421 is a prime number. **2.** No, it is divisible by 23.

1-2 Prime Factors

MAIN IDEA
Find the prime factorization of a composite number.

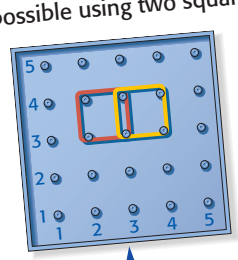
New Vocabulary
factor
prime number
composite number
prime factorization

Math Online
glencoe.com
- Concepts in Motion
- Extra Examples
- Personal Tutor
- Self-Check Quiz
- Reading in the Content Area

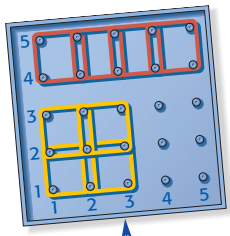
MINI Lab

Any given number of squares can be arranged into one or more different rectangles.

STEP 1 Use a geoboard to make as many different rectangles as possible using two squares. Then repeat with four squares.



Only one rectangle can be made using two squares. The dimensions of this rectangle are 1 × 2.



Two different rectangles can be made using four squares. The dimensions of these rectangles are 1 × 4 and 2 × 2.

STEP 2 Copy and complete the table using 2, 3, 4, . . ., 20 squares. Use a geoboard to help you.

Number of Squares	Dimensions of Each Rectangle
2	1 × 2
3	
4	1 × 4, 2 × 2

1. For what numbers can more than one rectangle be formed?
2. For what numbers can only one rectangle be formed?
3. For the numbers in which only one rectangle is formed, what do you notice about the dimensions of the rectangle?

When two or more numbers are multiplied, each number is called a **factor** of the product.

$1 \times 7 = 7$ (factors of 7)

$1 \times 6 = 6, 2 \times 3 = 6$ (factors of 6)

A whole number that has exactly two unique factors, 1 and the number itself, is a **prime number**. A number greater than 1 with more than two factors is a **composite number**.

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SIEVE OF ERATOSTHENES

One way of finding all the primes that are less than a given number is to eliminate those numbers that are not prime. This method was first used by the Greek mathematician Eratosthenes (ca. 230 B.C.E.) and is called the **Sieve of Eratosthenes.** Figure 4.11 illustrates the use of this method to find the primes that are less than 120.

We begin the process by crossing out 1, which is not a prime, and then circling 2 and crossing out all the remaining multiples of 2 (4, 6, 8, 10, . . .). Then 3 is circled and all the remaining multiples of 3 are crossed out (6, 9, 12, . . .). This process is continued by circling 5 and 7 and crossing out their multiples. Since every composite number less than 120 must have at least one prime factor less than 11 ($11 \times 11 = 121$), it is unnecessary to cross out the multiples of primes greater than 7. The numbers in the table that are not crossed out are prime.

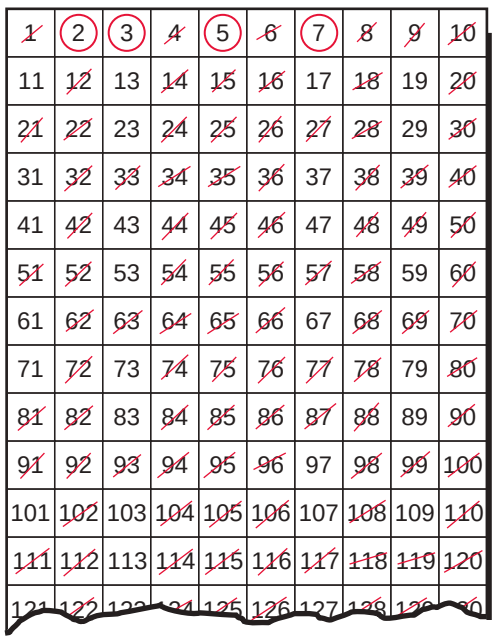


Figure 4.11

EXAMPLE L

Examine the locations of the primes in Figure 4.11.

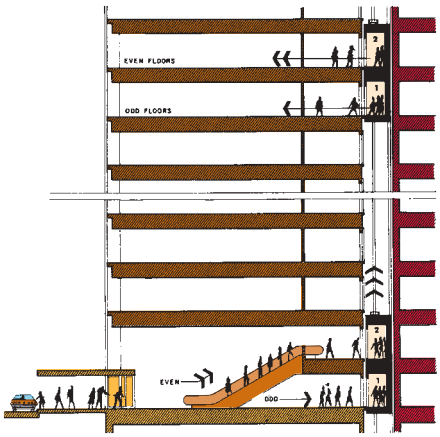
1. What patterns can you see?
2. What is the longest sequence of consecutive numbers that are not prime?

Solution **1.** Several primes occur in pairs, with one composite number between them: 3 and 5, 5 and 7, and 11 and 13 are the first few such pairs, and 107 and 109 are the largest such pair in this table. Such primes are called **twin primes.** You might also notice that except for 2 and 5, all the primes occur in columns 1, 3, 7, and 9. **2.** There are two sequences of seven consecutive numbers that are not prime; the first sequence is the numbers from 90 to 96, and the second sequence is the numbers from 114 to 120.*

*The online 4.1 Mathematics Investigation, Frequency of Primes (see website), prints the prime numbers between two numbers that are entered and counts the number of primes in the interval. There is also an option for printing twin primes.

Exercises and Problems 4.1

Use the following information in exercises 1 and 2. Some skyscrapers have double-deck elevators to minimize the number of elevator shafts required. People entering the building use the bottom deck of the elevator to arrive at odd-numbered floors, or they may take an escalator to and from a mezzanine to use the top deck of the elevator, which stops at even-numbered floors.



1. **a.** Suppose you had to deliver packages to floors 11, 26, 35, and 48. How could you do so with the least amount of elevator riding and walking only one flight of stairs?
   b. Describe an efficient scheme for delivering to the 11 floors from 20 to 30.

2. **a.** Suppose the escalator was not operating and you had to deliver to floors 32, 19, 47, 28, and 50. How could this be done with the least amount of elevator riding and walking only two flights of stairs?
   b. Describe an efficient scheme for delivering to any number of odd- and even-numbered floors if the escalator is working.

Classify each of the statements in exercises 3 and 4 as true or false.

3. **a.** $3|4263$
   b. $15|1670$
   c. $12 \nmid 84$

4. **a.** $5 \nmid 49$
   b. $1 \nmid 17$
   c. $13|315$

There are several ways of expressing in words the relationship between two numbers when one divides the other. Rewrite each statement in exercises 5 and 6 in the form $a|b$, using the *divides* relationship.

5. **a.** 7 is a factor of 63.
   b. 40 is a multiple of 8.
   c. 13 is a divisor of 39.
   d. 36 is divisible by 12.

6. **a.** 322 is divisible by 23.
   b. 13 is a divisor of 403.
   c. 30,000 is a multiple of 3000.
   d. 34 is a factor of 6324.

In exercises 7 and 8, illustrate statement **a** with a linear model and statement **b** with a rectangular model.

7. **a.** $6|54$ **b.** $12|60$

8. **a.** $7|42$ **b.** $43|516$

Color rods such as the Cuisenaire Rods shown below are often found in elementary schools.* They may be used as linear models to illustrate the concepts of factors and multiples.



A row in which all the rods are the same is called a **one-color train.** Here is a red one-color train representing 12. Use this model in exercises 9 and 10.

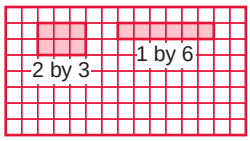


*Cuisenaire Rods is a registered trademark of Cuisenaire Company of America, Inc.

9. a. What other Cuisenaire Rods can be used to form a one-color train for 12? What information do these trains provide for the number 12?
b. What one-color trains can be used to represent 15? What information is illustrated by these trains?
c. How many different one-color trains of two or more rods will there be for each of the numbers 2, 3, 5, and 7? What information does this illustrate for prime numbers?

10. a. If an all-brown train is equal in length to an all-orange train, what can be said about the number of brown rods compared to the number of orange rods?
b. If a number can be represented by an all-red train, an all-green train, and an all-black train, it has at least eight factors. Name these factors.
c. What is the smallest number of red rods for which an all-red train is equal in length to an all-blue train?

Each rectangular array of squares (see Figures 4.8 and 4.9) gives information about the number of factors of a number. Two rectangles can be formed for the number 6, showing that 6 has factors of 2, 3, 1, and 6.



Copy the rectangular grid from the website and use it to sketch as many different rectangular (including square) arrays as possible for each of the following numbers: 15, 16, 30, 25, 17. Use these arrays for exercises 11 and 12.

11. a. What kinds of numbers will have only one rectangular array?
b. Which three of the given numbers have an even number of factors? Use your sketches to explain why.
c. If a number does not have a square array, explain why its factors will occur in pairs.
d. If a number has eight factors, how many different arrays will it have? Find the smallest such number.

12. a. Two of the given numbers have square arrays. Make a conjecture about the number of factors for square numbers. Explain how your sketches support your conjecture.
b. What kind of a number will have five factors? Find the smallest such number.
c. If any nonsquare number has four different rectangular arrays, how many factors does it have?

Which of the numbers in exercises 13 and 14 are divisible by 3? Determine the remainder when the number is divided by 3.

13. a. 465,076,800 **b.** 100,101,000

14. a. 907,116,341 **b.** 477,098,304

Which numbers in exercises 15 and 16 are divisible by 9? Determine the remainder when the number is divided by 9.

15. a. 48,276,348,114
b. 206,347,166,489

16. a. 2,136,479,180,022
b. 7,302,511,648,591

Explain your reasoning or give a counterexample to answer each question in exercises 17 and 18.

17. a. If a number is divisible by 3, is it divisible by 9?
b. If a number is divisible by 9, is it divisible by 3?

18. a. If a number is divisible by 12, is it divisible by 6?
b. If a number is divisible by 6, is it divisible by 12?

Which of the numbers in exercises 19 and 20 are divisible by 4? Determine the remainder when the number is divided by 4.

19. a. 47,382,729,162
b. 512,112,911,576

20. a. 14,710,816,558
b. 4,328,104,292

21. Sketch base-ten pieces for a four-digit number, and explain how they can be used to illustrate the divisibility-by-4 test.

22. Sketch base-ten pieces for a four-digit number that is not divisible by 4, and explain how the divisibility-by-4 test can be used to obtain the remainder when dividing by 4.

Write each statement in 23 and 24 in words, and classify it as true or false. If the statement is true, show a sketch to illustrate the divisibility property. (See illustrations of divisibility properties on page 219.) If the statement is false, show a counterexample.

23. a. If $a|b$ and $a \nmid c$, then $a \nmid (b - c)$.
b. If $a \nmid b$ and $a \nmid c$, then $a \nmid (b + c)$.
c. If $a|b$ and $b|c$, then $a|c$.

24. a. If $a|c$ and $b|c$, then $(a + b)|c$.
b. If $a|b$ and $a \nmid c$, then $a \nmid bc$.
c. If $a|b$ and $a|c$, then $a|(b - c)$.

25. The numbers 2, 3, 5, 7, 11, and 13 are not factors of 173. Explain why it is possible to conclude that 173 is prime without checking for more prime factors.

26. The first 10 prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29.
- **a.** Which of these prime numbers would you have to consider as possible factors of 367 in order to determine whether 367 is a prime or composite number?
- **b.** Is 367 prime or composite?

Which of the numbers in exercises 27 and 28 are prime?

27. **a.** 231 **b.** 277 **c.** 683

28. **a.** 187 **b.** 431 **c.** 391

Suppose Figure 4.11 was extended for answering questions 29 and 30.

29. Explain how the Sieve of Eratosthenes can be used to determine all the prime numbers less than 300.

30. Explain how the Sieve of Eratosthenes can be used to determine all the primes less than 400.

In exercises 31 and 32, test each number for divisibility by 11 by alternately adding and subtracting the digits from right to left, beginning with the units digit—that is, units digit minus tens digit plus hundreds digit, etc. If the result is divisible by 11, then the original number will be divisible by 11. (*Note:* Zero is divisible by 11.)

31.
- **a.** 63,011,454
- **b.** 19,321,488
- **c.** 4,209,909,682
- **d.** Will the test for divisibility by 11 work if the digits are alternately added and subtracted from left to right?

32.
- **a.** 9,874,684,259
- **b.** 8,418,470,316
- **c.** 7,197,183,232
- **d.** Sketch base-ten pieces for 341, and explain how they can be used to illustrate that 341 is divisible by 11.

33. The numbers in the following sequence increase by 2, then by 4, then by 6, etc. Continue this sequence until you reach the first number that is not a prime.

17 19 23 29 37

It is likely that no one will ever find a formula that will give all the primes less than an arbitrary number. The formulas in exercises 34 and 35 produce primes for a while, but eventually they produce composite numbers.

34. For which of the whole numbers $n = 2$ to 7 is $2^n - 1$ a prime?

35. The formula $n^2 - n + 41$ will give primes for $n = 1, 2, 3, \ldots, 40$ but not for $n = 41$. Which of the primes less than 100 are given by this formula?

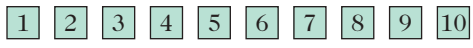
Two of the many conjectures involving primes are given in exercises 36 and 37.

36. The mathematician Christian Goldbach (1690–1764) conjectured that every odd number greater than 5 is the sum of three primes. Verify this conjecture for the following numbers: 21, 27, 31.

37. In 1845 the French mathematician Bertrand conjectured that between any whole number greater than 1 and its double there exists at least one prime. After 50 years this conjecture was proved true by the Russian mathematician Pafnuty Chebyshev. For the numbers greater than 5 and less than 15, is it true or false that there are at least two primes between every number and its double?

Reasoning and Problem Solving

38. **Featured Strategies: Using a Model, Making a Table, and Solving a Simpler Problem.** In a new school built for 1000 students, there were 1000 lockers that were all closed. As the students entered the school, they decided on the following plan. The first student who entered the building opened all 1000 lockers. The second student closed all lockers with even numbers. The third student changed all lockers that were numbered with multiples of 3 (that is, opened those that were closed and closed those that were open). The fourth student changed all lockers numbered with multiples of 4, the fifth changed all lockers numbered with multiples of 5, etc. After 1000 students had entered the building and changed the lockers according to this pattern, which lockers were left open?
- **a. Understanding the Problem.** To better understand this problem, think about what will happen as each of the first few students enter. For example, after the first three students, will locker 6 be open or closed?
- **b. Devising a Plan.** Simplifying a problem will sometimes help you to strike on an idea for the solution. Suppose there were only 10 lockers and 10 students. We could number 10 markers to represent the lockers and turn them upside down for open and right side up for closed.



Or we could form a table showing the state (open or closed) of each locker after each student passes through.

	Lockers									
	1	2	3	4	5	6	7	8	9	10
Student 1	O	O	O	O	O	O	O	O	O	O
Student 2		C		C		C		C		C
Student 3										

Which of the 10 lockers would be left open after 10 students passed through?

c. Carrying Out the Plan. Solving the problem for small numbers of students and lockers will help you to see a relationship between the number of each locker and whether it is left open or closed. What types of numbers will be on the lockers that are left open?

d. Looking Back. How many times will a locker be changed if it is numbered with a prime number?

39. The sum of the even numbers between 31 and 501 is how much less than the sum of the odd numbers between 32 and 502?

40. If a collection of pencils is placed in rows of 4, there are 2 pencils left; if placed in rows of 5, there are 3 left; and if placed in rows of 7, there are 5 left. What is the smallest possible number of pencils in the collection?

41. From the numbers 3, 5, 6, 7, 10, 11, 12, and 13, select five numbers that produce 15,015 when multiplied together.

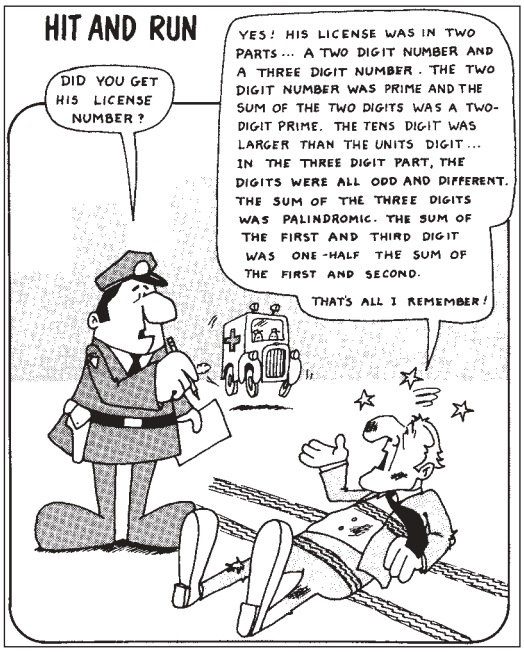
42. Joan's age was a factor of her grandfather's age for 6 consecutive years. What were her grandfather's ages during this time?

43. There are long sequences of consecutive whole numbers that include no primes. For example, the following five consecutive numbers are not prime. Can you see why without computing the products?

$$2 \times 3 \times 4 \times 5 \times 6 + 2$$
$$2 \times 3 \times 4 \times 5 \times 6 + 3$$
$$2 \times 3 \times 4 \times 5 \times 6 + 4$$
$$2 \times 3 \times 4 \times 5 \times 6 + 5$$
$$2 \times 3 \times 4 \times 5 \times 6 + 6$$

a. Construct a sequence of 10 consecutive whole numbers with no primes.

b. Explain how to construct a sequence of 100 consecutive whole numbers with no primes.



44. Use the information supplied by the victim in the cartoon to determine the license plate number of the hit-and-run car that left the scene.

45. Determine the license plate number of the hit-and-run car in the cartoon if the last condition for the two-digit part of the plate number is replaced by "The units digit was prime and the tens digit was one less," and the last condition for the three-digit part of the plate-number is replaced by, "The sum of the first and second digits was twice the sum of the second and third digits."

Teaching Questions

1. Two students were in disagreement as to whether or not 1 was a prime number. One student said 1 was a prime because like all the other prime numbers, 1 has only itself and 1 as a factor. The other student said 1 was not a prime according to her textbook. How would you resolve this problem?

2. Two students were discussing whether or not the sum of any three consecutive whole numbers was divisible by 3. One student said this was true because it worked for every example she had tried. The second student said it might not always be true because you can't prove something with examples. Write an explanation that begins with examples and then leads the students to understand that the sum of any three consecutive whole numbers is always divisible by 3.

3. When Ms. Lopez asked her class to use their tiles and make rectangular arrays for the counting numbers 1 to 12, Robert made only one rectangle for 4 (1 by 4) and one for 9 (1 by 9). He did not form a 2-by-2 square and a 3-by-3 square for these numbers because he said they were not rectangles. What would you say to Robert?

4. Sarah, a fifth-grade student, discovered that to test a number for divisibility by 8, multiply the digits to the left of the units digit by 2 and add the result to the units digit. If the resulting number is divisible by 8 then the original number is divisible by 8. For example, for 336, the test is $2 \times 33 + 6 = 72$, so 336 is divisible by 8 because 72 is divisible by 8. Use diagrams of base-ten number pieces to provide an explanation that would make sense to a middle school student as to why this method works.

Classroom Connections

1. In the **Grades 3–5 Standards—Number and Operations** (see inside front cover) under *Understand Numbers . . .*, read the expectation that involves describing "classes of numbers" and describe some classes that are determined by factors.

2. Arrays formed with tiles or graph paper are illustrated in this section on page 224 as well as in the **Spotlight on Teaching** at the beginning of this chapter. On page 227 the example from the **Elementary School Text** shows arrays that are formed on a geoboard. Discuss both the advantages and disadvantages of using geoboards as opposed to using tiles or graph paper to create arrays to introduce prime and composite number concepts.

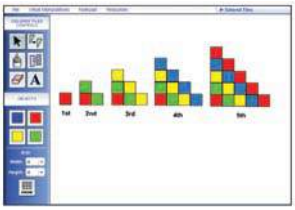
3. The one-page **Math Activity** at the beginning of this section uses base-five pieces to show why a number in base five is divisible by 4 if the sum of its digits is divisible by 4. Is a similar property true for divisibility by 5 in base six? Show sketches of base-six pieces to illustrate your answer.

4. The **Standards** quote on page 225 notes that students should recognize that different types of numbers have particular characteristics and it cites two examples. What characteristic involving factors does the number 1 have that sets it apart from all the other whole numbers?

5. Solve the two problems in the **Standards** statement on page 222. In your explanation of the first problem use base-ten sketches in your solution. (The second problem is related to the number 1001.)

MATH ACTIVITY 4.2

Virtual Manipulatives



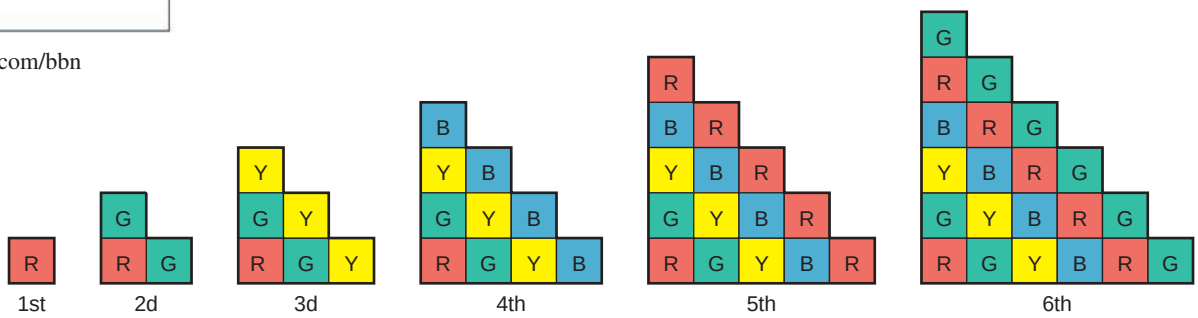
www.mhhe.com/bbn

Factors and Multiples from Tile Patterns

Purpose: Identify and extend factor and multiple patterns in color tile sequences.

Materials: Color Tiles in the Manipulative Kit or Virtual Manipulatives.

***1.** Find a pattern in this sequence, and use your tiles to build the seventh and eighth figures.

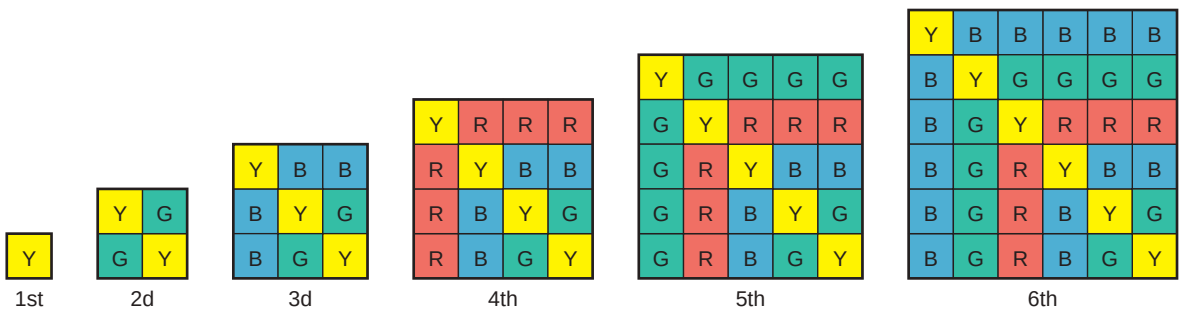


a. The diagonal edge of the fifth figure has red tiles, and the diagonal edge of the sixth figure has green tiles. Determine the color of the diagonal edge of the 20th figure and the 35th figure.

b. Describe a method for determining the color of the diagonal edge for any given figure in the sequence.

c. How many of each color of tile are required to build the 20th figure? The 35th figure? Describe your method.

2. Use your tiles to build the next figure in the following sequence.



a. Look closely at the sequence, and describe at least three patterns within the figures as the squares become larger.

b. How many of each color of tile will be needed to build the 10th figure? The 13th figure?

Section **4.2**

GREATEST COMMON FACTOR AND LEAST COMMON MULTIPLE



Adaptation of U.S. Census Clock

PROBLEM OPENER

One thousand raffle tickets for a school function are numbered with whole numbers from 1 to 1000. Each winning ticket has a number satisfying the following conditions: The number is even; there is one 7 in the numeral; the sum of the tens and units digits is divisible by 5; and the hundreds digit is greater than the units digit, which is greater than the tens digit. What are the numbers on the three winning tickets?

The census clock pictured here is an updated version of one that was once located in the lobby of the U.S. Department of Commerce Building. Four illustrated clock faces displayed the components of population change: births, deaths, immigrants, and emigrants. Plus and minus signs above the clock faces lighted to indicate when one of these components changed. The updated clock shows the projected U.S. population for 2010 and that there is a birth every 7 seconds, a death every 11 seconds, an arrival of an immigrant every 34 seconds, and the departure of an emigrant every 130 seconds.

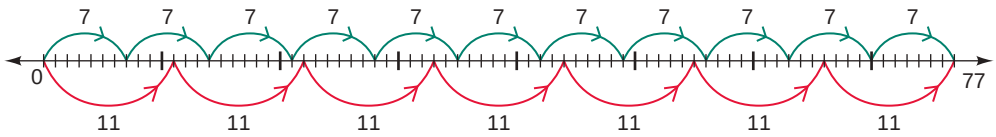
The times when these lights flashed together can be determined by using multiples of the different time periods of the various clocks.

EXAMPLE A

If the plus sign for the birth component of the census clock lights every 7 seconds, and the minus sign for the death component lights every 11 seconds, and both are seen to flash at the same moment, when will they flash together again?

Solution Using a *linear model* for multiples, we can draw a time line showing the flashing intervals for each component. The following line shows that 77 is a common multiple of 7 and 11 and that the lights for the birth and death components will flash again after 77 seconds.

Birth clock intervals of 7 seconds



Death clock intervals of 11 seconds

The purpose of this section is to develop the mathematical theory and skills to solve problems involving *common factors* and *common multiples* of numbers. Prime factorization is one approach used to solve such problems.

PRIME FACTORIZATIONS

Composite numbers can always be written as a product of primes. Such a product is called the **prime factorization** of a number. For example, the prime factorization of 12 is $2 \times 2 \times 3$. Since every whole number greater than 1 is either prime or a product of primes, prime numbers are often referred to as the *building blocks of the whole numbers.*

EXAMPLE B Find the prime factorization of each number.

1. 30 **2.** 40 **3.** 75

Solution **1.** $2 \times 3 \times 5$ **2.** $2 \times 2 \times 2 \times 5 = 2^3 \times 5$ **3.** $3 \times 5 \times 5 = 3 \times 5^2$

Notice that when a prime factor occurs more than once in the factorizations of numbers, such as in Example B, it is common to use exponents.

HISTORICAL HIGHLIGHT



Pierre de Fermat, 1601–1665

Pierre de Fermat has been called the greatest mathematician of the seventeenth century. His greatest achievement was establishing the foundations of number theory. At one time he wrote, "I have found a great number of exceedingly beautiful theorems." His famous *last theorem,* as it is called, states that there are no positive integers x, y, and z such that

$$x^n + y^n = z^n$$

for $n > 2$. Fermat wrote a brief note in the margin of a book saying that he had discovered a truly wonderful proof of this theorem, but the margin was too small to contain it. For more than 350 years, mathematicians searched for a proof of this theorem. Then in 1993, Andrew J. Wiles of Princeton University astonished the world of mathematics by announcing that he had proven Fermat's last theorem. An error was found, but a corrected proof was furnished by Wiles in 1994.*

*D. M. Burton, *The History of Mathematics,* 7th ed. (New York: McGraw-Hill, 2010), pp. 511–516.

The primes in the prime factorization of a composite number are unique except for their order. For example, any prime factorization of 30 will contain the factors 2, 3, and 5, although possibly not in that order. The uniqueness of prime factors for composite numbers is stated in the following important theorem.

Fundamental Theorem of Arithmetic Every composite whole number can be expressed as the product of primes in exactly one way except for the order of the factors in the product.

This theorem enables us to find the prime factorization of a number by first finding any two factors of the number and then continuing, if necessary, to find the factors of these numbers. Once we have only prime factors, the *Fundamental Theorem of Arithmetic* assures us that this is the only prime factorization.

Because of the Fundamental Theorem of Arithmetic, if two people find the prime factors of a number, we can be sure they both have found the same factors. Also, this theorem enables us to use various methods for finding the prime factors of a number because once they are found, these are the only prime factors. Let's look at two methods for finding prime factors.

Increasing Primes In this approach we begin by dividing a number by increasing primes, 2, 3, 5, 7, etc. Consider this approach for finding the prime factors of 300. Since 300 is even, we can divide by 2: $300 = 2 \times 150$. Then, since 150 is even, divide by 2 again: $300 = 2 \times 2 \times 75$. When it is no longer possible to divide by 2, try dividing by larger primes in order: 3, 5, 7, etc. In this case 3 divides 75, that is, $3 \times 25 = 75$, so $300 = 2 \times 2 \times 3 \times 25$, and from this we can see that the prime factorization of 300 is $2 \times 2 \times 3 \times 5 \times 5$.

This method of successively dividing by increasing primes can be carried out by the following algorithm:

$$\begin{array}{r|l} 2 & 300 \\ 2 & 150 \\ 3 & 75 \\ 5 & 25 \\ & 5 \end{array}$$

The *increasing primes* method also can be carried out by using a calculator to divide by 2, 3, 5, etc., and recording the list of primes that are factors. If you try a prime that is not a factor, you will obtain a quotient that is not a whole number.

Two Factors Another approach to finding the prime factors of a number is to begin with *any two factors,* not necessarily primes. This method is convenient when it is easy to see two such factors. For example, to find the prime factors of 300, we can replace 300 by 10×30. Then each of these two factors can be replaced by its factors, and so on. Since $10 = 2 \times 5$ and $30 = 3 \times 10$, we know $300 = 2 \times 5 \times 3 \times 10$. Finally, 10 can be replaced by 2×5, so

$$300 = 2 \times 5 \times 3 \times 2 \times 5 = 2^2 \times 3 \times 5^2$$

EXAMPLE C Find the prime factorization of 924.

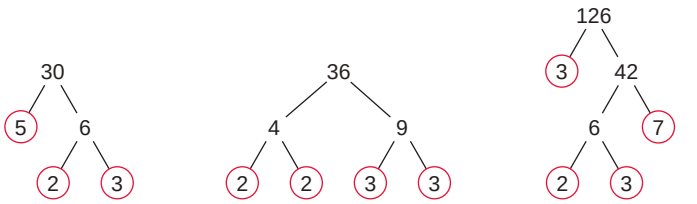
Solution Since two factors of 924 are not readily apparent, we find the prime factorization of 924 by using the algorithm of dividing by increasing primes. Notice in the third step of the following algorithm that 3 divides into 231 (see the test for divisibility by 3, pages 220–221).

$$\begin{array}{r|l} 2 & 924 \\ 2 & 462 \\ 3 & 231 \\ 7 & 77 \\ & 11 \end{array}$$

So, the prime factorization of 924 is $2 \times 2 \times 3 \times 7 \times 11$.

FACTOR TREES

Finding the prime factors of a number by first obtaining any two factors and then obtaining their factors is illustrated in Figure 4.12 with a diagram called a **factor tree.** Through a series of steps, a number is broken down into smaller and smaller factors until all the final factors are prime numbers. Three factor trees are shown, with their primes circled at the ends of the "branches."

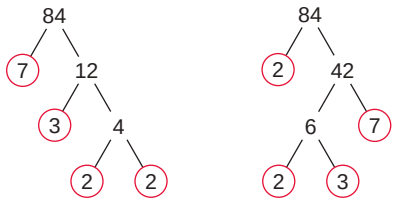


30 □ 5 □ 2 □ 3 36 □ 2 □ 2 □ 3 □ 3 126 □ 3 □ 2 □ 3 □ 7

Figure 4.12

EXAMPLE D Sketch a factor tree to find the prime factors of 84.

Solution The fundamental theorem of arithmetic guarantees that the prime factorization is unique, so a factor tree can be started with any two factors of a number. In the factor tree on the left, the first two factors are 7 and 12. In the factor tree on the right, the first two factors are 2 and 42.



$84 \square 7 \square 3 \square 2 \square 2$
$\square 2^2 \square 3 \square 7$

$84 \square 2 \square 2 \square 3 \square 7$
$\square 2^2 \square 3 \square 7$

Technology Connection

Factorizations

Do you know how to find the smallest number with a given number of factors? For example, the smallest number with exactly 6 factors? Use the *Mathematics Investigator* software to gather data and form a conjecture to predict the smallest number with a given number of factors.

Mathematics Investigation
Chapter 4, Section 2
www.mhhe.com/bbn

FACTORS OF NUMBERS

In Section 4.1 we classified numbers as *prime* or *composite* depending on their number of factors: prime numbers have only two factors, whereas composite numbers have three or more factors. We also saw that the factors of a number can be visualized by sketching rectangular arrays. The rectangles representing the factors of 24 are shown in Figure 4.13 on the next page. Notice that since 24 is not a square number, each rectangle produces a pair of factors. There are four pairs and a total of eight factors.

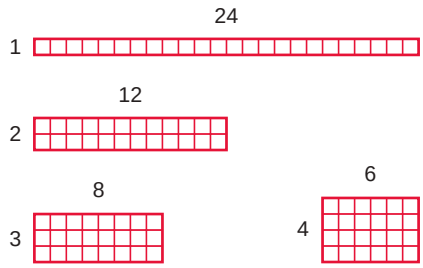


Figure 4.13

A list of *all* the factors of a number can be obtained by starting with 1 and then considering each whole number—2, 3, 4, etc.—in turn as a possible factor. For small numbers this can easily be done with mental calculations.

EXAMPLE E

List all the factors of each number.

1. 20 **2.** 34

Solution **1.** 1, 2, 4, 5, 10, 20 **2.** 1, 2, 17, 34

Notice that the solutions for Example E list *all* the factors of each number, not just the prime factors.

Another approach to finding the factors of a number is to first find its prime factorization and then use combinations of primes to find all the factors other than 1.

EXAMPLE F

The prime factorization of 273 is $3 \times 7 \times 13$. List all its factors.

Solution Since 1 is always a factor and each prime is a factor, we can begin the list with 1, 3, 7, and 13. Then we find the products of all pairs of primes: $3 \times 7 = 21$, $3 \times 13 = 39$, and $7 \times 13 = 91$. Finally, the product of all three primes $3 \times 7 \times 13 = 273$ is also a factor. So the factors of 273 are 1, 3, 7, 13, 21, 39, 91, and 273.

The fact that prime numbers can be used to obtain all the factors of a number is another reason why the prime numbers are called the building blocks of the whole numbers.*

PROBLEM-SOLVING APPLICATION

The following problem can be solved by *guessing and checking* or by finding the *prime factorization* of a number. Try to solve this problem before you read the solution.

Problem

The product of the ages of a group of teenagers is 10,584,000. Find the number of teenagers in the group and their ages.†

*The online 4.2 Mathematics Investigation, Factorizations (see website), tests any whole number less than 10 million to determine if it is prime or composite. If the number is composite, the computer prints the prime factorization and lists all its factors.

†"Problems of the Month," *The Mathematics Teacher* 82, no. 3: 189.

Understanding the Problem As a first step, it is helpful to list all the possible ages. **Question 1:** What are these ages?

Devising a Plan One approach is to *guess and check.* We might try dividing 10,584,000 by 13, then 14, etc. Another approach is to find the prime factorization of 10,584,000, since the prime factors can be used to build other factors. **Question 2:** What is the prime factorization of this number?

Carrying Out the Plan Expressed as a product of prime factors,

$$10{,}584{,}000 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 5 \times 5 \times 5 \times 7 \times 7$$

The ages of the teenagers can now be obtained by using combinations of these factors. Each 7 must be multiplied by 2, which means that there are two 14-year-olds, and each 5 must be multiplied by 3, which results in three 15-year-olds. **Question 3:** Using the remaining 2s, what is the age of the sixth teenager?

Looking Back The original problem can be varied by changing the number of teenagers to obtain a new product of ages. The prime factors suggest ways of obtaining new products. For example, we can cross out one 7 and one 2 or two 5s and two 3s. **Question 4:** Why can we not change the problem by crossing out only one 3 (or one 2)?

Answers to Questions 1–4 **1.** 13, 14, 15, 16, 17, 18, 19 **2.** The prime factorization is

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 5 \times 5 \times 5 \times 7 \times 7$$

3. 16 **4.** Crossing out only one 3 would leave a factor of 5 with which no number could be paired: 5×2 and 5×7 are not in the teens. Similarly, crossing out one 2 would leave a factor of 7 with which no number could be paired (7×3 and 7×5 are not in the teens), or it would leave three 2s and $2 \times 2 \times 2$ is not in the teens.

HISTORICAL HIGHLIGHT



Sophie Germain, 1776–1831

Sophie Germain (1776–1831), a Frenchwoman who won distinction in mathematics in the 1800s, has been called one of the founders of mathematical physics. She was self-educated in mathematics and physics from reading books in her parents' library. In 1801, when Gauss published *Disquisitiones arithmeticae,* a masterpiece on the theory of numbers, Germain sent him some of the results of her own mathematical investigations. Gauss was impressed with this work, and the two entered into an extensive correspondence. In 1816, her *Memoir on the Vibrations of Elastic Plates* earned her the prize offered by the French Academy of Sciences for the best essay on the mathematical laws of elastic surfaces. Winning the grand prize of the Academy elevated Germain to the ranks of the most noted mathematicians of the world. Germain published several other works dealing with the theory of elasticity, but she is best known for her work in the theory of numbers. Here she demonstrated the impossibility of solving Fermat's last theorem (see page 236) if x, y, and z are not divisible by an odd number.*

*L. M. Osen, *Women in Mathematics* (Cambridge, MA: The MIT Press, 1974), pp. 83–93.

GREATEST COMMON FACTOR

For any two numbers, there is always a number that is a factor of both. For example, the numbers 24 and 36 both have 6 as a factor. When a number is a factor of two numbers, it is called a **common factor** or **common divisor.**

EXAMPLE G

1. List all the factors of 24.
2. List all the factors of 36.
3. What are the common factors of 24 and 36?

Solution **1.** The factors of 24 are 1, 2, 3, 4, 6, 8, 12, and 24. **2.** The factors of 36 are 1, 2, 3, 4, 6, 9, 12, 18, and 36. **3.** The common factors of 24 and 36 are 1, 2, 3, 4, 6, and 12.

Among the common factors of two numbers there will always be a largest number, which is called the **greatest common factor.** The greatest common factor of 24 and 36 is 12. This is sometimes written GCF(24, 36) = 12.

Greatest Common Factor For any two nonzero whole numbers ***a*** and ***b*,** the **greatest common factor,** written GCF(***a*, *b***), is the greatest factor (divisor) of both ***a*** and ***b*.**

The concept of common factor of two numbers can be illustrated by representing each number by the same type of rod. For example, in Figure 4.14, both 20 and 30 can be represented by rods of length 2, and also by rods of length 5 and by rods of length 10. In this model, the GCF of the two numbers is the length of the longest rod that can be used to represent both numbers. Since the longest rod that can be used to represent both numbers is the 10-rod, the GCF of 20 and 30 is 10.

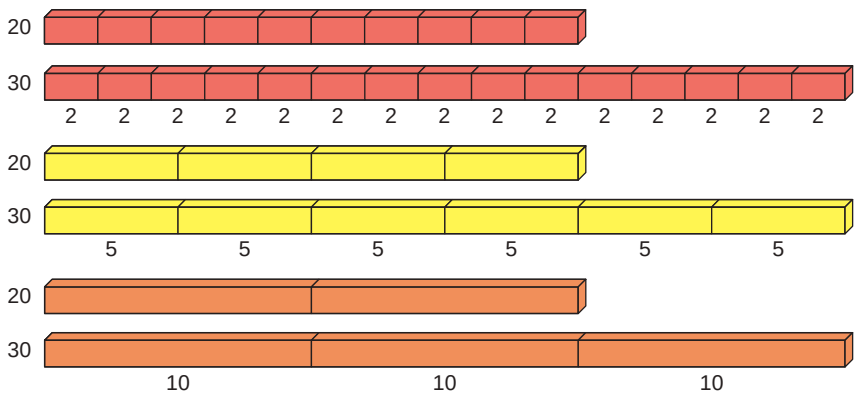
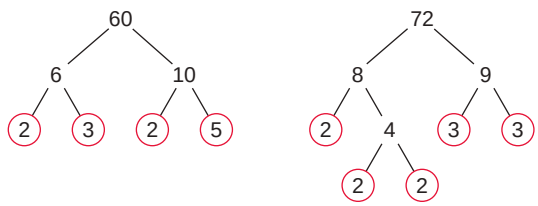


Figure 4.14

One method of finding the greatest common factor of two numbers is to list all the factors of both numbers, as was done in Example G for 24 and 36, and select the greatest one. A more convenient approach, especially for larger numbers, is to use prime factorizations. The GCF of two or more numbers can be built by using each prime factor the *minimum number of times* it occurs in each of the numbers.

EXAMPLE H

The following factor trees for 60 and 72 show the prime factors for both numbers.



Use these prime factors to obtain GCF(60, 72).

Solution We can see that 2 occurs as a prime factor two times for 60 and three times for 72. So the GCF of 60 and 72 will have the factor 2 occurring twice. Also, 3 occurs as a prime factor once in 60 and twice in 72. So the GCF of 60 and 72 will have the factor 3 occurring once. Since these are the only prime factors that are common to both 60 and 72, we can use these factors to build the GCF of 60 and 72, which is $2 \times 2 \times 3 = 12$.

Once you have found the prime factors of two numbers, whether by *factor trees* or some other method, the following method of listing the prime factors can be used to determine their GCF. In this method, the prime factors that are common to both numbers are listed beneath each other and the GCF is formed by the product containing one of each of the common factors. For example,

$$\begin{aligned} 60 &= 2 \times 2 \times 3 \times 5 \\ 72 &= 2 \times 2 \times 2 \times 3 \times 3 \\ \hline \text{GCF}(60, 72) &= 2 \times 2 \times 3 \end{aligned}$$

Notice how the primes that are common to both 60 and 72 are lined up. The GCF of 60 and 72 is the product determined from the columns of these lined-up factors: $2 \times 2 \times 3 = 12$.

EXAMPLE 1

Find the greatest common factors.

1. GCF(180, 220) **2.** GCF(92, 136) **3.** GCF(14, 34, 60)

Solution **1.** $180 = 2 \times 2 \times 3 \times 3 \times 5$ and $220 = 2 \times 2 \times 5 \times 11$. So GCF(180, 220) $= 2 \times 2 \times 5 = 20$, since 2 occurs as a factor twice in both 180 and 220 and 5 occurs as a factor once in both numbers. **2.** $92 = 2 \times 2 \times 23$ and $136 = 2 \times 2 \times 2 \times 17$. So GCF(92, 136) $= 2 \times 2 = 4$. **3.** $14 = 2 \times 7$ and $34 = 2 \times 17$ and $60 = 2 \times 2 \times 3 \times 5$. So GCF(14, 34, 60) $= 2$, since 2 is the only factor common to all three numbers.



Technology Connection

Calculators that display fractions and have a key for simplifying fractions can be used to determine the GCF of two numbers.* Let's see how this can be done by finding the GCF of 525 and 546. The keystrokes below show this process on a calculator that has the simplification key [SIMP]. In the first step, 525 is entered as the numerator of a fraction and 546 as the denominator. When [SIMP] is pressed the first time, the common factor 3 of 525 and 546 flashes briefly on the view screen and the fraction $\frac{525}{546}$ is replaced by $\frac{175}{182}$. When [SIMP] is pressed the second time, the common factor 7 of 175 and 182 flashes on the screen and the fraction $\frac{175}{182}$ is replaced by $\frac{25}{26}$. Further pressing of [SIMP] does not change $\frac{25}{26}$ because there are no common factors greater than 1 of 25 and 26. Since 3 and 7 are the only common factors greater than 1 for 525 and 546, their product, 21, is the GCF of 525 and 546.

Keystrokes	View Screens
525 [b/c] 546	$\frac{525}{546}$
[SIMP]	$3 \quad \frac{175}{182}$
[SIMP]	$7 \quad \frac{25}{26}$

If two whole numbers are entered into a calculator as the numerator and denominator of a fraction and the simplification key does not result in a fraction with smaller numbers, then the numbers have no common factor greater than 1. Two numbers whose GCF is 1 are **relatively prime.**

*Two fraction calculators with simplification keys are the CASIO fx-55 and the Texas Instruments TI-15.

EXAMPLE J

Use a fraction calculator or a factor tree to determine the GCF of each pair of numbers. Which pairs are relatively prime?

1. 245, 315 **2.** 550, 189 **3.** 232, 186 **4.** 156, 198

Solution **1.** GCF(245, 315) = 35. **2.** GCF(550, 189) = 1, so 550 and 189 are relatively prime. **3.** GCF(232, 186) = 2. **4.** GCF(156, 198) = 6.

Technology Connection

Fraction calculators such as those described on the previous page can be helpful in determining whether a number is prime. For example, rather than making separate checks to see if the primes 2, 3, 5, 7, or 11 are factors of 463, we can try simplifying the fraction $\frac{463}{2310}$, where 2310 is the product of 2, 3, 5, 7, and 11. The following keystrokes show that 463 and 2310 do not have a common factor greater than 1 because the SIMP key does not replace the fraction by a fraction having smaller numbers. In particular, this shows that 2, 3, 5, 7, and 11 are not factors of 463.

463 b/c 2310 ▯ $\frac{463}{2310}$ SIMP $\frac{463}{2310}$

Similarly, since the product of the next three primes (13, 17, and 19) is 4199, the next step is to see if $\frac{463}{4199}$ can be replaced by a fraction having smaller numbers. The following keystrokes show that 463 and 4199 have no common factors other than 1.

463 b/c 4199 ▯ $\frac{463}{4199}$ SIMP $\frac{463}{4199}$

Since 23×23 is greater than 463 and there are no primes less than 23 that are factors of 463, we know that 463 is a prime.

LEAST COMMON MULTIPLE

Every number has an infinite number of multiples. Here are the first few multiples of 5.

5 10 15 20 25 30 35 40 45 50 55 60 65 70 75

EXAMPLE K

Write the first few multiples of 7.

Solution 7 14 21 28 35 42 49 56 63 70 77 84 91 98 105

A number is called a **common multiple** of two numbers if it is a multiple of both. Notice that 35 and 70 occur in both of the preceding lists, so they are common multiples of 5 and 7. The first few multiples of 5 and 7 are shown on the number line in Figure 4.15. Beginning at zero, intervals of 5 units and intervals of 7 units do not coincide again until point 35 on the number line.

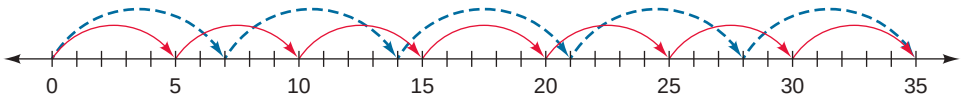


Figure 4.15

The next few common multiples of 5 and 7 are 70, 105, 140, 175, and 210. Every pair of nonzero whole numbers has an infinite number of common multiples. Among these common multiples will always be a smallest number, which is called the **least common**

multiple. The least common multiple of 5 and 7 is 35. This is sometimes written LCM(5, 7) = 35.

Least Common Multiple For any two nonzero whole numbers ***a*** and ***b*,** the **least common multiple,** written LCM(***a*, *b***), is the smallest multiple of both ***a*** and ***b*.**

Figure 4.16 uses rods to illustrate the concept of least common multiple. Notice that 5 rods of length 4 are required to equal 4 rods of length 5, and that 20 is the smallest length that can be formed by rods of both sizes. The fact that the vertical lines indicating the end of each rod line up only at the left end and the right end of Figure 4.16 shows that the least common multiple of 4 and 5 is 4 × 5 = 20.

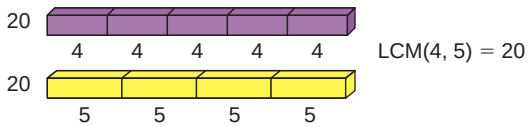


Figure 4.16

EXAMPLE L

Sketch rods for the following pairs of numbers to illustrate their least common multiple.

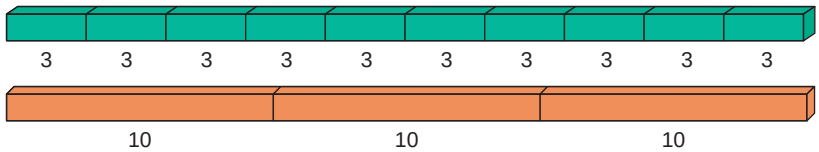
1. 3, 10 **2.** 4, 14 **3.** 6, 18

Research Statement

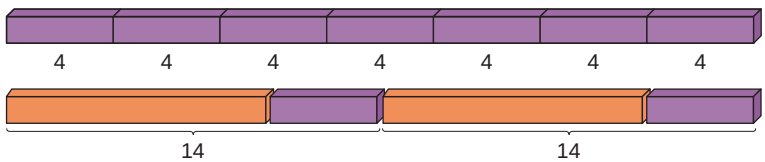
Because students often confuse factors and multiples, the greatest common factor and the least common multiple are difficult topics for students to grasp.

Graviss and Greaver

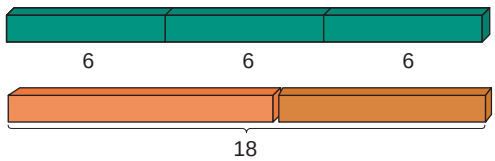
Solution **1.** LCM(3, 10) = 30



2. LCM(4, 14) = 28

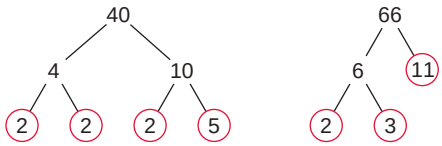


3. LCM(6, 18) = 18



One method of finding the least common multiple of two numbers is to first list some multiples of both numbers. For example, we found the least common multiple of 5 and 7 by listing the first few multiples of 5 and the first few multiples of 7. Another approach, which is more convenient for large numbers, is to use prime factorizations. The LCM of two or more numbers can be built from their prime factors by using each prime factor *the maximum number of times* it occurs in each of the numbers.

EXAMPLE M Factor trees for 40 and 66 are shown below. Use the prime factors to find LCM(40, 66).



Solution The factor trees show that 2 occurs three times as a factor of 40 and only once as a factor of 66, so 2 will have to occur three times as a factor in the LCM. Similarly, 5 is a factor of 40, and 3 and 11 are factors of 66, so these numbers will need to be included in the prime factors of the LCM. Therefore, the LCM of 40 and 66 is

$$2 \times 2 \times 2 \times 3 \times 5 \times 11 = 2^3 \times 3 \times 5 \times 11 = 1320$$

Once you have found the prime factors of two numbers, the method of listing the prime factors with the common factors lined up beneath each other, as was shown on page 242 for finding the GCF, can also be used to determine the LCM of two numbers. This is illustrated here for finding the GCF and LCM of 198 and 210. Notice that the common factors for both numbers are placed in columns under each other in both schemes. To find the GCF we form a product by listing one of each of the common factors, and to find the LCM we form the product by listing one of each of the common factors and each of the other factors. That is, the GCF has just the numbers with two or more in a column, and the LCM has only one of the numbers in each column.

$$\begin{array}{r} 198 = 2 \times 3 \times 3 \quad \times 11 \\ 210 = 2 \times 3 \quad \times 5 \times 7 \\ \hline \text{GCF}(198, 210) = 2 \times 3 \end{array} \qquad \begin{array}{r} 198 = 2 \times 3 \times 3 \quad \times 11 \\ 210 = 2 \times 3 \quad \times 5 \times 7 \\ \hline \text{LCM}(198, 210) = 2 \times 3 \times 3 \times 5 \times 7 \times 11 \end{array}$$

EXAMPLE N Find the least common multiples.

1. LCM(28, 44) **2.** LCM(21, 40) **3.** LCM(15, 36, 55)

Solution **1.** $28 = 2 \times 2 \times 7$ and $44 = 2 \times 2 \times 11$, so $\text{LCM}(28, 44) = 2 \times 2 \times 7 \times 11 = 308$. **2.** $21 = 3 \times 7$ and $40 = 2 \times 2 \times 2 \times 5$, so $\text{LCM}(21, 40) = 2 \times 2 \times 2 \times 3 \times 5 \times 7 = 840$. **3.** $15 = 3 \times 5$, $36 = 2 \times 2 \times 3 \times 3$, and $55 = 5 \times 11$, so $\text{LCM}(15, 36, 55) = 2 \times 2 \times 3 \times 3 \times 5 \times 11 = 1980$.

You may have noticed some special relationships in Example N for the LCM of two numbers. In part 1 of the example, the LCM of 28 and 44 could have been obtained by using all the factors in 28 and 44 and then dividing by the common factors of both numbers. That is,

$$\text{LCM}(28, 44) = \frac{(2 \times 2 \times 7) \times (2 \times 2 \times 11)}{2 \times 2} = \frac{28 \times 44}{\text{GCF}(28, 44)}$$

Similarly, you may have noticed in part 2 of Example N that 21 and 40 have no common prime factors—that is, they are relatively prime—so their LCM is the product of the prime factors from both numbers.

$$\text{LCM}(21, 40) = (3 \times 7) \times (2 \times 2 \times 2 \times 5) = 21 \times 40$$

These relationships are stated as a theorem for the LCM of two numbers.

For positive integers a and b,

$$\mathbf{LCM}(a, b) = \frac{a \times b}{\mathbf{GCF}(a, b)}$$

and when $\mathrm{GCF}(a, b) = 1$,

$$\mathbf{LCM}(a, b) = a \times b$$

EXAMPLE O

Determine the least common multiples.

1. LCM(17, 20) **2.** LCM(20, 33) **3.** LCM(138, 84)

Solution **1.** $\mathrm{GCF}(17, 20) = 1$, so $\mathrm{LCM}(17, 20) = 17 \times 20 = 340$. **2.** $\mathrm{GCF}(20, 33) = 1$, so $\mathrm{LCM}(20, 33) = 20 \times 33 = 660$. **3.** $\mathrm{GCF}(138, 84) = 6$, so $\mathrm{LCM}(138, 84) = \frac{(138 \times 84)}{6} = 1932$.

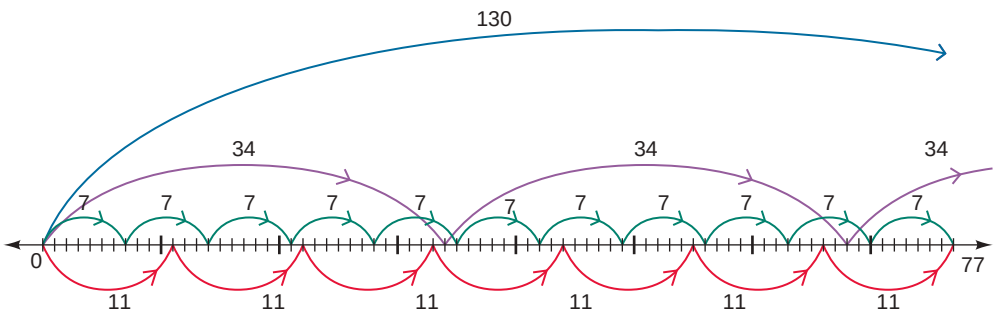
PROBLEM-SOLVING APPLICATION

Problem

The census clock described at the beginning of this section used flashing lights to indicate birth, death, immigration, and emigration rates. If all four lights flashed together at the same moment, how much time would pass before they would all flash together again?

Understanding the Problem The birth, death, immigration, and emigration lights flash every 7, 11, 34 and 130 seconds, respectively. It will take at least 130 seconds (2 minutes and 10 seconds) for all four lights to flash together again, because the emigration light flashes only every 130 seconds. **Question 1:** Which of the other lights will flash each 130 seconds?

Devising a Plan To obtain an idea for a plan, we can look at a *simpler problem.* At the beginning of this section in Example A, a sketch was used to show multiples of 7 and 11 on a number line and to determine that the birth and death lights flash together every 77 seconds. This suggests visualizing a number line with multiples of 7, 11, 34, and 130.



To find the first point beyond zero at which the multiples of 7, 11, 34, and 130 coincide, we need to obtain the LCM of these numbers. This requires finding their prime factorization. **Question 2:** What are the prime factorizations of 7, 11, 34, and 130?

Carrying Out the Plan The LCM of 7, 11, 34, and 130 can be built from the prime factors of these numbers.

$$7 = 7 \qquad 11 = 11 \qquad 34 = 2 \times 17 \qquad 130 = 2 \times 5 \times 13$$

The LCM must contain 2 as a factor once, 5 as a factor once, 7 as a factor once, 11 as a factor once, 13 as a factor once, and 17 as a factor once. So the LCM of 7, 11, 34, and 130 is

$$2 \times 5 \times 7 \times 11 \times 13 \times 17 = 170{,}170$$

Thus, every 170,170 seconds all four lights will flash together. **Question 3:** How long is this in hours and minutes?

Looking Back The number of people born in 170,170 seconds will be $170{,}170 \div 7 = 24{,}310$. Similarly, in 170,170 seconds the number of people who will die is $170{,}170 \div 11 = 15{,}470$, the number immigrating will be $170{,}170 \div 34 = 5005$, and the number emigrating will be $170{,}170 \div 130 = 1{,}309$. **Question 4:** What will be the total gain in population for each 170,170 seconds (1 day, 23 hours, 16 minutes, and 10 seconds)?

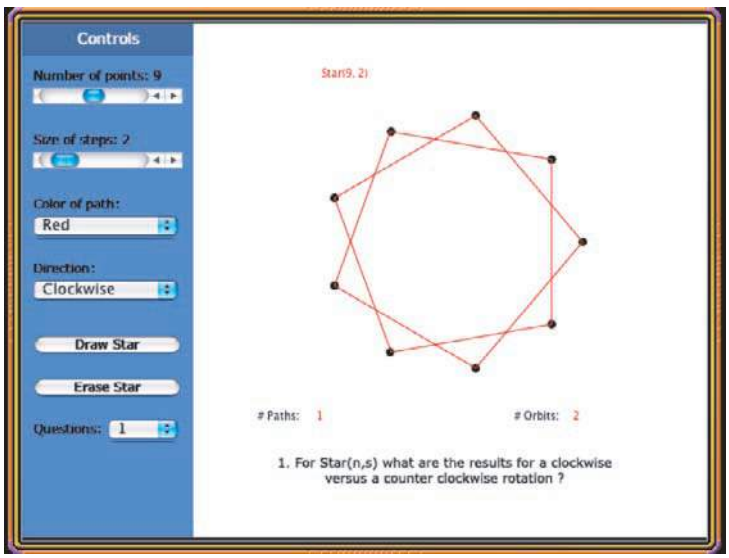
Answers to Questions 1–4 **1.** None, since neither 7, 11, nor 34 are factors of 130. **2.** The prime factorizations are

$$7 = 7 \qquad 11 = 11 \qquad 34 = 2 \times 17 \qquad 130 = 2 \times 5 \times 13$$

3. 170,170 seconds is equal to 1 day, 23 hours, 16 minutes, and 10 seconds (170,170 seconds = 2836 minutes and 10 seconds and 2836 minutes = 47 hours and 16 minutes). **4.** $24{,}310 - 15{,}470 + 5{,}005 - 1{,}309 = 12{,}536$. That is, approximately every 2 days, the U.S. population increases by over 12,000 people.

Technology Connection

In this applet you will draw star polygons by choosing the number of points and size of the steps. Star(9, 2) was drawn by beginning at one of 9 points and taking steps of size 2 around the circle of points until returning to the original point. What is the relationship between the numbers of points on a circle and sizes of steps? This applet enables you to form more than one star polygon on a given set of points to create colorful designs.



Analyzing Star Polygons Applet, Chapter 4
www.mhhe.com/bbn

Exercises and Problems 4.2

Find the prime factorization of each number in exercises 1 and 2.

1. a. 126 **b.** 308 **c.** 245

2. a. 663 **b.** 442 **c.** 858

Sketch a factor tree to find the prime factors of each number in exercises 3 and 4.

3. a. 400 **b.** 315 **c.** 825

4. a. 112 **b.** 385 **c.** 390

5. It has been estimated that life began on earth 1,000,000,000 (one billion) years ago. How can the Fundamental Theorem of Arithmetic be used to show that 7 is not a factor of this number?

List all the factors of each number in exercises 6 and 7.

6. a. 60 **b.** 182 **c.** 180

7. a. 500 **b.** 231 **c.** 245

List all the common factors for each pair of numbers in exercises 8 and 9.

8. a. 23, 64 **b.** 112, 84 **c.** 62, 116

9. a. 30, 40 **b.** 15, 22 **c.** 14, 56

Find the greatest common factor in exercises 10 and 11.

10. a. GCF(65, 60)
b. GCF(8, 30)
c. GCF(118, 7, 24)

11. a. GCF(280, 168)
b. GCF(12, 15, 125)
c. GCF(198, 165)

List the first five common multiples for each pair of numbers in exercises 12 and 13.

12. a. 50, 35 **b.** 14, 42 **c.** 19, 10

13. a. 4, 14 **b.** 6, 8 **c.** 12, 17

Find the least common multiple in exercises 14 and 15.

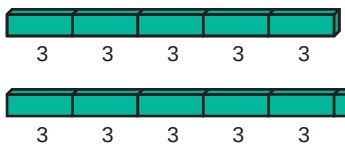
14. a. LCM(10, 40)
b. LCM(14, 15)
c. LCM(14, 5, 26)

15. a. LCM(22, 56)
b. LCM(6, 38, 16)
c. LCM(30, 42)

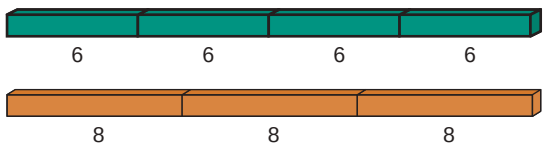
Cuisenaire Rods (see Exercises and Problems 4.1, exercises 9 and 10) are used in exercises 16 through 19 to illustrate common factors and common multiples.

Determine whether the rods in exercises 16 and 17 provide information about common factors or common multiples, and explain what information is illustrated by each diagram.

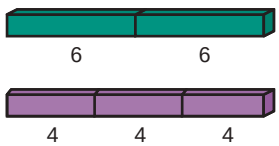
16. a.



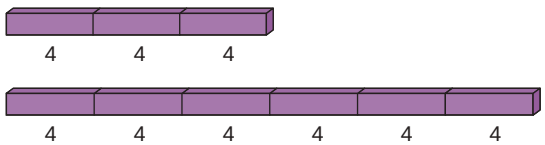
b.



17. a.



b.



18. a. If six yellow (5 units) rods are used to make a one-color train and eight yellow rods are used to make a second one-color train, what information about common factors or common multiples does this provide?

b. If the dark green rods (6 units) and the brown rods (8 units) are each used to form the shortest possible one-color train of matching length, how many dark green rods and how many brown rods will be required? What information about common factors or common multiples does this provide?

19. a. If the brown rods (8 units) and the orange rods (10 units) are each used to form the shortest possible one-color train of matching length, how many brown rods and how many orange rods will be required? What information about common factors or common multiples does this provide?

b. If five black (7 units) rods are used to make a one-color train and nine black rods are used to make a second one-color train, what information about common factors or common multiples does this provide?

For a U.S. Census clock based on the 2000 census, the birth light would flash every 10 seconds, the death light every 16 seconds, the immigration light every 81 seconds, and the emigration light every 900 seconds, to indicate gains and losses in population.

20. **a.** If the birth and death lights flashed at the same time, how many seconds would pass before they flashed together again?
b. During the time period calculated in part a, what is the gain in population due to births and deaths?
c. If the immigration light and the emigration light flashed together at the same time, how many hours and minutes would pass before they would flash together again?

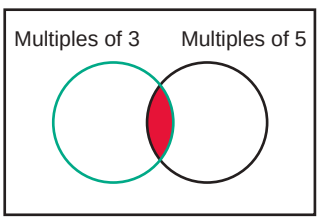
21. **a.** If the birth, death, and immigration lights flashed at the same time, how many seconds would pass before they flashed together again? Is this less than or greater than 1 hour?
b. For the time period in part a, what is the gain in population due to births, deaths, and immigration?
c. If all four lights flashed at the same time, what is the shortest time before they would all flash together again?

Reasoning and Problem Solving

22. **Featured Strategy: Drawing Venn Diagrams.** How many whole numbers from 1 to 300 are not multiples of 3 or 5?
a. **Understanding the Problem.** The number of multiples of 3 from 1 to 9 is $9 \div 3 = 3$.

1 2 ③ 4 5 ⑥ 7 8 ⑨

How many multiples of 3 are there from 1 to 300? How many multiples of 5 are there?
b. **Devising a Plan.** Since some of the multiples of 3 are also multiples of 5, we need to consider the intersection of the set of multiples of 3 and the set of multiples of 5. How many numbers less than or equal to 300 are multiples of both 3 and 5?
c. **Carrying Out the Plan.** The following Venn diagram helps us to visualize the information. Record the numbers of multiples in the appropriate regions of the diagram, and then determine how many numbers are not multiples of 3 or 5.



d. **Looking Back.** The Venn diagram helps to answer other questions. How many numbers less than or equal to 300 are multiples of 3 but not multiples of 5?

23. A school principal plans to form teams from 126 third-graders, 180 fourth-graders, and 198 fifth-graders so that there is the same number of students from each grade level on each team. If all students participate, what is the largest possible number of teams and how many students will there be on each team?

24. Shane has 72 inches of copper wire and 42 inches of steel wire.
a. What are the largest pieces he can cut these wires into so that each piece is the same length and all of the wire is used?
b. How many pieces of wire will he have?

25. A bakery has 300 chocolate chip cookies and 264 peanut butter cookies. The bakers wish to divide the chocolate chip cookies into piles and the peanut butter cookies into piles so that each pile has only one type of cookie, there is the same number of cookies in each pile, and each pile has the largest possible number of cookies.
a. How many cookies will there be in each pile?
b. How many piles of chocolate chip cookies will there be?
c. How many piles of peanut butter cookies will there be?

26. Janice and Bob both work night shifts. When they have the same night off, they go dancing together. If Janice has every fourth night off and Bob has every seventh night off, how often does Janice go dancing with Bob?

27. A clock shop has three cuckoo clocks on display. The cuckoos appear at different time intervals. One comes out every 10 minutes, one comes out every 15 minutes, and one comes out every 25 minutes. If they all appear at 5 o'clock, what is the next time they will all come out together?

28. For science day, a school is showing two films: one on volcanoes that is 24 minutes long and one on tornadoes that is 40 minutes long. If they both begin at 8 a.m. and run continuously until 3 p.m., at which times during the day will they both start at the same time?

29. Two sisters, Cindy and Nicole, bought a special 180-day health club membership. Cindy will use the club on every second day (days 2, 4, . . .), and Nicole will use the club on every third day (days 3, 6, . . .). If they go together on day 1, for how many of the 180 days will neither sister use the club?

30. A scientist receives signals from three quasars (distant sources of radio energy). The first quasar sends signals every 84 seconds, the second quasar every 42 seconds, and the third quasar every 30 seconds.

a. The scientist wishes to divide each of the three time intervals into the largest possible equal-size parts so that these parts are equal for all three time intervals. What is the size of these parts?

b. If all three signals are received at the same moment, what is the shortest length of time from that moment before two of the signals will again be received at the same time?

Determine the number of zeros at the right end of the numerals for the products in exercises 31 and 32.

31. $1 \times 2 \times 3 \times \ldots \times 98 \times 99 \times 100$

32. $50 \times 51 \times 52 \times \ldots \times 198 \times 199 \times 200$

The **proper factors** of a number are all its factors except the number itself. The Pythagoreans classified a number according to the sum of its proper factors. A number is **deficient** if the sum of its proper factors is less than the number, a number is **abundant** if the sum of its proper factors is greater than the number, and a number is **perfect** if the sum of its proper factors is equal to the number. Use this information in exercises 33 and 34.

33. **a.** Are there any perfect numbers less than 25? If so, what are they?

b. Among the whole numbers less than 25, are there more deficient numbers or abundant numbers?

34. **a.** Are there any perfect numbers between 25 and 50? If so, what are they?

b. Are there more deficient numbers or abundant numbers between 25 and 50?

Some calculators for middle school students have a SIMP key (see exercises 35 and 36), that replaces a fraction by an equal fraction having smaller numbers, if the numerator and denominator of the fraction have a common factor greater than 1. Find the missing fraction for each of the view screens. Find the GCF of the numerator and denominator of the original fraction entered into the calculator, and determine whether the pairs of numbers are relatively prime.

35. **a.**

Keystrokes	View Screen
99 b/c 105	$\frac{99}{105}$
SIMP	

b.

Keystrokes	View Screen
102 b/c 275	$\frac{102}{275}$
SIMP	

36. **a.**

Keystrokes	View Screen
98 b/c 429	$\frac{98}{429}$
SIMP	

b.

Keystrokes	View Screen
539 b/c 1260	$\frac{539}{1260}$
SIMP	

37. Selene entered the fraction $\frac{211}{30{,}030}$ into a fraction calculator and pressed the SIMP key (see exercises 35 and 36). Explain why she can conclude from the following view screens that 211 is a prime number.

Keystrokes	View Screen
211 b/c 30030	$\frac{211}{30030}$
SIMP	$\frac{211}{30030}$

38. Javier entered the fraction $\frac{143}{2310}$ into a fraction calculator and pressed the SIMP key (see exercises 35 and 36). Explain why he can conclude from the following view screens that 143 is *not* a prime number.

Keystrokes	View Screen
143 b/c 2310	$\frac{143}{2310}$
SIMP	$\frac{13}{210}$

Teaching Questions

1. Makenna asked her teacher why 1 was not a prime number because it had only one rectangle like 3, 5, 7, and 11. Give Makenna a good reason why 1 is not considered a prime number, besides saying it is not prime by definition.

2. Design an activity that uses tiles and the rectangular model for products to help school students discover that all square numbers have an odd number of factors and prime numbers have only two factors. Use diagrams to explain how your activity will help students to see why each of these statements is true.

3. As a teacher, some of your students have become confused over the concepts of greatest common factor (GCF) and least common multiple (LCM). One student said: "If the factors of a number are less than the multiples, why does the GCF ask for the greatest of the factors and the LCM ask for the least of the multiples?" Explain how you would answer this question.

4. A teacher used the following activity to convince one of his elementary school students that 1 is not a prime number. Using colored tiles he designated the following values for the first few primes: a red tile is worth 2; a yellow worth 3; blue worth 5; and green worth 7. He then asked the student to select a few tiles and calculate the product of their values. The student's product was 90 and the teacher, without seeing the tiles, was able to determine that the student had 1 red tile, 2 yellow tiles, and 1 blue. (a) How did the teacher know these were the tiles the student selected? (b) Explain by giving examples why it is not possible to determine the numbers and colors of tiles if 1 is considered to be a prime number and assigned a colored tile.

Classroom Connections

1. In the **Grades 6–8 Standards—Number and Operations** (see inside back cover) under *Understand numbers . . .* , read the sixth expectation and give an example of where each of the math concepts in this expectation are found in Section 4.2.

2. The **Research** statement on page 244 notes that students often confuse factors and multiples. One approach to helping students is to bring attention to the meanings of these words. Check the dictionary to see if the meaning and origin of "factor" and "multiple" might help students remember and understand their mathematical uses. Explain.

3. One very important reason for not classifying 1 as a prime number can be seen in the Fundamental Theorem of Arithmetic. Explain why there would be a problem with this theorem if 1 was considered to be a prime number.

4. The two **Historical Highlights** in this section feature Pierre de Fermat and Sophie Germain. In the 1600s Fermat wrote his famous margin note regarding what has become called "Fermat's Last Theorem," a theorem that remained unproven for over 350 years. Germain's work in the 1700s elevated her into the ranks of the world's most noted mathematicians. (a) Explain an important mathematical connection from the highlights between these two mathematicians. (b) This connection involves a proof by Germain. Give a few numerical examples of this proof.

CHAPTER 4 REVIEW

1. **Number theory relationships**
   a. **Number theory** is the study of whole numbers and their relationships.
   b. If one number is a **factor** of a second, then the second number is a **multiple** of the first.
   c. If a and b are whole numbers such that a is a factor of b, then a **divides** b, and we write $a|b$. If a does not divide b, we write $a \nmid b$.
   d. When a number is a factor of two numbers, it is called a **common factor** or **common divisor.**
   e. A number is called a **common multiple** of two numbers if it is a multiple of both numbers.

2. **Models for factors and multiples**
   a. The **linear model** uses number lines or rods to represent factors and multiples.
   b. In the **rectangular array** model, a number is represented by the number of squares or tiles in an array, and the dimensions of the array are factors of the number.

3. **Divisibility properties**
   a. For whole numbers a, b, and c, if $a|b$ and $a|c$, then $a|(b + c)$.
   b. For whole numbers a, b, and c, if $a|b$ and $a \nmid c$, then $a \nmid (b + c)$.
   c. For whole numbers a, b, and k, if $a|b$, then $a|bk$.

4. **Divisibility tests**
   a. A number is **divisible by 2** if and only if its units digit is divisible by 2.
   b. A number is **divisible by 3** if and only if the sum of its digits is divisible by 3.
   c. A number is **divisible by 4** if and only if the number represented by its tens and units digits is divisible by 4.
   d. A number is **divisible by 5** if and only if its units digit is divisible by 5.
   e. A number is **divisible by 6** if and only if it is divisible by 2 and 3.
   f. A number is **divisible by 9** if and only if the sum of its digits is divisible by 9.
   g. A number is **divisible by 11** if and only if the number obtained by alternately adding and subtracting its digits from right to left is divisible by 11.

5. **Prime and composite numbers**
   a. A nonzero whole number with exactly two factors is a **prime number.**
   b. A nonzero whole number with more than two factors is a **composite number.**
   c. The number 1 has only one factor and is neither prime nor composite.
   d. A number with an odd number of factors is a **square number.**
   e. The **prime number test** guarantees that for any whole number n and prime p such that $p^2 > n$, if there is no smaller prime that divides n, then n is a prime number.
   f. The **Sieve of Eratosthenes** is a systematic method of eliminating all the numbers less than a given number that are not prime.
   g. The product that expresses a number in terms of primes is called its **prime factorization.**
   h. The **Fundamental Theorem of Arithmetic** states that every composite whole number has one and only one prime factorization (if the order of the factors is disregarded).
   i. A **factor tree** is a diagram for finding the prime factors of a number.
   j. A list of all the factors of a number can be obtained by beginning with 1, listing the prime factors, and then multiplying combinations of the prime factors.

6. **GCF and LCM**
   a. The **greatest common factor** of two nonzero whole numbers a and b, written GCF(a, b), is the greatest factor the two numbers have in common.
   b. The **least common multiple** of two nonzero whole numbers a and b, written LCM(a, b), is the smallest multiple the two numbers have in common.
   c. If the GCF of two nonzero whole numbers is 1, the numbers are **relatively prime.**
   d. For positive integers a and b,

$$\mathbf{LCM}(a, b) = \frac{(a \times b)}{\mathbf{GCF}(a, b)}$$

   e. If two numbers are relatively prime, then their LCM is the product of the two numbers.
   f. Number lines and rods are common **linear models** for illustrating the concepts of GCF and LCM.

CHAPTER 4 TEST

1. Determine whether the following statements are true or false.
a. $3|48{,}025$ **b.** $2|3776$
c. $6 \nmid 7966$ **d.** $9|4576$

2. Rewrite each statement below in the form $a|b$.
a. 45 is divisible by 3. **b.** 12 is a factor of 60.
c. 20 divides 140. **d.** 102 is a multiple of 17.

3. Illustrate the fact that 78 is a multiple of 6, using:
a. A linear model
b. A rectangular array model

4. Rectangular arrays whose dimensions are whole numbers can be used to illustrate the factors of a number. What can be said about the number of rectangular arrays for each of the following types of numbers?
a. Prime number
b. Composite number
c. Square number

5. Determine whether the following statements are true or false.
a. If 3 divides the units digit of a number, then 3 divides the number.
b. If a number is divisible by 8, then it is divisible by 4.
c. If 2 divides the sum of the digits of a number, then 2 divides the number.
d. If a number is not divisible by 6, then it is not divisible by 3.

6. Which one of the following numbers is prime?
a. 331 **b.** 351 **c.** 371

7. Write the prime factorization of 1836.

8. List all the factors of 273.

9. Determine whether the following statements are true or false. If a statement is false, show a counterexample.
a. If $a|b$, then $a|(13 \times b)$.
b. If $a|(b + c)$, then $a|b$ or $a|c$.
c. If $a|b$ and $a \nmid c$, then $a \nmid (b + c)$.
d. If $a|bc$, then $a|b$.

10. List the following.
a. Four common factors of 30 and 40
b. Four common multiples of 15 and 20
c. Four common factors of 195 and 255
d. Five common multiples of 13 and 20

11. Find each GCF or LCM.
a. GCF(17, 30) **b.** LCM(14, 22)
c. LCM(12, 210) **d.** GCF(280, 165)
e. GCF(18, 28, 36) **f.** LCM(6, 15, 65)

12. Sketch linear models to illustrate:
a. The least common multiple of 3 and 8
b. The greatest common factor of 15 and 24

13. One lighthouse light flashes every 10 seconds, and a second lighthouse light flashes every 12 seconds. If they both flash at the same moment, how long will it be until they will flash together again?

14. How many whole numbers between 1 and 1000 are multiples of either 3 or 7?

15. Mike has 20 strips of wood molding that are each 70 inches long and 6 pieces that are each 28 inches long. He wants to cut all these strips so that each piece has the same length and no wood is left. What is the longest possible length that can be cut?

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CHAPTER

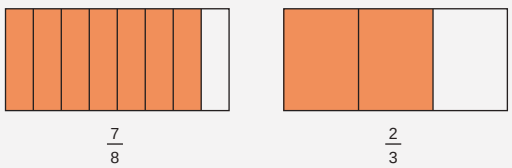
5 Integers and Fractions

Spotlight on Teaching

Excerpts from NCTM's Standards for School Mathematics Grades 6–8*

In the lower grades, students should have had experience in comparing fractions between 0 and 1 in relation to such benchmarks as 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and 1. In the middle grades, students should extend this experience to tasks in which they order or compare fractions, which many students find difficult. For example, fewer than one-third of the thirteen-year-old U.S. students tested in the National Assessment of Educational Progress (NAEP) in 1988 correctly chose the largest number from $\frac{3}{4}$, $\frac{9}{16}$, $\frac{5}{8}$, and $\frac{2}{3}$ (Kouba, Carpenter, and Swafford 1989). Students' difficulties with comparison of fractions have also been documented in more recent NAEP administrations (Kouba, Zawojewski, and Strutchens 1997). Visual images of fractions as fraction strips should help many students think flexibly in comparing fractions. As shown in Figure 6.2, a student might conclude that $\frac{7}{8}$ is greater than $\frac{2}{3}$ because each fraction is exactly "one piece" smaller than 1 and the missing $\frac{1}{8}$ piece is smaller than the missing $\frac{1}{3}$ piece. Students may also be helped by thinking about the relative locations of fractions and decimals on a number line.

Figure 6.2
A student's reasoning about the sizes of rational numbers



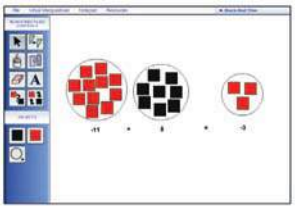
The $\frac{7}{8}$ portion is one piece less than a whole, and so is $\frac{2}{3}$. But the missing piece for $\frac{7}{8}$ is smaller than the missing piece for $\frac{2}{3}$. So $\frac{7}{8}$ is bigger than $\frac{2}{3}$.

**Principles and Standards for School Mathematics,* p. 216.

MATH ACTIVITY 5.1

Addition and Subtraction with Black and Red Tiles

Virtual Manipulatives

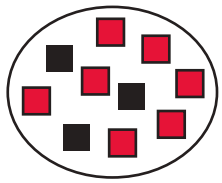


www.mhhe.com/bbn

Purpose: Use black and red tiles to model addition and subtraction of integers.

Materials: Black and Red Tiles in the Manipulative Kit or Virtual Manipulatives.

Black and red tiles are used to illustrate the integers . . . , $^-5$, $^-4$, $^-3$, $^-2$, $^-1$, 0, 1, 2, 3, 4, 5, ... Black tiles represent positive numbers, and red tiles negative numbers. Each pair of 1 black tile and 1 red tile represents 0, so each integer can be represented in many ways. The set shown at the right has 10 tiles, but the value of the set is $^-4$, since 3 black tiles can be matched with 3 red tiles.

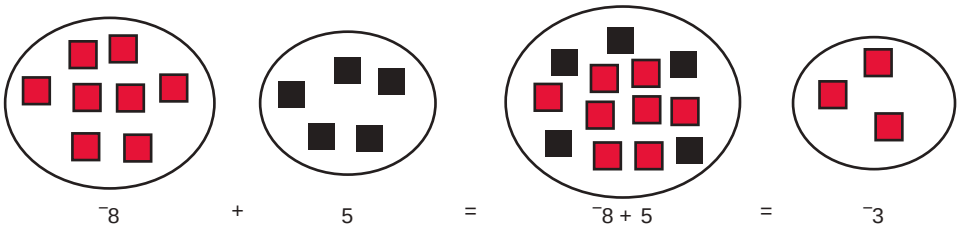


1. Select a small handful of black and red tiles, and drop them on the table. Record the numbers of black and red tiles and the value of the set.

a. Form a second set with a different number of black and red tiles that has the same value as the original set.

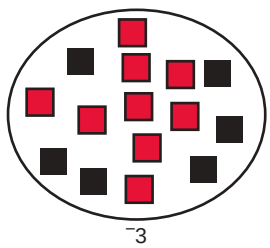
b. For each set you formed above, turn over each tile to its opposite side and determine the new value of the set. How is the value of the original set related to the value obtained by using the opposite sides of the tiles? Experiment with some other sets of tiles.

2. The sum of two integers can be computed by forming a set of tiles for each integer, combining the sets, and determining the value of the new set. The sum $^-8 + 5$ is illustrated here. Use your tiles to represent the following pairs of integers and to compute their sum. Show sketches both before and after you combine sets of tiles.



a. $^-7 + 5$ **b.** $12 + {}^-9$ **c.** $^-7 + {}^-6$ **d.** $^-8 + 8$

***3.** The difference of two integers can be computed by representing one of the integers by a set of tiles and using the *take-away* concept of subtraction. Use your tiles to form the set shown here, or some other set for $^-3$, and compute the following differences. Write subtraction equations for each difference.



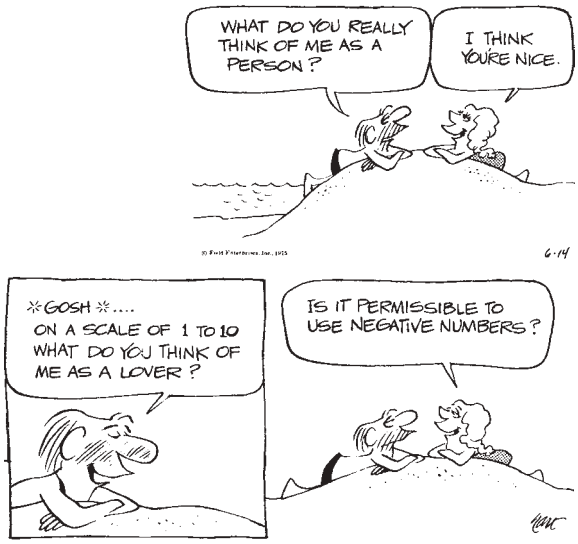
a. $^-3 - 4$ (remove 4 black tiles)

b. $^-3 - {}^-2$ (remove 2 red tiles)

c. $^-3 - {}^-6$ (remove 6 red tiles) **d.** $^-3 - 5$ (remove 5 black tiles)

4. Form the minimum collection of tiles to represent $^-3$. Explain how you can alter this set without changing its value, so that you can take away 2 black tiles to determine $^-3 - 2$. Use this example to show why taking away 2 black tiles is the same as adding 2 red tiles.

Section 5.1 INTEGERS



PROBLEM OPENER

Keeping the single-digit numbers from 1 to 9 in order,

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9$$

and inserting plus and/or minus signs, we can obtain a sum of 100 in several ways. For example,

$$1 + 2 + 3 - 4 + 5 + 6 + 78 + 9 = 100$$

Find another way.

The need for negative whole numbers ($^{-}1$, $^{-}2$, $^{-}3$, . . .) originated over 2000 years ago. As trading became more common, whole numbers were needed for two distinctly different uses: to indicate *credits* (or *gains*) and to indicate *debits* (or *losses*). Conventions were developed to permit the use of whole numbers in both cases. About 200 B.C.E. the Chinese were computing credits with red rods and debts with black rods (Figure 5.1). Similarly, in their writing they used red numerals and black numerals.*

Today it is customary to reverse the color scheme used by the Chinese. Banks often use red numerals to represent amounts below zero ("in the red" is negative). Black numerals



Rods for computing credits and debits

Figure 5.1

*D. E. Smith, *History of Mathematics,* 2d ed. (Lexington, MA: Ginn, 1925), pp. 257–258.

Research Statement

The premature use of the number line as a representation of positive integers can lead children to develop the incorrect notion that there are no numbers between the marked points.

Dufour-Janvier, Bednarz, and Belanger

are used to represent accounts above zero ("in the black" is positive). This is the convention that is used in this text.

POSITIVE AND NEGATIVE INTEGERS

When whole numbers became inadequate for the purposes of society, this number system was enlarged to include negative numbers. The whole numbers, 0, 1, 2, 3, 4, . . . , together with the negatives of the whole numbers, $^{-}1$, $^{-}2$, $^{-}3$, $^{-}4$, . . . , are called **integers.** A number line with a fixed reference point labeled 0 is a common model for visualizing the integers. The integers are assigned points on the number line that have been marked off in unit lengths to the right and left of 0. For each integer to the right of 0, there is a corresponding integer to the left of 0. These pairs of integers, 2 and $^{-}2$, 5 and $^{-}5$, 7 and $^{-}7$, etc., are called **opposites** or **negatives** of each other (Figure 5.2).

Sometimes the integers to the right of zero are labeled $^{+}1$, $^{+}2$, $^{+}3$, etc. to emphasize that they are **positive integers** as opposed to **negative integers** ($^{-}1$, $^{-}2$, $^{-}3$, etc.). Since minus signs are also used to represent *subtraction,* raising minus signs to indicate negative numbers helps to avoid confusion between the two uses of these symbols. In particular, raising minus signs to indicate negative integers is a common practice in elementary and middle school mathematics books.

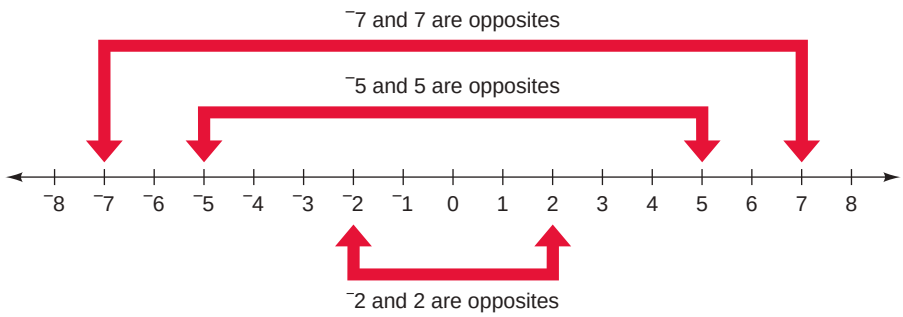


Figure 5.2

EXAMPLE A

Determine the opposite for each integer.

1. 14 **2.** 0 **3.** 1 **4.** $^{-}8$

Solution **1.** $^{-}14$ **2.** 0 **3.** $^{-}1$ **4.** 8

Technology Connection

Most calculators have a change-of-sign key such as [+/−] or [(−)] that can be used to obtain the opposite of the number in the view screen (see page 264). For example, if 35 is in the view screen, pressing the change-of-sign key changes the number to $^{-}35$, and pressing the key again produces 35.

HISTORICAL HIGHLIGHT

Traditionally, it has taken hundreds of years for a new type of number to prove itself necessary and earn a place beside the commonly accepted older numbers. This is especially true of negative numbers. By the seventh century, Hindu mathematicians

were using these numbers on a limited basis. They had symbols for negative numbers, such as $\overline{5}$ and $\overset{\circ}{5}$ for $^{-}5$, and rules for computing with them. However, it was another 1000 years before the Italian mathematician Jerome Cardan (1501–1576) gave the first significant treatment of negative numbers. Cardan called these new numbers *false* and represented each number by writing m: in front of the numeral. For example, he wrote m:3 for $^{-}3$. Other writers of this period called negative numbers *absurd numbers*. The resistance to negative numbers can be seen as late as 1796, when William Frend, in his text *Principles of Algebra,* argued against their use.

USES OF INTEGERS

The concept of number opposites in the form of positive and negative integers, also referred to as **positive and negative numbers,** is useful whenever we wish to count on both sides of a fixed point of reference. The positive integers indicate one direction, and the negative integers indicate the opposite direction.

Credits and Debts One common example of opposites is *credits,* which are represented by positive numbers, and *debts* or *deficits,* which are represented by negative numbers. The graph in Figure 5.3 shows the U.S. merchandise trade balance with Turkey, which is the difference between exports and imports. A minus sign denotes an excess of imports over exports from Turkey to the United States. For example, in 2002, the trade balance was approximately $^{-}400$ million dollars. This means the excess of imports to the U.S. from Turkey over the U.S. exports to Turkey amounted to 400 million dollars.*

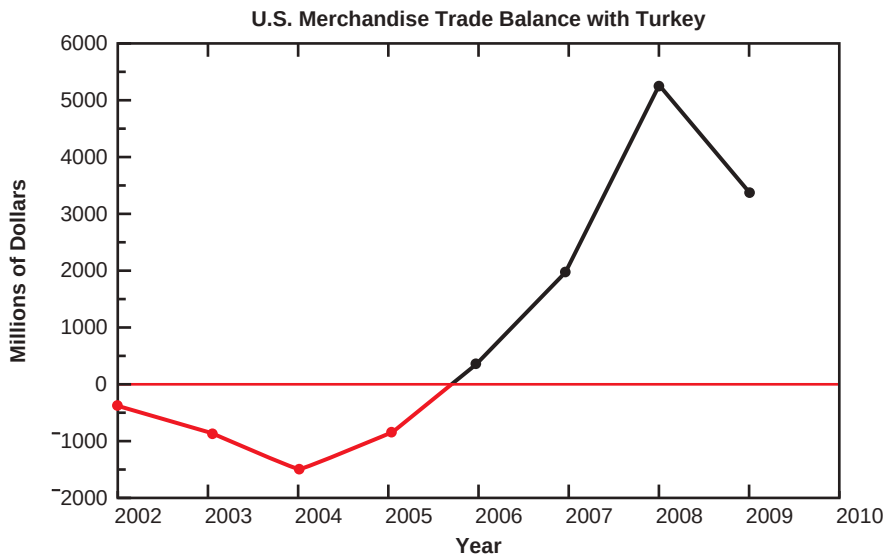


Figure 5.3

**Statistical Abstract of the United States,* 128th ed. (Washington, DC: Bureau of the Census, 2009), p. 783.

EXAMPLE B

1. Determine whether the U.S. trade balance with Turkey was positive or negative at the following times and interpret the results.

   a. 2004 **b.** 2009

2. Determine whether the U.S. trade balance with Turkey was increasing or decreasing for the following periods:

   a. 2002 to 2004 **b.** 2004 to 2006 **c.** 2008 to 2009

Solution **1. a.** ≈ -1500 million dollars. Overall, there were 1500 million dollars more in imports from Turkey than U.S. exports to Turkey. **b.** ≈ 3500 million dollars. Overall, exports exceeded imports by 3500 million dollars. **2. a.** Decreasing **b.** Increasing **c.** Decreasing

Temperature Measuring temperature is another familiar use for positive and negative numbers. The fixed reference point on the Celsius thermometer is 0 degrees, the temperature at which water freezes. On the Fahrenheit scale, water freezes at 32 degrees. On both scales, temperatures *above* zero are *positive* and those *below* zero are *negative.*

EXAMPLE C

Write the integer for each of the following temperatures.

1. 10° below zero on the Celsius thermometer
2. 20° below 32° on the Fahrenheit thermometer
3. 20° below zero on the Fahrenheit thermometer

Solution **1.** $-10°$ Celsius, or $-10°$C **2.** 12° Fahrenheit, or 12°F **3.** $-20°$F



Surveyor 1 scouted the lunar surface for future Apollo mission landing sites.

Sports In some sports it is convenient to use positive and negative numbers to indicate amounts from a given reference point. In golf this reference point is *par,* and a score of -4 represents four strokes *below par.* In football, the yard line at which plays begin is the reference point, and a *loss of yardage* is referred to as *negative yardage.*

Time Scientists often find it convenient to designate a given time as *zero time* and then refer to the *time before* and *time after* as being negative and positive, respectively. This practice is followed in the launching of rockets. If the time with respect to blastoff is -15 minutes, then it is 15 minutes before the launch.

Altitude Sea level is the common reference point for measuring altitudes. Charts and maps that label altitudes below and above sea level use negative and positive numbers. The chart in Figure 5.4 uses negative numbers to show altitudes above the floor of the Atlantic Ocean between South America and Africa.

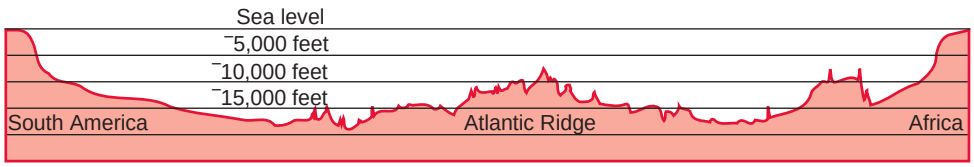


Figure 5.4

EXAMPLE D

Write each altitude as an integer.

1. 8500 feet below sea level
2. 300 feet above sea level
3. Sea level

Solution 1. $^{-}$8500 feet 2. 300 feet 3. 0 feet

MODELS FOR INTEGERS

There are many models for illustrating integers and operations on integers. The *number line model* and the *black and red chips model* will be used on the following pages.

Black and Red Chips Model The red and black rods used by the Chinese for positive and negative integers suggest a physical model for the integers. In place of rods we will use chips, and the color scheme will be reversed; that is, black chips will represent positive integers, and red chips negative integers. By establishing that each black chip together with a red chip represents 0 (think of each black chip as a \$1 credit and each red chip as a \$1 debt), we can represent every integer in an infinite number of ways. Three different sets that illustrate the number 3 are shown in Figure 5.5. In part b, the red chip is canceled by 1 black chip; in part c, 2 red chips are canceled by 2 black chips.

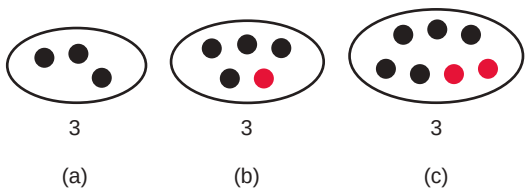


Figure 5.5

EXAMPLE E

Describe four different sets of chips that represent 0.

Solution Here are four possibilities: 4 red chips and 4 black chips; 1 red chip and 1 black chip; 3 red chips and 3 black chips; 7 red chips and 7 black chips.

ADDITION

Addition of integers can be illustrated by putting together (taking the union of) sets of black and red chips. Figure 5.6 shows sets for $^{-}$5 and 2 and their union, which contains 5 red chips and 2 black chips. The resulting set represents $^{-}$3, since 2 black chips can be matched with 2 red chips. This shows that $^{-}5 + 2 = {}^{-}3$.

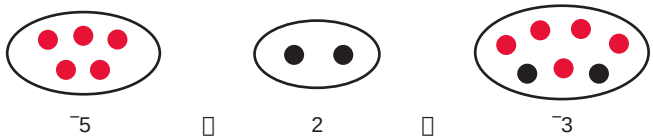


Figure 5.6

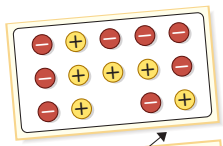
Combining a debt of \$5 with a credit of \$2 reduces the debt to \$3.

To add two integers with different signs, it is necessary to remove any zero pairs. A *zero pair* is a pair of counters that includes one positive counter and one negative counter.

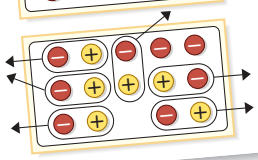
EXAMPLE Add Integers with Different Signs

3 Find $-8 + 6$.

METHOD 1 Use counters.

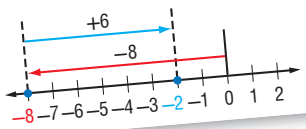


Place 8 negative counters and 6 positive counters on the mat.



Next, remove as many zero pairs as possible.

METHOD 2 Use a number line.



Start at 0. Move 8 units to the left to show −8. From there, move 6 units right to show +6.

So, $-8 + 6 = -2$.

CHOOSE Your Method

Add. Use counters or a number line if necessary.

d. $-6 + 3$ e. $4 + (-4)$ f. $+7 + (-3)$

Reading Math

Positive Integers A number without a sign is assumed to be positive.

The following rules are often helpful when adding integers.

Key Concept

Add Integers

Words The sum of two positive integers is always positive. The sum of two negative integers is always negative.

Examples $5 + 1 = 6$ $-5 + (-1) = -6$

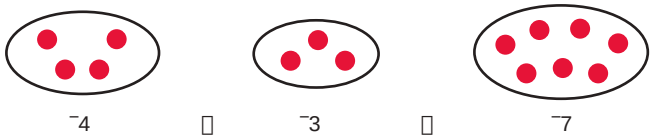
Words The sum of a positive integer and a negative integer is sometimes positive, sometimes negative, and sometimes zero.

Examples $5 + (-1) = 4$ $-5 + 1 = -4$ $-5 + 5 = 0$

From *Math Connects*, Grade 6, by Macmillan/McGraw-Hill. Copyright © 2009 by The McGraw-Hill Companies, Inc. Reprinted by permission of The McGraw-Hill Companies, Inc.

EXAMPLE F Sketch sets of chips to illustrate and compute $^{-}4 + {}^{-}3$.

Solution



Combining a debt of $4 with a debt of $3 produces a debt of $7.

The usual rules for addition can be discovered by using black and red chips to compute sums of positive and negative integers. For example, when a positive and a negative integer are added, the sum will be positive or negative depending on whether there are more black or red chips. The rules of signs for addition are shown below. In each of the three cases, a and b represent positive integers. Notice that in each of these cases, sums involving negative integers can be obtained by computing sums or differences of positive integers.

Rules of Signs for Addition Let a and b be positive integers:

1. Negative plus negative equals negative.

$$^{-}a + {}^{-}b = {}^{-}(a + b)$$
$$^{-}3 + {}^{-}7 = {}^{-}(3 + 7) = {}^{-}10$$

2. Positive plus negative equals positive if $a > b$.

$$a + {}^{-}b = a - b$$
$$13 + {}^{-}5 = 13 - 5 = 8$$

3. Positive plus negative equals negative if $a < b$.

$$a + {}^{-}b = {}^{-}(b - a)$$
$$6 + {}^{-}11 = {}^{-}(11 - 6) = {}^{-}5$$

The number line is a more abstract model for illustrating the addition of positive and negative numbers. To add two numbers, we begin by drawing an arrow from 0 to the point that corresponds to the first number. Then, if the second integer is positive, we move from that point to the right on the number line; and if it is negative, we move to the left. Two examples are shown in Figure 5.7.

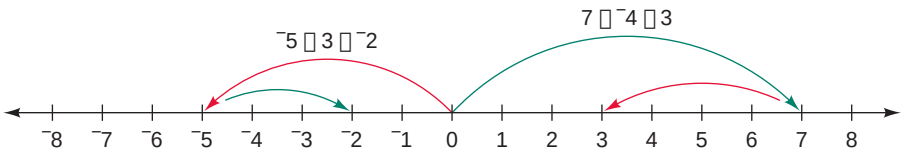


Figure 5.7

Technology Connection

Most calculators are designed to compute with negative as well as positive numbers. Some calculators have a key such as [(–)] for entering a negative number.* To compute $^{-}148 + {}^{-}315$ on such a calculator, the numbers and the addition operation are entered as they occur from left to right.

Keystrokes	View Screen
[(–)] 148	–148
[+]	–148
[(–)] 315	–315
[=]	–463

Other calculators have a change-of-sign key such as [+/–] or [+⟳–] to replace any number in the view screen by its opposite.† For instance, if 713 is on the view screen, pressing [+/–] will display $^{-}713$; and if $^{-}713$ is on the view screen, pressing [+/–] will display 713. Here is an example of the use of this key.

Keystrokes	View Screen
148 [+]	–148
[+]	–148
315 [+]	–315
[=]	–463

*Texas Instruments TI-15 and TI-73 have this key.

†CASIO fx-55 has this key.

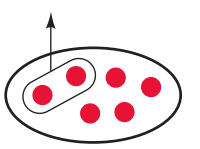
NCTM Standards

Computational fluency should develop in tandem with understanding of the role and meaning of arithmetic operations in number systems (Hiebert et al., 1997; Thornton 1990). p. 32

Or, you may prefer to compute with positive numbers and use the *rule of signs for addition* to choose the correct sign for the sum. In particular, using the rule $^{-}a + {}^{-}b = {}^{-}(a + b)$, you can determine $^{-}715 + {}^{-}643$ by computing $715 + 643 = 1358$ and using its opposite, $^{-}1358$.

SUBTRACTION

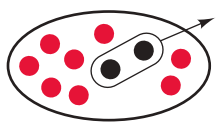
The *take-away model* can be used for subtraction of integers. In part a of Figure 5.8 we begin by representing $^{-}6$ by 6 red chips (debits) and then take away 2 red chips. In part b of Figure 5.8 we begin by representing $^{-}6$ by 8 red chips (debits) and 2 black chips (credits) and then take away 2 black chips.



$^{-}6 - {}^{-}2 = {}^{-}4$

Taking away a debt of $2 from a debt of $6 leaves a debt of $4.

(a)



$^{-}6 - 2 = {}^{-}8$

Taking away a credit of $2 from a total debit of $6 leaves a total debit of $8.

(b)

Figure 5.8

In the early grades, the take-away model for subtraction is used only when one whole number is subtracted from a larger one. However, it is still possible to use this model in cases such as $3 - 5$, where the number being subtracted is the larger one. This can be accomplished by using a suitable representation for 3. For example, instead of representing

3 by 3 black chips, as in set *A* of Figure 5.9, we can represent 3 by 5 black chips and 2 red chips, as in set *B*. Then 5 black chips can be taken away, leaving 2 red chips: $3 - 5 = {}^{-}2$.

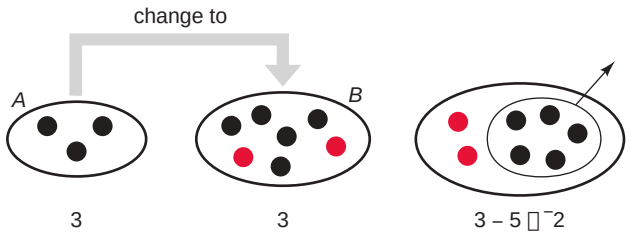


Figure 5.9 Having a credit of $3 and taking away a credit of $5 leaves a debt of $2.

Adding Opposites One common method of subtracting an integer is to add its opposite. If, instead of removing 5 black chips from set *B* in Figure 5.9, we put in 5 red chips, as in Figure 5.10, the final set will still represent ⁻2. In other words, *putting in 5 red chips* has the same effect as *taking away 5 black chips*. This suggests that subtracting 5 is the same as adding its opposite, ⁻5.

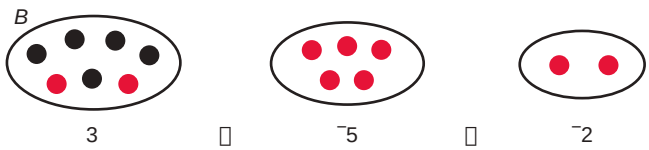


Figure 5.10 Having a credit of $5 taken away is like incurring a debt of $5.

This approach to subtraction is called **adding opposites.** It enables us to compute the difference of any two integers by computing a sum, as stated in the following definition.

Subtraction of Integers For any two integers a and b, $a - b$ is the sum of a plus the opposite of b,

$$a - b = a + {}^{-}b$$

EXAMPLE G

Compute the following differences.

1. $15 - 7$ **2.** $^{-}14 - 3$ **3.** $22 - {}^{-}5$

Solution **1.** $15 - 7 = 15 + {}^{-}7 = 8$ (The opposite of 7 is ⁻7.) **2.** $^{-}14 - 3 = {}^{-}14 + {}^{-}3 = {}^{-}17$ (The opposite of 3 is ⁻3.) **3.** $22 - {}^{-}5 = 22 + 5 = 27$ (The opposite of ⁻5 is 5.)

Technology Connection

A calculator change-of-sign key such as [+/−] and [+(−)] is handy if you wish to subtract negative integers. Here are the key strokes to compute $^{-}243 - {}^{-}109$:

Keystrokes	View Screen
243 [□□+]	–243
[−]	–243
109 [□□+]	–109
[□]	–134

Notice that the definition, $a - b = a + {}^{-}b$, allows us to replace ${}^{-}243 - {}^{-}109$ by ${}^{-}243 + 109$, which can be computed on a calculator by using the change-of-sign key to enter ${}^{-}243$ [+] 109, or by commuting the numbers to compute $109 - 243$.

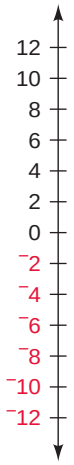


Figure 5.11

MULTIPLICATION

The familiar rules for multiplying with negative numbers, such as "a negative times a negative is a positive," are easy enough to remember but difficult to illustrate. There are many different approaches that attempt to justify the rules for multiplying with negative numbers in an intuitive manner. Two of these methods are explained in the following paragraphs, and a third approach using patterns is contained in Exercises and Problems 5.1.

Number Line Model Products of positive and negative integers can be illustrated on a number line. One number line model uses temperatures. Figure 5.11 shows a thermometer scale, a portion of a vertical number line. To illustrate products, we will use the following common conventions:

1. Temperature increases are represented by positive integers and temperature decreases by negative integers.
2. Time in the future is represented by positive integers and time in the past by negative integers.

EXAMPLE H

Compute each product by determining the new or old temperature. You may find it helpful to sketch a vertical number line.

1. If the temperature is now 0°, what will it be 4 hours from now if it increases 3° each hour?

4	×	3	=	□
4 hours from now (time in future)		3° increase		Temperature will be

2. If the temperature is now 0°, what will it be 5 hours from now if it decreases 2° each hour?

5	×	${}^{-}2$	=	□
5 hours from now (time in future)		2° decrease		Temperature will be

3. If the temperature is now 0°, what was it 3 hours ago if it has been increasing 5° each hour?

${}^{-}3$	×	5	=	□
3 hours ago (time in past)		5° increase		Temperature was

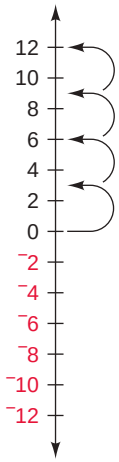
4. If the temperature is now 0°, what was it 4 hours ago if it has been decreasing 2° each hour?

${}^{-}4$	×	${}^{-}2$	=	□
4 hours ago (time in past)		2° decrease		Temperature was

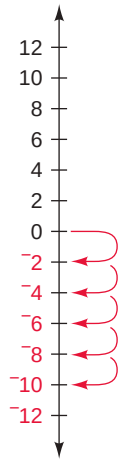
NCTM Standards

Negative integers should be introduced at this level [3–5] through the use of familiar models such as temperature or owing money. The number line is also an appropriate and helpful model. p. 151

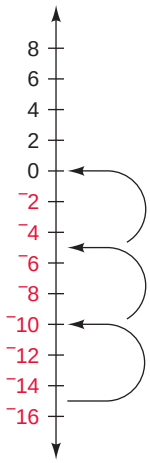
Solution **1.** The temperature will be 12°:
$4 \times 3 = 12$



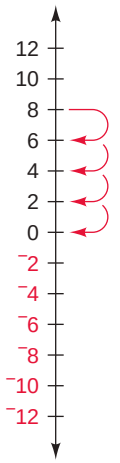
2. The temperature will be ⁻10°:
$5 \times {}^{-}2 = {}^{-}10$



3. The temperature was ⁻15°:
$^{-}3 \times 5 = {}^{-}15$



4. The temperature was 8°:
$^{-}4 \times {}^{-}2 = 8$



Black and Red Chips Model Multiplication by a positive integer can be illustrated by putting in groups of chips (repeated addition). Suppose that on 4 occasions you incur debts of \$2. This is represented in Figure 5.12 by 4 groups of 2 red chips; the figure illustrates $4 \times {}^{-}2$. Since there are 8 red chips, $4 \times {}^{-}2 = {}^{-}8$.

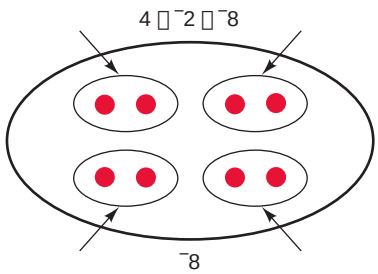
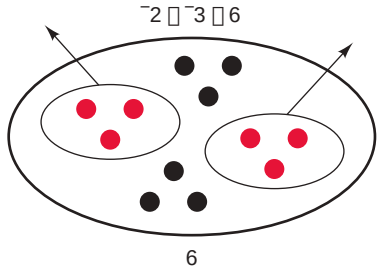


Figure 5.12 Receiving 4 debts of \$2 each is like receiving a debt of \$8.

Multiplication by a negative integer can be illustrated by removing groups of chips. First, let us assume you have \$6 and you have 2 debts, each for \$3. These are represented in Figure 5.13 by 6 black chips and 2 groups of 3 red chips and the value of all these chips is zero. Now suppose that these 2 debts are removed. Removing the 2 groups of 3 red chips illustrates the product $^{-}2 \times {}^{-}3$. Notice that removing 6 red chips changes the value of the set from zero to 6. This suggests that $^{-}2 \times {}^{-}3 = 6$, because 2 times, we are taking out 3 red chips.



Removing 2 debts of \$3 each is like removing a debt of \$6 or receiving a credit for \$6.
So, the value of the set increases from 0 to 6.

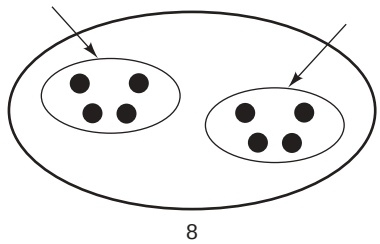
Figure 5.13

EXAMPLE 1

Sketch sets of chips to illustrate each computation. Describe the product in terms of debts and credits.

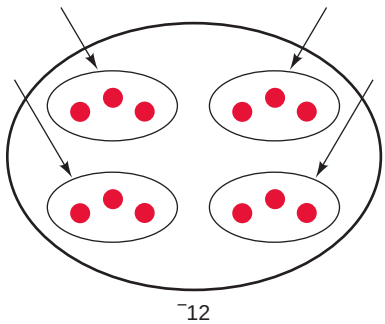
1. $2 \times 4 = 8$ **2.** $4 \times {}^{-}3 = {}^{-}12$ **3.** $^{-}5 \times {}^{-}3 = 15$

Solution **1.** $2 \times 4 = 8$ (2 times, put in 4 black chips; repeated addition.)



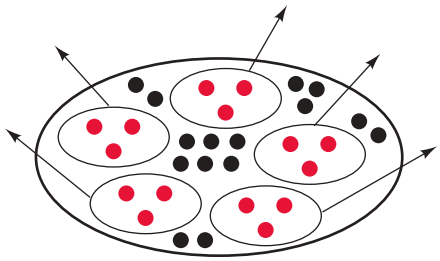
Receiving 2 credits of \$4 each is like receiving a credit for \$8.
So, the value of the set increases from 0 to 8.

2. $4 \times {}^{-}3 = {}^{-}12$ (4 times, put in 3 red chips; repeated addition.)



Receiving 4 debts of \$3 each is like receiving a debt for \$12.
So, the value of the set decreases from 0 to $^{-}12$.

3. $^{-}5 \times {}^{-}3 = 15$ (5 times, remove 3 red chips.)



15

Removing 5 debts of $3 each is like removing a debt of $15 or receiving a credit for $15. So, the value of the set increases from 0 to 15.

The black and red chips model and the number line model illustrate the reasonableness of the rules for multiplying with negative integers, which are shown in the following table. Notice these rules show that products involving negative integers can be obtained by computing products of positive integers.

Rules of Signs for Multiplication Let a and b be positive integers:

1. Positive times negative equals negative.

$$a \times {}^{-}b = {}^{-}(a \times b)$$
$$5 \times {}^{-}2 = {}^{-}(5 \times 2) = {}^{-}10$$

2. Negative times positive equals negative.

$${}^{-}a \times b = {}^{-}(a \times b)$$
$${}^{-}7 \times 3 = {}^{-}(7 \times 3) = {}^{-}21$$

3. Negative times negative equals positive.

$${}^{-}a \times {}^{-}b = a \times b$$
$${}^{-}4 \times {}^{-}5 = 4 \times 5 = 20$$

Technology Connection

Products involving negative integers can be computed on a calculator by using the (−) key or a change-of-sign key. The next example illustrates the use of +/− to compute $^{-}44 \times {}^{-}16$:

Keystrokes	View Screen
44 +/−	−44
×	−44
16 +/−	−16
=	704

However, it is much more convenient, even with a calculator, to use the rule of signs for multiplication to first compute the product of positive integers and then choose the correct sign. In particular, the rule $^{-}a \times {}^{-}b = a \times b$ allows us to determine the product $^{-}44 \times {}^{-}16$ by computing $44 \times 16 = 704$.

EXAMPLE J Compute each product.

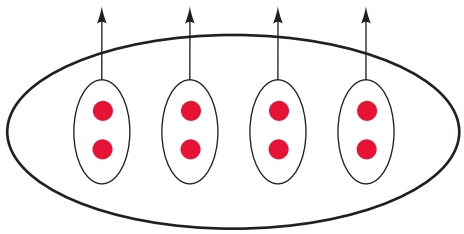
1. $^{-}426 \times 83$ **2.** $47 \times {}^{-}2876$ **3.** $^{-}106 \times {}^{-}17$

Solution **1.** Since $426 \times 83 = 35{,}358$ and a negative times a positive equals a negative, $^{-}426 \times 83 = {}^{-}35{,}358$. **2.** Since $47 \times 2876 = 135{,}172$ and a positive times a negative equals a negative, $47 \times {}^{-}2876 = {}^{-}135{,}172$. **3.** Since $106 \times 17 = 1802$ and a negative times a negative equals a positive, $^{-}106 \times {}^{-}17 = 1802$.

DIVISION

Both the *sharing (partitive)* and *measurement (subtractive)* concepts of division will be used in the following illustrations of division with the black and red chips model.

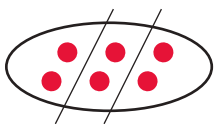
To show $^{-}8 \div {}^{-}2$, we begin with 8 red chips and then measure off, or subtract, as many groups of 2 red chips as possible (see Figure 5.14). Since there are 4 such groups, $^{-}8 \div {}^{-}2 = 4$. This illustration uses the *measurement* concept of division.



$^{-}8 \square {}^{-}2 \square 4$

Figure 5.14 A debt of $2 can be measured off (subtracted from) a debt of $8 a total of 4 times.

To show $^{-}6 \div 3$, we divide 6 red chips into 3 equal groups (see Figure 5.15). Since there are 2 red chips in each group, $^{-}6 \div 3 = {}^{-}2$. In this illustration the divisor 3 indicates the number of equal parts into which the set is divided. This illustration uses the *sharing* concept of division.



$^{-}6 \square 3 \square {}^{-}2$

Figure 5.15 Sharing a debt of $6 among 3 people gives each person a debt of $2.

In the preceding two examples we showed how two different concepts of division are meaningful for division involving negative and positive integers. In the following definition of division with integers, division is defined in terms of multiplication, just as it was for whole numbers.

Division of Integers For any integers a and b, with $b \neq 0$,

$$a \div b = k \quad \text{if and only if} \quad a = b \times k$$

for some integer k. In this case, b and k are factors of a.

EXAMPLE K

Calculate each quotient mentally, using the definition of division of integers.

1. $^{-}24 \div {}^{-}4$ **2.** $18 \div {}^{-}3$ **3.** $^{-}30 \div 6$

Solution **1.** $^{-}24 \div {}^{-}4 = 6$, since $^{-}24 = {}^{-}4 \times 6$. **2.** $18 \div {}^{-}3 = {}^{-}6$, since $18 = {}^{-}3 \times {}^{-}6$. **3.** $^{-}30 \div 6 = {}^{-}5$, since $^{-}30 = 6 \times {}^{-}5$.

The black and red chips model, and the inverse relationship between division and multiplication of integers, illustrate the reasonableness of the rules for dividing with negative integers, which are shown in the following table. Notice that quotients involving negative integers can be obtained by computing quotients of positive integers.

Rules of Signs for Division Let a and b be positive integers:

1. Negative divided by negative equals positive.

$$^{-}a \div {}^{-}b = a \div b$$

$$^{-}30 \div {}^{-}6 = 30 \div 6 = 5$$

2. Negative divided by positive equals negative.

$$^{-}a \div b = {}^{-}(a \div b)$$

$$^{-}14 \div 2 = {}^{-}(14 \div 2) = {}^{-}7$$

3. Positive divided by negative equals negative.

$$a \div {}^{-}b = {}^{-}(a \div b)$$

$$24 \div {}^{-}6 = {}^{-}(24 \div 6) = {}^{-}4$$

Technology Connection

As with multiplication of integers, division of integers can be performed on a calculator that displays negative numbers, or on any calculator by using the *rules of signs for division* to replace quotients involving negative integers by quotients of positive integers.

INEQUALITY

For any two integers on a number line, the number on the left is **less than** the number on the right, and the number on the right is **greater than** the number on the left. For example, in Figure 5.16, $^{-}8 < {}^{-}3$, $^{-}7 < {}^{-}4$, and $^{-}5 < 0$.

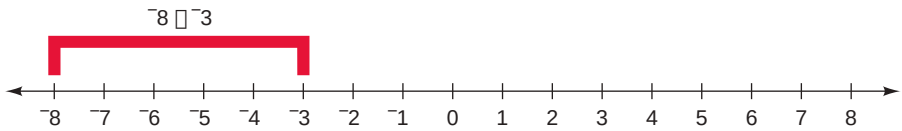


Figure 5.16

This property is stated more precisely in the following definition of inequality of integers. Although this definition is stated for *less than,* a corresponding statement holds for *greater than.*

Inequality of Integers For any two integers m and n, m is less than n, written $m < n$, if there is a positive integer k such that $m + k = n$.

In thinking about inequalities of negative numbers, you may find it helpful to recall the applications at the beginning of this section. Temperatures of $^-15$°C and $^-6$°C are both cold, but $^-15$°C is colder than $^-6$°C: $^-15 < {}^-6$ because there is a positive integer that can be added to $^-15$ to yield $^-6$ ($^-15 + 9 = {}^-6$). Similarly, an altitude of $^-8000$ feet is farther below sea level than an altitude of $^-5000$ feet: $^-8000 < {}^-5000$ because there is a positive integer that can be added to $^-8000$ to yield $^-5000$ ($^-8000 + 3000 = {}^-5000$).

EXAMPLE L

Write the appropriate inequality ($<$ or $>$) for each pair of numbers.

1. $^-37, {}^-55$ **2.** $^-110, 420$ **3.** $^-76, {}^-125$

Solution **1.** $^-37 > {}^-55$. **2.** $^-110 < 420$. **3.** $^-76 > {}^-125$.

PROPERTIES OF INTEGERS

Inverses for Addition Addition of integers has one property that addition of whole numbers does not have. For any integer, there is a unique integer, called its **opposite** or **inverse,** such that the integer plus its opposite is equal to zero. We refer to this property by saying that each integer has an **inverse for addition,** called the **additive inverse.**

EXAMPLE M

Find the integer that satisfies each equation.

1. $174 + \square = 0$ **2.** $^-351 + \square = 0$ **3.** $0 + \square = 0$

Solution **1.** $^-174$ **2.** 351 **3.** 0 (zero is its own inverse for addition).

Closure Properties The sum of any two integers is a unique integer, and the product of any two integers is also a unique integer. That is, the set of integers is closed for addition and multiplication.

Identity Properties Any integer added to 0 equals the given integer, and any integer multiplied by 1 equals the given integer. That is, 0 is the *identity for addition,* 1 is the *identity for multiplication,* and these are the only identity elements (they are unique) for addition and multiplication of integers.

Commutative Properties The operations of *addition and multiplication are commutative* for the integers. In particular, these properties hold for negative integers. For example,

$$^-37 + {}^-52 = {}^-52 + {}^-37 \quad \text{and} \quad {}^-37 \times {}^-52 = {}^-52 \times {}^-37$$

Associative Properties The operations of *addition and multiplication are associative* for the integers. These properties hold for any combination of three integers. For the integers $^-7$, $^-2$, and $^-6$,

$$(^-7 + {}^-2) + {}^-6 = {}^-7 + (^-2 + {}^-6) \quad \text{and} \quad (^-7 \times {}^-2) \times {}^-6 = {}^-7 \times (^-2 \times {}^-6)$$

Distributive Property The *distributive property of multiplication over addition* holds in the set of integers. (See Example N.)

EXAMPLE N

Show that the distributive property holds for the following equation by computing both sides of this equation:

$$^{-}2 \times (3 + {}^{-}7) = {}^{-}2 \times 3 + {}^{-}2 \times {}^{-}7$$

Solution Left side of equation: $^{-}2 \times (3 + {}^{-}7) = {}^{-}2 \times {}^{-}4 = 8$.
Right side of equation: $^{-}2 \times 3 + {}^{-}2 \times {}^{-}7 = {}^{-}6 + 14 = 8$.

MENTAL CALCULATIONS

Acquiring number sense is an important objective for elementary school students. Learning to do mental calculations can help them reach this objective. *Mental math techniques* encourage students to use number properties and discover computational shortcuts rather than perform rote calculations.

Compatible Numbers for Mental Calculation Finding combinations of compatible numbers is a technique for mental calculation that works with integers as well as with whole numbers.

EXAMPLE O

Do these computations using compatible numbers for mental calculations.

1. $50 + {}^{-}23 + {}^{-}60$
2. $15 + {}^{-}26 + 10 + {}^{-}5$
3. $^{-}5 \times 18 \times {}^{-}2$

Solution One approach to each problem is shown below; your selections of compatible numbers may differ from these: **1.** $50 + {}^{-}60 = {}^{-}10$, and $^{-}10 + {}^{-}23 = {}^{-}33$. **2.** $15 + {}^{-}5 = 10$, $10 + 10 = 20$, and $20 + {}^{-}26 = {}^{-}6$. **3.** $^{-}5 \times {}^{-}2 = 10$ and $10 \times 18 = 180$.

Substitutions for Mental Calculation Using the technique of substitution involves replacing a number by a sum, difference, product, or quotient that is more convenient to use in the computation.

EXAMPLE P

Do each computation mentally, using a convenient substitution.

1. $180 + {}^{-}37$
2. $^{-}6 \times 19$
3. $^{-}150 \div 6$
4. $12 \times {}^{-}54$

Solution Different substitutions may occur to you. **1.** $180 + {}^{-}37 = 180 + ({}^{-}30 + {}^{-}7)$, and by the associative property for addition this equals $150 + {}^{-}7 = 143$. **2.** $^{-}6 \times 19 = {}^{-}6 \times (20 + {}^{-}1)$, and by the distributive property this equals $^{-}120 + 6 = {}^{-}114$. **3.** Using the equal quotients technique described in Section 3.4, we can divide both 150 and 6 by 2. So $^{-}150 \div 6 = {}^{-}75 \div 3 = {}^{-}25$. Or both $^{-}150$ and 6 can be divided by 3: $^{-}150 \div 6 = {}^{-}50 \div 2 = {}^{-}25$. **4.** Using the equal products technique described in Section 3.3, we can divide 12 by 2 and multiply $^{-}54$ by 2: $12 \times {}^{-}54 = 6 \times {}^{-}108 = {}^{-}648$.

ESTIMATION

Computing estimations is one of the most powerful means of acquiring number sense. Estimation requires a knowledge of mental mathematics because one must choose approximate numbers that are convenient for mental calculations.

Rounding and Compatible Numbers for Estimation The mental calculating techniques of rounding and compatible numbers are often combined to obtain an estimation.

EXAMPLE Q

Obtain estimations by rounding and compatible numbers. (*Note:* In **4,** division should be performed before addition.)

1. $81 + {}^{-}32 + 21 + {}^{-}47$

2. ${}^{-}53 \times 142 \times {}^{-}2$

3. $12 \times 67 \times {}^{-}5$

4. ${}^{-}250 + 90 \div 32$

Solution You may find other estimations. **1.** $81 + {}^{-}32 + 21 + {}^{-}47 \approx (80 + 20) + ({}^{-}30 + {}^{-}50) = 100 + {}^{-}80 = 20$. **2.** ${}^{-}53 \times 142 \times {}^{-}2 \approx {}^{-}50 \times {}^{-}2 \times 142 = 100 \times 142 = 14{,}200$. **3.** $12 \times 67 \times {}^{-}5 = 12 \times {}^{-}5 \times 67 = {}^{-}60 \times 67 \approx {}^{-}60 \times 70 = {}^{-}4200$. **4.** ${}^{-}250 + 90 \div 32 \approx {}^{-}250 + 90 \div 30 = {}^{-}250 + 3 = {}^{-}247$.

Note: The number of negative integers in a product of numbers determines whether the product is positive or negative. Since there are two negative integers in **2,** the product of the three numbers is positive. Similarly, in **3** there is one negative integer, so the answer is negative.

PROBLEM-SOLVING APPLICATION

The following problem involves sums of positive and negative integers. Try to solve this problem before you read the solution. You may find it helpful to use the strategies of *solving a simpler problem* and *making an organized list.*

Problem

Consider the positive integers from 1 to 25 and their opposites.

$$\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \ldots, \pm 21, \pm 22, \pm 23, \pm 24, \pm 25$$

Describe all the different numbers that can be obtained by using each integer or its opposite exactly once to form sums of 25 integers.

Understanding the Problem Each sum must have 25 integers. One possibility is to use the opposites of 1, 2, and 3 and the positive integers from 4 to 25.

$${}^{-}1 + {}^{-}2 + {}^{-}3 + 4 + 5 + 6 + \ldots + 22 + 23 + 24 + 25 = 313$$

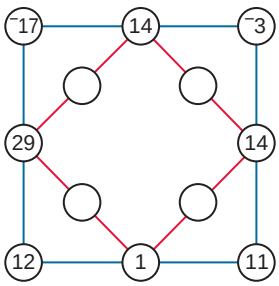
The greatest possible sum is 325. **Question 1:** What is the least possible sum?

Devising a Plan Let's solve a simpler problem by using the integers 1, 2, 3, 4, and 5 and their opposites. It is natural to *make an organized list* to consider all the possibilities. This

Technology Connection

Integer Differences

If any four integers are placed at the vertices of a square and their differences (larger minus smaller) are placed at the vertices of an inner square, and this process is continued, will a square with all zeros at the vertices eventually be obtained? The online 5.1 Mathematics Investigation will help you investigate this and similar questions where integers are placed at the vertices of other polygons.



Mathematics Investigation
Chapter 5, Section 1
www.mhhe.com/bbn

can be done by writing the sum $1 + 2 + 3 + 4 + 5$ and systematically replacing positive integers by their opposites.

$$\begin{aligned} 1 + 2 + 3 + 4 + 5 &= 15 \\ {}^{-}1 + 2 + 3 + 4 + 5 &= 13 \\ 1 + {}^{-}2 + 3 + 4 + 5 &= 11 \\ {}^{-}1 + {}^{-}2 + 3 + 4 + 5 &= 9 \end{aligned}$$

Question 2: Why can't a sum of 0 be obtained from such integers?

Carrying Out the Plan The list of sums, continued below, reveals a pattern.

$$\begin{aligned} {}^{-}1 + 2 + {}^{-}3 + 4 + 5 &= 7 \\ {}^{-}1 + 2 + 3 + {}^{-}4 + 5 &= 5 \\ {}^{-}1 + 2 + 3 + 4 + {}^{-}5 &= 3 \\ 1 + {}^{-}2 + 3 + 4 + {}^{-}5 &= 1 \\ 1 + 2 + {}^{-}3 + 4 + {}^{-}5 &= {}^{-}1 \\ 1 + 2 + 3 + {}^{-}4 + {}^{-}5 &= {}^{-}3 \\ {}^{-}1 + 2 + 3 + {}^{-}4 + {}^{-}5 &= {}^{-}5 \\ 1 + {}^{-}2 + 3 + {}^{-}4 + {}^{-}5 &= {}^{-}7 \\ 1 + 2 + {}^{-}3 + {}^{-}4 + {}^{-}5 &= {}^{-}9 \\ {}^{-}1 + 2 + {}^{-}3 + {}^{-}4 + {}^{-}5 &= {}^{-}11 \\ 1 + {}^{-}2 + {}^{-}3 + {}^{-}4 + {}^{-}5 &= {}^{-}13 \\ {}^{-}1 + {}^{-}2 + {}^{-}3 + {}^{-}4 + {}^{-}5 &= {}^{-}15 \end{aligned}$$

We obtained all the odd numbers from 15 to $^{-}15$. Perhaps you noticed a reason for this. Each time a positive integer is replaced by its opposite, the new sum differs by an even number. For example, if 1 is replaced by $^{-}1$, the new sum is decreased by 2. Similarly, replacing 2 by $^{-}2$ decreases the sum by 4; replacing 3 by $^{-}3$ decreases the sum by 6; etc. Thus, we obtain all the odd numbers from $^{-}15$ to 15. **Question 3:** What does this suggest about the solution to the original problem?

Looking Back The original problem considered the positive integers from 1 to 25 and their opposites. Suppose the original problem were changed to consider the integers from 1 to 24 and their opposites. **Question 4:** What sums would be obtained?

Answers to Questions 1–4 **1.** $^{-}325$ **2.** The sum of integers from 1 to 5 is 15, an odd number. Since replacing any one of the numbers from 1 to 5 by its opposite changes the sum by 2 times the number, which is an even number, such replacements will change the sum of 15 by an even number. Thus, the sum of all such integers is an odd number and therefore cannot be 0. **3.** The solution will include all odd numbers from 325 to $^{-}325$. **4.** All even numbers from 300 to $^{-}300$ would be obtained.

Exercises and Problems 5.1



Antarctica, the only polar continent, is centered near the South Pole and is covered by a huge ice dome reaching a height between 2 and 3 miles. One of the hazards of exploration are crevasses hidden by snow drifts.

1. If five daytime Fahrenheit temperatures recorded during an Antarctica summer are $^{-}18$, $^{-}17$, $^{-}24$, $^{-}34$, and $^{-}28$°F, what is the highest (warmest) of these temperatures?

2. Winter temperatures in Antarctica are usually below $^{-}100$°F. What is the lowest (coldest) of the following temperatures? $^{-}119$, $^{-}98$, $^{-}110$, $^{-}114$, and $^{-}108$°F

Find an integer that makes the left side of each equation in exercises 3 and 4 equal to the right side. Then write an inequality using $<$ or $>$ for each pair of numbers below the equations.

3. a. $^{-}3 + \square = {}^{-}2$
$^{-}3, {}^{-}2$

b. $^{-}14 + \square = 3$
$3, {}^{-}14$

c. $^{-}7 + \square = 1$
$^{-}7, 1$

4. a. $^{-}17 + \square = 31$
$^{-}17, 31$

b. $^{-}84 + \square = {}^{-}23$
$^{-}84, {}^{-}23$

c. $140 + \square = {}^{-}95$
$140, {}^{-}95$

Sketch a number line and locate each of the integers in exercises 5 and 6. Draw an arrow from each integer to its opposite.

5. 8, $^{-}3$, 5, and $^{-}1$

6. $^{-}7$, 5, 0, 2, and $^{-}4$

Answer each question in exercises 7 and 8 with an integer.

7. a. The temperature at 6 a.m. was $^{-}15$°F, and by noon it had warmed up 8°. What was the noontime temperature?

b. The nation's trade balance for the first quarter of a year was $^{-}\$23$ billion, and for the second quarter it was \$9 billion less. What was the second quarter trade balance in billions of dollars?

8. a. A submarine has an altitude of $^{-}5500$ feet, and it dives down 1500 feet. What is its new altitude in feet?

b. At $^{-}40$ minutes in the countdown for a space shuttle launch, technicians began a 7-minute check of the shuttle's compression system. What was the time in the countdown at the end of this check?

Use the black and red chips model in exercises 9 and 10 to sketch three different sets illustrating each integer.

9. a. $^{-}7$ **b.** 0 **c.** 3

10. a. 4 **b.** $^{-}9$ **c.** $^{-}1$

For exercises 11 and 12, show how to use a black and red chip model to illustrate each sum or difference, or show a replacement for the given set of chips to illustrate the difference. Explain your reasoning and complete each equation.

11. **a.** $^{-}7 + 4 =$ **b.** $^{-}6 + {}^{-}3 =$
c. $^{-}4 - {}^{-}3 =$ **d.** $4 - {}^{-}7 =$
e. $2 - 5 =$ **f.** $^{-}3 - 2 =$




12. **a.** $^{-}4 + 5 =$ **b.** $^{-}8 + {}^{-}4 =$
c. $^{-}5 - {}^{-}4 =$ **d.** $8 - {}^{-}3 =$
e. $4 - 9 =$ **f.** $^{-}8 - 2 =$

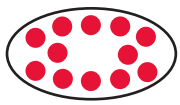
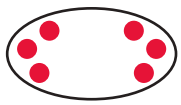



For exercises 13 and 14, use a black and red chip model or use the given set of chips to illustrate each product or quotient. Explain your reasoning and complete each equation.

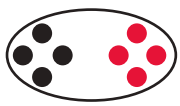
13. **a.** $^{-}2 \times 3 =$ **b.** $^{-}4 \times {}^{-}2 =$




c. $3 \times 4 =$ **d.** $3 \times {}^{-}2 =$
e. $^{-}12 \div 4 =$ **f.** $^{-}6 \div {}^{-}2 =$

14. **a.** $^{-}2 \times 2 =$ **b.** $^{-}2 \times {}^{-}4 =$




c. $3 \times {}^{-}4 =$ **d.** $2 \times 4 =$
e. $^{-}12 \div 3 =$ **f.** $^{-}10 \div {}^{-}2 =$




Answer each question in exercises 15 and 16, and then write a multiplication fact that the problem illustrates.

15. **a.** If the temperature is now 30°F, what was it 2 hours ago if it has been decreasing 6°F each hour?
b. If the temperature is now 12°F, what was it 6 hours ago if it has been increasing 4°F each hour?

16. **a.** If the temperature is now 0°F, what was it 5 hours ago if it has been increasing 3°F each hour?
b. If the temperature is now 10°F, what will it be 3 hours from now if it decreases 2°F each hour?

Draw a number line to illustrate each sum in exercises 17 and 18.

17. **a.** $6 + {}^{-}5$ **b.** $^{-}4 + 9$

18. **a.** $^{-}3 + {}^{-}4$ **b.** $^{-}5 + 5$

Compute each product or quotient in exercises 19 and 20.

19. **a.** $14 \times {}^{-}23$ **b.** $^{-}156 \div 13$
c. $^{-}278 \times {}^{-}46$ **d.** $^{-}1431 \div {}^{-}53$

20. **a.** $^{-}6 \times 2$ **b.** $8 \times {}^{-}3$
c. $24 \div {}^{-}6$ **d.** $^{-}20 \div 4$

Find the missing numbers for each equation in exercises 21 and 22.

21. **a.** $4 + \square = {}^{-}10$ **b.** $6 - \square = 10$
c. $^{-}6 \times \square = {}^{-}12$ **d.** $^{-}3 + \square = 0$

22. **a.** $^{-}15 \div \square = {}^{-}3$ **b.** $^{-}4 - \square = 7$
c. $^{-}1 \times \square = 1$ **d.** $24 \div \square = {}^{-}8$

Which number property of the integers is being used in each equality in exercises 23 and 24?

23. **a.** $(^{-}8 + 7) + 2 \times (^{-}6 \times {}^{-}5) =$
$(^{-}8 + 7) + (2 \times {}^{-}6) \times {}^{-}5$
b. $^{-}3 \times \dfrac{16 \times {}^{-}5}{{}^{-}14 + 2} = {}^{-}3 \times \dfrac{{}^{-}5 \times 16}{{}^{-}14 + 2}$

24. **a.** $^{-}4 \times \dfrac{16 + {}^{-}9}{{}^{-}6 + 13} = {}^{-}4 \times \dfrac{16 + {}^{-}9}{13 + {}^{-}6}$
b. $^{-}4 \times (^{-}3 + 3) + {}^{-}17 =$
$(^{-}4 \times {}^{-}3) + (^{-}4 \times 3) + {}^{-}17$

Determine whether the set is closed for the given operations in exercises 25 and 26. Explain your answer. If not closed give an example to show why not.

25. **a.** The set of integers for subtraction
b. The set of integers for division

26. **a.** The set of negative integers for multiplication
b. The set of negative integers for addition

Use *compatible numbers* or a *substitution* in exercises 27 and 28 to calculate each sum or difference mentally. Explain your method.

27. **a.** $^{-}125 + 17 + {}^{-}25 + 13$
b. $700 + {}^{-}298 + 135$

28. **a.** $70 + 43 + {}^{-}60$
b. $260 + {}^{-}49$

Use *equal products* or *equal quotients* in exercises 29 and 30 to do each computation mentally.

29. a. $24 \times {}^{-}25$ **b.** ${}^{-}90 \div 18$
c. ${}^{-}28 \times 5$ **d.** $400 \div {}^{-}16$

30. a. $16 \times {}^{-}35$ **b.** ${}^{-}900 \div {}^{-}45$
c. ${}^{-}1260 \div 140$ **d.** ${}^{-}28 \times 25$

Round each integer in exercises 31 and 32 to its leading digit and mentally approximate the sum of the numbers.

Example:

118	${}^{-}235$	${}^{-}190$	485	Approximate
Think	Think	Think	Think	sum
100	${}^{-}200$	${}^{-}200$	500	200

31. a. 78 ${}^{-}41$ 19 ${}^{-}38$
b. ${}^{-}23$ 51 ${}^{-}48$ ${}^{-}82$

32. a. ${}^{-}123$ 207 ${}^{-}315$ 186
b. ${}^{-}238$ 175 ${}^{-}103$ ${}^{-}214$

Replace numbers in exercises 33 and 34 by *compatible numbers* to obtain estimations. Show how you obtain your estimations.

33. a. ${}^{-}241 \div 60$ **b.** $64 \times {}^{-}11$

34. a. $26 + 59 \div {}^{-}3$ **b.** ${}^{-}31 \times 19$

In exercises 35 and 36, do not compute the answer, just determine whether it is positive or negative. Explain your reasoning.

35. a. ${}^{-}34 \times 46 \times 381 \times {}^{-}13$
b. ${}^{-}22 \times 17 \times 12 + 50$

36. a. $41 \times {}^{-}65 + 500$
b. $625 \div {}^{-}25 + {}^{-}250$

Extend the patterns in the columns of equations in exercises 37 and 38 by writing the next three equations. What multiplication rule for negative numbers is suggested by the last few equations that you write in each column?

37. a.
$5 \times 3 = 15$
$5 \times 2 = 10$
$5 \times 1 = 5$
$5 \times 0 = 0$

b.
$3 \times 6 = 18$
$2 \times 6 = 12$
$1 \times 6 = 6$
$0 \times 6 = 0$

38. a.
${}^{-}3 \times 3 = {}^{-}9$
${}^{-}3 \times 2 = {}^{-}6$
${}^{-}3 \times 1 = {}^{-}3$
${}^{-}3 \times 0 = 0$

b.
$3 \times 3 = 9$
$3 \times 2 = 6$
$3 \times 1 = 3$
$3 \times 0 = 0$

The computations in exercises 39 and 40 involve the change-of-sign key [+/−] on a calculator. Determine what computation is being performed, and find the answer for each.

39. a. 217 [+/−] − 366
b. 483 [+/−] + 225
c. 2257 ÷ 37 [+/−]
d. 1974 [+/−] ÷ 42

40. a. 16 [+/−] + 7 [+/−]
b. 25 [+/−] − 30 [+/−]
c. 54 × 35 [+/−]
d. 408 ÷ 17 [+/−]

Suppose you have a calculator that does not have a change-of-sign key. Explain how the rules of signs for operations with integers can be used to compute the expressions in exercises 41 and 42. Find each answer.

41. a. ${}^{-}487 + {}^{-}653$ **b.** $360 - {}^{-}241$
c. $32 \times {}^{-}14$ **d.** $336 \div {}^{-}16$

42. a. ${}^{-}1854 \div {}^{-}103$ **b.** ${}^{-}488 - {}^{-}179$
c. ${}^{-}133 \times 82$ **d.** $729 \div {}^{-}27$

Some calculators designed for elementary school students have a constant function that adds, subtracts, multiplies, and divides by constants. For example, pressing 2 [+] ${}^{-}3$ and repeatedly pressing [=] or a constant key will produce the arithmetic sequence ${}^{-}1, {}^{-}4, {}^{-}7, {}^{-}10, {}^{-}13, \ldots$. Assume that you have such a calculator, and write the sequences that will be obtained in exercises 43 and 44. (*Recall:* The change-of-sign key [+/−] is used to obtain opposites, not the key for subtraction [−].)

43. a. 0 [+] ${}^{-}17$ [=] [=] [=] [=] [=]
b. 2 [×] ${}^{-}26$ [=] [=] [=] [=] [=]

44. a. 10 [+] ${}^{-}43$ [=] [=] [=] [=] [=]
b. 1 [×] ${}^{-}13$ [=] [=] [=] [=] [=]

45. Use your calculator to try each of the following sequence of steps, where [+/−] is the change-of-sign key. Which sequence computes ${}^{-}15 - {}^{-}6$?

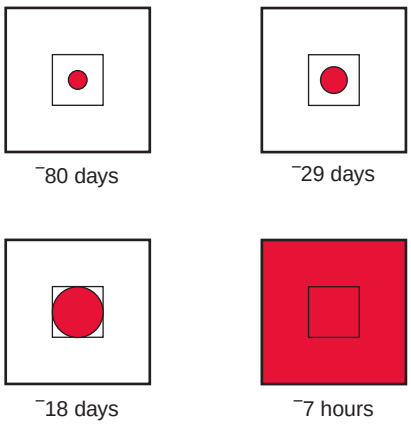
a. [−] 15 [−] 6 [=]
b. 15 [+/−] [−] 6 [+/−] [=]
c. [−] 15 [−] 6 [+/−] [=]

46. Use your calculator to try each of the following sequence of steps, where [+/−] is the change-of-sign key. Which sequence computes ${}^{-}245 - {}^{-}182$?

a. [−] 245 [−] 182 [+/−] [=]
b. [−] 245 [−] 182 [=]
c. 245 [+/−] [−] 182 [+/−] [=]

Reasoning and Problem Solving

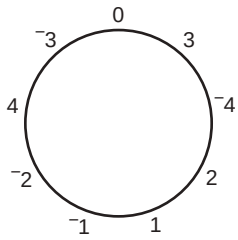
47. The closest approach by NASA's *Voyager 1* to Jupiter was 280,000 kilometers (about 168,000 miles). At $^{-}80$ days with respect to its time of closest approach, the spacecraft swiveled its narrow-angle television camera to begin its observational phase. The inner squares in the sketches here represent the camera's field of view at four different approach times.



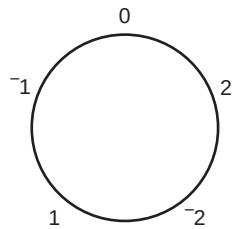
- **a.** How much time elapsed between the $^{-}80$-day view and the $^{-}29$-day view?
- **b.** How much time elapsed between the $^{-}18$-day view and the $^{-}7$-hour view?
- **c.** As *Voyager 1* moved away from Jupiter, it examined its moons: Io at $^{+}3$ hours, Europa at $^{+}5$ hours, and Ganymede at $^{+}14$ hours. Thirty-six hours after the $^{-}7$-hour view, *Voyager 1* examined Callisto. How many hours was this after its closest approach to Jupiter?

48. Featured Strategies: Solving a Simpler Problem and Guessing and Checking. How can the integers from $^{-}4$ to 4 be placed around a circle so that each integer from $^{-}10$ to 10 can be obtained by adding two or more neighbor numbers (numbers next to each other)?

- **a. Understanding the Problem.** The problem requires that the nine integers from $^{-}4$ to 4 be used and that the sums involve two or more adjacent integers. The following placement of integers will yield some of the numbers from $^{-}10$ to 10, but not all of them. For example, $0 = {}^{-}3 + 0 + 3$, and $1 = 0 + 3 + {}^{-}4 + 2$. Find some integers from $^{-}10$ to 10 that cannot be obtained by adding two or more adjacent integers from the circle at the top of the next column.



- **b. Devising a Plan.** This type of problem can be solved by *guessing and checking* the results. It may also help to *solve a simpler problem.* Consider placing the numbers $^{-}2$, $^{-}1$, 0, 1, and 2 around a circle to obtain all the sums from 3 to $^{-}3$. Show why it doesn't work to place each integer next to its opposite. How should these numbers be placed?



- **c. Carrying Out the Plan.** The solution to part b may suggest a solution to the original problem. The requirement that we be able to obtain a sum of 10 and a sum of $^{-}10$ completely determines the solution. Explain why. How can the integers from $^{-}4$ to 4 be placed to solve the problem?
- **d. Looking Back.** Another observation concerning the original problem is that each of the integers from $^{-}4$ to 4 can be routinely obtained by adding all the numbers on the circle except the negative of the given number. Explain why.

49. PS A radio station that keeps records of the low temperature each day obtained the following information. The temperature on Monday was below zero, and on Tuesday it was 8°F colder. Wednesday the temperature reading was twice as low as for Tuesday. Thursday the temperature was 4°F higher than on Wednesday. On Friday the temperature warmed up to $^{-}14$°F, which was just half of the temperature on Thursday. What was the recorded temperature for Monday?

50. PS In a city golf tournament each golfer played 18 holes of golf per day for 4 days. If a player **pars** a hole, that is, gets the ball in the hole in the number of strokes designed for the hole, the player receives a score of 0 for the hole. If the player gets the ball in the hole in one stroke under par, it is called a **birdie** and the score for the hole is $^{-}1$.

If the player gets the ball in the hole in one stroke over par, it is called a **bogey** and the score for the hole is $^{+}1$. The winner of the tournament had a 4-day total score of $^{-}13$ with 5 bogeys. If the winner had only birdies, pars, and bogeys, how many birdies did the player have?

51. Keeping the single-digit numbers from 1 to 7 in order,

PS

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$$

it is possible to insert five plus or minus signs to obtain a sum of 50.

$$^{-}12 + 3 - 4 + 56 + 7$$

Show how three plus or minus signs can be inserted to obtain a sum of 50.

52. Here are the positive and negative values of the consecutive whole numbers from 1 to 3.

PS

$$\pm 1 \quad \pm 2 \quad \pm 3$$

Consider selecting either the positive or the negative value of each integer. For example, we could select $^{-}1$, 2, $^{-}3$; or 1, 2, 3; or 1, 2, $^{-}3$; etc. By using all such possible selections, the following sums are possible: $^{-}6$, $^{-}4$, $^{-}2$, 0, 2, 4, and 6. For example, $^{-}6$ is obtained by using all the negative values $^{-}1 + ^{-}2 + ^{-}3$; the sum of $^{-}2$ is obtained from $^{-}3 + ^{-}1 + 2$; and the largest possible sum of 6 is obtained by adding all the positive values: $1 + 2 + 3$. What sums can be obtained by using either the positive or the negative value of each integer for these sequences?

a. $\pm 1 \quad \pm 2 \quad \pm 3 \quad \pm 4$

b. $\pm 1 \quad \pm 2 \quad \pm 3 \quad \pm 4 \quad \pm 5$

c. $\pm 1 \quad \pm 2 \quad \pm 3 \quad \pm 4 \quad \pm 5 \quad \pm 6$

d. Look for patterns and state a general conjecture regarding the sums for

$$\pm 1 \quad \pm 2 \quad \pm 3 \quad \pm 4 \quad \pm 5 \quad \pm 6 \quad \ldots \quad \pm n$$

where n is any whole number greater than 1.

Teaching Questions

1. Two students were discussing positive and negative numbers. One said positive numbers are for counting things you can see and negative numbers are for counting things you can't see. What would you say to help this student acquire a better understanding of negative numbers?

2. Suppose you were teaching about integers and integer operations to a middle school class and one of your students asked "What is the difference between the minus sign for subtraction and the minus sign for negative numbers?" How would you answer this question?

3. To help students gain ownership of their mathematics and to see that math symbols were invented by people, some teachers encourage students to develop their own symbols. If you were introducing integers by using the black and red chips model, explain why you would or why you would not encourage students to invent nonstandard symbols for these numbers.

4. Many elementary texts introduce integers using raised signs ($^{-}3$, $^{+}2$, etc.). Other texts do not use raised signs (-3, $+2$, etc.). Give a reason for using raised signs and a reason for not using raised signs. Make a case for which method you prefer.

Classroom Connections

1. On page 262, in the example from the **Elementary School Text,** two methods for modeling $^{-}8 + 6$ are illustrated. (a) Draw sketches to illustrate how you can determine $6 + ^{-}4$ with each model. (b) This example also says the rule "The sum of a positive integer and a negative integer is sometimes positive, sometimes negative, and sometimes zero" is often helpful when adding integers. Give an example where the sum of a positive integer and a negative integer is zero. Which of the two methods do you think would be the most helpful for illustrating this third case? Explain.

2. The **Standards** quote on page 267 notes that the number line is an appropriate model for introducing negative integers. Sketch a number line and label the integers from $^{-}10$ to 10. Devise a method of illustrating the following operations on your number line and include an explanation of how the operations of addition and subtraction will be carried out.

a. $3 - 4$ **b.** $^{-}2 + ^{-}3$

c. $3 + ^{-}5$ **d.** $^{-}3 - ^{-}2$

3. Look through the **PreK–2, Grades 3–5,** and **Grades 6–8 Standards—Number and Operations** (see inside front and back covers) and determine the level and the expectation that recommends integers be introduced into the curriculum. Then determine the level and the expectation that recommends introducing integer computations and integer operations.

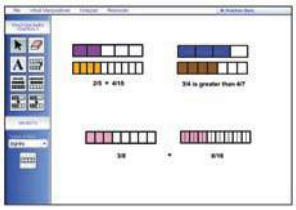
4. The **Historical Highlight** in this section documents some of the historical resistance to the acceptance of negative numbers. These numbers were called "absurd numbers" at one time. Do a search of math history books and/or the Internet for additional details on the history and development of negative numbers. Write a paragraph with details that would be interesting to school students.

5. The **Research** statement on page 258 brings up a problem with introducing the number line to model positive integers. Explain some ways to use the number line with elementary school students to avoid this problem.

MATH ACTIVITY 5.2

Equality and Inequality with Fraction Bars

Virtual Manipulatives

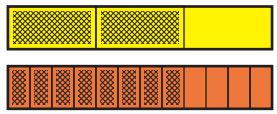


www.mhhe.com/bbn

Purpose: Use Fraction Bars to illustrate fraction equality, inequality and common denominators.

Materials: Fraction Bars in the Manipulative Kit or Virtual Manipulatives.

1. The fractions for these bars are equal because both bars have the same shaded amount. In the deck of 32 Fraction Bars there are 3 bars whose fractions equal $\frac{2}{3}$. Sort your deck of bars into piles so bars with the same shaded amounts are in the same pile.



$\frac{2}{3} = \frac{8}{12}$

a. Find all the fractions from the bars that equal the following fractions, and write equalities: $\frac{1}{2}$, $\frac{4}{6}$, $\frac{1}{4}$, $\frac{9}{12}$, and $\frac{1}{3}$.

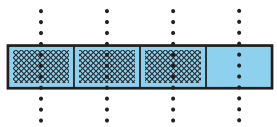
b. If a bar is all shaded, it is called a **whole bar** and its fraction is equal to 1. If a bar has no parts shaded, it is called a **zero bar** and its fraction is equal to 0. List all the fractions for the whole bars and zero bars.

2. A fraction is **not in lowest terms** if the numerator and denominator have a common factor greater than 1. List all the fractions from the deck other than those equal to 0 or 1 that are not in lowest terms. Then write each fraction in lowest terms.

***3.** A $\frac{3}{4}$ bar has more shading than a $\frac{2}{3}$ bar, so $\frac{3}{4} > \frac{2}{3}$. For each of the following pairs of fractions, find a fraction from the deck that is greater than the smaller fraction and less than the larger fraction.

a. $\frac{1}{2}$ and $\frac{2}{3}$ **b.** $\frac{1}{6}$ and $\frac{1}{3}$ **c.** $\frac{3}{4}$ and $\frac{11}{12}$ **d.** $\frac{2}{3}$ and $\frac{5}{6}$

4. Each part of this $\frac{3}{4}$ bar has been split into 2 equal parts. There are now 8 equal parts, and 6 of these are shaded. This illustrates the equality $\frac{3}{4} = \frac{6}{8}$.

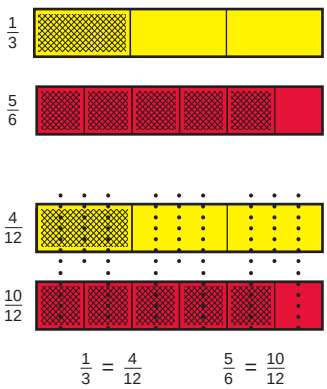


$\frac{3}{4} = \frac{6}{8}$

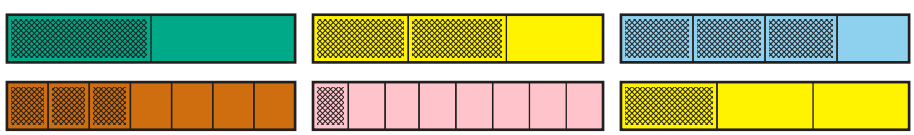
a. Find bars from the deck for each of the following fractions: $\frac{1}{4}$, $\frac{2}{3}$, $\frac{11}{12}$, $\frac{1}{2}$, $\frac{5}{6}$. Split each part of the bar into two equal parts to illustrate another fraction, and write equalities for these pairs of fractions.

b. Use the bars in part a and split each part of each bar into three equal parts to illustrate another fraction. Write equalities for these pairs.

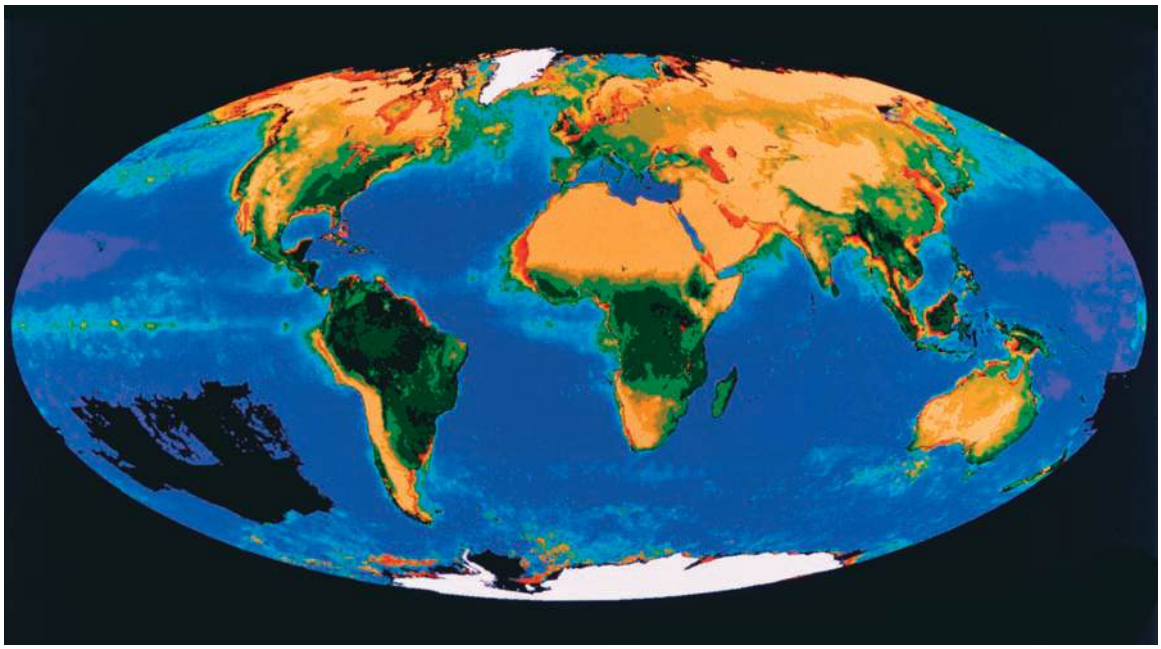
5. The top two bars shown at the left have different-size parts. Sometimes it is necessary to further subdivide the parts of a bar so that both bars have parts of the same size. If each part of the $\frac{1}{3}$ bar is divided into 4 equal parts and each part of the $\frac{5}{6}$ bar is divided into 2 equal parts, both bars will have 12 equal parts. Divide the parts of the following pairs of bars so that both bars have parts of the same size and write a pair of equations to show the fraction equalities (see examples at left). What fraction concept is being modeled by this activity?



$\frac{1}{3} = \frac{4}{12}$ $\frac{5}{6} = \frac{10}{12}$



Section 5.2 INTRODUCTION TO FRACTIONS



This composite image of the global biosphere was obtained by combining data from several satellites and shows the productive potential of the Earth's vegetation. The red and orange regions indicate microscopic sea plants that form the base of the marine food chain. The light and dark shades of green indicate tropical forests, farmlands, and savannas that have high potential for chlorophyll and leaf production, and the yellow shades show lower potential. The polar regions of the Earth's surface that involve ice, snow, glaciers, and frozen ground (permafrost) have barren conditions for chlorophyll production.

PROBLEM OPENER

Three tired and hungry people had a bag of apples. While the other two were asleep, one of the three awoke, ate one-third of the apples, and went back to sleep. Later a second person awoke, ate one-third of the remaining apples, and went back to sleep. Finally, the third person awoke and ate one-third of the remaining apples, leaving 8 apples in the bag. How many apples were in the bag originally?

Fractions are often used to describe the geographical characteristics of the Earth's surface. For example, if the Earth's surface were divided into 3 equal parts, 2 of these parts would be water. Glaciers and ice sheets cover $\frac{1}{10}$ of the Earth's land, and permanently frozen soil, called permafrost, occupies $\frac{1}{4}$ of the land in the Northern Hemisphere. One-third of the Earth's land is covered by seasonal snows, and snowfall in the United States accounts for $\frac{2}{3}$ of the annual precipitation.

HISTORICAL HIGHLIGHT



Egyptian leather scroll describing simple relations between fractions (ca. 1700 B.C.E.)

The Egyptians were using fractions before 2500 B.C.E. With the exception of $\frac{2}{3}$, all Egyptian fractions were **unit fractions,** that is, fractions with a numerator of 1 ($\frac{1}{3}$, $\frac{1}{4}$, etc.).

In hieratic (sacred) writings like the scroll shown here, the Egyptians placed a dot above a numeral to represent a unit fraction. For example, $\wedge$ represents the number 30, and $\dot{\wedge}$ represents the fraction $\frac{1}{30}$. It is interesting to note that as late as the 18th century, over 3000 years after the ancient Egyptians used such symbols, the symbols $\frac{\bullet}{2}$ and $\frac{\bullet}{4}$ were used in English books for the fractions $\frac{1}{2}$ and $\frac{1}{4}$.

Whereas the Egyptians used fractions with fixed numerators, the Babylonians (ca. 2000 B.C.E.) used only fractions with denominators of 60 and 60^2. The fraction $\frac{1}{60}$ was referred to as "the first little part" and $(\frac{1}{60})^2$ as "the second little part." Our use of the minute, $\frac{1}{60}$ of an hour, and second, $(\frac{1}{60})^2$ of an hour, was handed down to us from the Babylonians.

Research Statement

Students with an understanding of the written symbols for fractions are able to connect them to other representations, such as physical objects, pictorial representations, and spoken language.

Wearne and Kouba

FRACTION TERMINOLOGY

The word *fraction* comes from the Latin word *fractio,* a form of the Latin word *frangere,* meaning *to break.* The terms *broken number* and *fragment* frequently were used in the past as synonyms for *fraction.* Historically, fractions were first used for amounts less than a whole unit. This is how children first encounter fractions: one-half of a candy bar, one-third of a pizza, etc. Today fractions also include numbers that are greater than or equal to 1.

The term **fraction** is used to refer both to a number written in the form $\frac{a}{b}$ and to the numeral $\frac{a}{b}$ with $b \neq 0$. You need not be concerned about the distinction between $\frac{a}{b}$ as a number and $\frac{a}{b}$ as a numeral; the meaning will be clear from the context. For example, when we say the top number of a fraction is called the **numerator** and the bottom number is called the **denominator,** we are thinking of the fraction as a symbol or numeral with two parts. On the other hand, when we say, "Add the fractions $\frac{1}{2}$ and $\frac{1}{3}$," we are thinking of fractions as numbers.

Children in the early grades use fractions whose numerators and denominators are whole numbers. In the later grades fractions are encountered whose numerators and denominators are integers. Fractions whose numerators and denominators are integers are also called *rational numbers* (see Section 6.1). In general, the numerator and denominator of a fraction can be any numbers as long as the denominator is not zero. (See Sections 6.3 and 6.4 for examples of fractions involving decimals and irrational numbers.)

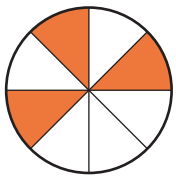
FRACTION CONCEPTS

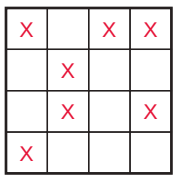
Three concepts of fractions will be illustrated in the following paragraphs: the *part-to-whole concept,* the *fraction-quotient concept,* and the *ratio concept.*

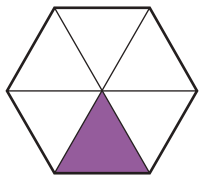
Part-to-Whole Concept The most common use of fractions involves the **part-to-whole concept,** that is, the use of a fraction to denote part of a whole. In the fraction $\frac{a}{b}$, the bottom number b indicates the number of equal parts in a whole, and the top number a indicates the number of parts being considered.

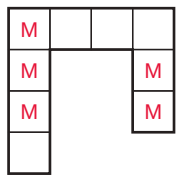
EXAMPLE A

Write the fraction for the shaded or lettered part of each figure.

1. 

2. 

3. 

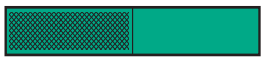
4. 

Solution 1. $\frac{3}{8}$ 2. $\frac{7}{16}$ 3. $\frac{1}{6}$ 4. $\frac{5}{9}$

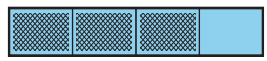
The Fraction Bars* model is a part-to-whole model for fractions in which the denominator of a fraction is represented by the number of equal parts in a bar and the numerator is the number of shaded parts.

EXAMPLE B

Write the fraction represented by the shaded portion of each bar.

1. 

2. 

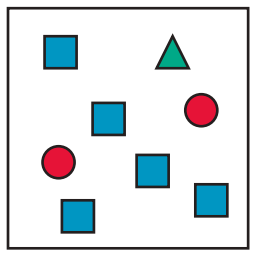
3. 

Solution 1. $\frac{1}{2}$ 2. $\frac{2}{3}$ 3. $\frac{3}{4}$

The *part-to-whole concept* of a fraction also is used in describing part of a set of individual objects.

EXAMPLE C

1. Write the fraction that shows what part of the following set is circles.
2. Write the fraction that shows what part of the set is squares.



Solution 1. $\frac{2}{8}$ of the objects are circles. 2. $\frac{5}{8}$ of the objects are squares.

Research Statement

In the 7th national mathematics assessment, fourth-grade students found representing a fraction to be easier if the unit was a region than if it was a set of objects.

Wearne and Kouba

Research Statement

Interpreting fractions and decimals on a number line is more difficult for students than interpreting fractions as part of geometric regions.

Kouba, Zawojewski, and Strutchens

Fractions can be located on a number line by using the part-to-whole concept. First select a **unit,** and then divide this interval into equal parts. To locate the fraction $\frac{r}{s}$, subdivide the unit interval into s equal parts, and beginning at 0, count off r of these parts. Figure 5.17 shows the sixths and tenths between $^{-}1$ and 1.

*Fraction Bars is a registered trademark of American Education Products, LLC

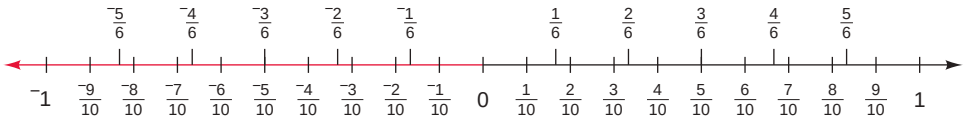
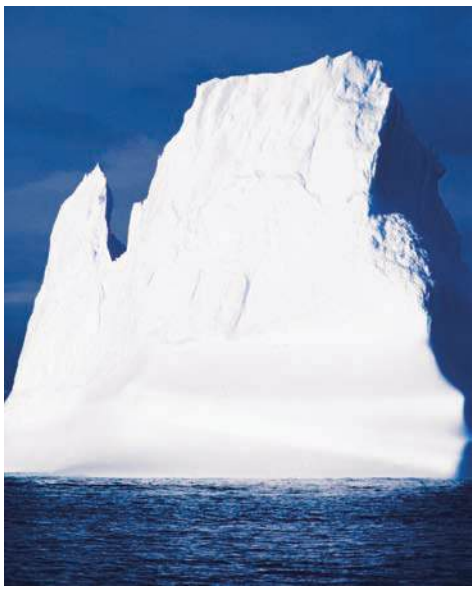


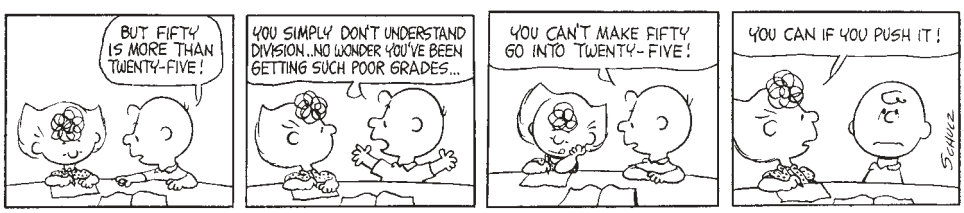
Figure 5.17

Applications of fractions that involve the part-to-whole concept are numerous. The fractional part of an iceberg that is under water is one example.



Eight-ninths of the volume of an iceberg is under water.

Fraction-Quotient Concept Another use of fractions arises from the division of one number by another. This is called the **fraction-quotient concept.** In the following cartoon Charlie Brown's little sister Sally has a problem. She is trying to divide 25 by 50. Charlie Brown's comment shows that he thinks of division only in terms of the *measurement concept,* that is, "How many 50s are in 25?" However, there is another approach to division. Remember that Section 3.4 discussed two concepts of division: *measurement (subtractive)* and *sharing (partitive).* With the *sharing concept,* dividing by 50 means there will be 50 parts. If we divide 25 objects of equal size, such as sticks of gum, into 50 equal parts, each part will be one-half of a stick: $25 \div 50 = \frac{1}{2}$.



Peanuts © UFS. Reprinted by Permission.

NCTM Standards

In addition to work with whole numbers, young children should also have some experience with simple fractions through connections to everyday situations. p. 82

Here is another illustration of the fraction-quotient concept. Figure 5.18 shows 3 whole bars placed end to end. To compute $3 \div 4$, we use the sharing concept of division and divide the 3 bars into 4 equal parts. This can be done by dividing the 3-bar in half and then dividing each half in half, as shown by the dashed lines. Comparing one of these four parts to a $\frac{3}{4}$ bar shows that $3 \div 4 = \frac{3}{4}$.

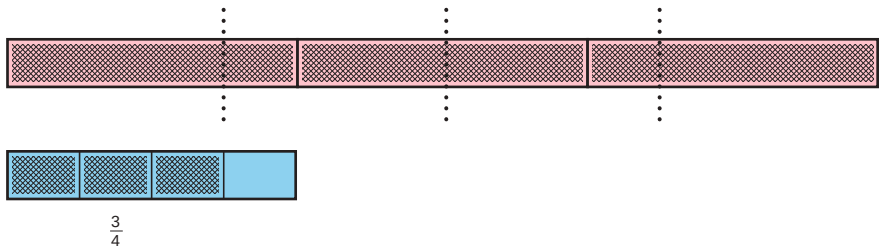


Figure 5.18

Both of the preceding examples show that a fraction results from the division of one whole number by another. This relationship between fractions and division of numbers is stated here:

> For any numbers a and b, with $b \neq 0$,
>
> $$\frac{a}{b} = a \div b$$

One of the major influences in the early development of fractions was the need to solve problems involving division of whole numbers. A problem found in Egyptian writings from 1650 B.C.E. requires that 4 loaves of bread be divided equally among 10 people. According to the above relationship, $4 \div 10 = \frac{4}{10}$, and each person would receive $\frac{4}{10}$ of a loaf of bread. Figure 5.19 shows 4 loaves of bread divided into 10 equal parts.

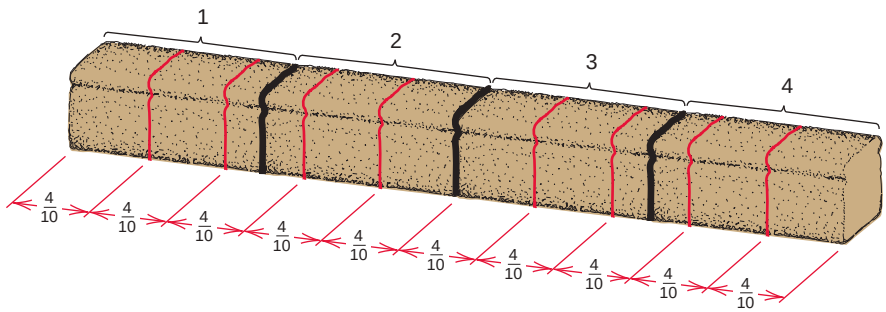


Figure 5.19

EXAMPLE D

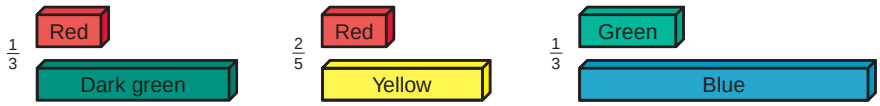
Answer each question with a fraction.

1. Two gallons of cider are poured into 7 containers in equal amounts. How much cider is there in each container?
2. Four acres of land are divided equally into 15 parts. How much land is in each part?

Solution 1. $\frac{2}{7}$ gallon 2. $\frac{4}{15}$ acre

Ratio Concept Another use of fractions involves the **ratio concept.** In this case, fractions are used to compare one amount to another. For example, we might say that a boy's height is one-third of his mother's height.

The *ratio concept* of fractions can be illustrated with Cuisenaire Rods by comparing the lengths of two rods (see Figure 5.20). It takes 3 red rods to equal the length of 1 dark green rod in part a, so the length of the red rod is $\frac{1}{3}$ the length of the dark green rod. If the length of the dark green rod is chosen as the unit, then the red rod represents $\frac{1}{3}$. If a different unit is selected, the red rod will represent a different fraction. For example, if the yellow rod in part b is the unit length, then the red rod represents $\frac{2}{5}$, because the length of the red rod is $\frac{2}{5}$ the length of the yellow rod. Several different rods can represent the same fraction, depending on the choice of unit. In part a, $\frac{1}{3}$ is represented by a red rod, but if the unit is the length of the blue rod, as in part c, then the green rod represents $\frac{1}{3}$.



(a) (b) (c)

Figure 5.20

Figure 5.21 provides another example of the ratio concept of a fraction. The length of the San Andreas Fault is compared to the length of California's coastal region. Ratios are discussed in greater detail in Section 6.3.

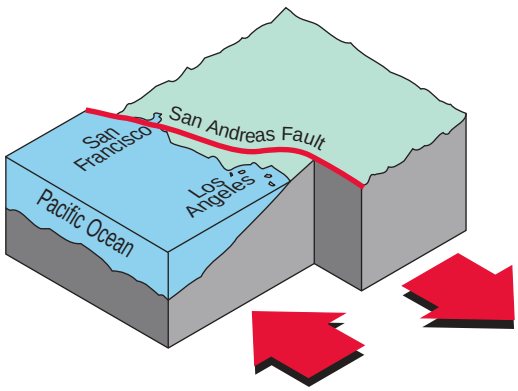


Figure 5.21
The San Andreas Fault runs three-fourths of the length of California's coastal region.

EXAMPLE E

It takes 4 red rods to equal the length of 1 brown rod and 5 white rods to equal the length of 1 yellow rod.

1. If the brown rod is the unit, what is the length of 1 red rod?
2. If the brown rod is the unit, what is the length of 3 red rods?
3. If the yellow rod is the unit, what is the length of 3 white rods?

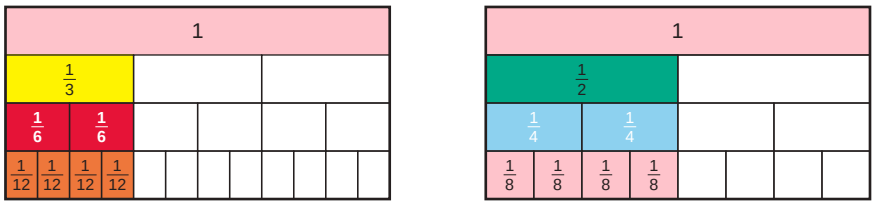
Solution 1. $\frac{1}{4}$ 2. $\frac{3}{4}$ 3. $\frac{3}{5}$

NCTM Standards

By using an area model in which part of a region is shaded, students can see how fractions are related to a unit whole, compare fractional parts of a whole, and find equivalent fractions. p. 150

EQUALITY OF FRACTIONS

Equality of fractions can be illustrated by comparing parts of figures. The charts in Figure 5.22 show that $\frac{1}{3} = \frac{2}{6}$, $\frac{1}{3} = \frac{4}{12}$, $\frac{1}{2} = \frac{2}{4}$, etc.



(a) (b)

Figure 5.22

EXAMPLE F Write three more equalities that are illustrated by each of the charts in Figure 5.22.

Solution Here are some possibilities:

Part a: $\frac{2}{3} = \frac{4}{6}$, $\frac{2}{3} = \frac{8}{12}$, $\frac{3}{6} = \frac{6}{12}$, $1 = \frac{3}{3}$, and $1 = \frac{6}{6}$.

Part b: $\frac{1}{2} = \frac{4}{8}$, $\frac{1}{4} = \frac{2}{8}$, $\frac{3}{4} = \frac{6}{8}$, $1 = \frac{4}{4}$, and $1 = \frac{8}{8}$.

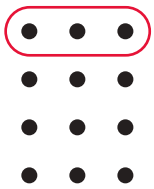
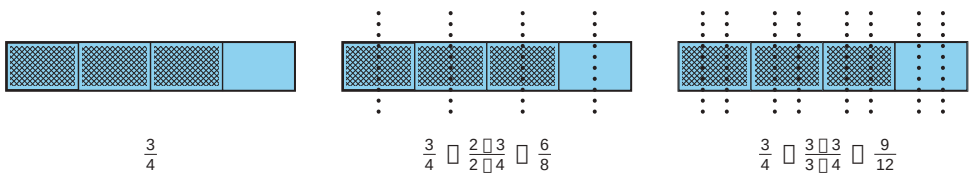


Figure 5.23

Equality of fractions can also be illustrated with sets of objects. For example, 3 out of 12, or $\frac{3}{12}$, of the points shown in Figure 5.23 are circled. Viewed in another way, $\frac{1}{4}$ of the points are circled, because there are 4 rows, each containing the same number of points, and 1 row is circled. So $\frac{3}{12}$ and $\frac{1}{4}$ are equivalent fractions representing the same amount.

For every fraction there are an infinite number of other fractions that represent the same number. The Fraction Bars in Figure 5.24 show one method of obtaining fractions equal to $\frac{3}{4}$. In part b, each part of the $\frac{3}{4}$ bar has been split into 2 equal parts to show that $\frac{3}{4} = \frac{6}{8}$. We see that doubling the number of parts in a bar also doubles the number of shaded parts. This is equivalent to multiplying both the numerator and denominator of $\frac{3}{4}$ by 2. Similarly, part c shows that splitting each part of a $\frac{3}{4}$ bar into 3 equal parts triples the number of parts in the bar and triples the number of shaded parts. This has the effect of multiplying both the numerator and denominator of $\frac{3}{4}$ by 3 and shows that $\frac{3}{4}$ is equal to $\frac{9}{12}$.



(a) (b) (c)

Figure 5.24

Research Statement

An understanding of equivalence of fractions is important in developing sense of relative size of fractions and helping students connect their intuitive understandings to more general methods.

Wearne and Kouba

The examples in Figure 5.24 illustrate the *fundamental rule for equality of fractions:* For any fraction, an **equal fraction** will be obtained by multiplying the numerator and denominator by a nonzero number.

Fundamental Rule for Equality of Fractions For any fraction $\frac{a}{b}$ and any number $k \neq 0$,

$$\frac{a}{b} = \frac{ka}{kb}$$

The rule for equality of fractions, $\frac{a}{b} = \frac{ka}{kb}$ when $k \neq 0$, holds because multiplying a number by 1 does not change the identity of the number, and for $k \neq 0$, $\frac{k}{k} = 1$; the identity for multiplication.

$$\frac{a}{b} = 1 \times \frac{a}{b} = \frac{k}{k} \times \frac{a}{b} = \frac{ka}{kb}$$

Simplifying Fractions The definition of equality of fractions justifies a process called **simplifying fractions.** For example, since 6 and 15 have a common factor of 3, $\frac{6}{15}$ may be written as a fraction with a smaller numerator and denominator:

$$\frac{6}{15} = \frac{6 \div 3}{15 \div 3} = \frac{2}{5}$$

Figure 5.25 illustrates this equality. If the figure is viewed as a bar with 15 equal parts, 6 of which are shaded, it represents $\frac{6}{15}$. Grouping the 15 parts of the bar into 5 equal groups of 3 parts each, as shown by the dotted lines, produces a bar with 5 equal parts, 2 of which are shaded, to show that $\frac{6}{15} = \frac{2}{5}$.

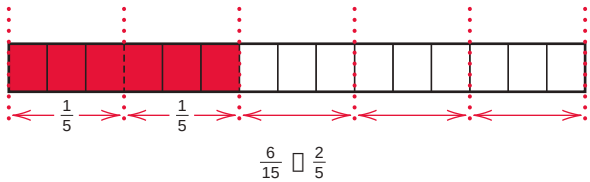


Figure 5.25

Whenever the numerator and denominator of a fraction have a common factor greater than 1, they can be divided by this factor to obtain a fraction with a smaller numerator and denominator. If the numerator and denominator of a fraction are divided by their greatest common factor (GCF), the resulting fraction is called the **simplest form** and the fraction is said to be in **lowest terms.**

EXAMPLE G

Write each fraction in simplest form.

1. $\frac{8}{12}$ 2. $\frac{24}{28}$ 3. $\frac{8}{16}$

Solution 1. $\frac{2}{3}$ 2. $\frac{6}{7}$ 3. $\frac{1}{2}$

Technology Connection

There are several different brands of calculators that display fractions. The keystrokes and view screens for two of these are shown in Figure 5.26.



Figure 5.26

Entering a fraction and pressing [=] (or Enter) will display the fraction in simplest form on most calculators designed for fractions. However, to help schoolchildren learn fraction concepts, some calculators like the one in Figure 5.26A have keys, such as [SIMP], for simplifying fractions. The following keystrokes show the use of the key [SIMP]. Look carefully at the view screens shown here on the right to see how this key simplifies the fraction $\frac{18}{24}$.

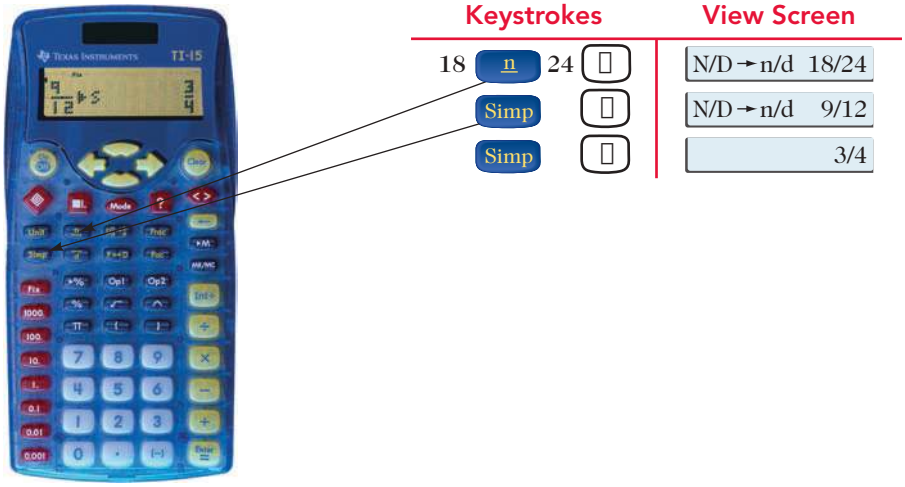


Figure 5.26A

You may have noticed from the above view screens that the fraction was simplified by a factor of 2 and then by a factor of 3. Each use of [SIMP] divides the numerator and denominator of the fraction by the smallest common factor greater than 1. Also, as can be seen in the above view screens, when the fraction is not in simplest form, this is indicated by N/D → n/d to show that further simplification is possible. It is also possible to select a common factor for simplifying a fraction. Suppose $\frac{12}{30}$ is on the calculator view screen: pressing [SIMP] [3] will replace the fraction by $\frac{4}{10}$; or pressing [SIMP] [6] will replace the fraction by $\frac{2}{5}$; but pressing [SIMP] 7 leaves $\frac{12}{30}$ unchanged.

Some calculators with a key for simplifying fractions display common factors of the numerator and denominator on the view screen (Figure 5.26B). The view screens shown below begin with $\frac{18}{24}$. The first time [SIMP] is pressed, the common factor 2 of 18 and 24 flashes briefly on the screen, and the fraction $\frac{9}{12}$ is displayed. The second time [SIMP] is pressed, the common factor 3 of 9 and 12 flashes briefly on the screen and $\frac{3}{4}$ is displayed.

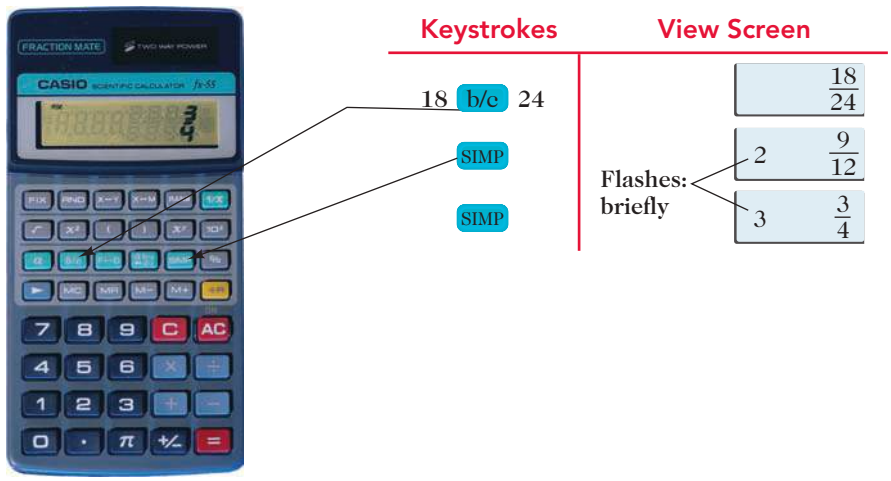


Figure 5.26B

When fractions are entered into some calculators, they appear on the view screen from left to right and several fractions can be seen at the same time (Figure 5.26C). If more fractions are entered than can be displayed on the view screen, previously entered fractions are moved off the screen but are retained internally in the calculator's memory.



Figure 5.26C

EXAMPLE H

Determine which fractions are in lowest terms. If a fraction is not in lowest terms, write the fraction in simplest form.

1. $\frac{4}{15}$ **2.** $\frac{-8}{24}$ **3.** $\frac{18}{30}$ **4.** $\frac{10}{21}$ **5.** $\frac{10}{-15}$

Solution **1.** $\frac{4}{15}$ is in lowest terms. **2.** GCF(8, 24) = 8, so $\frac{-8}{24} = \frac{-1}{3}$. **3.** GCF(18, 30) = 6, so $\frac{18}{30} = \frac{3}{5}$. **4.** $\frac{10}{21}$ is in lowest terms. **5.** GCF(10, 15) = 5, so $\frac{10}{-15} = \frac{2}{-3}$.

Research Statement

One implication for teachers from the 7th national mathematics assessment, is that there is a need to develop more fully the basic notions of rational numbers.

Wearne and Kouba

COMMON DENOMINATORS

One of the more important skills in the use of fractions is to replace two fractions with different denominators by two fractions with equal denominators. The fractions $\frac{1}{6}$ and $\frac{1}{4}$ have different denominators, and the Fraction Bars representing these fractions have different numbers of parts (see bars on left in Figure 5.27). If each part of the $\frac{1}{6}$ bar is split into 2 equal parts and each part of the $\frac{1}{4}$ bar is split into 3 equal parts, both bars will have 12 equal parts. The fractions for these new bars, $\frac{2}{12}$ and $\frac{3}{12}$, have a **common denominator** of 12.

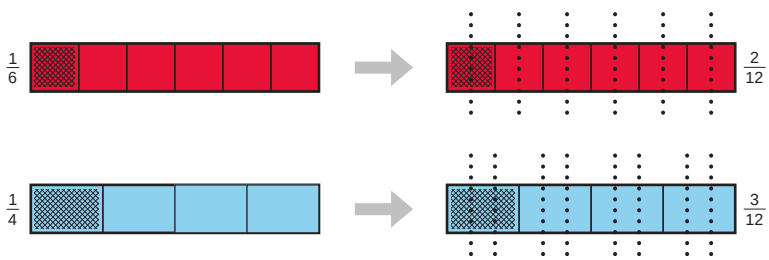


Figure 5.27

13-4 Equivalent Fractions

MAIN IDEA
I will find equivalent fractions.

New Vocabulary
equivalent fractions

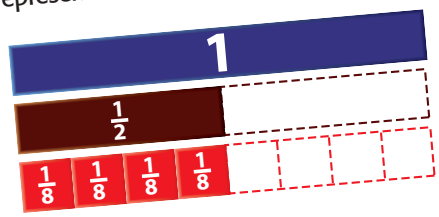
Math Online
macmillanmh.com
- Extra Examples
- Personal Tutor
- Self-Check Quiz

GET READY to Learn

Megan has 8 fish in an aquarium. Four fish are green. So, Megan says that $\frac{4}{8}$ of the fish are green. Megan could use another fraction to represent $\frac{4}{8}$.



The fraction models below show that $\frac{4}{8}$ is the same as $\frac{1}{2}$. Fractions that represent the same amount are **equivalent fractions**.



EXAMPLE 1 Find Equivalent Fractions

Find three fractions that are equivalent to $\frac{4}{8}$.

To find equivalent fractions, you can use multiplication or division.

One Way: Multiply			Another Way: Divide		
$\frac{4 \times 2}{8 \times 2} = \frac{8}{16}$	Multiply the numerator and the denominator by the same number, 2.		$\frac{4 \div 2}{8 \div 2} = \frac{2}{4}$	Divide the numerator and the denominator by the same number, 2.	
			$\frac{2 \div 2}{4 \div 2} = \frac{1}{2}$		

So, $\frac{8}{16}$, $\frac{2}{4}$, or $\frac{1}{2}$ are equivalent to $\frac{4}{8}$.

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NCTM Standards

Obtaining the same number of equal parts for two bars is a visual way of finding a common denominator. The *Curriculum and Evaluation Standards for School Mathematics* (p. 92) notes the importance of using models to understand the common ideas underlying different types of numbers:

> . . . to compare $\frac{2}{3}$ and $\frac{3}{4}$, students can use concrete materials to represent them as $\frac{8}{12}$ and $\frac{9}{12}$, respectively, and then conclude that $\frac{8}{12}$ is less than $\frac{9}{12}$, since 8 is less than 9. Thus, they learn that comparing fractions is like comparing whole numbers once common denominators have been identified.

Another method for finding common denominators of two fractions is to list the multiples of their denominators. The arrows in Figure 5.28 point to the common multiples of 6 and 4. The least common multiple (LCM) of 6 and 4 is 12. This is also the least common denominator for $\frac{1}{6}$ and $\frac{1}{4}$. In general, the **least common denominator** of two fractions is the least common multiple of their denominators.

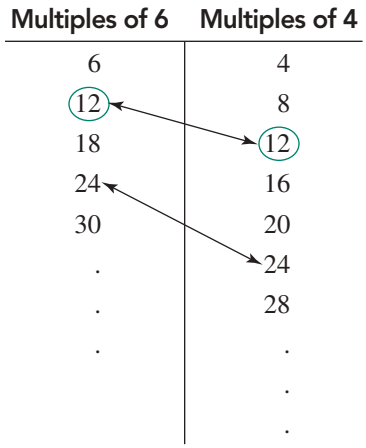


Figure 5.28

Once a common denominator has been found, two fractions can be replaced by fractions having the same denominator.

$$\frac{1}{6} = \frac{2 \times 1}{2 \times 6} = \frac{2}{12} \qquad \frac{1}{4} = \frac{3 \times 1}{3 \times 4} = \frac{3}{12}$$

EXAMPLE I

Replace the fractions in each pair by equal fractions having the least common denominator.

1. $\frac{3}{4}$ and $\frac{2}{5}$ **2.** $\frac{7}{12}$ and $\frac{-3}{8}$ **3.** $\frac{1}{6}$ and $\frac{5}{18}$ **4.** $\frac{1}{-4}$ and $\frac{3}{7}$

Solution **1.** LCM(4, 5) = 20; $\frac{3}{4} = \frac{15}{20}$ and $\frac{2}{5} = \frac{8}{20}$. **2.** LCM(12, 8) = 24; $\frac{7}{12} = \frac{14}{24}$ and $\frac{-3}{8} = \frac{-9}{24}$. **3.** LCM(6, 18) = 18; $\frac{1}{6} = \frac{3}{18}$, $\frac{5}{18}$ does not need to be replaced. **4.** First replace $\frac{1}{-4}$ by a fraction with a positive denominator $\left(\frac{-1}{4}\right)$. LCM(4, 7) = 28; $\frac{-1}{4} = \frac{-7}{28}$ and $\frac{3}{7} = \frac{12}{28}$.

Sometimes it is convenient to replace a fraction involving negative signs by equal fractions. In problem 4 of Example I, a fraction with a negative denominator was replaced by a fraction with a positive denominator. The rules for making such substitutions are shown on the next page and follow directly from the *rules of signs for division of integers* on page 271.

Rules of Signs for Fractions For a and b with $b \neq 0$,

$$\frac{a}{-b} = \frac{-a}{b} = -\left(\frac{a}{b}\right) \quad \text{and} \quad \frac{-a}{-b} = \frac{a}{b}$$

$$\frac{5}{-12} = \frac{-5}{12} = -\left(\frac{5}{12}\right) \quad \text{and} \quad \frac{-3}{-10} = \frac{3}{10}$$

We can determine whether two fractions are equal by obtaining their common denominators. Consider the fractions $\frac{19}{62}$ and $\frac{8}{27}$. Using the fundamental rule for equality of fractions, we can multiply the numerator and denominator of $\frac{19}{62}$ by 27 and the numerator and denominator of $\frac{8}{27}$ by 62.

$$\frac{19}{62} = \frac{27 \times 19}{27 \times 62} \quad \text{and} \quad \frac{8}{27} = \frac{62 \times 8}{62 \times 27}$$

These fractions now have a common denominator, and they are equal if and only if their numerators, 27×19 and 8×62, are equal. Are the fractions equal? This approach to determining equality suggests the following test for equality of fractions.

Test for Equality of Fractions For any fractions $\frac{a}{b}$ and $\frac{c}{d}$,

$$\frac{a}{b} = \frac{c}{d} \quad \text{if and only if} \quad ad = bc$$

EXAMPLE J

Use the test for equality of fractions to determine which pairs of fractions are equal.

1. $\frac{42}{105}$ and $\frac{14}{35}$ **2.** $\frac{-2}{3}$ and $\frac{2}{-3}$ **3.** $\frac{8}{71}$ and $\frac{5}{43}$ **4.** $\frac{3}{-9}$ and $\frac{4}{-12}$

Solution **1.** $42 \times 35 = 105 \times 14 = 1470$, so $\frac{42}{105} = \frac{14}{35}$. **2.** $(-2)(-3) = (3)(2) = 6$, so $\frac{-2}{3} = \frac{2}{-3}$.
3. $8 \times 43 = 344 \neq 71 \times 5 = 355$, so $\frac{8}{71} \neq \frac{5}{43}$. **4.** $3 \times -12 = -9 \times 4 = -36$, so $\frac{3}{-9} = \frac{4}{-12}$.

NCTM Standards

In grades 3–5 students should be able to reason about numbers by, for instance, explaining that $\frac{1}{2} + \frac{3}{8}$ must be less than 1 because each addend is less than or equal to $\frac{1}{2}$. p. 33

INEQUALITY

A tower of bars such as the one in Figure 5.29 illustrates many different inequalities of fractions.

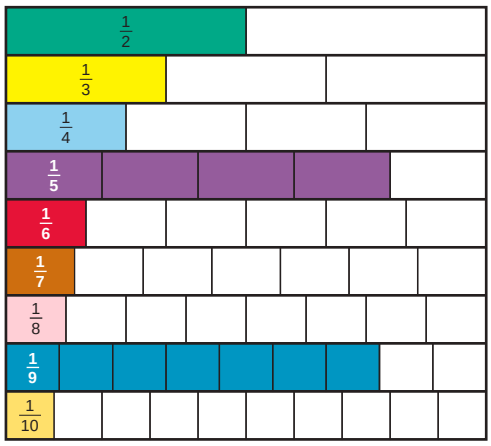


Figure 5.29

EXAMPLE K

Place the edge of a piece of paper on the bars in Figure 5.29 on the preceding page to determine an inequality, either $<$ or $>$, for each of the following pairs of fractions. The first pair of fractions has been colored in for you in Figure 5.29.

1. $\frac{4}{5}$ and $\frac{7}{9}$ **2.** $\frac{5}{8}$ and $\frac{2}{3}$ **3.** $\frac{2}{7}$ and $\frac{3}{10}$ **4.** $\frac{3}{4}$ and $\frac{5}{7}$

Solution **1.** $\frac{4}{5} > \frac{7}{9}$ **2.** $\frac{5}{8} < \frac{2}{3}$ **3.** $\frac{2}{7} < \frac{3}{10}$ **4.** $\frac{3}{4} > \frac{5}{7}$

EXAMPLE L

List a few other inequalities of fractions that are illustrated in Figure 5.29. What patterns of inequalities are there?

Solution Here are a few of the many inequalities:

$$\frac{1}{4} < \frac{1}{3} \quad \frac{2}{3} < \frac{3}{4} \quad \frac{1}{2} < \frac{2}{3} \quad \frac{5}{8} < \frac{6}{9} \quad \frac{1}{6} > \frac{1}{7} \quad \frac{4}{5} > \frac{3}{4}$$

The left edge of Figure 5.29 shows a pattern of decreasing inequalities

$$\frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \frac{1}{5} > \frac{1}{6} > \frac{1}{7} > \frac{1}{8} > \frac{1}{9} > \frac{1}{10}$$

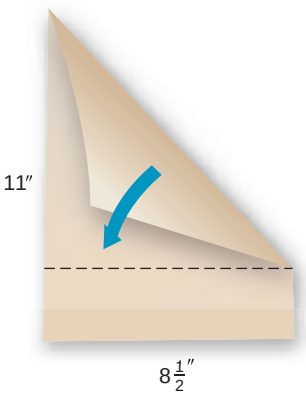
and the right edge shows an increasing pattern

$$\frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} < \frac{5}{6} < \frac{6}{7} < \frac{7}{8} < \frac{8}{9} < \frac{9}{10}$$

Laboratory Connection

Paper Folding

If you had a standard $8\frac{1}{2}$-inch by 11-inch sheet of paper and no ruler, could you obtain the whole number lengths from 1 to 10 inches by paper folding? This investigation will help you find whole and mixed number lengths by paper folding. Two folds, the first of which is shown here, bends the top right corner down to the left edge of the paper. A second fold can be used to produce a length of 6 inches.



Mathematics Investigation
Chapter 5, Section 2
www.mhhe.com/bbn

One of the reasons for finding a common denominator for two fractions is to be able to determine the greater fraction. It is difficult to determine whether $\frac{5}{8}$ or $\frac{3}{5}$ is greater without first replacing them by fractions having a common denominator. The least common multiple of 8 and 5 is 40. Replacing both $\frac{5}{8}$ and $\frac{3}{5}$ by fractions having a denominator of 40, we see that $\frac{5}{8}$ is the greater fraction.

$$\frac{5}{8} = \frac{5 \times 5}{5 \times 8} = \frac{25}{40} \qquad \frac{3}{5} = \frac{8 \times 3}{8 \times 5} = \frac{24}{40}$$

In general, an inequality for two fractions $\frac{a}{b}$ and $\frac{c}{d}$ with positive denominators can be determined by replacing them with fractions having a common denominator

$$\frac{ad}{bd} \quad \text{and} \quad \frac{bc}{bd}$$

and comparing their numerators ad and bc.

Test for Inequality of Fractions For any fractions $\frac{a}{b}$ and $\frac{c}{d}$, with b and d positive,

$$\frac{a}{b} < \frac{c}{d} \quad \text{if and only if} \quad ad < bc$$

and

$$\frac{a}{b} > \frac{c}{d} \quad \text{if and only if} \quad ad > bc$$

EXAMPLE M

Determine an inequality for each pair of fractions.

1. $\frac{3}{7}$ and $\frac{4}{10}$ **2.** $\frac{3}{4}$ and $\frac{7}{9}$ **3.** $\frac{^-3}{7}$ and $\frac{^-2}{5}$ **4.** $\frac{1}{2}$ and $\frac{1}{^-3}$

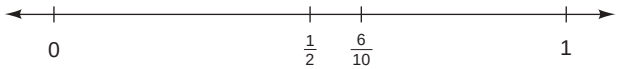
Solution **1.** $3 \times 10 > 7 \times 4$, so $\frac{3}{7} > \frac{4}{10}$. **2.** $3 \times 9 < 4 \times 7$, so $\frac{3}{4} < \frac{7}{9}$. **3.** $^-3 \times 5 < 7 \times {}^-2$, so $\frac{^-3}{7} < \frac{^-2}{5}$. **4.** $\frac{1}{2} > \frac{1}{^-3}$ because $\frac{1}{2}$ is positive and $\frac{1}{^-3}$ is negative.

DENSITY OF FRACTIONS

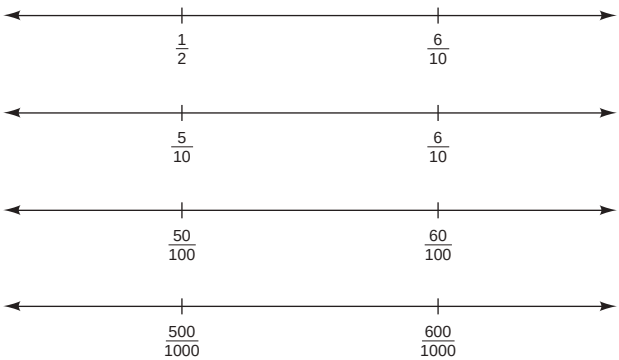
The integers are evenly spaced on the number line, and for any integer there is a "next" integer, both to its right and to its left. For fractions, however, this is not true.

EXAMPLE N

There is no single fraction that is the *next one* greater than $\frac{1}{2}$. Find a fraction that is between $\frac{1}{2}$ and $\frac{6}{10}$.



Solution One method of finding fractions between two given fractions is to express both fractions with a larger common denominator. The following figure shows several sections of a number line with fractions that are equal to $\frac{1}{2}$ and $\frac{6}{10}$. By increasing the denominators, we can easily find fractions between $\frac{1}{2}$ and $\frac{6}{10}$. For example, the third number line shows that the 9 fractions $\frac{51}{100}, \frac{52}{100}, \ldots, \frac{59}{100}$ are between $\frac{1}{2}$ and $\frac{6}{10}$, and the fourth number line shows that the 99 fractions $\frac{501}{1000}, \frac{502}{1000}, \ldots, \frac{599}{1000}$ are between $\frac{1}{2}$ and $\frac{6}{10}$.



Similarly, there is no fraction next to zero. To state this another way, there is no smallest fraction greater than zero. The following sequence of fractions gets closer and closer to zero, but no matter how far we go in this sequence, these fractions will always be greater than zero.

$$\frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{8} \quad \frac{1}{16} \quad \frac{1}{32} \quad \frac{1}{64} \quad \frac{1}{128} \quad \cdots$$

These examples are special cases of the more general fact that between any two fractions there is always another fraction. We refer to this property by saying that the fractions are **dense.** Because of this property of denseness, there are an infinite number of fractions between any two fractions.

MIXED NUMBERS AND IMPROPER FRACTIONS

Historically, a fraction stood for part of a whole and represented a number less than 1. The idea that there could be fractions such as $\frac{4}{4}$ or $\frac{5}{4}$ with numerators greater than or equal to the denominator was uncommon even as late as the sixteenth century. Such fractions are called **improper fractions,** and as their name indicates, at one time they were not thought of as authentic fractions.

When improper fractions are written as a combination of whole numbers and fractions, they are called **mixed numbers.** The numbers here are examples of mixed numbers:

$$1\frac{1}{5} \qquad 2\frac{3}{4} \qquad 4\frac{2}{3} \qquad 15\frac{1}{8}$$

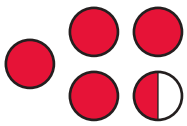
Placing a whole number and a fraction side by side, as in mixed numbers, indicates the sum of the two numbers. For example, $1\frac{1}{5}$ means $1 + \frac{1}{5}$ and $^{-}2\frac{3}{4} = {}^{-}2 + \frac{^{-}3}{4}$. This fact is used in converting a mixed number to an improper fraction. For example,

$$1\frac{1}{5} = 1 + \frac{1}{5} = \frac{5}{5} + \frac{1}{5} = \frac{6}{5} \quad \text{and} \quad {}^{-}2\frac{3}{4} = {}^{-}2 + \frac{^{-}3}{4} = \frac{^{-}8}{4} + \frac{^{-}3}{4} = \frac{^{-}11}{4}$$

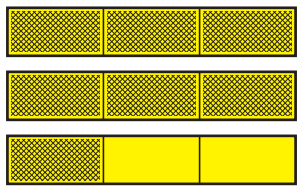
EXAMPLE O

Write a mixed number and an improper fraction to express the shaded amount of each figure. In part 1, each disk represents 1, and in part 2, each bar represents 1.

1.



2.



Solution **1.** $4\frac{1}{2}$ or $\frac{9}{2}$ (Since each whole disk has 2 halves, a total of 9 halves are shaded).

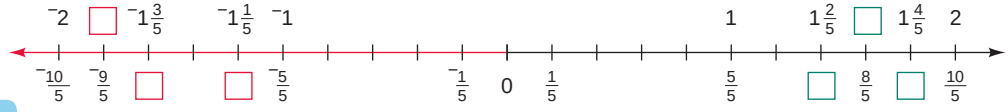
$$4\frac{1}{2} = 4 + \frac{1}{2} = \frac{8}{2} + \frac{1}{2} = \frac{9}{2}$$

2. $2\frac{1}{3}$ or $\frac{7}{3}$ (Since each whole bar has 3 thirds, a total of 7 thirds are shaded).

$$2\frac{1}{3} = 2 + \frac{1}{3} = \frac{6}{3} + \frac{1}{3} = \frac{7}{3}$$

EXAMPLE P

Write the missing mixed number above each improper fraction on the following number line, and write the missing improper fraction below each mixed number on the number line.



Solution $\frac{^{-}9}{5} = {}^{-}1\frac{4}{5}$, $^{-}1\frac{3}{5} = \frac{^{-}8}{5}$, $^{-}1\frac{1}{5} = \frac{^{-}6}{5}$, $1\frac{2}{5} = \frac{7}{5}$, $\frac{8}{5} = 1\frac{3}{5}$, and $1\frac{4}{5} = \frac{9}{5}$

Notice that in a negative mixed number the negative sign is only on the whole number part and yet the fraction is also understood to be negative. For example, $^{-}1\frac{1}{5}$ is equal to $^{-}1 + {}^{-}\left(\frac{1}{5}\right)$.

Research Statement

Performance suggests that many students are not proficient at locating points on the number line when mixed numbers are required.

Blume and Heckman

The number line in Example P on the preceding page indicates that the improper fractions $\frac{5}{5}$ and $\frac{10}{5}$ are equal to whole numbers. One method of illustrating such equalities is to use the relationship between a fraction and the division of whole numbers (the fraction-quotient concept on page 286) as shown below. For example, $\frac{5}{5}$ can be illustrated by 5 divided into 5 equal parts, as shown in Figure 5.30, and $\frac{10}{5}$ by 10 divided into 5 equal parts.

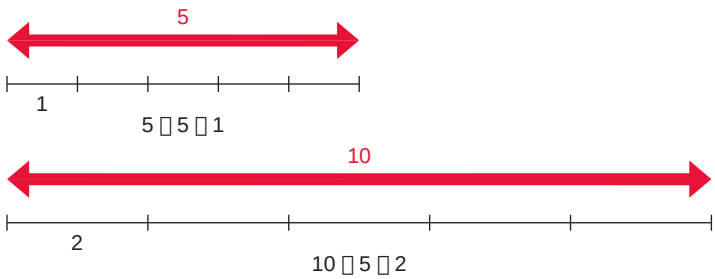


Figure 5.30

Technology Connection

Calculators that display fractions will convert improper fractions to mixed numbers. Some calculators will automatically replace an improper fraction that has been entered by a mixed number whose fraction is in simplified form. For example, the keystrokes shown here yield the fraction $\frac{30}{18}$, and pressing $=$ replaces the improper fraction by $1\frac{2}{3}$.

Keystrokes	View Screen
30 b/c 18	$\frac{30}{18}$
=	$1\frac{2}{3}$

In this next sequence of keystrokes, we see that once an improper fraction is entered and $=$ is pressed, it is replaced by a mixed number or integer, but it will not necessarily be in simplified form. These keystrokes also show the use of the key SIMP for simplifying fractions.

Keystrokes	View Screen
30 n 18 =	N/D→n/d $1\frac{12}{18}$
SIMP =	N/D→n/d $1\frac{6}{9}$
SIMP =	$1\frac{2}{3}$

Some calculators have a key for converting back and forth from an improper fraction to a mixed number, as illustrated below.

Keystrokes	View Screen
5 b/c 3	$\frac{5}{3}$
A b/c ↔ d/c	$1\frac{2}{3}$
A b/c ↔ d/c	$\frac{5}{3}$

MENTAL CALCULATIONS

NCTM Standards

The *Curriculum and Evaluation Standards for School Mathematics* (p. 58) states,

> Fraction symbols, such as $\frac{1}{4}$ and $\frac{2}{3}$, should be introduced only after children have developed the concepts and oral language necessary for symbols to be meaningful and should be carefully connected to both the models and oral language.

One reason for the use of models is to build number sense for mental calculations. Inequalities of fractions can often be determined by appealing to visual models. For example, the chart in Figure 5.31 shows why it is easy to determine an inequality for two unit fractions: *The more parts in a bar, the smaller the parts.*

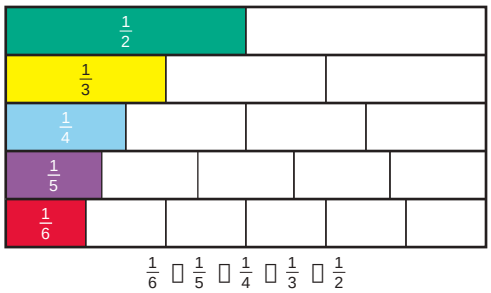


Figure 5.31

EXAMPLE Q

Use the observation in the preceding paragraph to mentally determine an inequality for each of the following pairs of fractions.

1. $\frac{1}{12}$ and $\frac{1}{5}$ **2.** $\frac{1}{9}$ and $\frac{1}{20}$ **3.** $\frac{1}{11}$ and $\frac{1}{3}$

Solution **1.** $\frac{1}{12} < \frac{1}{5}$ because a bar with 12 equal parts has smaller parts than a bar with 5 equal parts. **2.** $\frac{1}{9} > \frac{1}{20}$ because a bar with 9 equal parts has larger parts than a bar with 20 equal parts. **3.** $\frac{1}{11} < \frac{1}{3}$ because a bar with 11 equal parts has smaller parts than a bar with 3 equal parts.

It is also easy to compare two fractions mentally when the numerator of each fraction is 1 less than the denominator of the fraction. For example, to compare $\frac{4}{5}$ and $\frac{2}{3}$ (see Figure 5.32), we note that a $\frac{4}{5}$ bar is within $\frac{1}{5}$ of being a whole bar and a $\frac{2}{3}$ bar is within $\frac{1}{3}$ of being a whole bar. Since $\frac{1}{5} < \frac{1}{3}$, a $\frac{4}{5}$ bar is closer in length to a whole bar than a $\frac{2}{3}$ bar. So $\frac{4}{5} > \frac{2}{3}$.

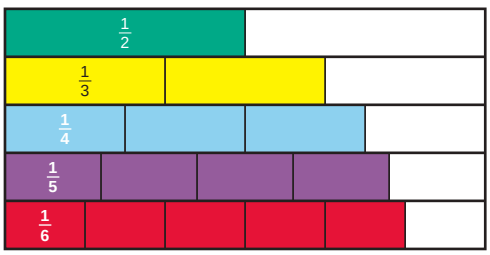


Figure 5.32

EXAMPLE R Mentally determine an inequality for each of the following pairs of fractions.

1. $\frac{9}{10}$ and $\frac{2}{3}$ **2.** $\frac{3}{4}$ and $\frac{14}{15}$ **3.** $\frac{19}{20}$ and $\frac{5}{6}$

Solution **1.** $\frac{9}{10} > \frac{2}{3}$ because $\frac{1}{10} < \frac{1}{3}$. **2.** $\frac{3}{4} < \frac{14}{15}$ because $\frac{1}{4} > \frac{1}{15}$. **3.** $\frac{19}{20} > \frac{5}{6}$ because $\frac{1}{20} < \frac{1}{6}$.

Comparing fractions to $\frac{1}{2}$ is also a convenient method of determining inequalities for some pairs of fractions. Consider the fractions $\frac{3}{7}$ and $\frac{5}{9}$. Figure 5.33 shows that if the numerator is less than one-half the denominator, the bar for the fraction is less than half shaded and the fraction is less than $\frac{1}{2}$. If the numerator is equal to one-half the denominator, the bar for the fraction is half shaded and the fraction is equal to $\frac{1}{2}$. If the numerator is greater than one-half the denominator, more than half of the bar for the fraction is shaded and the fraction is greater than $\frac{1}{2}$. Since $\frac{3}{7} < \frac{1}{2}$ and $\frac{5}{9} > \frac{1}{2}$, we know that $\frac{3}{7} < \frac{5}{9}$.

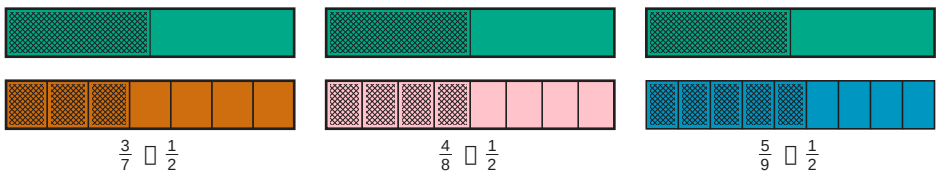


Figure 5.33

EXAMPLE S Mentally determine an inequality for each pair of fractions by comparing each fraction to $\frac{1}{2}$.

1. $\frac{5}{8}$ and $\frac{4}{11}$ **2.** $\frac{7}{15}$ and $\frac{5}{9}$ **3.** $\frac{9}{20}$ and $\frac{3}{5}$

Solution **1.** $\frac{5}{8} > \frac{1}{2}$ and $\frac{4}{11} < \frac{1}{2}$, so $\frac{5}{8} > \frac{4}{11}$. **2.** $\frac{7}{15} < \frac{1}{2}$ and $\frac{5}{9} > \frac{1}{2}$, so $\frac{7}{15} < \frac{5}{9}$. **3.** $\frac{9}{20} < \frac{1}{2}$ and $\frac{3}{5} > \frac{1}{2}$, so $\frac{9}{20} < \frac{3}{5}$.

ESTIMATION

Estimations involving fractions are often obtained by rounding. Rounding fractions and mixed numbers to the nearest whole number involves comparing fractions to $\frac{1}{2}$. If the fraction is less than $\frac{1}{2}$, it is rounded to 0; and if it is greater than $\frac{1}{2}$, it is rounded to 1. If the fraction is equal to $\frac{1}{2}$, it may be rounded up or down; in this text we will round such fractions up. Similarly, a mixed number is rounded down or up depending on whether the fraction rounds to 0 or to 1.

Rounding can be thought of by considering which whole number on a number line a given number is closest to (see Figure 5.34). For example, $2\frac{5}{8}$ is closer to 3 than to 2, so it rounds to 3; and $\frac{3}{8}$ is closer to 0 than to 1, so it rounds to 0.

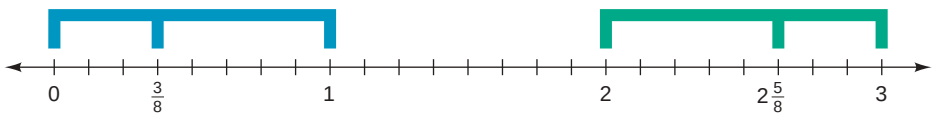


Figure 5.34

EXAMPLE T Round each fraction or mixed number to the nearest whole number.

1. $\frac{2}{5}$ **2.** $4\frac{3}{7}$ **3.** $\frac{8}{3}$ **4.** $7\frac{3}{6}$

Solution **1.** 0 **2.** 4 **3.** 3 **4.** 8

Another method of approximating fractions is to replace a fraction by a close approximation that is in simpler form. For example,

$$\frac{7}{22} \approx \frac{7}{21} = \frac{1}{3} \qquad \textit{and} \qquad 1\frac{4}{9} \approx 1\frac{3}{9} = 1\frac{1}{3}$$

EXAMPLE U Replace each fraction or mixed number by a close approximation that is in simpler form.

1. $4\frac{17}{30}$ **2.** $12\frac{5}{26}$ **3.** $\frac{4}{13}$ **4.** $\frac{19}{80}$

Solution Here are some possible replacements. Others may occur to you. **1.** $4\frac{17}{30} \approx 4\frac{15}{30} = 4\frac{1}{2}$. **2.** $12\frac{5}{26} \approx 12\frac{5}{25} = 12\frac{1}{5}$. **3.** $\frac{4}{13} \approx \frac{4}{12} = \frac{1}{3}$. **4.** $\frac{19}{80} \approx \frac{20}{80} = \frac{1}{4}$.

PROBLEM-SOLVING APPLICATION

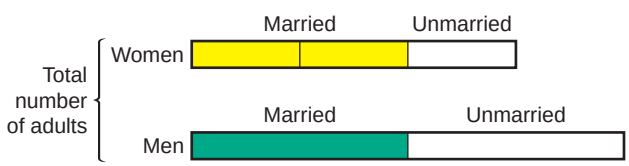
The following problem can be solved by *guessing and checking* or by *making a drawing*. Try to solve this problem before you read the solution. If you need assistance, read the paragraph "Understanding the Problem" and then try to find the solution.

Problem

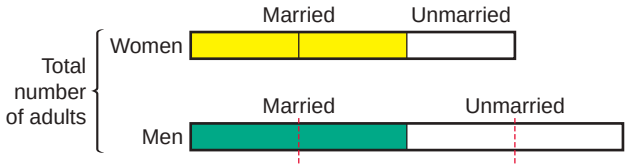
In a certain community two-thirds of the women are married and one-half of the men are married. No one in the community is married to a person outside the community. What fraction of the adults are unmarried?

Understanding the Problem Let's *guess* at some numbers to obtain a better understanding of the problem. For the number of women, we want to select a number that is divisible by 3 (12, 15, 18, 24, 30, 60, etc.) so that we can find $\frac{2}{3}$ of it. Suppose we choose a total of 24 women. In this case 16 women ($\frac{2}{3}$ of 24) are married. Since for each married woman there is a married man, there are 16 married men and thus 32 men in the community. **Question 1:** For this example, what fraction of the adults are married and what fraction are unmarried?

Devising a Plan Another approach is to *make a drawing* to see if it leads to a solution. The married women can be represented by shading $\frac{2}{3}$ of a figure and the married men by shading $\frac{1}{2}$ of a figure, as shown here. These two figures represent the total number of adults. **Question 2:** Why is the shaded region for the men equal to the shaded region for the women?



Carrying Out the Plan Let's continue the visual approach. If we divide in halves both the shaded and unshaded parts of the figure representing the men as shown here, each new part is the same size as each part of the figure representing the women. **Question 3:** How does this show that $\frac{4}{7}$ of the number of adults are married and $\frac{3}{7}$ are not married?

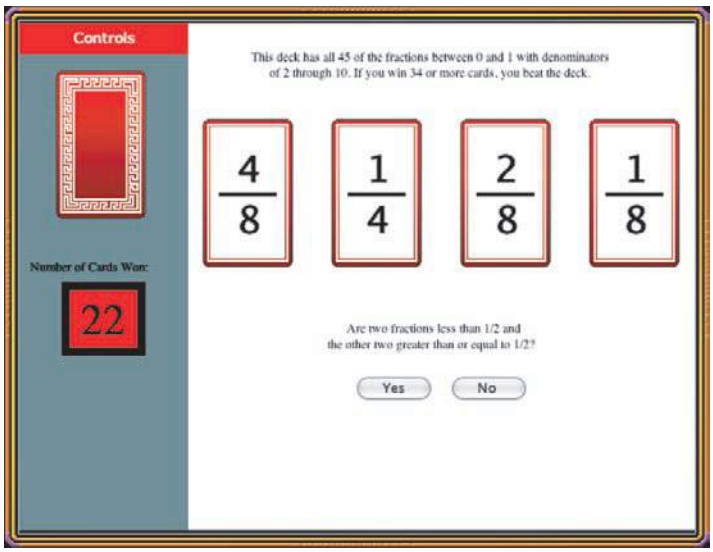


Looking Back The visual approach suggests that it is not necessary to know the number of women and men in the community; we will obtain the same answer regardless of the number of women used in a numerical approach. Suppose we select 30 as the number of women. Then there will be 20 married women ($\frac{2}{3}$ of 30 is 20) and 20 married men. Since $\frac{1}{2}$ of 40 is 20, there will be 40 men in the community and a total of 70 people. **Question 4:** In this case, how many people are not married, and what fraction of the people are not married?

Answers to Questions 1–4 **1.** $\frac{32}{56} = \frac{4}{7}$ are married and $\frac{24}{56} = \frac{3}{7}$ are unmarried. **2.** There is the same number of married men as married women in the community. **3.** There are a total of 7 equal parts: the 4 shaded parts represent the married people, so $\frac{4}{7}$ are married; the 3 unshaded parts represent the unmarried people, so $\frac{3}{7}$ are not married. **4.** 30 people are not married, so $\frac{30}{70} = \frac{3}{7}$ are not married.

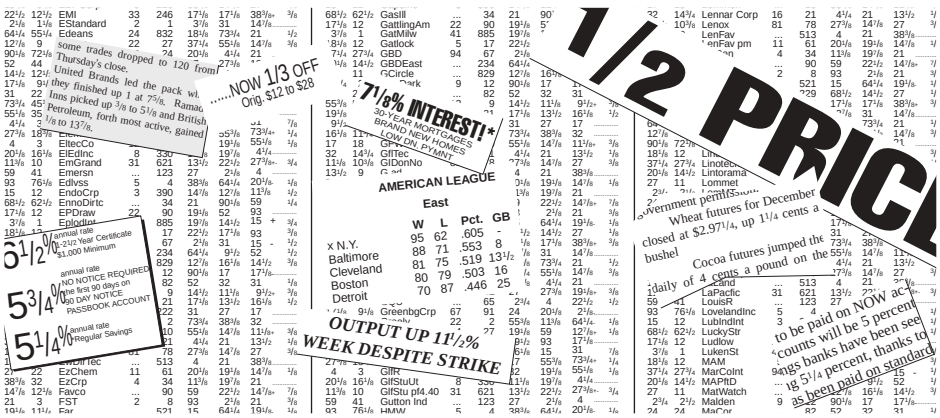
Technology Connection

This is a game played with a deck of fraction cards between 0 and 1 having denominators of 2 through 10. As cards are turned up, you make decisions involving inequalities of two fractions, or four fractions if you decide to take a chance. In this example, how would you answer the question to win four cards? If you win 35 cards, you beat the deck.



Taking A Chance Applet, Chapter 5
www.mhhe.com/bbn

Exercises and Problems 5.2



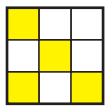
Use the collage shown here in exercises 1 and 2. Only a few different denominators occur in the fractions that appear frequently in newspapers, magazines, sale ads, etc.

1. **a.** Name four fractions with different denominators from the collage.
   b. The four fractions you selected in part a can be paired in six ways. Write these six pairs of fractions and their common denominators.

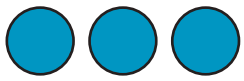
2. **a.** There are many mixed numbers in the collage. Name five of them.
   b. Write the mixed numbers you selected in part a as improper fractions.

For the figures in exercises 3a and 4a, write two fractions: one to indicate what part of the figure is shaded and one to indicate what part of the figure is unshaded. For the figures in exercises 3b and 4b, complete the equation and describe how the answer can be obtained from the figure.

3. **a.**

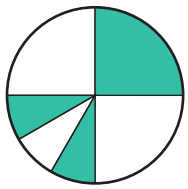


b.

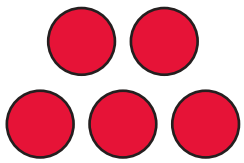


3 □ 4

4. **a.**



b.



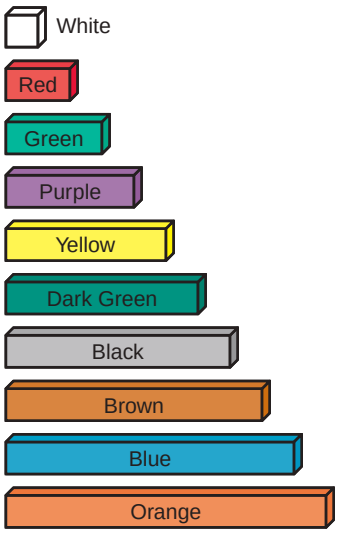
5 □ 3

Three concepts of a fraction were illustrated in this section: *part-to-whole, fraction-quotient,* and *ratio*. Answer the questions in exercises 5 and 6 by writing fractions in lowest terms and write the name of the fraction concept that is involved in each question.

5. **a.** If one ship has a length of 600 feet and another ship has a length of 400 feet, the length of the shorter ship is what fraction of the length of the longer ship?
   b. Maria cut a 6-foot length of ribbon into 7 equal parts. What fraction of a foot is each piece of ribbon?
   c. In a collection of beetles there are 3 Japanese beetles and 12 Asian beetles. What fraction of the collection is Asian beetles?

6. **a.** A 14 mile route is divided into 6 equal parts for a relay race. What fraction of a mile is the length of each part of the race?
   b. On a school debate team there are 6 boys and 8 girls. What fraction of the debate team is boys?
   c. If the height of a girl is 4 feet and the height of a basketball rim above the floor is 10 feet, the height of the girl is what fraction of the height of the basketball rim above the floor?

Cuisenaire Rods can represent various fractions, depending on the choice of the unit rod. For example, if the unit rod is brown, then the purple rod represents $\frac{1}{2}$. Use the Cuisenaire Rods in exercises 7 and 8.



7. a. If the dark green rod represents $\frac{3}{4}$, what is the unit rod?
b. What fraction is represented by the yellow rod if the black rod is the unit rod?

8. a. What is the unit rod if the purple rod represents $\frac{2}{3}$?
b. If the unit rod is the orange rod, what fraction is represented by the black rod?

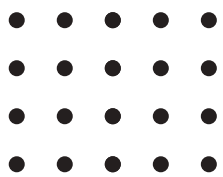
9. Use the array of dots to illustrate each equality.



a. $\frac{1}{3} = \frac{4}{12}$ **b.** $\frac{6}{12} = \frac{1}{2}$

c. $\frac{2}{3} = \frac{8}{12}$ **d.** $\frac{9}{12} = \frac{3}{4}$

10. Use the array of dots to illustrate each equality.



a. $\frac{1}{4} = \frac{5}{20}$ **b.** $\frac{3}{5} = \frac{12}{20}$

c. $\frac{4}{5} = \frac{16}{20}$ **d.** $\frac{3}{4} = \frac{15}{20}$

Write the missing numbers for the fractions in exercises 11 and 12.

11. a. $\frac{3}{7} = \frac{12}{\square}$ **b.** $\frac{\square}{8} = \frac{25}{40}$

c. $\frac{5}{6} = \frac{\square}{30}$ **d.** $\frac{9}{\square} = \frac{27}{12}$

12. a. $\frac{7}{8} = \frac{\square}{32}$ **b.** $\frac{\square}{3} = \frac{40}{24}$

c. $\frac{2}{3} = \frac{12}{\square}$ **d.** $\frac{5}{\square} = \frac{20}{24}$

Write each fraction in exercises 13 and 14 in lowest terms.

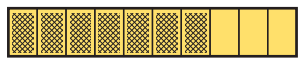
13. a. $\frac{4}{18}$ **b.** $\frac{12}{27}$

c. $\frac{4}{12}$ **d.** $\frac{-16}{24}$

14. a. $\frac{18}{15}$ **b.** $\frac{-20}{42}$

c. $\frac{63}{70}$ **d.** $\frac{99}{154}$

Split the parts of the Fraction Bars in exercises 15 and 16 to illustrate the given equalities.

15. a.  $\frac{7}{10} \square \frac{14}{20}$

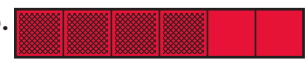
b.  $\frac{6}{7} \square \frac{18}{21}$

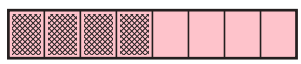
16. a.  $\frac{1}{9} \square \frac{3}{27}$

b.  $\frac{1}{4} \square \frac{2}{8}$

Group the parts of each bar in exercises 17 and 18 to show why the fraction on the left side of the equation equals the fraction on the right side.

17. a.  $\frac{9}{12} \square \frac{3}{4}$

b.  $\frac{4}{6} \square \frac{2}{3}$

18. a.  $\frac{4}{8} \square \frac{1}{2}$

b.  $\frac{8}{10} \square \frac{4}{5}$

Complete the equations in exercises 19 and 20 so that each pair of fractions has the least common denominator.

19. a. $\frac{2}{3} =$ $\quad$ $\frac{4}{5} =$ $\quad$ **b.** $\frac{1}{6} =$ $\quad$ $\frac{-7}{12} =$

20. a. $\frac{3}{15} =$ $\quad$ $\frac{5}{6} =$ $\quad$ **b.** $\frac{5}{8} =$ $\quad$ $\frac{3}{-10} =$

Determine an inequality ($<$ or $>$) for each pair of fractions in exercises 21 and 22.

21. a. $\frac{3}{7}$ and $\frac{5}{9}$ **b.** $\frac{1}{4}$ and $\frac{1}{6}$ **c.** $\frac{-5}{6}$ and $\frac{-7}{8}$

22. a. $\frac{1}{4}$ and $\frac{2}{9}$ **b.** $\frac{3}{8}$ and $\frac{1}{3}$ **c.** $\frac{4}{7}$ and $\frac{-5}{9}$

Between any two fractions there are an infinite number of fractions. Find a fraction that is between the fractions in each pair in 23 and 24.

23. a. $\frac{1}{20}$ and $\frac{1}{10}$ **b.** $\frac{1}{2}$ and $\frac{5}{8}$

24. a. $\frac{1}{3}$ and $\frac{1}{4}$ **b.** $\frac{8}{9}$ and $\frac{9}{10}$

Write each of the fractions in exercises 25 and 26 as a mixed number or a whole number.

25. a. $\frac{5}{3}$ **b.** $\frac{8}{8}$ **c.** $\frac{25}{6}$

26. a. $\frac{-21}{7}$ **b.** $\frac{17}{5}$ **c.** $\frac{18}{2}$

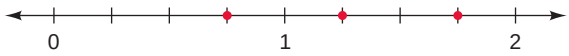
Write each of the mixed numbers in exercises 27 and 28 as a fraction.

27. a. $1\frac{3}{4}$ **b.** $-2\frac{1}{5}$ **c.** $4\frac{2}{3}$

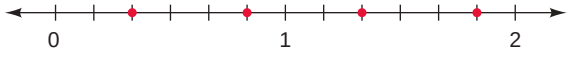
28. a. $2\frac{5}{6}$ **b.** $1\frac{3}{7}$ **c.** $-3\frac{1}{2}$

For each point indicated on the number lines in exercises 29 and 30, write the fraction or improper fraction that identifies it below the line. Then write the mixed number for each improper fraction above the line.

29.



30.



Mentally determine an inequality for each pair of fractions in exercises 31 and 32, and explain your reasoning by using a drawing.

31. a. $\frac{1}{12}$ and $\frac{1}{20}$ **b.** $\frac{7}{8}$ and $\frac{9}{10}$ **c.** $\frac{5}{12}$ and $\frac{6}{11}$

32. a. $\frac{-1}{50}$ and $\frac{-1}{30}$ **b.** $\frac{5}{9}$ and $\frac{3}{7}$ **c.** $\frac{11}{12}$ and $\frac{19}{20}$

Round each fraction or mixed number in exercises 33 and 34 to the nearest integer.

33. a. $\frac{4}{10}$ **b.** $1\frac{1}{3}$ **c.** $3\frac{1}{2}$

34. a. $\frac{4}{7}$ **b.** $-2\frac{3}{4}$ **c.** $\frac{2}{5}$

Some calculators for middle school students have a key for simplifying fractions. The key SIMP is used for this purpose in exercises 35 and 36. Determine each fraction for the given steps if the numerator and the denominator of each fraction are divided by their smallest common factor other than 1 in moving from one calculator view screen to the next.

35.

	Keystrokes	View Screen
a.	60 b/c 630 =	60/630
b.	SIMP ▯	
c.	SIMP ▯	
d.	SIMP ▯	

36.

	Keystrokes	View Screen
a.	126 b/c 210 =	126/210
b.	SIMP ▯	
c.	SIMP ▯	
d.	SIMP ▯	

Assume that you have a calculator for fractions with the key [A b/c ↔ d/c] and that pressing this key replaces an improper fraction by a mixed number in lowest terms. Determine the view screens in exercises 37 and 38 for such a calculator.

37.

	Keystrokes	View Screen
a.	88 b/c 32 ▯	$\frac{88}{32}$
	A b/c ↔ d/c ▯	
b.	154 b/c 70 ▯	$\frac{154}{70}$
	A b/c ↔ d/c ▯	

38.

	Keystrokes	View Screen
a.	156 b/c 117 ▯	$\frac{156}{117}$
	A b/c ↔ d/c ▯	
b.	924 b/c 546 ▯	$\frac{924}{546}$
	A b/c ↔ d/c ▯	

Reasoning and Problem Solving

39. A furlong equals $\frac{1}{8}$ mile, and it is used for measuring distances on horse racetracks. If a horse runs 1 mile in 1 minute 30 seconds, how long will it take the horse to run 6 furlongs at the same rate?

40. A buyer's guide for single-lens reflex cameras gives the following shutter speeds in seconds. Mila needs a shutter speed opening that is less than $\frac{1}{200}$ second. Which of these shutter speeds can she use?

$$\frac{1}{90} \quad \frac{1}{250} \quad \frac{1}{60} \quad \frac{1}{1000} \quad \frac{1}{100}$$

41. Suppose you need to drill a $\frac{3}{8}$-inch hole, but the set of drill bits is measured in sixteenths of an inch. What size drill bit should you use?

42. To pass inspection in a certain state, a car's tire treads must have a depth of at least $\frac{1}{16}$ inch. If the tires on a car have tread depths in inches of $\frac{1}{2}$, $\frac{1}{8}$, $\frac{1}{32}$, and $\frac{1}{4}$, which tire will not pass inspection?

43. One-fiftieth of the Earth's crust is magnesium, and $\frac{1}{20}$ of the Earth's crust is iron. Is there more iron or more magnesium in the Earth's crust?

44. Pure gold is quite soft. To make it more useful for such items as rings and jewelry, jewelers mix it with other metals, such as copper and zinc. Pure gold is marked 24k (24 karat). What fraction of a ring is pure gold if it is marked 14k?

45. Some health authorities say that we should have 1 gram of protein per day for each kilogram of weight. There are about 40 grams of protein in 1 liter of milk. A liter of fat-free milk weighs about 1040 grams. What fraction of the milk's weight is protein?

Use the information about the camera shown here to answer questions 46 through 48. Turning the control dial reveals the following shutter speeds: 2000, 1000, 500, 250, 125, 60, 30, 15, 10, 8, 6, and 3. The reciprocal of each of these numbers is the length of time the shutter stays open to let light into the film. For example, 250 is a shutter speed of $\frac{1}{250}$ seconds. The fastest opening for this camera is $\frac{1}{2000}$ seconds. For lengths of time longer than those in the preceding list, there is a different system of numbers that enables shutter openings of up to 30 seconds.



46. The less light that is available, the longer the shutter must stay open for the film to be properly exposed. Will a shutter setting of 15 allow more or less light than a setting of 60?

47. If a shutter speed setting of 250 doesn't allow quite enough light, the control dial should be turned to what number?

48. The fastest flash-sync speed for this camera is $\frac{1}{90}$ seconds. What are the shutter speeds from the above list that are slower than this flash-sync speed?

Use these two recipes to determine if the amounts in exercises 49 through 51 are enough to meet the requirements for the given ingredients in the recipe.

Blueberry Crepes

- 2 cups all-purpose flour
- $1\frac{2}{3}$ cups skim milk
- $1\frac{2}{3}$ cups cold water
- 4 large eggs
- 3 tablespoons canola oil
- 2 cups frozen small, wild blueberries
- 2 teaspoons cornstarch
- $\frac{3}{4}$ cup sugar
- 1 teaspoon vanilla extract
- $\frac{1}{2}$ teaspoon lemon juice

Maple-Orange Custard

- $\frac{1}{2}$ cup sugar
- $\frac{1}{4}$ cup maple syrup
- 1 tablespoon unsalted butter
- 1 tablespoon finely grated orange rind
- 2 eggs, separated
- 3 tablespoons all-purpose flour
- $\frac{1}{3}$ cup orange juice
- 1 cup skim milk, blended with $\frac{1}{2}$ cup nonfat dry powdered milk

49. **a.** $1\frac{3}{4}$ cups of skim milk for Blueberry Crepes
b. $\frac{1}{4}$ cup of orange juice for Maple-Orange Custard

50. **a.** $\frac{1}{3}$ cup of maple syrup for Maple-Orange Custard
b. $\frac{7}{8}$ cup of sugar for Blueberry Crepes

51. **a.** $\frac{1}{3}$ teaspoon lemon juice for Blueberry Crepes
b. $\frac{3}{8}$ cup of sugar for Maple-Orange Custard

52. **Featured Strategies: Guessing and Checking, Solving a Simpler Problem, and Finding a Pattern.** Five jars numbered 1 through 5 contain a total of 92 candy bars. If each jar contains two more candy bars than the previous jar, how much candy is in each jar?



1 2 3 4 5

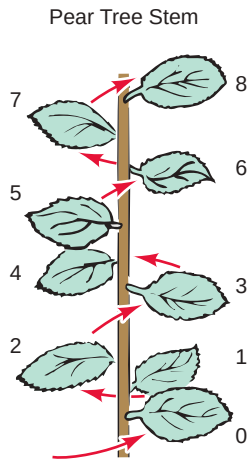
a. Understanding the Problem. Sometimes a problem such as this one can be solved by *guessing* and then adjusting the next guess. Even if no solution is found, you may obtain a better understanding of the problem. Try a few numbers, and describe what you learn by guessing.

b. Devising a Plan. Your guesses in part a should suggest that the answers are not whole numbers. Let's simplify the problem by changing the number of candy bars. How many bars will there be in each jar if there is a total of 100 bars?

c. Carrying Out the Plan. Continue by determining the number of bars in each jar in each of the following cases, assuming there are 100 candy bars.
(1) Each jar has 3 more than the previous jar.
(2) Each jar has 4 more than the previous jar.
(3) Each jar has 5 more than the previous jar.
Look for a pattern in your answers. What general approach to solving this type of problem is suggested? What is the solution to the original problem with 92 candy bars?

d. Looking Back. Solve this problem for 7 jars containing 92 candy bars, assuming each jar has 2 more bars than the previous jar.

53. The Fibonacci numbers 1, 1, 2, 3, 5, 8, 13, . . . occur as the numerators and denominators of fractions associated with patterns of leaf arrangements on trees and plants. These numbers are used in problems 53 and 54. This drawing shows the leaf arrangement from a branch of a pear tree.



If you begin with a leaf on a pear tree branch and count the number of leaves until you reach a leaf that grows from a point just above the first, the number of leaves, not counting the first (zero leaf), will be the Fibonacci number 8. Furthermore, in passing around the branch from a leaf to the leaf directly above, you will make 3 complete turns. This means that each leaf is $\frac{3}{8}$ of a turn from its adjacent leaves. The pattern described by this fraction is called *phyllotaxis* (or leaf divergence). For any given species of tree, the appropriate fraction can be found by counting the leaves (or branches) and the turns around the stem. Fill in the missing numbers in the fractions for phyllotaxis in the following table by finding a pattern for the numerator and denominator in each of the first three fractions.

Tree	Phyllotaxis
Apple	$\frac{2}{5}$
Pear	$\frac{3}{8}$
Beech	$\frac{1}{3}$
Almond	$\frac{5}{\square}$
Elm	$\frac{\square}{2}$
Willow	$\frac{3}{\square}$

54. The seeds of sunflowers and daisies, the scales of pinecones and pineapples, and the leaves on certain vegetables are arranged in two spirals. In these cases the numerator and denominator of the fraction for phyllotaxis give the number of clockwise and counterclockwise spirals. Find a pattern for the numerator and denominator of each fraction in the table shown below and, given that the fraction is not equal to 1, fill in the missing numbers in the fractions for phyllotaxis.

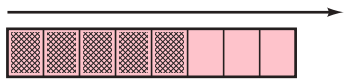
Plant	Phyllotaxis
White pinecone	$\frac{5}{8}$
Pineapple	$\frac{8}{13}$
Daisy	$\frac{21}{34}$
Cauliflower	$\frac{\square}{3}$
Celery	$\frac{1}{\square}$
Medium sunflower	$\frac{55}{\square}$
Large sunflower	$\frac{\square}{144}$

Teaching Questions

1. Suppose one of your students said $\frac{0}{6}$ is in lowest terms because his calculator simplifies fractions and when he tries to simplify $\frac{0}{6}$ he gets an error message. How would you explain to the student that $\frac{0}{6}$ is not in lowest terms?
2. Suppose you were teaching and some students asked you if there is a smallest positive number. Write a sequence of questions that you would ask the students to help convince them that there is no smallest positive number.
3. Anna likes working with fractions and understands the equality and inequality lessons with Fraction Bars. After class she asks her teacher how to tell which is greater, $\frac{4}{9}$ or $\frac{1}{3}$, because she does not have a Fraction Bar for $\frac{4}{9}$. Devise a way to help her using the Fraction Bars model she understands.
4. Two students were discussing unit fractions, that is, fractions like $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, etc., with 1 in the numerator. One student thought the fractions become larger as the denominators increase and the other thought the fractions become smaller. Explain what you would do to help them with this disagreement.

Classroom Connections

1. On page 292, in the example from the **Elementary School Text,** a fraction piece model is used to show that $\frac{4}{8}$ is the same as $\frac{1}{2}$ and the methods "multiply" and "divide" are used to find fractions equivalent to $\frac{4}{8}$. (a) Use Fraction Bar sketches to illustrate why the "multiply" method works. Explain your ideas. (b) Use Fraction Bar sketches to illustrate why the "divide" method works. Explain your ideas.
2. Discuss the representation of fractions by regions as opposed to the representation of fractions by sets of objects and explain each of the advantages of using regions that are cited in the **Standards** quote on page 288.
3. The **Research** statement on page 284 notes the difficulty students have interpreting fractions on a number line as compared to fractions from a region model. Explain how a bar like the one shown here can be used to transfer an understanding of fractions from regions to fractions on a number line. Illustrate with a few diagrams and show how fractions less than 1 and fractions greater than 1 can be located.



4. Several of the **Standards** and **Research** statements in this section refer to the fact that students prefer the region model for fractions over other models. Describe some of the difficulties of using sets of objects and/or number lines as opposed to the region model for learning about fractions.
5. In the **Grades 3–5 Standards—Number and Operations** (see inside front cover) under *Understand Numbers . . .* , read the third expectation and then give an example of each of the uses of fractions mentioned in this expectation.

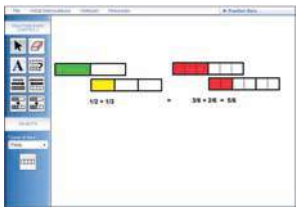
MATH ACTIVITY 5.3

Operations with Fraction Bars

Purpose: Use Fraction Bars to model fraction addition, subtraction, multiplication, and division.

Materials: Fraction Bars in the Manipulative Kit or Virtual Manipulatives.

Virtual Manipulatives

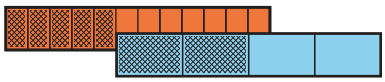


www.mhhe.com/bbn

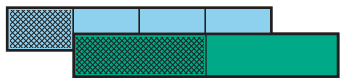
1. The sum of two fractions is modeled by placing the shaded amounts of their bars end to end. The bars at the right show that the total shaded amount is $\frac{5}{6}$. Write addition equations for the fractions represented by the following pairs of bars.



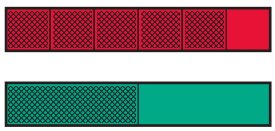
$\frac{2}{6} + \frac{1}{2} = \frac{5}{6}$

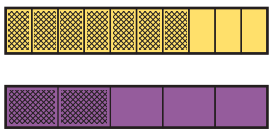
a. 

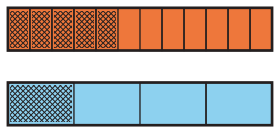
b. 

c. 

***2.** The difference of two fractions is modeled by lining up their bars and comparing their shaded amounts. Write subtraction equations for the fractions represented by the following pairs of bars.

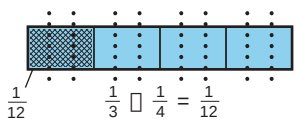
a. 

b. 

c. 

3. Turn the shaded amounts of the Fraction Bars face down and select any two bars. If any fractions equal 0 or 1, place them aside and select others. Write equations for the sum and difference of these fractions. Then repeat this activity for three other pairs of bars.

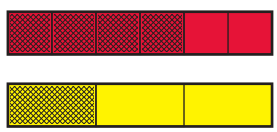
4. The product $\frac{1}{3} \times \frac{1}{4}$ means $\frac{1}{3}$ of $\frac{1}{4}$ and can be illustrated by splitting each part of a $\frac{1}{4}$ bar into 3 equal parts. One-third of the shaded amount is $\frac{1}{12}$, so $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$. Draw sketches to illustrate and compute each of the following products.



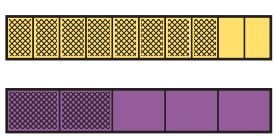
$\frac{1}{12}$ $\frac{1}{3} \square \frac{1}{4} = \frac{1}{12}$

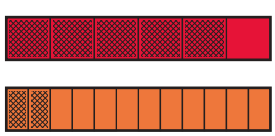
a. $\frac{1}{2} \times \frac{1}{2}$ b. $\frac{1}{3} \times \frac{1}{7}$ c. $\frac{1}{4} \times \frac{1}{2}$ d. $\frac{2}{3} \times \frac{1}{5}$

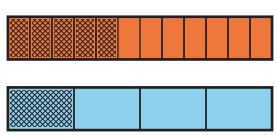
***5.** The quotient of two fractions is modeled by lining up their bars and determining how many times greater the shaded amount of one is than the other. The bars at the right show that $\frac{1}{3}$ can be subtracted from (or "fits into") $\frac{4}{6}$ twice. Write division equations for the fractions represented by the following pairs of bars.



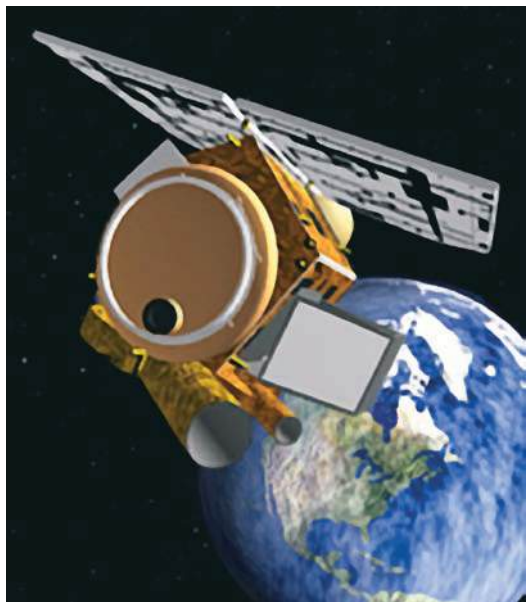
$\frac{4}{6} \square \frac{1}{3} = 2$

a. 

b. 

c. 

Section 5.3 OPERATIONS WITH FRACTIONS



In January 2005, the spacecraft *Deep Impact* (DI) was launched to gather information about the comet Tempel 1. This photo shows a graphic depiction of DI as it began its 268-million-mile trip to the comet. DI arrived at the potato-shaped comet that was about half the size of Manhattan and traveling at 23,000 miles per hour. It launched a barrel-sized container, called Impactor, that crashed into the comet. Photos and samples of debris from the impact enabled scientists to obtain information about the early origins of our solar system. The mission also gave scientists information about how they might someday stop a comet if one threatens Earth.

PROBLEM OPENER

Your father gives one-half of the money in his pocket to your mother, one-fourth of what is left to your brother, and one-third of what then remains to your sister. He then splits the rest with you. If you get $2, how much did your father start with?

Fractions are used to indicate how much smaller a diagram or model is than the life-size object. For example, the dimensions of the model in Figure 5.35 are one-fourth of the dimensions of the *Deep Impact* spacecraft. This means that each length on the model can be obtained by multiplying the corresponding length on the spacecraft by $\frac{1}{4}$. In this section we will examine the four basic operations with fractions.

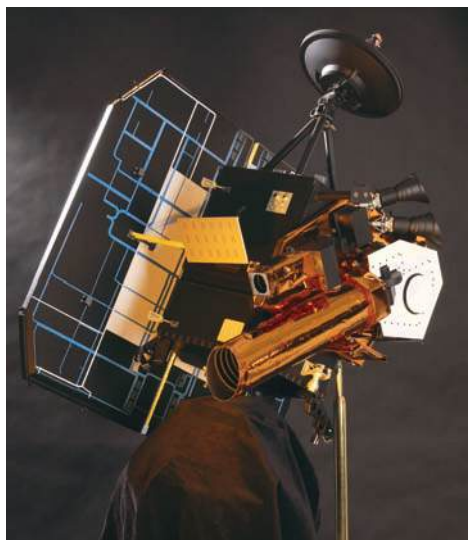


Figure 5.35
This is one of the models used for the development of *Deep Impact* (DI). The ratio of the lengths from this model to the corresponding lengths of the spacecraft is 1 to 4. Models played an important part in the design and testing of DI.

HISTORICAL HIGHLIGHT



Maria Gaetana Agnesi, 1718–1799

Italy's Maria Gaetana Agnesi, a brilliant linguist, philosopher, and mathematician, was the first of 21 children of a professor of mathematics at the University of Bologna. By her 13th birthday she was fluent in Latin, Greek, Hebrew, French, Spanish, and German as well as her native Italian. In 1748 she published *Instituzioni Analitiche,* two huge volumes containing a complete and unified treatment of algebra, analysis, and recent advances in calculus. This is the first surviving mathematical work written by a woman. Widely acclaimed as a model of clarity and exposition, it was translated into French (1775) and English (1801). In recognition of her exceptional accomplishments, Pope Benedict XIV appointed Agnesi to the chair of mathematics and natural philosophy at the University of Bologna in 1750. Maria Agnesi is said to have been the first woman professor of mathematics on a university faculty.*

*D. M. Burton, *The History of Mathematics,* 7th ed. (New York: McGraw-Hill, 2010), p. 430.

ADDITION

The concept of addition is the same for fractions as for whole numbers. The addition of whole numbers is illustrated by putting together, or combining, two sets of objects. Similarly, the addition of fractions can be illustrated by combining two amounts.

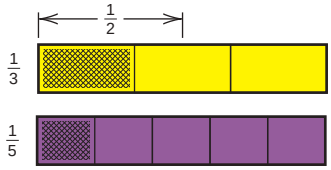
EXAMPLE A

Suppose that parts of two school days are used for a national testing program: $\frac{1}{3}$ day is required for test A, and $\frac{1}{5}$ day is required for test B. Approximately what part of a whole day is used for testing?

NCTM Standards

The development of rational number concepts is a major goal for grades 3–5, which should lead to informal methods for calculating with fractions. p. 33

Solution One approach to solving this problem is to sketch a figure to represent each amount. The following figures represent $\frac{1}{3}$ and $\frac{1}{5}$, and we can see that when the shaded amounts are combined, the total is approximately $\frac{1}{2}$.



Finding the sum of two fractions is easy when they have the same denominator. Figure 5.36 shows Fraction Bars for $\frac{4}{6}$ and $\frac{3}{6}$. If the shaded parts of each bar are placed end to end, the total shaded amount is 1 whole bar and $\frac{1}{6}$ bar. So $\frac{4}{6} + \frac{3}{6} = \frac{7}{6}$, or $1\frac{1}{6}$.



$\frac{4}{6}$ □ $\frac{3}{6}$ □ $\frac{7}{6}$ or $1\frac{1}{6}$

Figure 5.36

Addition of fractions can also be illustrated on a number line (Figure 5.37 on the next page) by placing arrows for the fractions end to end.

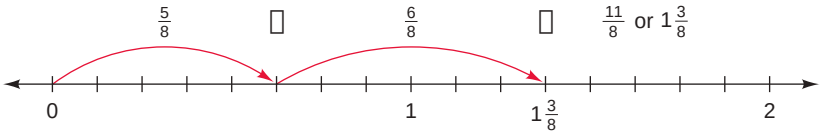


Figure 5.37

EXAMPLE B

Write each sum as a fraction or a mixed number.

1. $\frac{4}{5} + \frac{3}{5}$ **2.** $\frac{2}{9} + \frac{6}{9}$ **3.** $\frac{5}{3} + \frac{2}{3}$

Solution **1.** $\frac{7}{5} = 1\frac{2}{5}$ **2.** $\frac{8}{9}$ **3.** $\frac{7}{3} = 2\frac{1}{3}$

Unlike Denominators The difficulty in adding fractions occurs when the denominators are unequal. The $\frac{2}{5}$ bar and the $\frac{1}{3}$ bar in part a of Figure 5.38 show that $\frac{2}{5} + \frac{1}{3}$ is greater than $\frac{3}{5}$ but less than $\frac{4}{5}$. To determine this sum exactly, we must replace the two fractions by fractions having the same denominator. Since the least common multiple of 5 and 3 is 15, this is the **least common denominator** of $\frac{2}{5}$ and $\frac{1}{3}$ and these fractions can be replaced by $\frac{6}{15}$ and $\frac{5}{15}$, as shown in part b of Figure 5.38. The sum of these two fractions is $\frac{11}{15}$, so $\frac{2}{5} + \frac{1}{3} = \frac{11}{15}$.

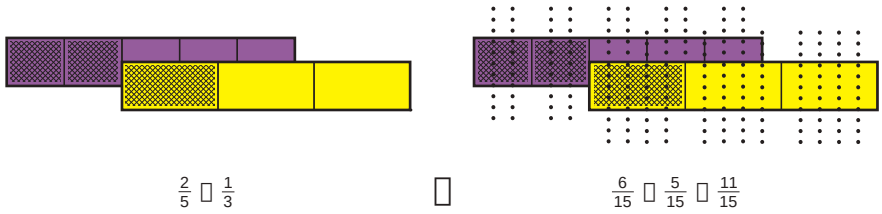


Figure 5.38

Multiplying the denominators of two fractions by each other will always yield a common denominator (but not necessarily the smallest one). Once two fractions have a common denominator, their sum is computed by adding the numerators and retaining the denominator. This is stated in the following rule for adding fractions.

Addition of Fractions For any fractions $\frac{a}{b}$ and $\frac{c}{d}$,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$$

EXAMPLE C

Find each sum.

1. $\frac{1}{4} + \frac{3}{7}$ **2.** $\frac{5}{6} + \frac{-1}{8}$ **3.** $\frac{7}{10} + \frac{9}{20}$

Solution **1.** $\frac{1}{4} + \frac{3}{7} = \frac{1 \times 7}{4 \times 7} + \frac{3 \times 4}{7 \times 4} = \frac{7}{28} + \frac{12}{28} = \frac{19}{28}$.

2. $\frac{5}{6} + \frac{-1}{8} = \frac{5 \times 4}{6 \times 4} + \frac{-1 \times 3}{8 \times 3} = \frac{20}{24} + \frac{-3}{24} = \frac{17}{24}$.

3. $\frac{7}{10} + \frac{9}{20} = \frac{7 \times 2}{10 \times 2} + \frac{9}{20} = \frac{14}{20} + \frac{9}{20} = \frac{23}{20} = 1\frac{3}{20}$.

Explore 5-4

Math Lab
Unlike Denominators

MAIN IDEA
Use models to add and subtract fractions with unlike denominators.

Math Online
glencoe.com
• Concepts in Motion

In this lab, you will use fraction strips to add and subtract fractions with *unlike* denominators.

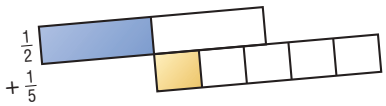
ACTIVITY

1 **Use fraction strips to find $\frac{1}{2} + \frac{1}{5}$.**

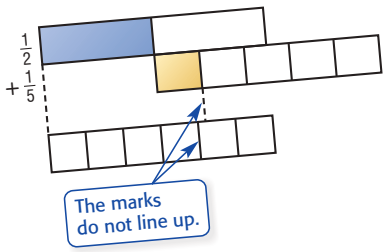
STEP 1 Model each fraction.



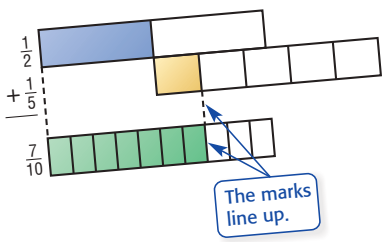
STEP 2 To add, line up the end of the shaded part of the first strip with the beginning of the second strip.



STEP 3 Test different fraction strips below the model, lining up each with the beginning of the first strip. Do the marks line up? If not, try another strip.



STEP 4 Once the correct strip is found, shade the sections between the beginning of the strip to the point where they line up.



So, $\frac{1}{2} + \frac{1}{5} = \frac{7}{10}$.

CHECK Your Progress **Use fraction strips to add.**

a. $\frac{1}{10} + \frac{2}{5}$ b. $\frac{1}{6} + \frac{1}{2}$ c. $\frac{1}{2} + \frac{3}{4}$

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Mixed Numbers Mixed numbers are combinations of whole numbers and fractions. The sum of two mixed numbers can be found by adding the whole numbers and fractions separately. If the denominators of the fractions are unequal, as in problem 2 in Example D, a common denominator is found before the fractions are added.

EXAMPLE D

Compute each sum.

1. $6\frac{2}{5} + 3\frac{4}{5}$

2. $3\frac{1}{2} + 5\frac{2}{3}$

Solution 1. $6\frac{2}{5} + 3\frac{4}{5} = 9\frac{6}{5} = 10\frac{1}{5}$

2. $3\frac{1}{2} = 3\frac{3}{6}$, $5\frac{2}{3} = 5\frac{4}{6}$, $8\frac{7}{6} = 9\frac{1}{6}$

SUBTRACTION

The concepts of subtraction are the same for fractions as for whole numbers. That is, the *take-away concept,* the *comparison concept,* and the *missing-addend concept* apply to subtraction of fractions. For the bars in part a of Figure 5.39, the *comparison concept* shows the difference between $\frac{1}{2}$ and $\frac{1}{6}$ is $\frac{2}{6}$, and the *missing-addend concept* shows that $\frac{2}{6}$ must be added to $\frac{1}{6}$ to obtain $\frac{1}{2}$. Part b of the figure illustrates subtraction on a number line and shows that $\frac{11}{12}$ take away $\frac{7}{12}$ equals $\frac{4}{12}$, or that $\frac{4}{12}$ must be added to $\frac{7}{12}$ to obtain $\frac{11}{12}$.

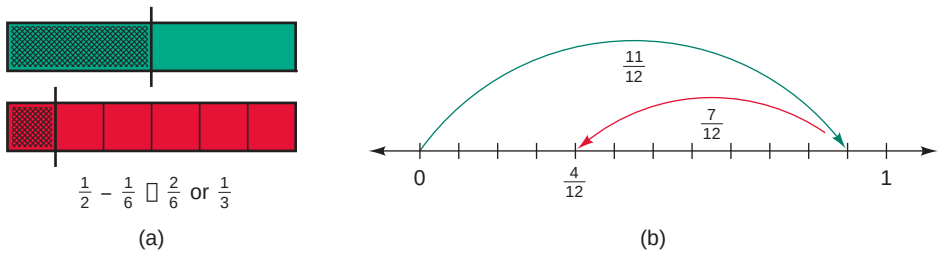


Figure 5.39

NCTM Standards

The NCTM K–4 Standard, *Fractions and Decimals* in the *Curriculum and Evaluation Standards for School Mathematics* (p. 59), advises that

> Physical materials should be used for exploratory work in adding and subtracting basic fractions, solving simple real-world problems, and partitioning sets of objects to find fractional parts of sets and relating this activity to division. For example, children learn that $\frac{1}{3}$ of 30 is equivalent to "30 divided by 3," which helps them relate operations with fractions to earlier operations with whole numbers.

Unlike Denominators The bars in part a of Figure 5.40 on the next page show that the difference between $\frac{5}{6}$ and $\frac{1}{4}$ is greater than $\frac{3}{6}$ and less than $\frac{4}{6}$. To compute this difference, we can replace these fractions by fractions having a common denominator. The least common denominator of $\frac{5}{6}$ and $\frac{1}{4}$ is 12, so these fractions can be replaced by $\frac{10}{12}$ and $\frac{3}{12}$, as shown in part b of the figure. The difference between these two fractions is $\frac{7}{12}$, so $\frac{5}{6} - \frac{1}{4} = \frac{7}{12}$.

NCTM Standards

Students who have a solid conceptual foundation in fractions should be less prone to committing computational errors than students who do not have such a foundation. p. 218

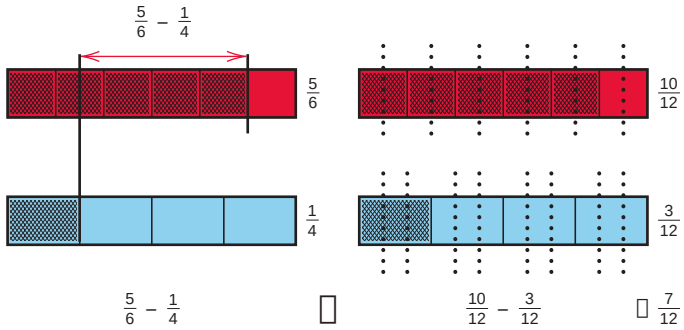


Figure 5.40

The general rule for subtracting fractions is usually stated using a common denominator that is the product of the two denominators, even though this may not be the least common denominator. Once two fractions have a common denominator, their difference is computed by subtracting the numerators and retaining the denominator. The resulting fraction may sometimes be simplified if its numerator and denominator are found to have a factor in common.

Subtraction of Fractions For any fractions $\frac{a}{b}$ and $\frac{c}{d}$,

$$\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad - bc}{bd}$$

EXAMPLE E

Find each difference.

1. $\frac{7}{8} - \frac{1}{3}$ 2. $\frac{1}{2} - \frac{4}{5}$ 3. $\frac{2}{3} - \frac{-1}{4}$

Solution

1. $\frac{7}{8} - \frac{1}{3} = \frac{7 \times 3}{8 \times 3} - \frac{1 \times 8}{3 \times 8} = \frac{21}{24} - \frac{8}{24} = \frac{13}{24}.$

2. $\frac{1}{2} - \frac{4}{5} = \frac{1 \times 5}{2 \times 5} - \frac{4 \times 2}{5 \times 2} = \frac{5}{10} - \frac{8}{10} = \frac{-3}{10}.$

3. $\frac{2}{3} - \frac{-1}{4} = \frac{2 \times 4}{3 \times 4} - \frac{-1 \times 3}{4 \times 3} = \frac{8}{12} - \frac{-3}{12} = \frac{8}{12} + \frac{3}{12} = \frac{11}{12}.$

Mixed Numbers The difference between two mixed numbers can be found by subtracting the whole-number parts and the fractions separately. Sometimes regrouping (borrowing) is necessary before this can be done, as in problem 1 in Example F. If the denominators of the fractions in the mixed numbers are unequal, the fractions must be replaced by fractions having a common denominator. In some cases, both regrouping (borrowing) and changing denominators will be necessary before mixed numbers can be subtracted, as in problem 2 in Example F.

EXAMPLE F

Compute each difference.

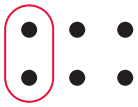
1. $4\frac{1}{5} - 1\frac{2}{5}$

2. $5\frac{1}{6} - 2\frac{3}{4}$

Solution 1. $4\frac{1}{5} = 3\frac{6}{5}$
$-1\frac{2}{5} = -1\frac{2}{5}$
$2\frac{4}{5}$

2. $5\frac{1}{6} = 5\frac{2}{12} = 4\frac{14}{12}$
$-2\frac{3}{4} = -2\frac{9}{12} = -2\frac{9}{12}$
$2\frac{5}{12}$

MULTIPLICATION



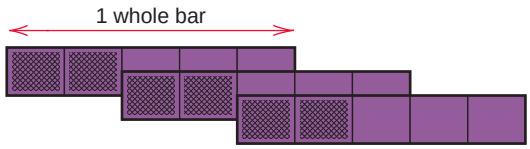
$\frac{1}{3}$ of 6 is 2

Figure 5.41

Given a product of two whole numbers $m \times n$, we can think of the first number m as indicating "how many" of the second number. For example, 2×3 means 2 of the 3s. Multiplication of fractions may be viewed in a similar manner. In the product $\frac{1}{3} \times 6$, the first number tells us "how much" of the second; that is, $\frac{1}{3} \times 6$ means $\frac{1}{3}$ of 6 (Figure 5.41).

There is a major difference between the outcome of multiplying by a whole number and the outcome of multiplying by a fraction. When we multiply by a whole number greater than 1, the product is greater than the second number being multiplied. However, when we multiply by a fraction less than 1, the product is less than the second number being multiplied. This is often a problem for schoolchildren who have been accustomed to multiplying by whole numbers before encountering products with fractions.*

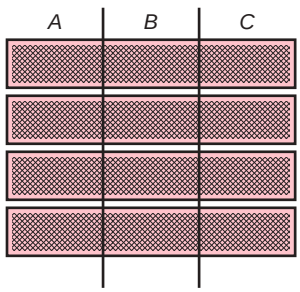
Whole Number Times a Fraction In this case we can interpret multiplication to mean *repeated addition*, just as we did with multiplication of whole numbers in Chapter 3. The whole number indicates the number of times the fraction is to be added to itself. Figure 5.42 illustrates $3 \times \frac{2}{5}$ and shows that this product equals $1\frac{1}{5}$.



3 of $\frac{2}{5}$ is $\frac{2}{5} + \frac{2}{5} + \frac{2}{5} = \frac{6}{5}$ or $1\frac{1}{5}$

Figure 5.42

Fraction Times a Whole Number The product $\frac{1}{3} \times 4$ means $\frac{1}{3}$ of 4. This product can be illustrated by using 4 whole bars and dividing the whole set into 3 equal parts (Figure 5.43). The vertical lines split these bars into three equal parts A, B, and C. Part A, which is one-third of the 4 bars, consists of 4 one-thirds, or 4 thirds.



$\frac{1}{3}$ of 4 is $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{4}{3}$

Figure 5.43

The product $\frac{1}{3} \times 4$ can also be illustrated by beginning with a 4-bar, that is, 4 whole bars placed end to end, as shown in Figure 5.44 on the next page, and dividing it into 3 equal parts (see dotted lines). Each part is $1\frac{1}{3}$ whole bars, so $\frac{1}{3} \times 4 = 1\frac{1}{3}$.

*According to D. E. Smith, in his *History of Mathematics,* 2d ed. (Lexington, MA: Ginn, 1925), p. 225, as early as the fifteenth and sixteenth centuries, writers on the subject of fractions expressed concern over this problem.

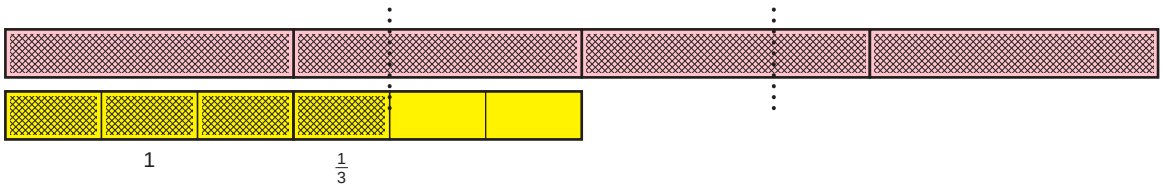


Figure 5.44

These examples suggest the following rule for products involving a whole number and a fraction.

Whole Number Times a Fraction and Fraction Times a Whole Number For any whole number k and fraction $\frac{a}{b}$,

$$k \times \frac{a}{b} = \frac{ka}{b} \quad \text{and} \quad \frac{a}{b} \times k = \frac{ak}{b}$$

EXAMPLE G

Compute the following products.

1. $3 \times \frac{2}{7}$ **2.** $\frac{2}{5} \times {}^{-}24$ **3.** $6 \times \frac{3}{8}$

Solution **1.** $\frac{6}{7}$ **2.** $\frac{{}^{-}48}{5} = {}^{-}9\frac{3}{5}$ **3.** $\frac{18}{8} = 2\frac{2}{8}$

Fraction Times a Fraction The product $\frac{1}{3} \times \frac{1}{5}$ means $\frac{1}{3}$ of $\frac{1}{5}$. This can be illustrated by using a rectangular model or Fraction Bar model. To do this, Figure 5.45a begins with a rectangle that has 1 part out of 5 shaded (see vertical lines) and then has been split horizontally into 3 equal parts A, B, and C. Figure 5.45b begins with a $\frac{1}{5}$ Fraction Bar and each part is split vertically into 3 equal parts. The darker parts of the rectangle and the Fraction Bar are both $\frac{1}{3}$ of $\frac{1}{5}$. Each of the new small parts is $\frac{1}{15}$ of the whole, so $\frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$.

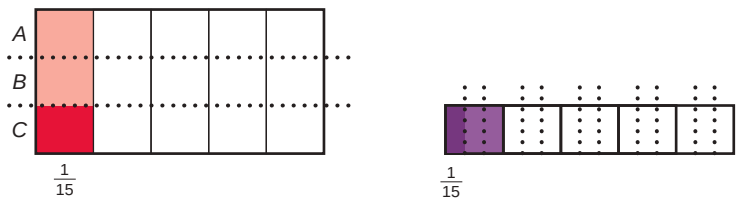


Figure 5.45

The product $\frac{2}{3} \times \frac{4}{5}$ means $\frac{2}{3}$ of $\frac{4}{5}$, so we will begin with a rectangle that has 4 parts out of 5 shaded (Figure 5.46a) and a $\frac{4}{5}$ Fraction Bar (Figure 5.46b), and take $\frac{2}{3}$ of each shaded part. In order to do this, Figure 5.46a has been split horizontally into 3 equal parts A, B, and C and in Figure 5.46b, each part has been split vertically into 3 equal parts. Each of the new small parts is $\frac{1}{15}$ of the whole figure. The 8 darker parts of each figure represent $\frac{8}{15}$, so $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$.

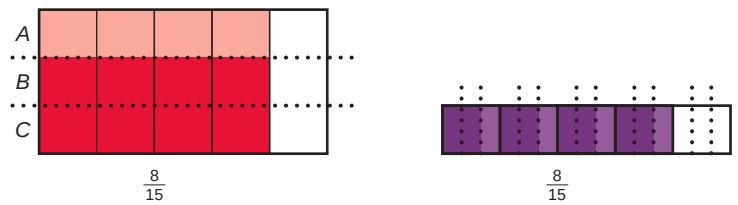


Figure 5.46

The rule for computing the product of two fractions is suggested in the preceding illustrations. In each of these examples, the product can be found by *multiplying numerator by numerator and denominator by denominator.*

Multiplication of Fractions For any fractions $\frac{a}{b}$ and $\frac{c}{d}$,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

Notice that since the whole number k is equal to the fraction $\frac{k}{1}$, a product involving a whole number and a fraction is a special case of the rule for multiplication of fractions.

EXAMPLE H

Compute each product.

1. $\frac{1}{2} \times \frac{8}{9}$ **2.** $\frac{4}{7} \times \frac{-2}{5}$ **3.** $6 \times \frac{4}{5}$

Solution **1.** $\frac{8}{18}$ or $\frac{4}{9}$ **2.** $\frac{-8}{35}$ **3.** $\frac{24}{5}$ or $4\frac{4}{5}$

Products involving a mixed number can be computed by replacing the mixed number by an improper fraction as shown in Example I.

EXAMPLE I

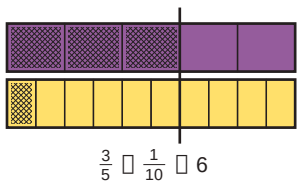
Compute each product.

1. $\frac{3}{4} \times 6\frac{1}{5}$ **2.** $3\frac{1}{8} \times -2\frac{1}{3}$

Solution **1.** $\frac{3}{4} \times 6\frac{1}{5} = \frac{3}{4} \times \frac{31}{5} = \frac{93}{20} = 4\frac{13}{20}$. **2.** $3\frac{1}{8} \times -2\frac{1}{3} = \frac{25}{8} \times \frac{-7}{3} = \frac{-175}{24} = -7\frac{7}{24}$.

DIVISION

Division of fractions can be viewed in much the same way as division of whole numbers. One of the meanings of division of whole numbers is represented by the measurement (subtractive) concept. For example, to explain $15 \div 3$, we often say, "How many times can we subtract 3 from 15?" Similarly, for $\frac{3}{5} \div \frac{1}{10}$ we can ask, "How many times can we subtract $\frac{1}{10}$ from $\frac{3}{5}$?" Figure 5.47 shows that the shaded amount of a $\frac{1}{10}$ bar can be subtracted from the shaded amount of a $\frac{3}{5}$ bar 6 times. Or, viewed in terms of multiplication, the shaded amount of the $\frac{3}{5}$ bar is 6 times the shaded amount of the $\frac{1}{10}$ bar.



$\frac{3}{5}$ □ $\frac{1}{10}$ □ 6

Figure 5.47

This interpretation of division of fractions continues to hold even when the quotient is not a whole number. Figure 5.48 on the next page shows that the shaded amount of the $\frac{1}{3}$ bar can be subtracted from the shaded amount of the $\frac{5}{6}$ bar 2 times, and there is a remainder. Just as in whole number division, the remainder is then compared to the divisor using a fraction. In this example, the remainder is $\frac{1}{2}$ as big as the divisor, so the quotient is $2\frac{1}{2}$.

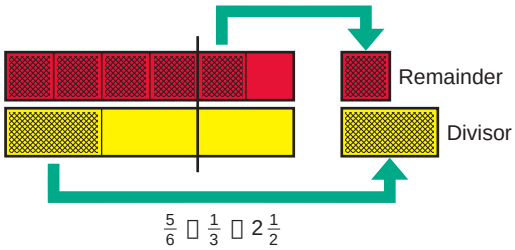


Figure 5.48

One of the early methods of dividing one fraction by another was to replace both fractions by fractions having a common denominator. When this is done, the quotient can be obtained by disregarding the denominators and dividing the two numerators. For example,

$$\frac{2}{5} \div \frac{3}{4} = \frac{8}{20} \div \frac{15}{20} = 8 \div 15 = \frac{8}{15}$$

One method currently taught to divide one fraction by another is to *invert the divisor and multiply.*

$$\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} = \frac{3}{2} = 1\frac{1}{2}$$

Figure 5.49 helps to show why the method of *inverting and multiplying* works. We saw in Section 3.4 the mental calculating technique called *equal quotients,* whereby we recognize that the relative size of two sets does not change if both sets are halved or both sets are tripled. In general, the quotient of two numbers can be replaced by dividing or multiplying both numbers by the same number. The relative size of the shaded amounts of the two bars in part a of Figure 5.49 is unchanged by tripling these amounts to obtain the bars in part b. The new bars in part b show that the total shaded amount of the top two bars is $1\frac{1}{2}$ times the total shaded amount of the lower bar.

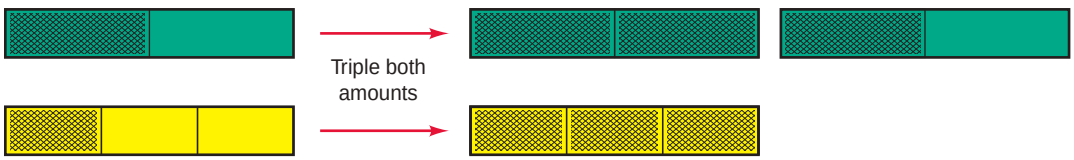


Figure 5.49

This illustration of tripling the amounts in Figure 5.49 is recorded in the following equations by multiplying both fractions by 3.

$$\frac{1}{2} \div \frac{1}{3} = \left(\frac{1}{2} \times 3\right) \div \left(\frac{1}{3} \times 3\right) = \frac{3}{2} \div 1 = \frac{3}{2}$$

In a similar manner, to divide $\frac{a}{b}$ by $\frac{c}{d}$, the *equal-quotients* technique enables us to multiply both numbers by *bd.*

$$\frac{a}{b} \div \frac{c}{d} = (bd)\frac{a}{b} \div (bd)\frac{c}{d} = (\cancel{b}d)\frac{a}{\cancel{b}} \div (b\cancel{d})\frac{c}{\cancel{d}} = \frac{ad}{bc}$$

This is the result produced by the invert-and-multiply rule for finding the quotient of two fractions.

Division of Fractions For any fractions $\frac{a}{b}$ and $\frac{c}{d}$, with $\frac{c}{d} \neq 0$,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

EXAMPLE J

Compute each quotient.

1. $\frac{1}{2} \div \frac{1}{3}$ **2.** $\frac{7}{8} \div \frac{2}{5}$ **3.** $\frac{2}{3} \div \frac{^-1}{4}$

Solution **1.** $\frac{1}{2} \div \frac{1}{3} = \frac{1}{2} \times \frac{3}{1} = \frac{3}{2} = 1\frac{1}{2}$. **2.** $\frac{7}{8} \div \frac{2}{5} = \frac{7}{8} \times \frac{5}{2} = \frac{35}{16} = 2\frac{3}{16}$.

3. $\frac{2}{3} \div \frac{^-1}{4} = \frac{2}{3} \times \frac{^-4}{1} = \frac{^-8}{3} = {}^-2\frac{2}{3}$.

One method of computing the quotient of two mixed numbers is to replace each mixed number by an improper fraction and use the definition of division of fractions.

EXAMPLE K

Compute each quotient.

1. $5\frac{1}{3} \div 1\frac{1}{8}$ **2.** $8\frac{3}{4} \div 2\frac{1}{2}$

Solution **1.** $5\frac{1}{3} \div 1\frac{1}{8} = \frac{16}{3} \div \frac{9}{8} = \frac{16}{3} \times \frac{8}{9} = \frac{128}{27} = 4\frac{20}{27}$.

2. $8\frac{3}{4} \div 2\frac{1}{2} = \frac{35}{4} \div \frac{5}{2} = \frac{35}{4} \times \frac{2}{5} = \frac{70}{20} = \frac{7}{2} = 3\frac{1}{2}$.

Technology Connection

The four basic operations can be performed on fractions and mixed numbers with calculators that display fractions. Each of these operations is illustrated in the following examples.

$\left(\frac{1}{2} + \frac{3}{10}\right)$	1 b/c 2 □ 3 b/c 10 □	$\frac{8}{10}$
$\left(\frac{5}{6} - \frac{1}{3}\right)$	5 b/c 6 − 1 b/c 3 □	$\frac{3}{6}$
$\left(\frac{5}{6} \times \frac{1}{3}\right)$	5 b/c 6 □ 1 b/c 3 □	$\frac{5}{18}$
$\left(\frac{7}{8} \div \frac{1}{2}\right)$	7 b/c 8 □ 1 b/c 2 □	$1\frac{6}{8}$

On many calculators designed for schoolchildren, the results from fraction operations will not be displayed in simplest form as shown in three of the four preceding calculator screens. The purpose is to help students follow the processes involved in computing. The fractions and mixed numbers can then be simplified, if needed, by calculator keys. The four preceding view screens would appear as follows for a calculator that displays fractions in simplest form.

$\frac{4}{5}$ $\frac{1}{2}$ $\frac{5}{18}$ $1\frac{3}{4}$

NCTM Standards

The division of fractions has traditionally been quite vexing for students. Although "invert and multiply" has been a staple of conventional mathematics instruction and although it seems to be a simple way to remember how to divide fractions, students have for a long time had difficulty doing so. p. 219

NUMBER PROPERTIES

The properties for addition and multiplication of integers (the inverse property for addition and the closure, identity, commutative, associative, and distributive properties) also hold for addition and multiplication of fractions. Furthermore, multiplication of fractions has an additional property: *inverses for multiplication.* The rules for adding and multiplying fractions will be used to illustrate each of the following properties.

Closure for Addition The sum of any two fractions is another unique fraction. This property results from the closure and uniqueness properties for addition and multiplication of integers. Notice how closure for multiplication and closure for addition of integers (black arrows) are needed to obtain closure for addition of fractions (green arrow).

$$\frac{-2}{3} + \frac{4}{5} = \frac{-2 \times 5}{3 \times 5} + \frac{4 \times 3}{5 \times 3} = \frac{-10}{15} + \frac{12}{15} = \frac{-10 + 12}{15} = \frac{2}{15}$$

Closure for Multiplication The product of any two fractions is another unique fraction. This property is a direct result of the closure and uniqueness property for multiplication of integers. The black arrow here shows where the closure property for multiplication of integers is needed.

$$\frac{-2}{3} \times \frac{4}{5} = \frac{-2 \times 4}{3 \times 5} = \frac{-8}{15}$$

Identity for Addition The sum of any fraction and 0 is the given fraction. This property follows from the corresponding property for integers, which states that 0 plus any integer is the given integer.

$$\frac{5}{6} + 0 = \frac{5}{6} + \frac{0}{6} = \frac{5 + 0}{6} = \frac{5}{6}$$

Identity for Multiplication The product of any fraction and 1 is the given fraction. This property is a result of the corresponding property for integers, which states that 1 times any integer is the given integer.

$$1 \times \frac{-4}{5} = \frac{1 \times -4}{5} = \frac{-4}{5}$$

Addition Is Commutative Two fractions that are being added can be interchanged (commuted) without changing the sum.

$$\frac{7}{8} + \frac{3}{5} = \frac{35}{40} + \frac{24}{40} = \frac{35 + 24}{40} = \frac{24 + 35}{40} = \frac{24}{40} + \frac{35}{40} = \frac{3}{5} + \frac{7}{8}$$

This property is illustrated on the number lines in Figure 5.50.

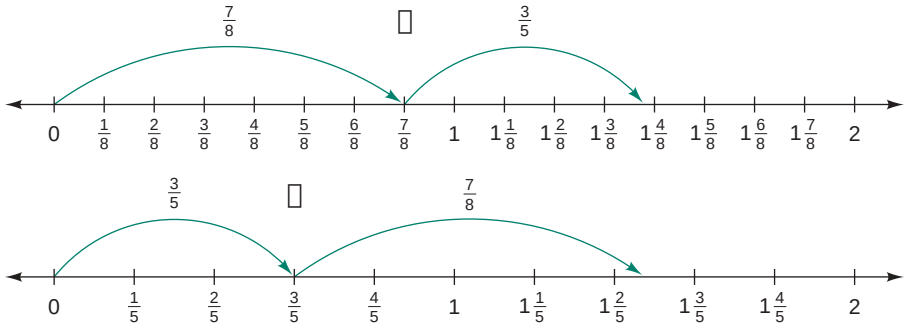


Figure 5.50

Addition Is Associative In a sum of three fractions, the middle number may be grouped (associated) with either of the other two numbers as shown in Example L.

EXAMPLE L

Compute the sum on each side of the equation to illustrate the associative property for addition. (The sums inside the parentheses should be computed first.)

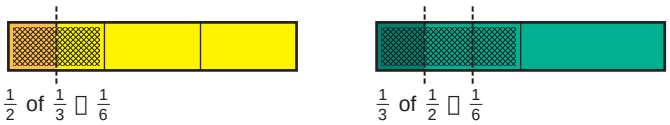
$$\left(\frac{1}{3} + \frac{1}{4}\right) + \frac{1}{6} = \frac{1}{3} + \left(\frac{1}{4} + \frac{1}{6}\right)$$

Solution Left side: $\left(\frac{1}{3} + \frac{1}{4}\right) + \frac{1}{6} = \frac{7}{12} + \frac{1}{6} = \frac{9}{12}$ Right side: $\frac{1}{3} + \left(\frac{1}{4} + \frac{1}{6}\right) = \frac{1}{3} + \frac{5}{12} = \frac{9}{12}$

Multiplication Is Commutative Two fractions that are being multiplied can be interchanged (commuted) without changing the product. This follows from the corresponding property for integers.

$$\frac{1}{2} \times \frac{1}{3} = \frac{1 \times 1}{2 \times 3} = \frac{1 \times 1}{3 \times 2} = \frac{1}{3} \times \frac{1}{2}$$

An illustration of this property is interesting because the processes of taking $\frac{1}{2}$ of something and taking $\frac{1}{3}$ of something are quite different. To take $\frac{1}{2}$ of $\frac{1}{3}$, we begin with a $\frac{1}{3}$ bar, as shown in part a of Figure 5.51; and to take $\frac{1}{3}$ of $\frac{1}{2}$, we use a $\frac{1}{2}$ bar, as shown in part b of Figure 5.51. In each representation the darker part of the bar is $\frac{1}{6}$ of a whole bar, so $\frac{1}{2} \times \frac{1}{3} = \frac{1}{3} \times \frac{1}{2}$.



(a) (b)

Figure 5.51

Multiplication Is Associative In a product of three fractions, the middle number may be grouped with either of the other two numbers as shown in Example M.

EXAMPLE M

Compute the product on each side of the equation to illustrate the associative property for multiplication.

$$\left(\frac{1}{2} \times \frac{3}{4}\right) \times \frac{1}{5} = \frac{1}{2} \times \left(\frac{3}{4} \times \frac{1}{5}\right)$$

Solution Left side: $\left(\frac{1}{2} \times \frac{3}{4}\right) \times \frac{1}{5} = \frac{3}{8} \times \frac{1}{5} = \frac{3}{40}$ Right side: $\frac{1}{2} \times \left(\frac{3}{4} \times \frac{1}{5}\right) = \frac{1}{2} \times \frac{3}{20} = \frac{3}{40}$

Multiplication Is Distributive over Addition When a sum (or difference) of two fractions is multiplied by a third number, we can add (or subtract) the two fractions and then multiply, or we can multiply both fractions by the third number and then add (or subtract).

EXAMPLE N

Perform the calculations on each side of the equation to illustrate the distributive property of multiplication over addition.

$$\frac{1}{2} \times \left(\frac{3}{4} + \frac{7}{10}\right) = \left(\frac{1}{2} \times \frac{3}{4}\right) + \left(\frac{1}{2} \times \frac{7}{10}\right)$$

Solution Left side: $\frac{1}{2} \times \left(\frac{3}{4} + \frac{7}{10}\right) = \frac{1}{2} \times \left(\frac{15}{20} + \frac{14}{20}\right) = \frac{1}{2} \times \frac{29}{20} = \frac{29}{40}$

Right side: $\left(\frac{1}{2} \times \frac{3}{4}\right) + \left(\frac{1}{2} \times \frac{7}{10}\right) = \frac{3}{8} + \frac{7}{20} = \frac{15}{40} + \frac{14}{40} = \frac{29}{40}$

Inverses for Addition For every fraction there is another fraction, called its **opposite** or **inverse for addition,** such that the sum of the two fractions is 0. The fractions $\frac{3}{4}$ and $\frac{-3}{4}$ are inverses for addition. Notice the use of integer inverses; $^{+}3 + {}^{-}3 = 0$.

$$\frac{3}{4} + \frac{{}^{-}3}{4} = \frac{3 + {}^{-}3}{4} = \frac{0}{4} = 0$$

Inverses for Multiplication For every fraction not equal to zero, there is a nonzero fraction, called its **reciprocal** or **inverse for multiplication,** such that the product of the two numbers is 1. The reciprocal of the fraction $\frac{3}{8}$ is $\frac{8}{3}$. Note that the closure property of integers is required to obtain inverses for fraction multiplication.

$$\frac{3}{8} \times \frac{8}{3} = \frac{3 \times 8}{8 \times 3} = \frac{24}{24} = 1$$

MENTAL CALCULATIONS

The mental calculating techniques that we have used for computing with whole numbers—*compatible numbers, substitutions, equal differences, add-up,* and *equal quotients*—are also appropriate for fractions. We will look at examples of these techniques and point out some of the number properties that make them possible.

Compatible Numbers for Mental Calculations Compatible fractions are numbers that can be conveniently combined in a given computation.

EXAMPLE O

Perform each calculation mentally by finding compatible fractions.

1. $2\frac{1}{5} + \frac{2}{3} + 1\frac{4}{5}$
2. $5\frac{5}{6} - \frac{3}{4} + 2\frac{1}{6} + 4$
3. $\frac{1}{3} \times 14 \times 9 \times \frac{1}{2}$

Solution

1. $2\frac{1}{5} + \frac{2}{3} + 1\frac{4}{5} = 2\frac{1}{5} + 1\frac{4}{5} + \frac{2}{3} = 4 + \frac{2}{3} = 4\frac{2}{3}$.
2. $5\frac{5}{6} - \frac{3}{4} + 2\frac{1}{6} + 4 = 5\frac{5}{6} + 2\frac{1}{6} + 4 - \frac{3}{4} = 8 + 4 - \frac{3}{4} = 11\frac{1}{4}$.
3. $\frac{1}{3} \times 14 \times 9 \times \frac{1}{2} = \frac{1}{3} \times 9 \times 14 \times \frac{1}{2} = 3 \times 7 = 21$.

Note: The rearrangements of numbers in Example O require the use of the commutative and associative properties of addition and multiplication of fractions.

Products involving fractions and whole numbers in which the denominator of the fraction divides the whole number can be calculated mentally by dividing the whole number by the denominator. To compute $\frac{2}{3} \times 24$, we can divide 24 by 3 and multiply the result by 2, as shown in the following equations. We use the fact that $\frac{2}{3} = 2 \times \frac{1}{3}$ in the first equation and the associative property for multiplication in the second equation.

$$\frac{2}{3} \times 24 = \left(2 \times \frac{1}{3}\right) \times 24 = 2 \times \left(\frac{1}{3} \times 24\right) = 2 \times 8 = 16$$

The steps in the preceding equations can be shortened considerably by dividing the denominator of the fraction and the whole number by their greatest common factor.

$$\frac{2}{3} \times 24 = \frac{2}{\cancel{3}_{1}} \times \cancel{24}^{8} = 16$$

EXAMPLE P

Calculate each of the following products mentally, using compatible fractions.

1. $\frac{5}{8} \times 32$ **2.** $^{-}54 \times \frac{5}{9}$ **3.** $\frac{4}{3} \times 18$

Solution **1.** $\frac{5}{8} \times 32 = 5 \times \left(\frac{1}{8} \times 32\right) = 5 \times 4 = 20$ $\left(\text{or } \frac{5}{\cancel{8}} \times \overset{4}{\cancel{32}} = 20\right)$.

2. $^{-}54 \times \frac{5}{9} = \left(^{-}54 \times \frac{1}{9}\right) \times 5 = {}^{-}6 \times 5 = {}^{-}30$ $\left(\text{or } \overset{-6}{\cancel{-54}} \times \frac{5}{\cancel{9}} = {}^{-}30\right)$.

3. $\frac{4}{3} \times 18 = 4 \times \left(\frac{1}{3} \times 18\right) = 4 \times 6 = 24$ $\left(\text{or } \frac{4}{\cancel{3}} \times \overset{6}{\cancel{18}} = 24\right)$.

Substitutions for Mental Calculations Sometimes it is possible to substitute one form of a number for another to obtain compatible fractions. For example,

$$4 \times 2\frac{6}{7} = 4 \times \left(3 - \frac{1}{7}\right) = 12 - \frac{4}{7} = 11\frac{3}{7}$$

EXAMPLE Q

Find a convenient substitution in order to perform each calculation mentally.

1. $2\frac{7}{8} + \frac{1}{4}$ **2.** $4\frac{5}{6} - \frac{1}{2}$ **3.** $7 \times 2\frac{9}{10}$

Solution **1.** $2\frac{7}{8} + \frac{1}{4} = 2\frac{7}{8} + \frac{1}{8} + \frac{1}{8} = 3 + \frac{1}{8} = 3\frac{1}{8}$. **2.** $4\frac{5}{6} - \frac{1}{2} = 4\frac{2}{6} + \frac{3}{6} - \frac{1}{2} = 4\frac{2}{6}$.

3. $7 \times 2\frac{9}{10} = 7 \times \left(3 - \frac{1}{10}\right) = 21 - \frac{7}{10} = 20\frac{3}{10}$.

[Notice the use of the distributive property to multiply 7 times $\left(3 - \frac{1}{10}\right)$ in problem 3.]

Equal Differences and Add-Up for Mental Calculations Changing two numbers by adding the same amount to both results in the same difference between the two numbers. This technique is very useful in subtracting fractions because in some cases it avoids the need for regrouping (borrowing). For example, to mentally compute $5 - 2\frac{4}{5}$, we can increase both numbers by $\frac{1}{5}$ to obtain the difference easily.

$$5 - 2\frac{4}{5} = 5\frac{1}{5} - 3 = 2\frac{1}{5}$$

The reason we can add $\frac{1}{5}$ to both numbers is that in so doing we are really adding 0 (using the identity property for addition), as shown in the first of the next few equations.

$$5 - 2\frac{4}{5} = (5 + 0) - 2\frac{4}{5} = 5 + \left(\frac{1}{5} - \frac{1}{5}\right) - 2\frac{4}{5}$$

$$= \left(5 + \frac{1}{5}\right) - \left(2\frac{4}{5} + \frac{1}{5}\right) = 5\frac{1}{5} - 3 = 2\frac{1}{5}$$

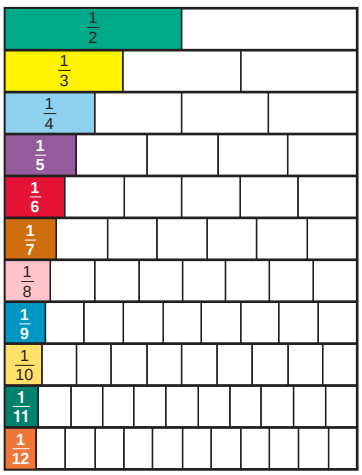
Or the difference can be obtained by adding up from the smaller number to the larger:

$$2\frac{4}{5} + \frac{1}{5} = 3 \quad \text{and} \quad 3 + 2 = 5 \quad \text{so} \quad 5 - 2\frac{4}{5} = 2\frac{1}{5}$$

Laboratory Connection

Fraction Patterns

This tower of bars represents all the fractions with denominators of 2 through 12. Surprising relationships can be found involving addition, subtraction, multiplication, and equality of fractions by placing a ruler on the bars and comparing adjacent bars. Explore these relationships in this investigation.



Mathematics Investigation
Chapter 5, Section 3
www.mhhe.com/bbn

EXAMPLE R

Calculate each difference mentally, using the *equal-differences* technique.

1. $3\frac{2}{8} - 1\frac{7}{8}$ **2.** $5\frac{3}{10} - 3\frac{9}{10}$ **3.** $6\frac{1}{5} - 2\frac{3}{5}$

Solution **1.** $3\frac{2}{8} - 1\frac{7}{8} = 3\frac{3}{8} - 2 = 1\frac{3}{8}$ $\left(\text{Increase both by } \frac{1}{8}\right)$.

2. $5\frac{3}{10} - 3\frac{9}{10} = 5\frac{4}{10} - 4 = 1\frac{4}{10}$ $\left(\text{Increase both by } \frac{1}{10}\right)$.

3. $6\frac{1}{5} - 2\frac{3}{5} = 6\frac{3}{5} - 3 = 3\frac{3}{5}$ $\left(\text{Increase both by } \frac{2}{5}\right)$.

Note: The add-up method is also convenient for computing each of these differences.

Equal Quotients for Mental Calculations The mental calculation technique called *equal quotients* is especially convenient when we are dividing by unit fractions—fractions whose numerator is 1. This is illustrated in the first two quotients in Example S.

EXAMPLE S

Calculate each quotient mentally by using the *equal-quotients* technique.

1. $\frac{7}{9} \div \frac{1}{4}$ **2.** $\frac{5}{4} \div \frac{1}{3}$ **3.** $\frac{6}{7} \div \frac{2}{3}$

Solution **1.** $\frac{7}{9} \div \frac{1}{4} = 4\left(\frac{7}{9}\right) \div 4\left(\frac{1}{4}\right) = \frac{28}{9} \div 1 = 3\frac{1}{9}$.

2. $\frac{5}{4} \div \frac{1}{3} = 3\left(\frac{5}{4}\right) \div 3\left(\frac{1}{3}\right) = \frac{15}{4} \div 1 = 3\frac{3}{4}$.

3. $\frac{6}{7} \div \frac{2}{3} = 3\left(\frac{6}{7}\right) \div 3\left(\frac{2}{3}\right) = \frac{18}{7} \div 2 = \left(\frac{1}{2}\right)\frac{18}{7} \div \left(\frac{1}{2}\right)2 = \frac{9}{7} \div 1 = 1\frac{2}{7}$.

ESTIMATION

Skill at estimating with fractions is especially important, since approximations by whole numbers or compatible fractions are often all that is needed. One of the most common estimating techniques is *rounding*.

NCTM Standards

Students in grades 3–5 will need to be encouraged to routinely reflect on the size of an anticipated solution. If $\frac{3}{8}$ of a cup of sugar is needed for a recipe and the recipe is doubled, will more or less than one cup of sugar be needed? p. 156

Rounding The sum or difference of mixed numbers and fractions can be estimated by rounding each number to the nearest whole number.

$$6\frac{1}{3} + 2\frac{3}{4} + 1\frac{1}{5} \approx 6 + 3 + 1 = 10$$

$$8\frac{1}{3} - 2\frac{3}{5} \approx 8 - 3 = 5$$

Estimating a product by rounding two mixed numbers to the nearest whole number may produce a good estimation, as in the following example.

$$6\frac{3}{4} \times 8\frac{1}{3} \approx 7 \times 8 = 56 \qquad \text{The actual product is } 56\frac{1}{4}.$$

Or it may give a rough estimation, as in the next example.

$$6\frac{1}{2} \times 8\frac{1}{3} \approx 7 \times 8 = 56 \qquad \text{The actual product is } 54\frac{1}{6}.$$

A more reliable estimation for $6\frac{1}{2} \times 8\frac{1}{3}$ can be obtained by multiplying the two whole numbers, 6×8, and then adding the product of $\frac{1}{2} \times 8$ and the product of $6 \times \frac{1}{3}$.

$$\begin{aligned} 6\frac{1}{2} \times 8\frac{1}{3} &\approx (6 \times 8) + \left(\frac{1}{2} \times 8\right) + \left(6 \times \frac{1}{3}\right) \\ &= 48 + 4 + 2 \\ &= 54 \end{aligned}$$

Figure 5.52 shows why this method produces a good estimation. The actual product is represented by the region whose dimensions are $6\frac{1}{2}$ by $8\frac{1}{3}$. The estimation is represented by the 6×8 red region, the $6 \times \frac{1}{3}$ green region, and the $\frac{1}{2} \times 8$ blue region. The $\frac{1}{2} \times \frac{1}{3}$ gray region in the lower right corner represents the difference between the actual product and the estimation, which we obtained in the preceding example. This shows that the difference between the product and the estimation is small.

The distributive property is used below to obtain the four partial products that correspond to the four regions in Figure 5.52. It is used once in going from step 1 to step 2 and twice in going from step 2 to step 3.

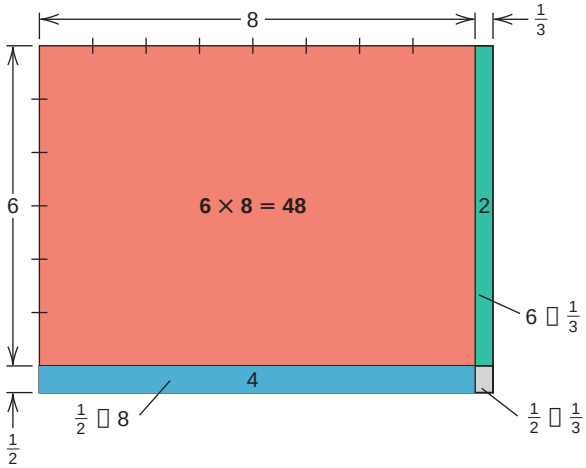


Figure 5.52

Step 1 $6\frac{1}{2} \times 8\frac{1}{3} = \left(6 + \frac{1}{2}\right) \times \left(8 + \frac{1}{3}\right)$

Step 2 $= \left(6 + \frac{1}{2}\right)8 + \left(6 + \frac{1}{2}\right)\frac{1}{3}$

Step 3 $= (6 \times 8) + \left(\frac{1}{2} \times 8\right) + \left(6 \times \frac{1}{3}\right) + \left(\frac{1}{2} \times \frac{1}{3}\right)$

Step 4 $= 48 + 4 + 2 + \frac{1}{6}$

Step 5 $= 54\frac{1}{6}$

EXAMPLE T Estimate $10\frac{1}{2} \times 6\frac{1}{5}$ by multiplying the two whole numbers and then adding the products of the fractions and the whole numbers as illustrated above.

Solution $10\frac{1}{2} \times 6\frac{1}{5} \approx (10 \times 6) + \left(\frac{1}{2} \times 6\right) + \left(10 \times \frac{1}{5}\right) = 60 + 3 + 2 = 65.$

Compatible Numbers for Estimation Replacing a fraction by a reasonably close and compatible fraction can be very useful in obtaining an estimation. Similarly, if a product involves a whole number and a fraction, we can often obtain an estimation by replacing the whole number with a reasonably close whole number that is divided evenly by the denominator of the fraction.

$$\frac{2}{7} \times 20 \approx \frac{2}{7} \times 21 = \frac{2}{\cancel{7}} \times \overset{3}{\cancel{21}} = 6$$

Or in a sum or difference, we can obtain an estimation by replacing a fraction by a compatible fraction.

$$3\frac{5}{8} + 9\frac{1}{7} \approx 3\frac{5}{8} + 9\frac{1}{8} = 12\frac{6}{8} = 12\frac{3}{4}$$

EXAMPLE U

Use compatible numbers to estimate each computation mentally.

1. $\frac{3}{4} \times 31$ **2.** $4\frac{7}{10} + 2\frac{2}{11}$ **3.** $6 \times 2\frac{1}{7}$

Solution **1.** $\frac{3}{4} \times 31 \approx \frac{3}{4} \times 32 = \frac{3}{\cancel{4}} \times \overset{8}{\cancel{32}} = 24$. **2.** $4\frac{7}{10} + 2\frac{2}{11} \approx 4\frac{7}{10} + 2\frac{2}{10} = 6\frac{9}{10}$.

3. $6 \times 2\frac{1}{7} \approx 6 \times 2\frac{1}{6} = 12 + 1 = 13$ (Notice in this solution that the distributive property is needed to multiply 6 times $2\frac{1}{6}$, since $6 \times 2\frac{1}{6} = 6 \times \left(2 + \frac{1}{6}\right)$).

PROBLEM-SOLVING APPLICATION

When two people or machines can accomplish a task at different rates, sometimes we need to determine how much time will be required for them to do the job together. The solutions to such problems often require fractions, and the information can be illustrated by diagrams.

Problem

Mary and Bill have the responsibility of mowing their school's soccer field. Mary can mow it in 4 hours with her lawn mower, and Bill can mow it in 6 hours with his lawn mower. How long will it take them if they work together?

Research Statement

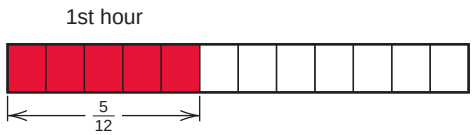
The 6th national mathematics assessment found that only 25% of the eighth-grade students could write a word problem involving division of a whole number and a fraction.

Kouba, Zawojewski, and Strutchens

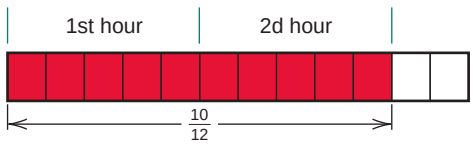
Understanding the Problem We know it will require less than 4 hours because Mary can do the job alone in 4 hours. It will require more than 1 hour because Mary can mow $\frac{1}{4}$ of the field in 1 hour and Bill can mow $\frac{1}{6}$ of the field in 1 hour. **Question 1:** What fraction of the field can they mow in 1 hour, working together?

Devising a Plan One approach is to *make a drawing*. Since Mary and Bill can mow $\frac{1}{4} + \frac{1}{6} = \frac{5}{12}$ of the field in 1 hour, let's consider a figure to illustrate this part of the total field. The following figure has 12 equal parts, and the 5 shaded parts represent the amount they

can mow in 1 hour. **Question 2:** According to this diagram, approximately how long (to the nearest hour) will it take them, working together, to mow the field?



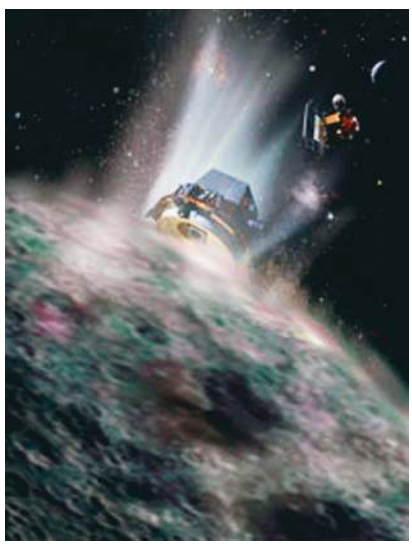
Carrying Out the Plan The next figure shows that in 2 hours they can mow $\frac{10}{12}$ of the field and that 2 of the 12 parts remain to be mowed. **Question 3:** How long will it take them, working together, to mow the entire field?



Looking Back Mary and Bill can mow the entire field in $2\frac{2}{5}$ hours (2 hours 24 minutes). **Question 4:** If Bill gets his mower sharpened and can then mow the field in 5 hours, can Mary and Bill, working together, mow the field in less than 2 hours?

Answers to Questions 1–4 **1.** $\frac{5}{12}$ **2.** 2 hours **3.** $2\frac{2}{5}$ hours, or 2 hours 24 minutes. **4.** No. Since Mary can mow $\frac{1}{4}$ of the field in 1 hour and Bill can mow $\frac{1}{5}$ in 1 hour, together they can mow $\frac{1}{4} + \frac{1}{5} = \frac{9}{20}$ in 1 hour. Thus, in 2 hours they can only mow $\frac{9}{20} + \frac{9}{20} = \frac{18}{20}$ of the field.

Exercises and Problems 5.3



This image shows a graphical depiction of the spacecraft *Impactor* as it crashed into the comet Tempel 1. Use the fact that each dimension of one of the *Deep Impact* models was $\frac{1}{10}$ the dimensions of the spacecraft in exercises 1 and 2.

1. a. If the *Impactor* for the model has a diameter of 3.6 inches, what is the diameter of the life-size *Impactor*?

b. If the spacecraft has a length of 10 feet, what is the length of the model?

2. a. If the antenna on the model has a diameter of 3 inches, what is the diameter of the antenna on the spacecraft?

b. If the solar panel on the spacecraft has a length of 90 inches and a width of 90 inches, what is the length and width of the solar panel on the model?

The fraction below each photo of the animals shown here tells what portion of the life-size object the picture is, and the lengths below in exercises 3 and 4 were obtained by placing a ruler on each photo.

3. a. The Artic Fox, also called the blue fox, has a life span of about 5 years. Its photo height here is $1\frac{1}{4}$ inches. What is the height of the corresponding life-size fox?



Artic Fox ($\frac{1}{12}$)

b. The Bald Eagle has an average life span of 15 to 20 years. The photo length of one of its wings is $\frac{3}{8}$ inch. What is the length of one of the life-size eagle wings?



Bald Eagle ($\frac{1}{32}$)

4. a. The Mute Swan was introduced from Europe to North America and can live up to 25 years. Its photo height here is $\frac{3}{4}$ inch. What is the height of the corresponding life-size swan?



Mute Swan ($\frac{1}{32}$)

b. The female eagle shown in 3b is slightly larger than the male. The photo length of the right wing (length from tip of wing to back center of eagle) is $\frac{3}{4}$ inch. What is the wingspan (length from tip of one wing to tip of the other) of the life-size eagle?

Sketch Fraction Bars in exercises 5 and 6 to illustrate each computation.

5. a. $\frac{3}{10} + \frac{2}{5} = \frac{7}{10}$ **b.** $\frac{5}{6} - \frac{1}{3} = \frac{3}{6}$

c. $\frac{2}{3} \div \frac{1}{6} = 4$ **d.** $\frac{1}{4} \times 3 = \frac{3}{4}$

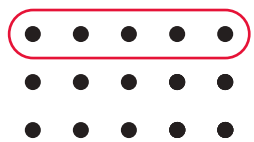
e. $\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$ **f.** $\frac{1}{3} \times \frac{1}{6} = \frac{1}{18}$

6. a. $\frac{1}{3} + \frac{1}{2} = \frac{5}{6}$ **b.** $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

c. $2 \div \frac{1}{3} = 6$ **d.** $\frac{5}{6} - \frac{1}{4} = \frac{7}{12}$

e. $\frac{1}{3} \times \frac{3}{7} = \frac{1}{7}$ **f.** $\frac{9}{12} \div \frac{1}{3} = 2\frac{1}{4}$

The next figure can be used to show that $\frac{1}{3} \times 15 = 5$, since 15 dots can be divided into 3 equal groups of 5 dots each.

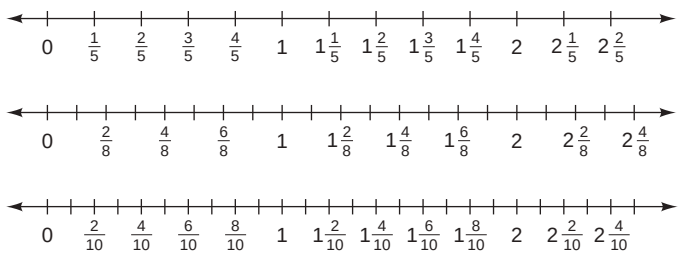


Sketch sets of dots to illustrate and determine the products in exercises 7 and 8.

7. a. $\frac{3}{8} \times 24$ **b.** $\frac{2}{5} \times 30$

8. a. $\frac{5}{6} \times 18$ **b.** $\frac{3}{7} \times 28$

Sums of fractions can be approximated on number lines. For example, place the edge of a piece of paper on the eighths line below, and mark off the length $1\frac{1}{8}$. Then place the beginning of this marked-off length at the $1\frac{1}{5}$ point on the fifths line to approximate the sum $1\frac{1}{5} + 1\frac{1}{8}$. A different approximation can be obtained by marking off the sum on the eighths line.



Use the preceding number lines to approximate the sums in exercises 9 and 10. Write the number that the sum is closest to on the given line. Then compute these sums and compare them to your approximations.

9. a. $\frac{4}{5} + 1\frac{3}{8}$ (fifths line)

b. $\frac{5}{8} + \frac{7}{10}$ (eighths line)

10. a. $\frac{3}{10} + 1\frac{4}{5}$ (tenths line)

b. $1\frac{1}{5} + 1\frac{1}{8}$ (fifths line)

Differences of fractions can be approximated on number lines. For example, place the edge of a piece of paper on the fifths line on the preceding page, and mark off the length $\frac{3}{5}$. Then place the end of this marked-off length at the point $1\frac{7}{8}$ on the eighths line on the preceding page, to approximate the difference $1\frac{7}{8} - \frac{3}{5}$. Use the preceding number lines to approximate the differences in exercises 11 and 12. Write the number that the difference is closest to on the given line. Then compute these differences and compare them with your approximations.

11. a. $1\frac{4}{5} - \frac{5}{8}$ (fifths line)

b. $1\frac{1}{8} - \frac{7}{10}$ (eighths line)

12. a. $\frac{9}{10} - \frac{3}{5}$ (tenths line)

b. $1\frac{7}{8} - \frac{3}{5}$ (eighths line)

Perform the operations in exercises 13 and 14. Replace all improper fractions in your answers with whole numbers or mixed numbers, and write all fractions in lowest terms.

13. a. $\frac{2}{3} + \frac{3}{4}$ **b.** $\frac{1}{6} + \frac{3}{8}$ **c.** $\frac{2}{3} \times 6$

d. $\frac{-3}{4} \times \frac{2}{5}$ **e.** $2\frac{1}{4} + 1\frac{1}{3}$ **f.** $-1\frac{5}{6} + 3\frac{1}{2}$

g. $2\frac{1}{4} \times 3\frac{1}{2}$ **h.** $14\frac{1}{2} \div 2\frac{1}{4}$ **i.** $\frac{7}{8} - \frac{1}{3}$

14. a. $\frac{3}{4} - \frac{2}{5}$ **b.** $\frac{3}{4} \div \frac{1}{10}$ **c.** $\frac{2}{3} \div \frac{1}{5}$

d. $3\frac{1}{4} - 1\frac{1}{8}$ **e.** $5\frac{1}{3} - 2\frac{1}{2}$ **f.** $-3 \div \frac{1}{5}$

g. $4 \times 5\frac{1}{8}$ **h.** $3\frac{1}{4} + 7\frac{5}{6}$ **i.** $2\frac{1}{3} \times 6\frac{1}{4}$

Write the opposite (inverse for addition) and the reciprocal (inverse for multiplication) of each number in the tables in exercises 15 and 16.

15.

Number	$\frac{7}{8}$	-4	$\frac{-1}{2}$	10
Opposite				
Reciprocal				

16.

Number	$\frac{12}{5}$	27	$\frac{3}{8}$	-249
Opposite				
Reciprocal				

Error Analysis In computing with fractions, several types of errors frequently occur. In the following example, $3\frac{2}{5}$ should have been replaced by $2\frac{7}{5}$. Instead, the 1 that was regrouped from the 3 was erroneously placed in the numerator to form $2\frac{12}{5}$.

$$\begin{array}{rr} 3\frac{2}{5} = & 2\frac{12}{5} \\ \underline{-\frac{4}{5}} = & \underline{-\frac{4}{5}} \\ & 2\frac{8}{5} = 3\frac{3}{5} \end{array}$$

Find plausible reasons for the errors in the computations in exercises 17 and 18.

17. a. $\frac{1}{3} + \frac{1}{4} = \frac{6}{8} + \frac{5}{8} = \frac{11}{8}$ **b.** $5\frac{3}{7} - \frac{1}{4} = 5\frac{2}{3}$

c. $\frac{1}{2} \times \frac{4}{11} = \frac{8}{11}$ **d.** $\frac{11}{12} \div \frac{1}{3} = \frac{11}{3}$

18. a. $\frac{1}{4} + \frac{5}{6} = \frac{6}{10}$ **b.** $\frac{7}{8} - \frac{2}{3} = \frac{5}{5}$

c. $\frac{1}{2} \times \frac{3}{8} = \frac{6}{8}$ **d.** $\frac{3}{4} \div \frac{1}{5} = \frac{4}{15}$

State the number property in exercises 19 and 20 that is being used in each of these equalities.

19. a. $\frac{3}{7} + \left(\frac{2}{9} + \frac{1}{3}\right) = \frac{3}{7} + \left(\frac{1}{3} + \frac{2}{9}\right)$

b. $\frac{3}{7} + \left(\frac{2}{9} \times \frac{9}{2}\right) = \frac{3}{7} + 1$

c. $\frac{2}{9} + \left(\frac{3}{7} + \frac{1}{3}\right) = \left(\frac{2}{9} + \frac{3}{7}\right) + \frac{1}{3}$

20. a. $\frac{5}{6} \times \left(\frac{3}{4} + \frac{1}{2}\right) = \left(\frac{3}{4} + \frac{1}{2}\right) \times \frac{5}{6}$

b. $\frac{3}{4} \times \frac{5}{6} + \frac{1}{2} \times \frac{5}{6} = \left(\frac{3}{4} + \frac{1}{2}\right) \times \frac{5}{6}$

c. $\frac{7}{8} + \left(\frac{-2}{3} + \frac{2}{3}\right) = \frac{7}{8} + 0$

Determine whether the set in exercises 21 and 22 is closed for the given operation. Explain your answer. If not closed, give an example to show why not.

21. a. The set of positive fractions for addition
b. The set of negative fractions for multiplication

22. a. The set of positive fractions for division
b. The set of fractions between $^-10$ and 10 for subtraction

Exercises 23 and 24 contain compatible fractions. Use mental calculations to find exact answers, and show or explain your method.

23. a. $^-2\frac{2}{3} + 5\frac{1}{2} + 6\frac{1}{3} + 1\frac{1}{2}$ **b.** $5\frac{5}{8} + 2\frac{1}{2} - 1\frac{1}{8}$

c. $\frac{-1}{3} \times \frac{2}{5} \times 12 \times 20$ **d.** $\frac{7}{4} \times 24 + 8$

24. a. $\frac{1}{4} \times 17 \times 120$ **b.** $\frac{3}{7} + 2\frac{1}{5} + 1\frac{4}{7}$

c. $\frac{2}{3} \times 14 \times 18 \times \frac{1}{7}$ **d.** $15\frac{1}{2} + 5\frac{1}{3} - 4\frac{1}{2}$

Find convenient substitutions in exercises 25 and 26, and determine exact answers by performing each calculation mentally. Show your substitution.

25. a. $215\frac{1}{5} - 10\frac{4}{5}$ **b.** $4 \times 9\frac{6}{7}$ **c.** $86\frac{11}{12} + 10\frac{1}{2}$

26. a. $3 \times 3\frac{4}{5}$ **b.** $6\frac{7}{10} - 1\frac{1}{2}$ **c.** $12\frac{5}{6} + \frac{1}{3}$

Compute each difference or quotient in exercises 27 and 28 by using *equal differences, add-up,* or *equal quotients.* Show your method.

27. a. $8 - 3\frac{6}{7}$ **b.** $15\frac{3}{10} - 10\frac{9}{10}$

c. $\frac{5}{8} \div \frac{1}{3}$

28. a. $\frac{7}{4} \div \frac{1}{6}$ **b.** $7\frac{1}{8} - 3\frac{3}{4}$

c. $\frac{4}{7} \div \frac{1}{5}$

Round each mixed number in exercises 29 and 30 to the nearest whole number to estimate the sum of numbers in each row.

Example:

$7\frac{2}{3}$	$2\frac{3}{4}$	$4\frac{1}{8}$	$5\frac{9}{10}$	Approximate sum
Think 8	Think 3	Think 4	Think 6	21

29. a. $1\frac{2}{3}$ $3\frac{1}{4}$ $1\frac{1}{2}$ $2\frac{1}{10}$

b. $3\frac{4}{5}$ $2\frac{1}{6}$ $3\frac{2}{5}$ $4\frac{1}{2}$

30. a. $10\frac{1}{3}$ $5\frac{1}{2}$ $2\frac{7}{8}$ $5\frac{1}{6}$

b. $6\frac{1}{2}$ $2\frac{3}{4}$ $1\frac{1}{4}$ $2\frac{5}{6}$

Approximate the product of each pair of mixed numbers in exercises 31 and 32 by multiplying the whole numbers and adding the products of the fractions and the whole numbers (as shown on page 326). Show each step in obtaining the answer.

31. a. $4\frac{1}{3} \times 6\frac{1}{2}$ **b.** $5\frac{1}{4} \times 8\frac{2}{5}$

32. a. $3\frac{1}{4} \times 4\frac{2}{3}$ **b.** $10\frac{1}{3} \times 6\frac{1}{2}$

Estimate each computation in exercises 33 and 34, using *compatible numbers.* Show your compatible-number replacement.

33. a. $\frac{6}{7} \times 34$ **b.** $9\frac{4}{5} + 5\frac{1}{6}$

34. a. $8 \times 4\frac{1}{7}$ **b.** $\frac{3}{4} \times 81$

Reasoning and Problem Solving

Draw a diagram to illustrate the given information and the solution for 35 through 40.

35. What fractional amount of the Earth's surface is covered by oceans or glaciers if $\frac{2}{3}$ is covered by water and $\frac{1}{10}$ by glaciers? (*Note:* Glaciers occur only over land.)

36. An experiment calls for $8\frac{1}{2}$ ounces of sulfate, but the classroom chemistry kit has only $3\frac{1}{5}$ ounces. How much more sulfate is needed?

37. In 1897, 48 million pounds of blue shad was caught in the ocean between Maine and Florida. The yearly catch is now $\frac{1}{6}$ of the 1897 catch. How many pounds of blue shad are caught yearly now?

38. A school's enrollment decreased by $\frac{1}{4}$ because of a reorganization of districts. The new enrollment is 270. What was the school's enrollment before the change?

39. On Wednesday it rained $1\frac{1}{2}$ inches and on Thursday it rained only $\frac{1}{3}$ as much as it had on Wednesday. What fraction of an inch did it rain on Thursday?

40. Sound travels $\frac{1}{5}$ mile in 1 second. How many seconds will it take a sound wave to travel 2 miles?

The sequences of numbers and keys in exercises 41 and 42 are for a calculator that displays fractions. For example, $\frac{2}{7}$ is entered by the keystrokes 2 [b/c] 7. Determine the fraction or mixed number in lowest terms for each computation.

41. a. 2 [b/c] 7 [+] 3 [b/c] 4 [=]

b. 7 [b/c] 8 [−] 2 [b/c] 3 [=]

c. 9 [b/c] 14 [×] 4 [b/c] 9 [=]

d. 5 [b/c] 6 [÷] 7 [b/c] 8 [=]

42. a. 8 [b/c] 9 [+] 5 [b/c] 7 [=]

b. 4 [b/c] 11 [−] 1 [b/c] 6 [=]

c. 3 [b/c] 22 [×] 4 [b/c] 7 [=]

d. 2 [b/c] 3 [÷] 4 [b/c] 9 [=]

43. Jan and Carl were using their calculators to compute $80 \div \frac{3}{5}$. What quotient will be obtained for each sequence of steps if their calculators are designed to follow the *order of operations?* Whose sequence of calculator steps will not produce the correct answer? Explain why.

Jan's Steps	Carl's Steps
1. 80	1. 80
2. [÷]	2. [÷]
3. (3 [÷] 5)	3. 3 [÷] 5
4. [=]	4. [=]

44. Annette and Sharon computed $\frac{3}{5} \div \frac{1}{8}$ on their calculators by using the steps below. What quotient will be obtained for each sequence of steps if their calculators are designed to follow the *order of operations?* Whose sequence of calculator steps will produce the correct quotient?

Annette's Steps	Sharon's Steps
1. 3 [÷] 5	1. 3 [÷] 5
2. [÷]	2. [÷]
3. 1 [÷] 8	3. (1 [÷] 8)
4. [=]	4. [=]

45. A taxpayer computes her federal income tax report using the following method. Some of the amounts are shown. Compute the missing amounts.

a.	$\frac{1}{3}$ of base income of $22,000	_____
b.	$\frac{1}{5}$ of averageable income of $12,000	_____
c.	Line 1 plus line 2	_____
d.	Tax on line 3	$2318
e.	Tax on line 1	$1630
f.	Line 4 minus line 5	_____
g.	Line 6 multiplied by 4	_____
h.	Line 4 plus line 7 (total tax)	_____

Use these two recipes to answer questions 46 and 47.

Millet Spoon Bread

- $\frac{1}{3}$ cup dry millet
- $\frac{3}{4}$ cup water
- $\frac{1}{2}$ cup cornmeal
- 2 cups low-fat buttermilk
- $\frac{1}{2}$ teaspoon baking soda
- 2 eggs, beaten
- 1 tablespoon safflower oil

Roquefort Dressing

- $\frac{1}{2}$ cup nonfat cottage cheese
- $\frac{1}{2}$ cup nonfat plain yogurt
- $\frac{1}{3}$ cup low-fat buttermilk
- $\frac{1}{4}$ cup crumbled Roquefort cheese
- 1 teaspoon white pepper

46. a. The recipe for Millet Spoon Bread is for 6 servings. What is the amount of each ingredient in order to double the recipe for 12 servings?

b. What is the total amount of the nonfat cottage cheese, the low-fat buttermilk, and the Roquefort cheese in the Roquefort Dressing recipe?

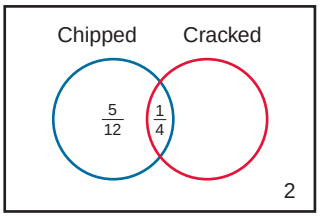
47. a. The recipe for Roquefort Dressing makes $1\frac{1}{2}$ cups. What is the amount of each ingredient in order to obtain 6 cups of this dressing?

b. If the dry millet, the water, and the cornmeal are poured into a 2-cup container, what fraction of a cup of the container will be unfilled?

48. Featured Strategies: Guessing and Checking, and Drawing Venn Diagrams. Mr. Hash bought some plates at a yard sale. After arriving home he found that $\frac{2}{3}$ of the plates were chipped, $\frac{1}{2}$ were cracked, and $\frac{1}{4}$ were both chipped and cracked. Only 2 plates were without chips or cracks. How many plates did he buy in all?

a. Understanding the Problem. Let's guess a number to become more familiar with the problem. If Mr. Hash bought 48 plates, how many were cracked and how many were chipped?

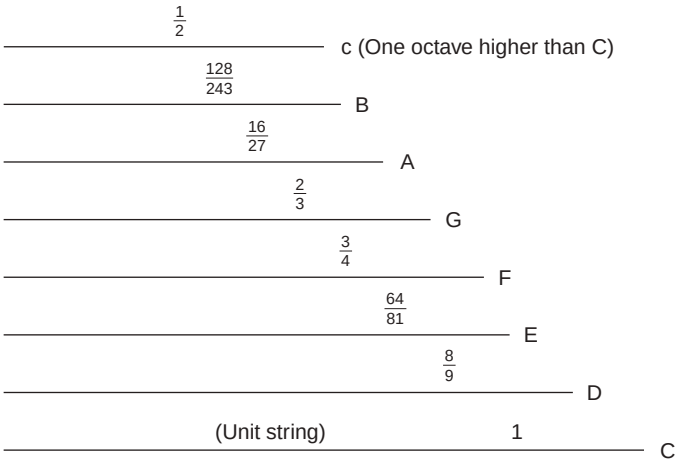
b. Devising a Plan. We could continue guessing and checking. Another approach is to *draw a Venn diagram.* The following figure uses circles to represent the chipped plates and the cracked plates. Two plates were neither chipped nor cracked. Since $\frac{2}{3}$ were chipped and $\frac{1}{4}$ were chipped and cracked, $\frac{2}{3} - \frac{1}{4} = \frac{5}{12}$ were chipped but not cracked. What fraction of the plates were cracked but not chipped?



c. Carrying Out the Plan. After determining the fraction of the cracked plates that were not chipped, we can add the three fractions to find the fraction of the plates that were chipped and/or cracked. How can this information lead to the solution? How many plates did Mr. Hash buy?

d. Looking Back. Suppose that instead of 2 plates, there were 3 plates that were neither chipped nor cracked. In this case what would be the total number of plates purchased?

49. Musical notes produced from two strings or wires of equal diameter and tension will vary according to the lengths of the strings. Different fractions of the length of the unit string (the longest string in the illustration) can be used to produce the notes C, D, E, F, G, A, B, and c (do, re, mi, fa, so, la, ti, and do). In particular, if a string is one-half as long as another, its tone or note will be an octave higher than the longer string.*



*See C. F. Linn, *The Golden Mean* (New York: Doubleday), pp. 9–13, for an elementary explanation of the origin of these fractions.

a. With the exception of two strings, each string from the unit string to the top string is $\frac{8}{9}$ the length of the previous string. For example, the G string is $\frac{8}{9}$ the length of the F string since $\frac{8}{9} \times \frac{3}{4} = \frac{2}{3}$. Which other strings are $\frac{8}{9}$ of the length of the preceding strings?

b. The white piano keys pictured below are the notes of the scale from C to c. All but two of these keys are separated by black keys. How is this observation related to the answer for part a?



50. One painter can paint a room in 2 hours; another painter requires 4 hours to paint the same room. How long will it take them to paint the room together?

51. Two-thirds of Mrs. Hoffman's fifth-grade students are boys. To make the number of boys and girls equal, 4 boys go to the other fifth-grade class, and 4 girls come from that class into Mrs. Hoffman's class. Now one-half of her students are boys. How many students are in Mrs. Hoffman's class?

52. You have two candles. One is blue and 8 inches tall and the other is yellow and 12 inches tall. The blue candle burns $\frac{1}{4}$ inch every hour and the yellow candle burns $\frac{1}{2}$ inch every hour. If the yellow candle is lighted 6 hours after the blue candle is lighted and both candles burn continuously, which candle will burn out first? After the first candle has burned out, how much longer will the other candle burn?

53. Paula removed $\frac{4}{5}$ of the DVDs from a new box of read/write DVDs and Sam removed $\frac{7}{10}$ of the DVDs that remained in the box. If Sam removed 28 DVDs from the box, how many DVDs did Paula remove?

54. Two antenna cables differ in length by 164 inches. The shorter cable is $\frac{1}{3}$ the length of the other. Samir needs 350 inches of antenna cable. If he splices the two cables together, will he have enough cable?

55. A common mistake of schoolchildren in adding fractions is to add numerator to numerator and denominator to denominator. However, one student noticed that for the fractions $\frac{1}{3}$ and $\frac{4}{5}$, the fraction obtained by this method, $\frac{1+4}{3+5} = \frac{5}{8}$, is greater than $\frac{1}{3}$ and less than $\frac{4}{5}$. Are there other pairs of fractions for which this process will produce a third fraction that is between the original two fractions? Form a conjecture and show examples to support your reasoning.

Teaching Questions

1. One of your students multiplies fractions by getting a common denominator and multiplying the numerators (for example, $\frac{2}{3} \times \frac{3}{4} = \frac{8}{12} \times \frac{9}{12} = \frac{72}{12}$). The student says she does this because she adds fractions by getting a common denominator and then adding the numerators. How can you help this student understand why her method of multiplying fractions is incorrect and why her answer is unreasonable?
2. Two students were discussing results from operations with fractions, and both agreed that "multiplication makes bigger" and "division makes smaller." Write an explanation using visual models that would make sense to students to show their beliefs involving multiplication and division of fractions are incorrect.
3. The **Standards** say that informal methods of calculating with fractions (see quote on page 311) should precede rules and algorithms for such calculations. Using a visual model to represent fractions, illustrate each of the following operations without rules and algorithms and provide brief explanations that would make sense to middle school students.

$$\frac{1}{2} + \frac{1}{8} = \frac{5}{8} \qquad \frac{3}{4} - \frac{3}{8} = \frac{3}{8} \qquad 2 \times \frac{3}{4} = 1\frac{1}{2}$$

$$\frac{1}{2} \div 4 = \frac{1}{8} \qquad \frac{1}{2} \div \frac{1}{8} = 4$$

4. An understanding of division involving fractions and/or whole numbers can be enhanced by creating and solving word problems that are meaningful. Write a word problem for each of the following situations: division of a whole number by a fraction; division of a fraction by a fraction; and division of a fraction by a whole number.
5. A student conjectures that it is always possible to find a fraction between two fractions by adding the numerators and denominators. For example, $\frac{1+2}{3+5} = \frac{3}{8}$ is between $\frac{1}{3}$ and $\frac{2}{5}$. Does this conjecture seem to be true? How would you respond to the student?
6. One student wanted to know why the number 2 can not be canceled in the expression $\frac{2(3)+7}{2}$. Explain how the distributive property and multiplication by $\frac{1}{2}$ shows why this cancellation is not possible. How else could the student be shown that this cancellation is incorrect?

Classroom Connections

1. On page 313 the example from the **Elementary School Text** illustrates fraction addition. (a) Explain how this example follows the recommendation in the **Standards** quote on page 311. (b) Give an example of how the fraction pieces used on this elementary text page might provide a conceptual foundation to help students avoid computational errors, as mentioned in the **Standards** quote on page 315.
2. One of the **National Assessment of Educational Progress (NAEP)** test questions asks 13-year-old students to approximate $\frac{12}{13} + \frac{7}{8}$, and over 50 percent responded with the incorrect answers of 19 or 21. Explain how they might have obtained these answers. What does the **Standards** quote on page 315 say would help students avoid such unreasonable results?
3. In the **Grades 3–5 Standards—Number and Operations** (see inside front cover) under *Understand Numbers . . .* , read the fifth expectation and cite some examples of how this is satisfied in Section 5.3.
4. The **Spotlight on Teaching** at the beginning of Chapter 5 recommends the use of rectangular regions to compare fractions. This spotlight also notes that on one of the NAEP tests, fewer than $\frac{1}{3}$ of the 13-year-olds correctly chose the largest number from $\frac{3}{4}$, $\frac{9}{16}$, $\frac{5}{8}$, and $\frac{2}{3}$. Use the visual model and the reasoning suggested in the spotlight to determine the order of these four fractions. How would you explain the reasoning in this spotlight to school students?

CHAPTER 5 REVIEW

1. Integers

a. The negative whole numbers together with the whole numbers are called **integers.**

b. For each integer, there is another integer, called its **opposite,** such that their sum is zero.

c. For any two integers m and n, m is **less than** n, written $m < n$, if there is a positive integer k such that $m + k = n$.

d. The integers less than zero are called **negative integers** ($^-1$, $^-2$, $^-3$, . . .). Often they are denoted by a raised minus sign to avoid confusion with the operation of subtraction.

e. The integers greater than zero are called **positive integers.** They are sometimes denoted by a raised plus sign to emphasize that they are positive.

2. Operations on integers

a. Addition Positive plus positive equals positive. Negative plus negative equals negative. Negative plus positive may be positive ($^-5 + 8 = 3$), negative ($^-5 + 2 = {}^-3$), or zero ($^-5 + 5 = 0$).

b. Subtraction For any two integers n and s, $n - s$ is the sum of n plus the opposite of s.

c. Multiplication Positive times positive equals positive. Positive times negative equals negative. Negative times positive equals negative. Negative times negative equals positive.

d. Division For any integers n and s, with $s \neq 0$, $n \div s = k$ if and only if $n = s \times k$ for some integer k.

3. Fractions

a. A **fraction** is a number in the form $\frac{a}{b}$, where a and b are any numbers except $b \neq 0$. In this chapter a and b are integers.

b. In the fraction $\frac{a}{b}$, a is called the **numerator** and b is called the **denominator.**

c. There are three concepts of fractions: the **part-to-whole concept,** the **fraction-quotient concept,** and the **ratio concept.**

d. For any fraction $\frac{a}{b}$ and any number $k \neq 0$, $\frac{a}{b}$ is equal to $\frac{ka}{kb}$.

e. To **simplify a fraction** $\frac{a}{b}$, divide a and b by GCF(a, b).

f. If GCF(a, b) = 1, then $\frac{a}{b}$ is said to be in **lowest terms** or **simplified.**

g. The **least common denominator** of $\frac{a}{b}$ and $\frac{c}{d}$ is LCM(b, d).

h. Between any two fractions there is always another fraction. This property is referred to by saying the fractions are **dense.**

i. If the numerator of a fraction is greater than or equal to the denominator, the fraction is called an **improper fraction.**

j. A number that is written as a whole number and a fraction is called a **mixed number.**

4. Fraction operations

a. Addition For fractions $\frac{a}{b}$ and $\frac{c}{d}$,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

b. Subtraction For fractions $\frac{a}{b}$ and $\frac{c}{d}$,

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

c. Multiplication For fractions $\frac{a}{b}$ and $\frac{c}{d}$,

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

d. Division For fractions $\frac{a}{b}$ and $\frac{c}{d}$, with $\frac{c}{d} \neq 0$,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

5. Models

a. Black and red chips are used to illustrate integers, with black chips representing positive integers (credits) and red chips representing negative integers (debts).

b. Number lines illustrate integers, with negative integers below zero and positive integers above zero.

c. Fraction Bars and **sets of dots** illustrate the part-to-whole concept of fractions.

d. Cuisenaire Rods illustrate the ratio concept of fractions.

e. The part-to-whole concept of fractions can be illustrated with **number lines** by dividing unit intervals into equal numbers of parts.

6. Number properties

Properties a through j hold for addition and multiplication of integers and fractions. Property k holds for multiplication of fractions.

a. A set is closed for addition if the sum of any two numbers from the set is in the set.

b. A set is closed for multiplication if the product of any two numbers from the set is in the set.

c. Identity for addition

d. Identity for multiplication

e. Addition is commutative.

f. Multiplication is commutative.

g. Addition is associative.

h. Multiplication is associative.

i. Multiplication distributes over addition and subtraction.
j. Every number has a unique **inverse for addition** called its **inverse** such that the sum of the two numbers is zero.
k. Every fraction not equal to zero has a unique **inverse for multiplication** called its **reciprocal** such that the product of the two fractions is 1.

7. **Mental calculations**
   a. **Compatible numbers** is the technique of using pairs of numbers that are especially convenient for mental calculation.
   b. **Substitution** is the technique of breaking a number into a convenient sum, difference, product, or quotient.
   c. **Equal differences** is the technique of increasing or decreasing both numbers in a difference by the same amount.
   d. **Add-up** is the technique of finding a difference by adding up from the smaller number to the larger.
   e. **Equal quotients** is a type of substitution that uses the fact that the quotient of two numbers remains the same when both numbers are divided by the same number.

8. **Estimation**
   a. **Rounding** is the technique of replacing one or more numbers in a sum, difference, product, or quotient by an approximate number to obtain an estimation. Often fractions are rounded to the nearest whole number.
   b. **Compatible numbers** is the technique of computing estimations by replacing one or more numbers with convenient approximate numbers.

CHAPTER 5 TEST

1. Sketch sets of black and red chips to illustrate each operation, and determine each answer.
   a. $8 + {}^{-}5$ b. ${}^{-}7 - {}^{-}3$ c. $3 \times {}^{-}4$
   d. ${}^{-}20 \div {}^{-}4$ e. $6 - 2$ f. ${}^{-}15 \div 3$

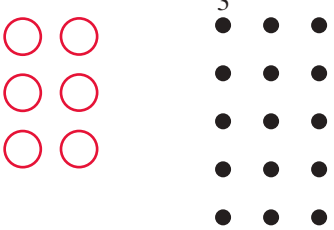
2. Sketch a number line to illustrate each sum.
   a. ${}^{-}8 + 3$ b. ${}^{-}8 + {}^{-}6$

3. Compute each product or quotient.
   a. ${}^{-}7 \times {}^{-}6$ b. $30 \div {}^{-}5$
   c. $8 \times {}^{-}10$ d. ${}^{-}40 \div {}^{-}8$

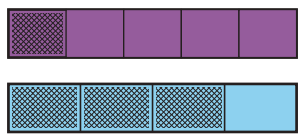
4. Use *equal products* or *equal quotients* to calculate each answer mentally. Explain your method.
   a. ${}^{-}16 \times 25$ b. $800 \div {}^{-}16$

5. Use *compatible numbers* to obtain an estimation. Show your replacement.
   a. ${}^{-}271 \div 30$ b. $\frac{1}{8} \times 55$
   c. $4 \times 6\frac{1}{5}$ d. $11 \times {}^{-}34$

6. Use each figure to illustrate the operation or equality. Determine the answer for each operation.
   a. $6 \div 4$ b. $\frac{1}{3} \times 15$



   c. $\frac{2}{3} \times \frac{1}{5}$
   d. $\frac{3}{4} = \frac{6}{8}$



7. Complete each equation so that the pairs of fractions have the least common denominator.
   a. $\frac{3}{14} =$ b. $\frac{1}{24} =$
   $\frac{5}{16} =$ $\frac{{}^{-}7}{8} =$

8. Determine an inequality for each pair of fractions.
   a. $\frac{6}{11}$ and $\frac{5}{9}$ b. $\frac{3}{5}$ and $\frac{6}{11}$ c. $\frac{{}^{-}4}{9}$ and $\frac{{}^{-}3}{7}$

9. Mentally determine an inequality for each pair of fractions, and explain your reasoning.
   a. $\frac{1}{8}$ and $\frac{1}{10}$ b. $\frac{4}{7}$ and $\frac{5}{12}$
   c. $\frac{1}{2}$ and $\frac{7}{12}$ d. $\frac{5}{6}$ and $\frac{7}{8}$

10. Compute these sums and differences.
    a. $2\frac{1}{6} + 4\frac{1}{3}$ b. $5\frac{2}{3} - 1\frac{1}{5}$
    c. $6\frac{5}{8} + 1\frac{2}{3}$ d. $10\frac{1}{5} - 4\frac{5}{6}$

11. Compute these products and quotients.

a. $2\frac{1}{4} \times 6\frac{1}{3}$ **b.** $8\frac{4}{5} \div 2\frac{1}{8}$

c. $\frac{2}{3} \times {}^{-}14$ **d.** $6 \div 1\frac{1}{2}$

12. State the number property that is being used in each of these equations.

a. $6 + \left(\frac{3}{5} \times \frac{5}{3}\right) = 6 + 1$

b. $\frac{3}{4} \times \left(7 + \frac{{}^{-}1}{2}\right) = \frac{3}{4} \times 7 + \frac{3}{4} \times \frac{{}^{-}1}{2}$

c. $\frac{2}{3} \times \left(\frac{1}{5} + \frac{1}{2}\right) = \left(\frac{1}{5} + \frac{1}{2}\right) \times \frac{2}{3}$

d. $8 + \left(\frac{4}{5} + \frac{{}^{-}4}{5}\right) = 8 + 0$

13. True or false?

a. Subtraction of integers is commutative.

b. The set of positive integers is closed for multiplication.

c. The set of positive fractions is closed for multiplication.

d. The set of integers is closed for division.

e. Subtraction of fractions is associative.

14. The habitat of the Silky Shark is worldwide oceanic as well as coastal waters. Its name comes from its unusually smooth skin. The photo length of this specimen is $\frac{1}{72}$ the length of the life-size fish.



a. The photo length of this fish is $1\frac{3}{4}$ inches. What is the length of the life-size fish?

b. The photo distance between the tips of its tail fins is $\frac{3}{8}$ inch. What is this distance for the life-size fish?

15. A school's new pump can fill the swimming pool in 5 hours; the old pump takes 10 hours. How long will it take to fill the pool if both pumps are used together?

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CHAPTER

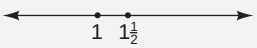
6 Decimals: Rational and Irrational Numbers

Spotlight on Teaching

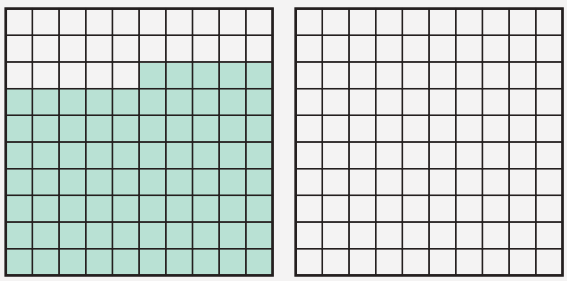
Excerpts from NCTM's Standards for School Mathematics Grades 6–8*

In the middle grades, students should become facile in working with fractions, decimals, and percents. Teachers can help students deepen their understanding of rational numbers by presenting problems, such as those in Figure 6.1, that call for flexible thinking.

a. If  is 3/4, draw the fraction strip for 1/2, for 2/3, for 4/3, and for 3/2. Be prepared to justify your answers.

b. 

Using the points you are given on the number line above, locate 1/2, 2 1/2, and 1/4. Be prepared to justify your answers

c. 

Use the drawing above to justify in as many different ways as you can that 75% of the square is equal to 3/4 of the square. You may reposition the shaded squares if you wish.

Figure 6.1

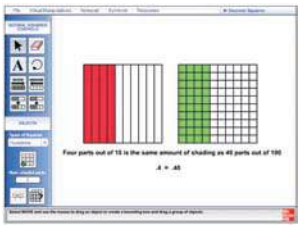
At the heart of flexibility in working with rational numbers is a solid understanding of different representations for fractions, decimals, and percents. In grades 3–5, students should have learned to generate and recognize equivalent forms of fractions, decimals, and percents, at least in some simple cases. In the middle grades, students should build on and extend this experience to become facile in using fractions, decimals, and percents meaningfully. Students can develop a deep understanding of rational numbers through experiences with a variety of models, such as fraction strips, number lines, 10 × 10 grids, area models, and objects.

**Principles and Standards for School Mathematics*, p. 215.

MATH ACTIVITY 6.1

Decimal Place Value with Base-Ten Pieces and Decimal Squares

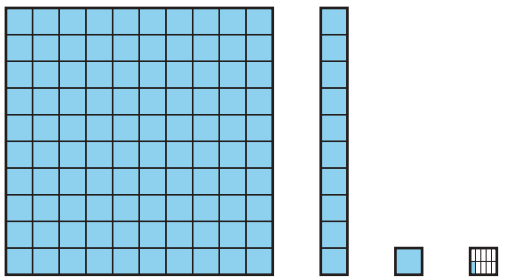
Virtual Manipulatives



www.mhhe.com/bbn

Purpose: Explore decimal place value concepts using Decimal Squares and base-ten pieces.

Materials: Base-Ten Pieces and Decimal Squares in the Manipulative Kit or Virtual Manipulatives.

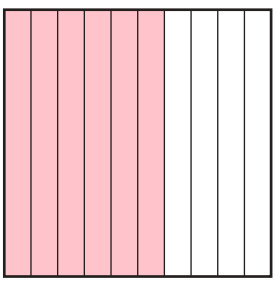


***1.** When the largest base-ten piece in your kit represents the unit, the other base-ten pieces take on the values shown here. Notice that the hundredths piece is divided into 10 equal parts to represent thousandths, and 1 part is shaded to represent 1 thousandth.

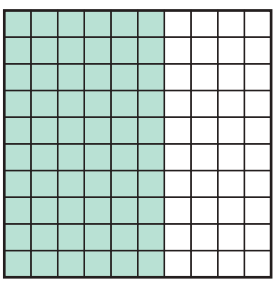
a. Take out your base-ten pieces and investigate the relationship between the pieces; list some of the relationships you discovered between these four types of pieces.

b. Form the collection of 1 unit piece, 4 tenths pieces, and 12 hundredths pieces. By using only your base-ten pieces and exchanging (trading) the pieces, it is possible to represent this collection in many different ways. Record some of these in a place value table like the one shown here.

Units	Tenths	Hundredths	Thousandths
1	4	12	



.6



.60

2. In the Decimal Squares model the unit square is divided into 10, 100, and 1000 equal parts to represent tenths, hundredths, and thousandths (respectively). Sort your deck of Decimal Squares into three piles according to color.

a. Determine the smallest and largest decimal represented in each pile.

b. How do the shaded amounts of each type of Decimal Square increase?

c. List some relationships between the three types of Decimal Squares.

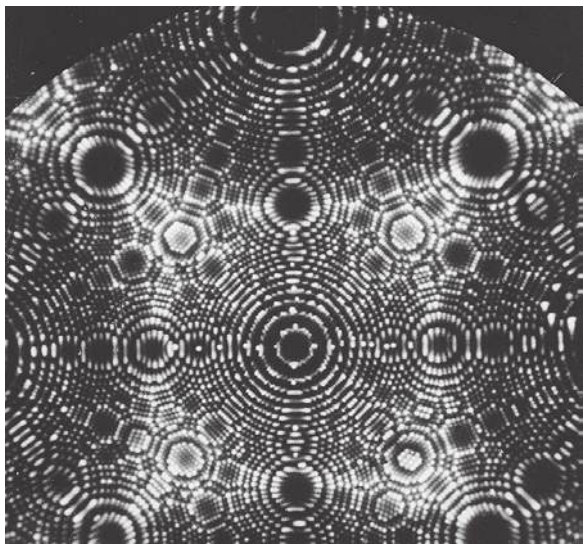
3. The two Decimal Squares shown at the left illustrate .6 = .60 because both squares have the same amount of shading. In the deck of Decimal Squares there are three squares whose decimals equal .6. Sort your deck of Decimal Squares into piles so squares with the same shaded amount are in the same pile.

a. Find all the decimals from the Decimal Squares that equal the following: .5, .35, .9, and .10 and write each corresponding equality statement.

b. The two-place decimal .65 is not equal to a one-place decimal, such as .6 or .7. List all the other two-place decimals from your deck that are not equal to a one-place decimal.

c. The three-place decimal .375 is not equal to a two-place decimal. List all the other three-place decimals from your deck that are not equal to a two-place decimal.

Section 6.1 DECIMALS AND RATIONAL NUMBERS



Circular patterns of atoms in an iridium crystal, magnified more than 1 million times by a field ion microscope.

PROBLEM OPENER

A carpenter agrees that during a specified 30-hour period he be paid $15.50 every hour he works and that he pay $16.60 every hour he does not work. At the end of 30 hours, he finds he has earned $47.70. How many hours did he work?*

Each dot in the remarkable photograph above is an atom in an iridium crystal. The circular patterns show the order and symmetry governing atomic structures. The diameters of atoms, and even the diameters of electrons contained in atoms, can be measured by decimals. Each atom in this picture has a diameter of .000000027 centimeter, and the diameter of an electron is .00000000000056354 centimeter.

The use of decimals is not restricted to describing small objects. The gross national product (GNP) and the national income (NI) for selected 5-year periods are expressed to the nearest tenth of a billion dollars in Figure 6.1.†

Figure 6.1

	1980	1985	1990	1995	2000	2005
GNP (billions)	$2631.7	$4053.1	$5803.2	$7400.5	$9855.9	$12,488.0
NI (billions)	$2116.6	$3229.9	$4642.1	$5876.7	$8795.2	$10,883.0

In our daily lives we encounter decimals in representations of dollar amounts: $17.35, $12.09, $24.00, etc. In elementary school, pennies, dimes, and dollars are commonly used for teaching decimals.

*"Problems of the Month," *Mathematics Teacher*.

†Adapted from *Statistical Abstract of the United States,* 124th ed. (Washington, DC: Bureau of the Census, 2007).

HISTORICAL HIGHLIGHT

The person most responsible for our use of decimals is Simon Stevin, a Dutchman. In 1585 Stevin wrote *La Disme,* the first book on the use of decimals. He not only stated the rules for computing with decimals but also pointed out their practical applications. Stevin showed that business calculations with decimals can be performed as easily as those involving only whole numbers. He recommended that the government adopt the decimal system and enforce its use.

As decimals gained acceptance in the sixteenth and seventeenth centuries, a variety of notations were used. Many writers used a vertical bar in place of a decimal point. Here are some examples of how 27.847 was written during this period.

27 | 847	27(847)	27 | $\underline{847}$	27 847
27847 . . . ③	$27,8^{i}4^{ii}7^{iii}$	27847 o	27 ⓪ 8 ① 4 ② 7 ③

DECIMAL TERMINOLOGY AND NOTATION

The word *decimal* comes from the Latin *decem,* meaning *ten.* Technically, any number written in base-ten positional numeration can be called a decimal. However, **decimal** most often refers only to numbers such as 17.38 and .45, which are expressed with **decimal points.** A combination of a whole number and a decimal, such as 17.38, is also called a **mixed decimal.**

There are currently many variations in decimal notation. In England, the decimal point is placed higher above the line than in the United States. In other European countries, a comma is used in place of a decimal point. A comma and a raised numeral denote a decimal in Scandinavian countries.

United States	England	Europe	Scandinavian countries
82.17	82·17	82,17	$82,^{17}$

The number of digits to the right of the decimal point is called the **number of decimal places.** There are two decimal places in 7.08 and one decimal place in 104.5. The positions of the digits to the left of the decimal point represent **place values** that are increasing powers of 10 (1, 10, 10^2, 10^3, . . .). The positions to the right of the decimal point represent **place values** that are decreasing powers of 10 (10^{-1}, 10^{-2}, 10^{-3}, . . .), or reciprocals of powers of 10 ($\frac{1}{10}$, $\frac{1}{10^2}$, $\frac{1}{10^3}$, . . .). In the decimal 5473.286 (Figure 6.2), the 2 represents $\frac{2}{10}$, the 8 represents $\frac{8}{100}$, and the 6 represents $\frac{6}{1000}$. Notice the similarity in pairs of names to the right and left of the units digit, for example, tens and tenths, hundreds and hundredths, etc. The convention in the preceding Historical Highlight of placing a small zero under the units digit helped to focus attention on these pairs of names.

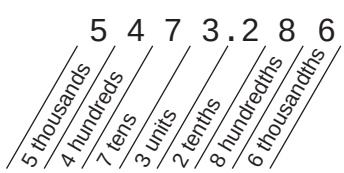


Figure 6.2

EXAMPLE A

Express the value of the digit marked by the arrow as a fraction whose denominator is a power of 10.

1. 47.35 **2.** 6.089 **3.** 14.07

Solution **1.** $\frac{5}{100}$ **2.** $\frac{9}{1000}$ **3.** $\frac{0}{10}$

Like whole numbers, decimals can be written in expanded form to show the powers of 10 (Figure 6.3).

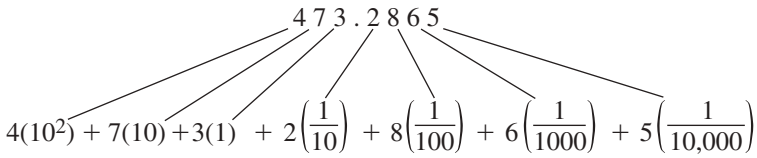
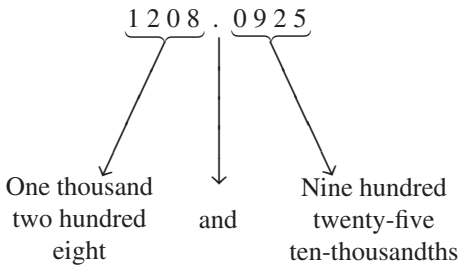


Figure 6.3

NCTM Standards

In grades 3–5, students should have learned to think of decimal numbers as a natural extension of the base-ten place-value system to represent quantities less than 1. In grades 6–8 they should also understand decimals as fractions whose denominators are powers of 10. p. 216

Reading and Writing Decimals The digits to the left of the decimal point are read as a whole number, and the decimal point is read *and.* The digits to the right of the point are also read as a whole number, after which we say the name of the place value of the last digit. For example, 1208.0925 is read "one thousand two hundred eight and nine hundred twenty-five ten-thousandths."



EXAMPLE B

Write the name of each decimal.

1. 3.472 **2.** 16.14 **3.** .3775

Solution **1.** Three and four hundred seventy-two thousandths. **2.** Sixteen and fourteen hundredths. **3.** Three thousand seven hundred seventy-five ten-thousandths.

One place where you may see the names of numbers is on bank checks. When writing an amount of money, some people write the decimal part of a dollar in words. Notice that on the bank check in Figure 6.4 on the next page it is unnecessary to write *dollars* or *cents.* The amount is in terms of dollars, and this unit is printed at the end of the line on which the amount of money is written. Some people write the decimal part of a dollar as a fraction. For example, the amount of this check might have been written "one hundred seventy-seven and $\frac{24}{100}$."

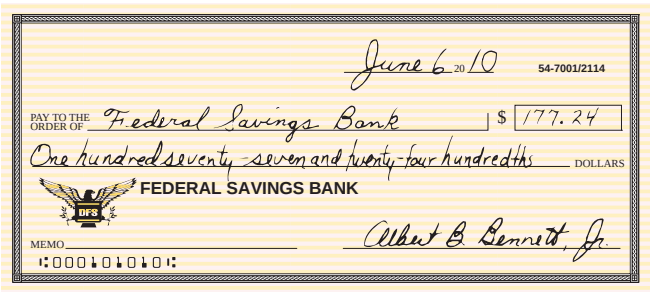

June 6 20 10 54-7001/2114
PAY TO THE ORDER OF Federal Savings Bank $ 177.24
One hundred seventy-seven and twenty-four hundredths DOLLARS
FEDERAL SAVINGS BANK
MEMO
Albert B. Bennett, Jr.

Figure 6.4

Research Statement

As with fractions, an understanding of the symbolism for representing decimals is essential to developing understanding of operations with decimals.

Resnick et al.

MODELS FOR DECIMALS

Models are important for providing conceptual understanding and insight into the use of decimals.

> It is important that decimals be thought of as numbers and the ability to relate them to models should assist in this. We should be spending more time having children become familiar with decimals, their meanings and uses, before rushing directly to decimal computation. Think of the time we spend with counting objects and modeling whole numbers before formal operations with whole numbers are introduced.*

Decimal Squares The Decimal Squares model illustrates the part-to-whole concept of decimals and place value. Unit squares are divided into 10, 100, and 1000 equal parts (Figure 6.5), and the decimal tells what part of the square is shaded.†

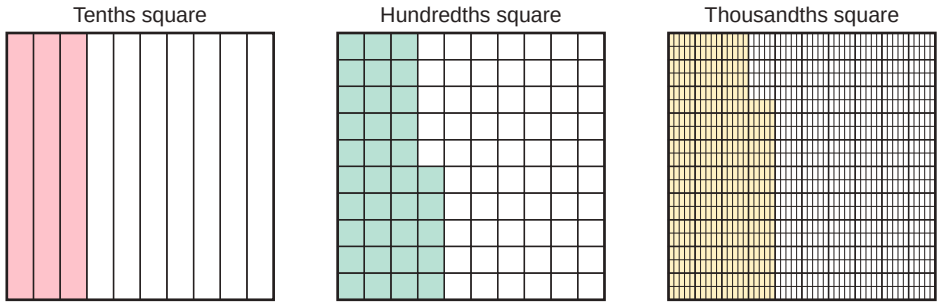


Figure 6.5

Each decimal in Figure 6.5 can be obtained by beginning with the fraction for the shaded amount of the square and obtaining the expanded form of the decimal. For example, the fraction for the square representing 375 parts out of 1000 is $\frac{375}{1000}$.

$$\frac{375}{1000} = \frac{300}{1000} + \frac{70}{1000} + \frac{5}{1000} = \underbrace{\frac{3}{10} + \frac{7}{100} + \frac{5}{1000}} = .375$$

	Tenths	Hundredths	Thousandths
Place value table	3	7	5

Similarly, the decimal for $\frac{3}{10}$ is .3, and the decimal for $\frac{35}{100}$ is .35.

*T. P. Carpenter, H. Kepner, M. K. Corbitt, M. M. Lindquist, and R. E. Reys, "Decimals: Results and Implications from National Assessment," *Arithmetic Teacher*.

†Decimal Squares is a registered trademark of American Education Products, LLC

14-1 Tenths and Hundredths

MAIN IDEA

I will identify, read, and write tenths and hundredths as decimals and fractions.

New Vocabulary

- **decimal**
- **decimal point**
- **tenth**
- **hundredth**

Math Online

macmillanmh.com

- Extra Examples
- Personal Tutor
- Self-Check Quiz

GET READY to Learn

It costs 85 cents for a child to ride the light rail system. Can you write this part of a dollar as a fraction and as a decimal?



A **decimal** is a number that uses place value and a **decimal point** to show part of a whole.

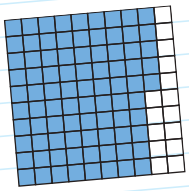
EXAMPLE 1 Read and Write Decimals

MONEY **Write 85 cents as a fraction and as a decimal.**

The amount 85 cents means 85 pennies out of 1 dollar.

One Way: Use a Model

Draw a hundreds model. Shade 85 out of 100 parts to show 85 cents.



Read eighty-five hundredths

Write $\frac{85}{100}$ or 0.85

Another Way: Place Value

Hundreds	Tens	Ones	Tenths	Hundredths
		0	8	5

Read eighty-five hundredths

Write $\frac{85}{100}$ or 0.85

So, 85 cents is $\frac{85}{100}$ as a fraction and 0.85 as a decimal.

From *Math Connects*, Grade 5, by Macmillan/McGraw-Hill. Copyright © 2009 by The McGraw-Hill Companies, Inc. Reprinted by permission of The McGraw-Hill Companies, Inc.

EXAMPLE C Describe the square that would represent each fraction, and write the decimal for each fraction.

1. $\frac{4728}{10{,}000}$ 2. $\frac{6}{100}$

Solution 1. A square with 4728 parts shaded out of 10,000

$$\frac{4728}{10{,}000} = \frac{4000}{10{,}000} + \frac{700}{10{,}000} + \frac{20}{10{,}000} + \frac{8}{10{,}000}$$
$$= \frac{4}{10} + \frac{7}{100} + \frac{2}{1000} + \frac{8}{10{,}000}$$
$$= .4728$$

2. A square with 6 parts shaded out of 100

$$\frac{6}{100} = \frac{0}{10} + \frac{6}{100} = .06$$

Technology Connection

Example C shows that it is easy to obtain the decimal for a fraction whose denominator is a power of 10; it is just a matter of locating the decimal point. Try the example on your calculator by dividing 4728 by 10,000 and 6 by 100. It is also instructive to enter 4728 into a calculator and then repeatedly divide by 10. Each time the decimal point moves one digit to the left.

Keystrokes	View Screen
4728 ÷ 10 =	472.8
÷ 10 =	47.28
÷ 10 =	4.728
÷ 10 =	.4728

In general, to divide an integer by a power of 10, *begin with the units digit and, for each factor of 10, count off a digit to the left to locate the decimal point.*

Number Line The number line is a common model for illustrating decimals. One method of marking off a unit from 0 to 1 is to use the edge of a Decimal Square, as shown in Figure 6.6. This approach shows the relationship between a *region model* for a unit (the Decimal Square) and a *linear model* for a unit (the edge of a square). The Decimal Square can be used repeatedly to mark off tenths on the number line from 0 to 1, 1 to 2, etc.

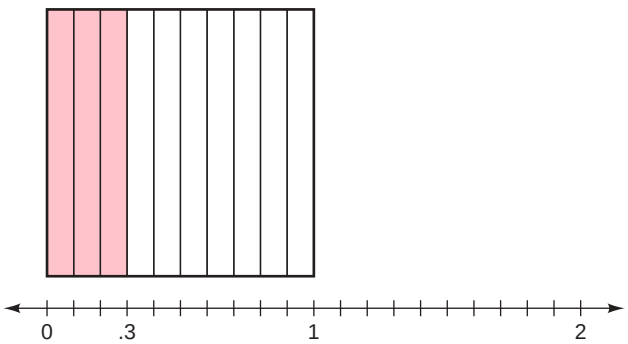


Figure 6.6

Consider locating the point for .372 on a number line. One approach is to use the expanded form of the decimal

$$.372 = \frac{3}{10} + \frac{7}{100} + \frac{2}{1000}$$

and locate the point in several steps, as shown in Figure 6.7. First, the point for $\frac{3}{10}$ (.3) is located at the end of the third interval, as in Figure 6.6. Second, the expanded form shows that we must add $\frac{7}{100}$, so the interval from .3 to .4 is divided into 10 equal parts, which are hundredths. To add $\frac{7}{100}$ (.07), we begin at .3 and go to the end of the seventh interval. This is the point for .37. Finally, the interval from .37 to .38 is divided into 10 equal parts, which are thousandths. To add $\frac{2}{1000}$, we begin at .37 and go to the end of the second interval. This is the point for .372.

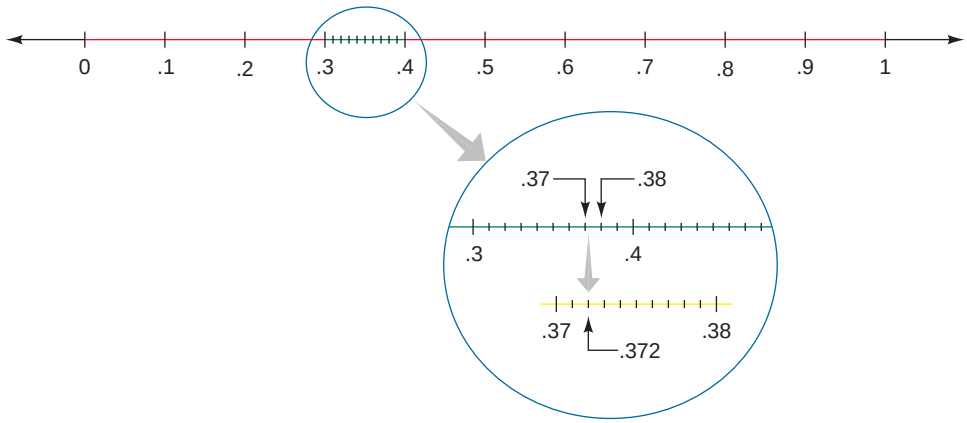


Figure 6.7

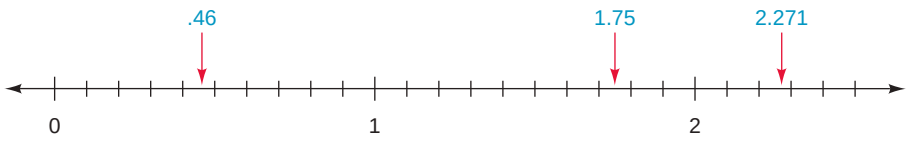
EXAMPLE D Sketch a number line and mark the approximate location of each decimal.

1. .46 **2.** 1.75 **3.** 2.271

NCTM Standards

Students' understanding and ability to reason will grow as they represent fractions and decimals with physical materials and on number lines. . . . p. 33

Solution



Decimals are used for negative as well as positive numbers. The graph in Figure 6.8 on the next page represents increasing and decreasing changes from year to year in the producer price index for crude materials, in tenths of a percent, from 1985 to 2007.* The producer price index decreased in 1985 and 1986, and increased from 1987 to 1990 and from 2003 to 2007.

**Statistical Abstract of the United States,* 128th ed. (Washington, DC: Bureau of the Census, 2009), p. 473.

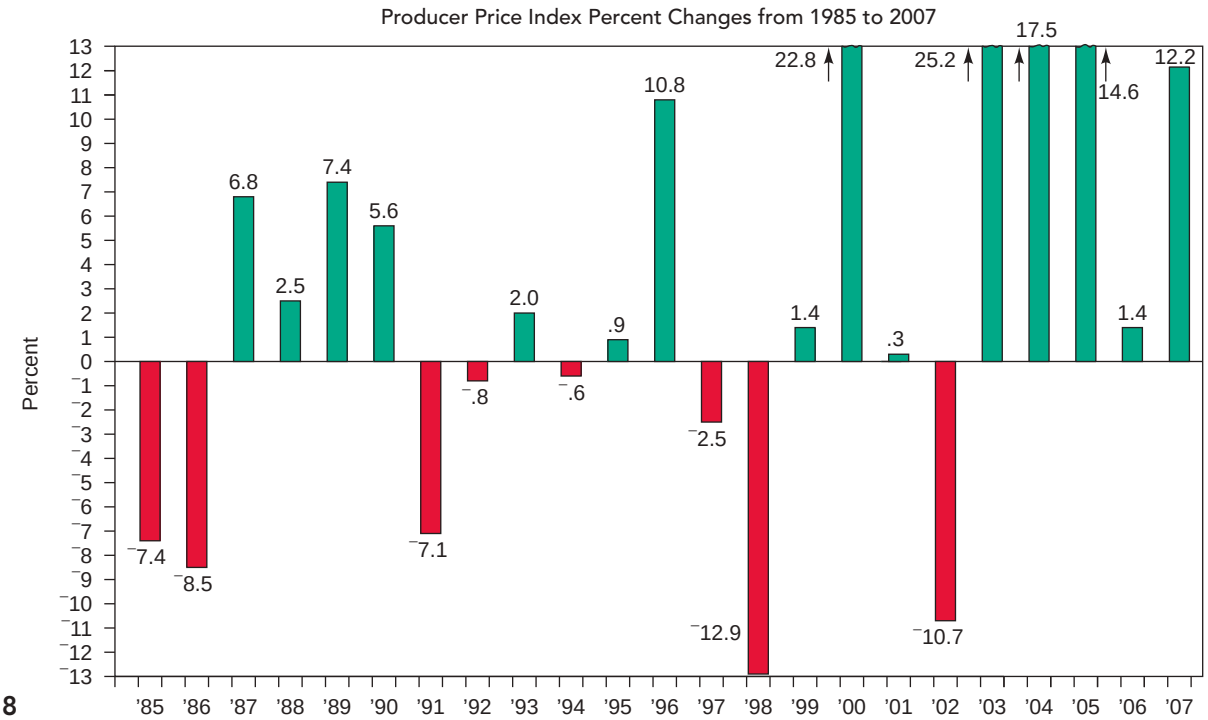


Figure 6.8

The decimals 2.5 and −2.5 for the years of 1988 and 1997 are opposites. For every decimal, whether positive or negative, there is a corresponding decimal called its **opposite** (or **inverse for addition**) such that the sum of the two decimals is zero. Several decimals and their opposites are shown on the number line in Figure 6.9.

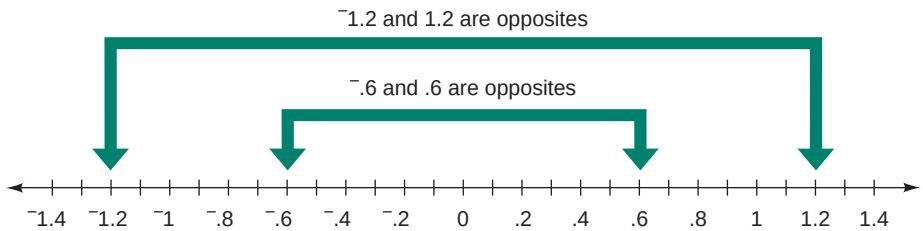


Figure 6.9

EQUALITY OF DECIMALS

Equality of decimals can be illustrated visually by comparing the shaded amounts in their Decimal Squares. Figure 6.10 shows that 4 parts out of 10, 40 parts out of 100, and 400 parts out of 1000 are all represented by the same amount of shading—in each Decimal Square, four columns are shaded. This illustrates that

$$.4 = .40 = .400$$

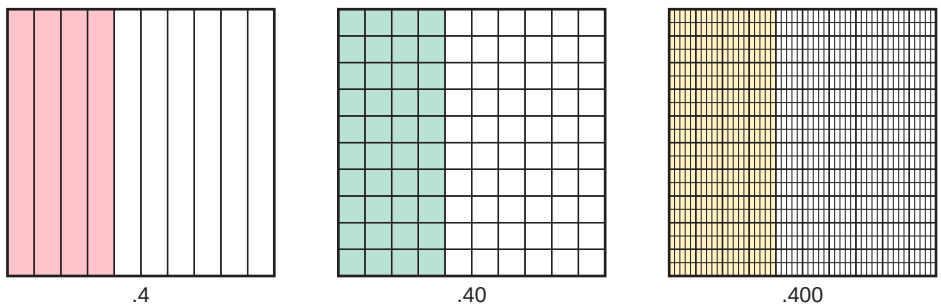


Figure 6.10

EXAMPLE E

Complete each equation by writing the indicated decimal, and describe the square representing each decimal in the equation.

1. .35 = _____ (thousandths)
2. .670 = _____ (hundredths)
3. .600 = _____ (tenths)

Solution **1.** .35 = .350 (35 parts out of 100 and 350 parts out of 1000 are shaded). **2.** .670 = .67 (670 parts out of 1000 and 67 parts out of 100 are shaded). **3.** .600 = .6 (600 parts out of 1000 and 6 parts out of 10 are shaded).

NCTM Standards

Students in these grades [3–5] should use models and other strategies to represent and study decimal numbers. For example, they should count by tenths verbally or use a calculator. p. 150

Decimal Squares also give us a visual model for place value. Consider the Decimal Square for .475 in Figure 6.11. The 4 full columns that are shaded (400 thousandths) represent $\frac{4}{10}$ or .4 ($\frac{400}{1000} = \frac{4}{10}$); the 7 small squares that are shaded (70 thousandths) represent $\frac{7}{100}$ or .07 ($\frac{70}{1000} = \frac{7}{100}$); and the 5 small parts that are shaded (5 thousandths) represent $\frac{5}{1000}$ or .005. Thus, the decimal .475 can be thought of as 4 tenths, 7 hundredths, and 5 thousandths.

$$.475 = .4 + .07 + .005$$

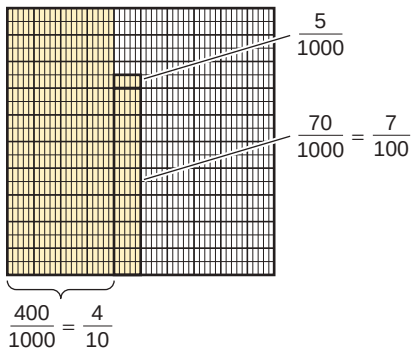


Figure 6.11

INEQUALITY OF DECIMALS

Research indicates that students from elementary school through college often have difficulty determining inequalities for decimals. One source of confusion is to think of the digits in the decimal as representing whole numbers (see Example F on the following page).

Figure 6.12 shows that .47 < .6. Even though 47 is greater than 6, a smaller amount of the square is shaded for .47 than for .6.

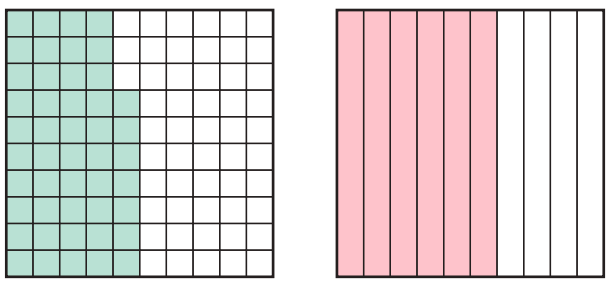


Figure 6.12

We can also see that $.47 < .6$ by noting that in the Decimal Square for .47, 4 full columns and part of another are shaded, whereas in the Decimal Square for .6, 6 full columns are shaded. In other words, the digit in the tenths place for .47 is less than the digit in the tenths place for .6. In general, the following *place value test* determines inequalities for decimals.

Place Value Test for Inequality of Decimals The greater of two positive decimals that are both less than 1 will be the decimal with the greater digit in the tenths place. If these digits are equal, this test is applied to the hundredths digits, etc.

The question in the next example is from a test given as part of a nationwide testing program in schools every 4 years.* Over one-half of the 13-year-olds who took the test selected an incorrect answer.

EXAMPLE F

Which number is the greatest?

.19 .036 .195 .2

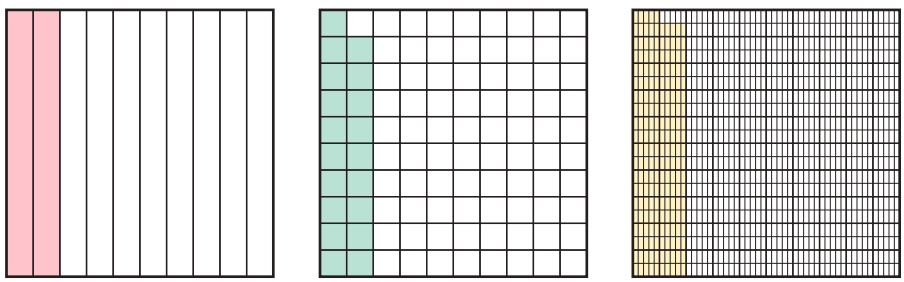
NCTM Standards

Without a solid conceptual foundation, students often think of decimal numbers incorrectly; they may, for example, think that 3.75 is larger than 3.8 because 75 is more than 8 (Resnick et al. 1989). p. 216

Solution One approach is to use the place value test for inequality of decimals. Since 2 is the greatest of the digits in the tenths place of these four decimals, .2 is the greatest number. Another approach is to change each decimal to thousandths. This will show that 200 thousandths is the greatest number of thousandths among these four decimals.

.190 .036 .195 .200

A visual approach with Decimal Squares illustrates that 2 full columns of shading (or 2 parts shaded out of 10) is more than 19 parts shaded out of 100 or 195 parts shaded out of 1000.



Note: Of the 13-year-olds tested, 47 percent selected .195 for the answer. This error may be due to the fact that 195 is greater than 19, 36, or 2.

RATIONAL NUMBERS

Up to this point, numbers written in the form $\frac{a}{b}$, where a and b are integers with $b \neq 0$, have been called *fractions.* This is the terminology commonly used in elementary and middle

*T. P. Carpenter, M. K. Corbitt, H. S. Kepner, M. M. Lindquist, and R. E. Reys, *Results from the Second Mathematics Assessment of the National Assessment of Educational Progress*.

schools. However, the word *fraction* has a more general meaning and includes the quotient of any two numbers, integers or not, as long as the denominator is not zero. Specifically, fractions $\frac{a}{b}$ where a and b are integers and $b \neq 0$ are called *rational numbers*.

Rational Number Any number that can be written in the form $\frac{a}{b}$, where $b \neq 0$ and a and b are integers, is called a **rational number.**

For example, $\frac{1}{9}$, $\frac{-3}{7}$, $\frac{1}{5}$, $\frac{7}{10}$, and $\frac{-1}{3}$ are rational numbers. When the denominator of a rational number equals 1, the rational number equals an integer: $\frac{6}{1} = 6$, $\frac{-4}{1} = -4$, $\frac{12}{1} = 12$, etc. Therefore, integers are also rational numbers.

Research Statement

Research has shown that most middle-grades children were unsuccessful with a set of tasks that mixed fraction and decimal notation.

Markovits and Sowder

Rational numbers can be expressed by many different number symbols or numerals. For example, $\frac{3}{10}$ is a rational number and $\frac{3}{10} = .3$, so .3 is also a rational number. In the following paragraphs we will show that all rational numbers $\frac{a}{b}$ can be written as decimals.

We have seen that it is easy to convert a fraction to a decimal if the denominator is a power of 10:

$$\frac{64}{100} = .64 \quad \frac{7283}{1000} = 7.283 \quad \frac{54}{10{,}000} = .0054$$

Sometimes when the denominator is not a power of 10, the fraction can be replaced by an equal fraction whose denominator is a power of 10. For example, $\frac{1}{4}$ can be replaced by $\frac{25}{100}$ because 100 is a multiple of 4:

$$\frac{1}{4} = \frac{25 \times 1}{25 \times 4} = \frac{25}{100} = .25$$

Since $10 = 2 \times 5$, we know by the Fundamental Theorem of Arithmetic that any power of 10 will have only 2 and 5 as prime factors.

$$10 = 2 \times 5 \qquad 100 = 10^2 = 2^2 \times 5^2 \qquad 1000 = 10^3 = 2^3 \times 5^3 \text{ etc.}$$

Consider replacing $\frac{3}{8}$ by a fraction whose denominator is a power of 10. Since $8 = 2^3$, we need to multiply the numerator and denominator of $\frac{3}{8}$ by 5^3.

$$\frac{3}{8} = \frac{3}{2^3} = \frac{3 \times 5^3}{2^3 \times 5^3} = \frac{375}{10^3} = .375$$

EXAMPLE G

Convert each fraction to a decimal by first writing a fraction whose denominator is a power of 10.

1. $\frac{7}{20}$ **2.** $\frac{3}{25}$ **3.** $\frac{11}{16}$ **4.** $\frac{3}{40}$

Solution **1.** $\frac{7}{20} = \frac{35}{100} = .35$ **2.** $\frac{3}{25} = \frac{12}{100} = .12$ **3.** $\frac{11}{16} = \frac{11}{2^4} = \frac{11 \times 5^4}{2^4 \times 5^4} = \frac{6875}{10^4} = .6875$
4. $\frac{3}{40} = \frac{3}{2^3 \times 5} = \frac{3 \times 5^2}{2^3 \times 5^3} = \frac{75}{10^3} = .075$

In the examples on the preceding pages, all the decimals had a finite number of digits. Such decimals are called **terminating** (or **finite**) **decimals.** However, there are decimals that are not terminating. There is no power of 10 that has 3 as a factor, so $\frac{1}{3}$ cannot be written as a fraction whose denominator is a power of 10. In general, we have the following rule.

Terminating Decimal If a nonzero rational number $\frac{a}{b}$ is in simplest form, it can be written as a terminating decimal if and only if b has only 2s and/or 5s in its prime factorization.

EXAMPLE H

Which of these rational numbers can be written as terminating decimals?

1. $\frac{5}{6}$ **2.** $\frac{1}{80}$ **3.** $\frac{9}{15}$ **4.** $\frac{3}{14}$

Solution **1.** 6 has a factor of 3, so $\frac{5}{6}$ cannot be written as a terminating decimal. **2.** 80 has only factors of 2 and 5, so $\frac{1}{80}$ can be written as a terminating decimal. **3.** $\frac{9}{15} = \frac{3}{5}$, and since the denominator of the fraction in simplest form has only 5 as a factor, $\frac{9}{15}$ can be written as a terminating decimal. **4.** 14 has a factor of 7, so $\frac{3}{14}$ cannot be written as a terminating decimal.

Technology Connection

Repeating Decimals

How many digits are there in the repeating part (repetend) of the decimal for $\frac{3}{17}$? Use the online 6.1 Mathematics Investigation to obtain repeating decimals for rational numbers and discover patterns to predict the lengths of their repetends.

Mathematics Investigation
Chapter 6, Section 1
www.mhhe.com/bbn

Let's consider finding a decimal for $\frac{1}{3}$. We know from the fraction-quotient concept in Section 5.2 that $\frac{1}{3} = 1 \div 3$. Figure 6.13 illustrates the first few steps in dividing 1 by 3. Part a shows a unit square with 10 tenths and illustrates that $1 \div 3$ is .3 with .1 remaining. In part b, the remaining .1 is replaced by 10 hundredths, and dividing by 3 produces .03 with .01 remaining. In part c, the remaining 1 hundredth is replaced by 10 thousandths, and dividing by 3 produces .003 with .001 remaining. The three steps of the division process in Figure 6.13 produce .3, .03, and .003, which give a total shaded amount of .333 with .001 remaining.

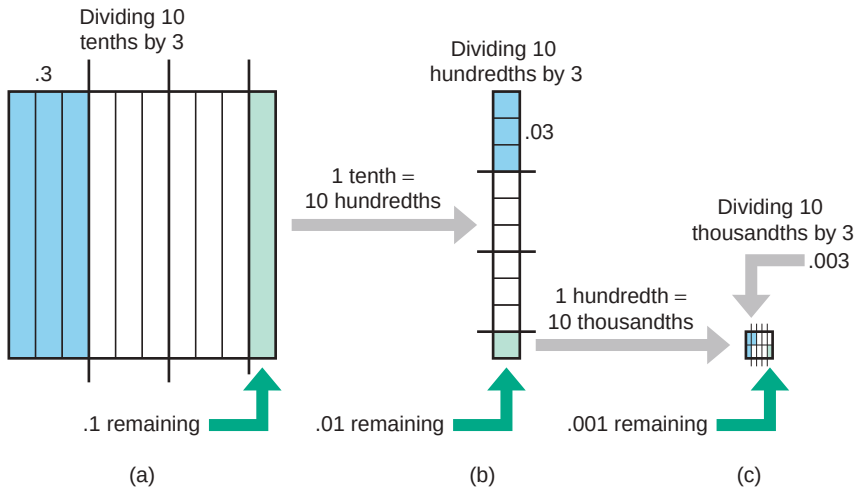


Figure 6.13

Continuing this process of splitting the remaining part by 10 and dividing by 3 shows that the decimal for $\frac{1}{3}$ has a repeating pattern of 3s. When a decimal does not terminate and contains a repeating pattern of digits, it is called a **repeating,** or **infinite, decimal.***

The step-by-step visual illustration in Figure 6.13 has a corresponding numerical division algorithm. The first step is to divide 10 tenths by 3. The result is .3 with .1 remaining.

*The Mathematics Investigation, Repeating Decimals, (see website) prints the decimal for positive rational numbers $\frac{a}{b}$ with $a < 10{,}000$ and $b < 10{,}000$. If the decimal is repeating, it counts the number of digits in the nonrepeating part (if any) and the number of digits in the repeating part (repetend).

The second step is to divide 10 hundredths by 3. The result of the first two steps is .33 with .01 remaining. In the third step, 10 thousandths are divided by 3. The result of the first three steps is a quotient of .333 with .001 remaining. This process can be continued to obtain any number of 3s in the decimal approximation for $\frac{1}{3}$.

NCTM Standards

The study of rational numbers in the middle grades should build on students' prior knowledge of whole-number concepts and skills and their encounters with fractions, decimals, and percents in lower grades and in everyday life. p. 215

Step 1

$$\begin{array}{r} .3 \\ 3\overline{)1.0} \\ \underline{9} \\ \text{One-tenth} \rightarrow 1 \end{array}$$

Step 2

$$\begin{array}{r} .33 \\ 3\overline{)1.00} \\ \underline{9} \\ 10 \\ \underline{9} \\ \text{One-hundredth} \rightarrow 1 \end{array}$$

Step 3

$$\begin{array}{r} .333 \\ 3\overline{)1.000} \\ \underline{9} \\ 10 \\ \underline{9} \\ 10 \\ \underline{9} \\ \text{One-thousandth} \rightarrow 1 \end{array}$$

The division algorithm can be used to obtain a terminating or repeating decimal for any rational number. For example, when the numerator of $\frac{3}{8}$ is divided by its denominator, the division algorithm shows that the decimal terminates after three digits.

$$\begin{array}{r} .375 \\ 8\overline{)3.000} \\ \underline{24}00 \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \end{array}$$

On the other hand, when the numerator of $\frac{4}{7}$ is divided by its denominator, the quotient does not terminate but repeats the same arrangement of six digits (571428) over and over. In this case the decimal is repeating. The reason for this can be seen by looking at the remainders 5, 1, 3, 2, 6, and 4, which are circled below. These six numbers, plus 0, are all the possible remainders when a number is divided by 7. So after six steps in the process of dividing 4 by 7, the numbers in the decimal quotient repeat. Notice the use of the bar above the six digits in the quotient to indicate the repeating pattern. The block of digits that is repeated over and over is called the **repetend.**

$$\begin{array}{r} .\overline{5714285} \\ 7\overline{)4.0000000} \\ \underline{35} \\ \textcircled{5}0 \\ \underline{49} \\ \textcircled{1}0 \\ \underline{7} \\ \textcircled{3}0 \\ \underline{28} \\ \textcircled{2}0 \\ \underline{14} \\ \textcircled{6}0 \\ \underline{56} \\ \textcircled{4}0 \\ \underline{35} \end{array}$$

The preceding example illustrates why every rational number $\frac{r}{s}$ can be represented by either a repeating decimal or a terminating decimal. When r is divided by s, the remainders are always less than s (see the Division Algorithm Theorem, page 194). If a remainder of 0 occurs in the division process, as it does when we divide 3 by 8, then the decimal terminates. If there is no zero remainder, then eventually a remainder will be repeated, in which case the digits in the quotient will also start repeating.



Technology Connection

Calculators are convenient for finding decimal representations of fractions. For fractions represented by repeating decimals, such as $\frac{7}{12} = .5833333333\ldots$, the number in the calculator view screen is an approximation because it shows only a few of the digits. For many applications this is sufficient accuracy.

Most calculators use several more digits than are shown in their view screens. When the number of digits in a decimal exceeds the space in a calculator's view screen, the calculator may keep several hidden digits that are used internally for greater accuracy. To determine if your calculator uses hidden digits, try the following keystrokes to obtain a decimal approximation for $\frac{1}{17}$.

Keystrokes	View Screen
1 ÷ 17 =	0.0588235
× 100,000 =	5882.3529
− 5882 =	0.3529411

Notice that the final view screen shows five hidden digits 2, 9, 4, 1, and 1 that were not in the first view screen. The purpose of multiplying by a power of 10 and subtracting the whole number part of the product is to move the digits in the view screen to the left to make room for digits that may be hidden. Did you "uncover" hidden digits beyond the last digit that was initially displayed for $\frac{1}{17}$ by your calculator?

EXAMPLE I

Write the decimal for each rational number. Use a bar to show the repetend (repeating digits).

1. $\frac{3}{11}$ **2.** $\frac{5}{6}$ **3.** $\frac{5}{12}$

Solution **1.** $.\overline{27}$ **2.** $.8\overline{3}$ **3.** $.41\overline{6}$

Notice in the solutions to Example I that the repeating pattern in $.8\overline{3}$ does not begin until the hundredths digit and the repeating pattern in $.41\overline{6}$ does not begin until the thousandths digit.

Every rational number is the quotient of two integers, and we have seen that such numbers can be written as a terminating or repeating decimal. Conversely, every terminating or repeating decimal can be written as the quotient of two integers. An example of a terminating decimal written as a quotient of two integers is shown on the next page. A method for writing repeating decimals as the quotient of two integers will be shown in Section 6.2.

Terminating decimals can be written as fractions whose denominators are powers of 10. The following equations show why .378 equals $\frac{378}{1000}$. In the first equation, .378 is

written in expanded form. The least common denominator for the fractions is 1000, and their sum is $\frac{378}{1000}$.

$$.378 = \frac{3}{10} + \frac{7}{100} + \frac{8}{1000}$$
$$= \frac{300}{1000} + \frac{70}{1000} + \frac{8}{1000} = \frac{378}{1000}$$

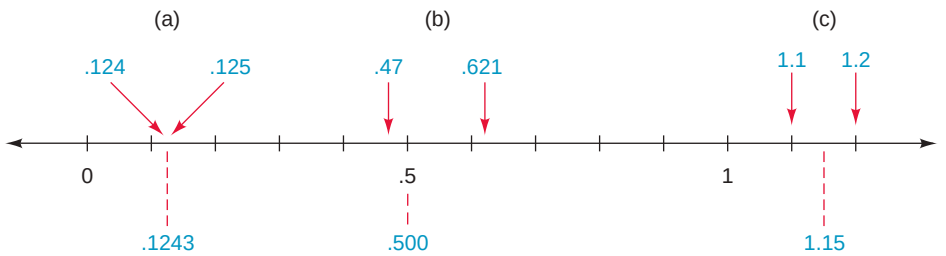
DENSITY OF RATIONAL NUMBERS

In Chapter 5, on page 296, we saw examples of the fact that the rational numbers, when written as fractions, are *dense*. That is, between any two such numbers there is always another. To show this, we looked at a method for finding a fraction between two given fractions. In the following example, we will consider ways of finding a decimal between any two given decimals.

EXAMPLE J Sketch a number line and mark the location of each pair of decimals. Then find another decimal between them.

1. .124 and .125 **2.** .47 and .621 **3.** 1.1 and 1.2

Solution **1.** One method of finding decimals between two given decimals is to express both decimals with a greater number of decimal places. For example, .124 = .1240 and .125 = .1250, and the 9 four-place decimals .1241, .1242, . . . , .1249 are between .124 and .125. Also, .124 = .12400 and .125 = .12500 and the 99 five-place decimals .12401, .12402, . . . , .12499 are between .124 and .125. Similarly, the process can be continued as there are 999 six-place decimals between .124000 and .125000, etc. **2.** Since .47 = .470, any of the three-place decimals between .47 and .621 may be selected, and by increasing the number of decimal places for .47 and .621, more decimals can be found between these numbers. **3.** 1.1 = 1.10 and 1.2 = 1.20, so the 9 decimals 1.11, 1.12, . . . , 1.19 are between 1.1 and 1.2. Also, 1.1 = 1.100 and 1.2 = 1.200, so the 99 decimals 1.101, 1.102, . . . , 1.199 are between 1.1 and 1.2, etc.



ESTIMATION

Often a calculator view screen will be filled with the digits of a decimal, when some approximate value is all that is necessary. Rounding to a given place value is the most common method of obtaining decimal estimations.

Rounding Decimals can be rounded to the nearest whole number, nearest tenth, nearest hundredth, etc. Before we look at a rule for rounding decimals, let's consider some visual illustrations.

Squares for three decimals are shown in Figure 6.14 on the next page. Consider rounding these decimals to the nearest tenth. The decimal .648 rounds to .6 because 6 full columns and *less than one-half* of the next column are shaded; .863 rounds to .9 because 8 full columns and *more than one-half* of the next column are shaded; .35 can be either *rounded*

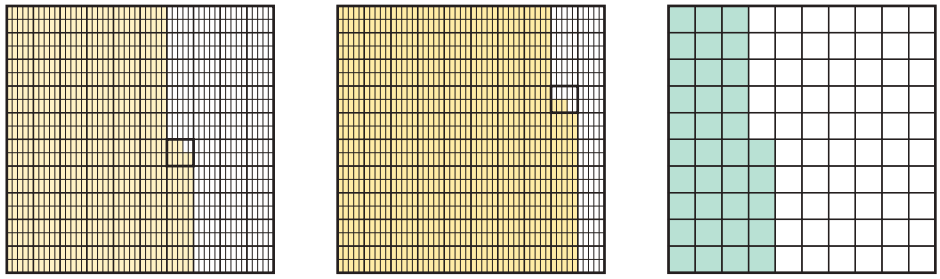


Figure 6.14

up to .4 or *rounded down* to .3, because 3 full columns and *one-half* of the next column are shaded. In this text we adopt the policy of *rounding up.*

Next consider rounding the decimals in Figure 6.14 to the nearest hundredth. The square for .648 shows that 64 hundredths are shaded (6 full columns and 4 hundredths in the next column); the remaining 8 thousandths are more than one-half of the next hundredth, so .648 rounds to .65. The Decimal Square for .863 shows that 86 hundredths are shaded (8 full columns and 6 hundredths in the next column); the remaining 3 thousandths are less than one-half of the next hundredth, so .863 rounds to .86.

EXAMPLE K

Round each decimal to the nearest tenth and to the nearest hundredth.

1. .283 **2.** .068 **3.** 14.649

Solution **1.** .283 rounded to the nearest tenth is .3 and to the nearest hundredth is .28. **2.** .068 rounded to the nearest tenth is .1 and to the nearest hundredth is .07. **3.** 14.649 rounded to the nearest tenth is 14.6 and to the nearest hundredth is 14.65.

Notice in Example K that 14.649 rounded to the nearest tenth is not 14.7. You can confirm this by visualizing a Decimal Square for .649: 6 full columns are shaded and less than one-half of the next column (49 thousandths) is shaded.

The preceding examples are special cases of the following general rule for rounding decimals. Notice that this is similar to the rule that was stated for rounding whole numbers on page 132 of Chapter 3.

Rule for Rounding Decimals

1. Locate the place value to which the number is to be rounded, and check the digit to its right.
2. If the digit to the right is 5 or greater, then all digits to the right are dropped and the digit with the given place value is increased by 1.
3. If the digit to the right is 4 or less, then all digits to the right of the digit with the given place value are dropped.

EXAMPLE L

Round 1.6825 to the given number of decimal places.

1. Two decimal places (round to hundredths)

2. One decimal place (round to tenths)

3. Three decimal places (round to thousandths)

Hundredths Tenths Thousandths

Solution **1.** 1.6825 → 1.68 **2.** 1.6825 → 1.7 **3.** 1.6825 → 1.683

Technology Connection

Some calculators automatically round decimals that exceed the space in the view screen. On such calculators, if 2 is divided by 3, the decimal 0.66 . . . 667 will be displayed, where the last digit in the view screen is rounded from 6 to 7. Almost all calculators that round off at a digit that is followed by 5 will increase this digit, as described in the preceding rule for rounding numbers. For example, $\frac{55}{99}$ is equal to the repeating decimal .5555 . . . . If 55 is divided by 99 on a calculator that rounds, 0.55 . . . 556 will show in the display. Try this on your calculator.

PROBLEM-SOLVING APPLICATION

Problem

The price of a single pen is 39 cents. This price is reduced if pens are purchased in quantity. The price per pen is always a whole number of cents and never less than 2 cents. If we know that a person bought all the pens in a box for $22.91, we can determine the number of pens. What is this number?

Understanding the Problem Assume the cost per pen is less than 39 cents and it must be a whole number of cents. **Question 1:** What are the possibilities for the reduced price?

Devising a Plan Since there is a reasonably small number of possible prices, one approach, if a calculator is available, is to *guess and check.* This can be done by replacing $22.91 by 2291 and dividing by the whole numbers 38, 37, 36, etc. until a whole number quotient is obtained; or we can divide $22.91 by the decimals .38, .37, .36, etc. If we view this problem as a matter of whole numbers (whole numbers of pennies) and divisibility, we arrive at another approach: finding the prime factorization of 2291. **Question 2:** How does the fact that there is only one solution tell us that the reduced price in cents is a prime number?

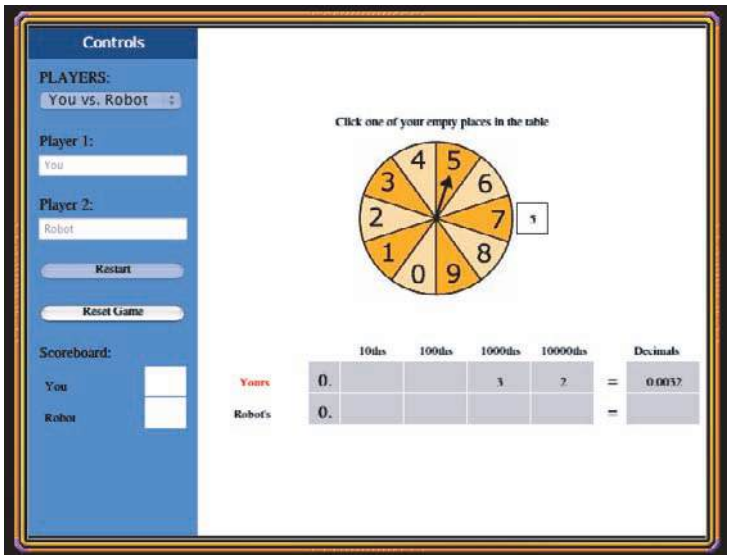
Carrying Out the Plan A quick use of the divisibility tests for 2, 3, and 5 shows that these numbers are not factors of 2291. Since the prime is less than 39, we need only check the primes 7, 11, 13, 17, 19, 23, 29, 31, and 37. **Question 3:** Which prime divides 2291, and what is the number of pens in the box?

Looking Back Suppose we kept the conditions of the problem the same but changed the reduced price for the entire box to $15.17. **Question 4:** How many pens would be in the box?

Answers to Questions 1–4 **1.** Whole numbers of cents from 2 to 38. **2.** If a composite number less than 39 divides 2291, then the factors of the composite number will divide 2291 and there will be more than one possibility for the cost of each pen. **3.** 29 is a factor of 2291, so the cost of each pen is 29 cents and there are 79 pens in the box. **4.** The first prime less than 39 that divides 1517 is 37. Since $37 \times 41 = 1517$, the number of pens in the box is 41.

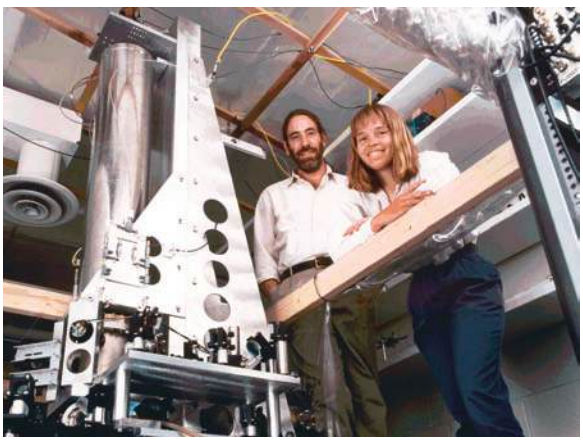
Technology Connection

This is a two-person game in which each player tries to form the largest possible four-place decimal by placing digits from the spinner into the place value table. In the following example, you are playing against the Robot. In which box of the place value table would you place the "5" from the spinner? As you play this game, try to determine the Robot's strategy in placing digits in the table.



Competing At Place Value Applet, Chapter 6, Section 1
www.mhhe.com/bbn

Exercises and Problems 6.1



This laser-cooled atomic fountain clock from the National Institute of Standards and Technology (NIST) in Boulder, Colorado, is one of the world's most accurate clocks. If it ran for 60 million years it would not be off by more than one second. Extremely accurate timing information is critical for such things as cell phone networks and Global Positioning Systems. Shown here are NIST scientists Dawn Meekhof and Steve Jefferts, who led the team that built this clock.

The clock shown here can be adjusted for time intervals shorter than one thousand trillionths of a second. What is the decimal for each of the time intervals in exercises 1 and 2?

1. a. 1 thousandth of a second
b. 1 millionth of a second

2. a. 1 billionth of a second
b. 1 trillionth of a second

The first book about decimals was written by Simon Stevin in 1585. He used small circled numerals between the digits to indicate decimal places. Among his examples he wrote $27 + \frac{847}{1000}$ as 27 ⓪ 8 ① 4 ② 7 ③. Use Stevin's method of notation in exercises 3 and 4.

3. a. How would the number 7.46 be written, using this system?
b. Explain how the decimal point could have evolved from this system.

4. a. How would the number 245.639 be written, using this system?

b. Consider the number in Stevin's example. In England this decimal is written as 27·847 and in the United States as 27.847. What advantage is there in the English location of the decimal point?

For each indicated digit in exercises 5 and 6, write a fraction whose denominator is a power of 10.

5. a. $23.\underset{\downarrow}{1}78$ **b.** $7.\underset{\downarrow}{2}016$

c. $.00\underset{\downarrow}{3}3$ **d.** $.\underset{\downarrow}{9}999$

6. a. $.0\underset{\downarrow}{3}8$ **b.** $47.\underset{\downarrow}{2}93$

c. $.55\underset{\downarrow}{5}6$ **d.** $.6\underset{\downarrow}{5}5$

Find the decimal for each fraction in exercises 7 and 8, and write the name of the decimal.

7. a. $\frac{33}{100}$ **b.** $\frac{392}{10{,}000}$ **c.** $\frac{54}{1000}$ **d.** $\frac{7481}{10}$

8. a. $\frac{42}{100}$ **b.** $\frac{64{,}193}{10{,}000}$ **c.** $\frac{9}{1000}$ **d.** $\frac{436}{10}$

Write the name of each dollar amount in exercises 9 and 10.

9. a. \$347.96 **b.** \$23.50

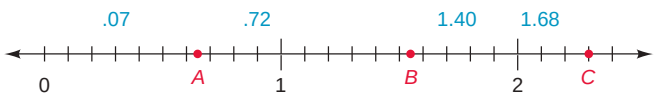
c. \$1144.03

10. a. \$502.85 **b.** \$1,372,500

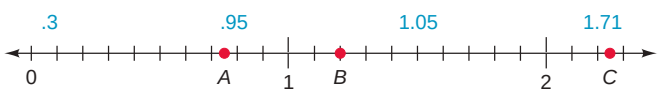
c. \$6035.25

Draw an arrow from each number to its approximate location on the number line in exercises 11 and 12. Also, write the number that corresponds to each point on the line that is labeled with a letter.

11.



12.



Write an equal decimal having the given number of decimal places in exercises 13 and 14. Briefly describe the Decimal Squares for both decimals in the equation.

13. a. .4 = _____ (hundredths)

b. .47 = _____ (thousandths)

14. a. .300 = _____ (tenths)

b. .270 = _____ (hundredths)

Describe Decimal Squares for each side of the equality or inequality in exercises 15 and 16 to explain why each statement is true.

15. a. .7 = .70

b. .43 = .430

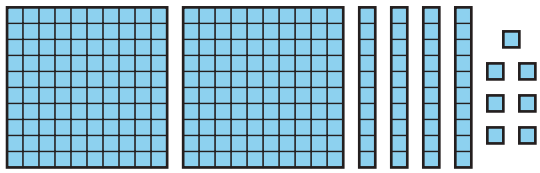
c. .45 < .6

16. a. .3 > .295

b. .085 < .13

c. .19 = .190

In exercises 17 and 18, what number is represented by the following figures for each of the given units?



17. a. Each small square represents 1 unit.

b. Each large square represents 1 unit.

18. a. Each 1 × 10 strip represents 1 unit.

b. Each small square represents 10 units.

Sketch Decimal Squares for each fraction in exercises 19 and 20, and describe how to find the decimal for the fraction. (Copy Blank Decimal Squares from the website.)

19. a. $\frac{1}{4}$ **b.** $\frac{1}{5}$

20. a. $\frac{1}{8}$ **b.** $\frac{1}{6}$ (to three decimal places)

Write each fraction in exercises 21 and 22 as a decimal. If the decimal is repeating, use a bar to show the repeating digits.

21. a. $\frac{3}{8}$ **b.** $\frac{2}{11}$

c. $\frac{345}{990}$ **d.** $\frac{15}{4}$

22. **a.** $\frac{5}{9}$ **b.** $\frac{13}{5}$ **c.** $\frac{439}{900}$ **d.** $\frac{6}{37}$

Match each rational number in Set *A* with an equal rational number in Set *B* in exercises 23 and 24.

23.

Set A				Set B			
.25	$.41\overline{6}$	.875	$.06\overline{}$	$.8\overline{3}$	$.\overline{34}$	$.\overline{3}$	.3
$\frac{1}{3}$	$\frac{5}{6}$	$\frac{3}{10}$	$\frac{34}{99}$	$\frac{7}{8}$	$\frac{1}{15}$	$\frac{1}{4}$	$\frac{5}{12}$

24.

Set A				Set B			
.35	$.3\overline{7}$	.625	$.762\overline{}$	$.583\overline{}$	$.\overline{62}$	.9	.1875
$\frac{3}{16}$	$\frac{9}{10}$	$\frac{62}{99}$	$\frac{7}{12}$	$\frac{34}{90}$	$\frac{5}{8}$	$\frac{686}{900}$	$\frac{7}{20}$

Which of the fractions in exercises 25 and 26 have terminating decimals, and which have repeating decimals?

25. **a.** $\frac{7}{17}$ **b.** $\frac{2}{9}$ **c.** $\frac{6}{15}$

26. **a.** $\frac{1}{50}$ **b.** $\frac{3}{18}$ **c.** $\frac{5}{13}$

For each fraction in exercises 27 and 28, use a Decimal Square with 100 equal parts and explain how the first two digits in the decimal for the fraction can be represented by shading. Then explain how the decimal can be rounded to two decimal places. (Copy Blank Decimal Squares from the website.)

27. **a.** $\frac{1}{3}$ **b.** $\frac{1}{9}$ **c.** $\frac{1}{8}$

28. **a.** $\frac{1}{6}$ **b.** $\frac{1}{7}$ **c.** $\frac{1}{12}$

Replace each decimal in exercises 29 and 30 with an approximation that is obtained from the leading nonzero digit. Then determine the approximation if the decimal is *rounded* to the leading nonzero digit.

29. **a.** .0045 **b.** .408 **c.** .074 **d.** .00263

30. **a.** .062 **b.** .0027 **c.** .165 **d.** .228

One method of comparing two fractions for inequalities is to find their decimal representations. Use this method in exercises 31 and 32 to rearrange the fractions in increasing order from left to right.

31. $\frac{16}{20}, \frac{19}{34}, \frac{38}{52}, \frac{21}{25}, \frac{11}{17}$

32. $\frac{5}{7}, \frac{26}{30}, \frac{32}{51}, \frac{10}{13}, \frac{5}{6}$

Find the decimal for each fraction in exercises 33 and 34, and round it to the nearest ten-thousandth.

33. **a.** $\frac{1}{16}$ **b.** $\frac{3}{32}$ **c.** $\frac{7}{64}$ **d.** $\frac{35}{64}$

34. **a.** $\frac{1}{13}$ **b.** $\frac{5}{7}$ **c.** $\frac{2}{3}$ **d.** $\frac{7}{9}$

Round each decimal in exercises 35 and 36 to the given place value.

35. **a.** .3728 (hundredths)
b. .084 (tenths)
c. 14.3716 (thousandths)
d. .349 (tenths)

36. **a.** .384615 (thousandths)
b. .35294 (tenths)
c. 2.893 (hundredths)
d. 6.043478 (ten-thousandths)

37. **a.** Which of the following decimals is the smallest? Sketch or describe Decimal Squares to support your answer.

.07 1.003 .08 .075 .3

b. The preceding question was part of a test on decimals that was given to over 7000 students entering college.* Approximately $\frac{2}{5}$ of these students chose the incorrect answer of .075. What confusion about decimals might have motivated this choice?

38. **a.** Which of the following decimals is the greatest? Sketch or describe Decimal Squares to support your answer.

.19 .036 .195 .2

b. The preceding question was given to students aged 13 as part of the Second Math Assessment of the National Assessment of Educational Progress.† More students chose the incorrect answer .195 than the correct answer. What confusion about decimals might have led to this choice?

*A. S. Grossman, "Decimal Notation: An Important Research Finding," *The Arithmetic Teacher* 30, pp. 32–33.

†T. P. Carpenter, H. Kepner, M. K. Corbitt, M. M. Lindquist, and R. E. Reys, "Decimals: Results and Implications from National Assessment," *Arithmetic Teacher* 28, pp. 34–37.

39. Match each rational number in Set *A* with an equal rational number from Set *B*.

Set A				Set B			
.05	.125	.4	$.58\overline{3}$	$\frac{4}{15}$	$\frac{7}{12}$	$.3\overline{8}$	$.\overline{6}$
$.\overline{532}$	$.2\overline{6}$	$\frac{2}{3}$	$\frac{7}{18}$	$\frac{532}{999}$	$\frac{1}{8}$	$\frac{1}{20}$	$\frac{2}{5}$

Assume that a calculator which operates with three hidden digits (three more digits than are shown on the view screen) is used in exercises 40 and 41. Determine the next three hidden digits for the decimals in each of the view screens.

40. a. 1 ÷ 31 = 0.0322580

b. 8 ÷ 23 = 0.3478260

41. a. 2 ÷ 19 = 0.1052631

b. 14 ÷ 29 = 0.4827586

Reasoning and Problem Solving

42. Money amounts are often rounded to values that can be paid in standard currency. Use the following values in parts a to c: \$45.789, \$45.443, \$45.375, \$45.4650, \$45.6749.

a. Round each amount to the nearest hundredth of a dollar (nearest cent).

b. Some mortgage lenders round up any fraction of a cent. Using this method, round these amounts up to the nearest one-hundredth of a dollar.

c. Some lenders even round up to the nearest dime. Using this method, what are these amounts rounded to the nearest tenth of a dollar?

43. a. A catalog lists the following camera weights in ounces: 17.31, 16.25, 15.90, 28.06, and 22.55. What is each weight rounded to the nearest tenth of an ounce?

b. In one year .20 of the injuries to players of the National Football League were to the knees and .025 were to the arms. Were there more injuries to the knees or to the arms?

c. A pointer on a water meter is halfway between .8 and .9. What is the decimal position of the pointer?

44. a. A sewing machine has attachments called *throat plates* for making eyelets. If the holes on a certain throat plate are .218, .14, .2, and .196 inch in diameter, which is the largest hole?

b. Physicists have calculated that one cubic foot of air weighs 1.29152 ounces. What is this weight rounded to the nearest tenth of an ounce?

c. One kilometer is 1000 meters. The length of a soccer field for international matches is 110 meters. What decimal part of a kilometer equals 110 meters?

45. Fiona has a set of drill bits that are printed in fractions of an inch with the following sizes:

$$\frac{5}{32} \quad \frac{1}{8} \quad \frac{3}{16} \quad \frac{1}{4} \quad \frac{1}{16} \quad \frac{3}{32}$$

a. If the blueprints for her cabinet call for drilling holes of .13 inch, which drill bit is the closest size?

b. Her plans call for drilling holes of .32 inch. Does she have a drill bit large enough?

c. She needs to drill 20 very small holes in the cabinet for inserting finishing nails. What is her smallest drill bit?

The *Guinness Book of Records* documents the evolution of sports records in the twentieth century. Four records for the high jump are shown in exercises 46 and 47. Convert the feet and inches to feet, rounding to two decimal places.

46. a. M. Sweeney, 6 feet $5\frac{5}{8}$ inches, United States, 1895

b. Zhu Jianhua, 7 feet $9\frac{1}{4}$ inches, China, 1983

47. a. Lester Steers, 6 feet 11 inches, United States, 1941

b. Heike Henkel, 6 feet $9\frac{1}{2}$ inches, Germany, 1982

48. Featured Strategies: Solving a Simpler Problem and Making a Table. A piece of paper is cut in half, and one piece is placed on top of the other. Then the two pieces are cut in half, and one half is placed on top of the other, forming a stack with four pieces. If this process is carried out a total of 25 times and the original piece of paper is .003 inch thick, what is the height of the stack to the nearest foot?

a. Understanding the Problem. Simplifying the problem may suggest what needs to be done to obtain a solution. Suppose each piece of paper were $\frac{1}{2}$ inch thick. How thick would the stack be after three cuts?

b. Devising a Plan. Forming a table will help you see a pattern between the number of cuts and the number of pieces of paper. What numbers should appear in the blank lines of the table on the top of the next page? What is the number of pieces of paper after 25 cuts, expressed as a power of 2?

Number of Cuts	Number of Pieces
1	2
2	4
3	8
4	—
5	—
.	.
.	.
.	.
25	—

c. Carrying Out the Plan. Our simplification in part a suggests that we must multiply the thickness of the paper (.003, or $\frac{3}{1000}$ inch) by the total number of pieces of paper. What is the final height of the stack to the nearest foot?

d. Looking Back. Another way to think of the height of the final stack is to divide the number of feet by 5280, the number of feet in 1 mile. How high is the stack to the nearest tenth of a mile?

PS Describe some of the patterns among the equations in 49 and 50, and determine if the patterns continue to hold when the equations are extended. Are new patterns formed as the equations are extended?

49. $\frac{1}{9} = .111111\ldots$ $\frac{2}{9} = .222222\ldots$

$\frac{3}{9} = .333333\ldots$

50. $\frac{1}{11} = .090909\ldots$ $\frac{2}{11} = .181818\ldots$

$\frac{3}{11} = .272727\ldots$

Teaching Questions

1. Suppose an elementary school student asked you why writing zeros to the right of a whole number increases the value of the number, but when zeros are written to the right of a decimal point the value of the number does not increase. Write a response that would make sense to this student.
2. When asked to find a number between 1.4 and 1.5, one of your students responded $1.4\frac{1}{2}$. Is she correct? What decimal do you think she means by this notation? How can you use the Decimal Squares model to help her write the number as a decimal?
3. Read the Standards quote on page 350 and then describe a method that you feel would provide a "solid conceptual foundation" to enable school students to understand inequality of decimals with different numbers of decimal places. Illustrate your method with diagrams of how you would help students understand why 3.75 is less than 3.8.
4. What does the finding in the Research statement on page 351 show about students' problem-solving ability when both fractions and decimals are involved? Describe an activity that would help students to recognize some of the common equivalences of fractions and decimals.
5. Explain how you can use Decimal Squares to show that between two decimals there are more decimals. Use the decimals .47 and .48 as an example. Discuss how you can extend this idea to .374 and .375 using Decimal Squares.

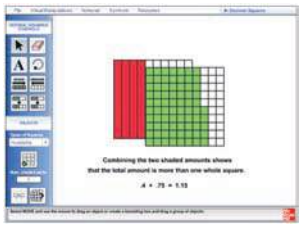
Classroom Connections

1. Discuss the connections between the second expectation under *Understand numbers* . . . in the **Grades 6–8 Standards—Number and Operations** (see inside back cover) and both the **Standards** quote on page 347 and the text material on pages 347–348.
2. The **Spotlight on Teaching** at the beginning of this chapter shows models for fractions and decimals and notes that students need such models to develop flexible thinking with these kinds of numbers. Give an example involving fractions and decimals of what you think is meant by "flexible thinking."
3. What does the **Research** statement on page 344 say about symbolism for decimals? List some reasons why our notation for decimals might be more difficult for students to understand than the notation for fractions.
4. On page 345 the example from the **Elementary School Text** uses the Decimal Square part-to-whole model to show 85 hundredths and illustrates how it can be written using decimal and fraction notations and named with written words. (a) If you have a thousandths Decimal Square with 143 parts shaded, write it as a decimal and as a fraction and give the decimal using written words. (b) How does the model on the school page satisfy the recommendation in the **Standards** quote on page 343 as to how students' should think of decimals?
5. The **Standards** quote on page 353 discusses teaching decimals by building on "students' prior knowledge of whole-number concepts. . . ." What evidence is there in the **Historical Highlight** on page 342 that writers in the sixteenth and seventeenth centuries saw decimals as an extension of whole numbers?

MATH ACTIVITY 6.2

Decimal Operations with Decimal Squares

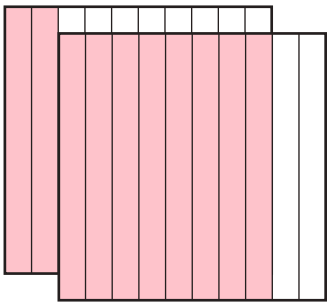
Virtual Manipulatives



www.mhhe.com/bbn

Purpose: Use Decimal Squares to model the four basic operations on decimals.

Materials: Copies of Blank Decimal Squares from the website and Decimal Squares in the Manipulative Kit or Virtual Manipulatives.



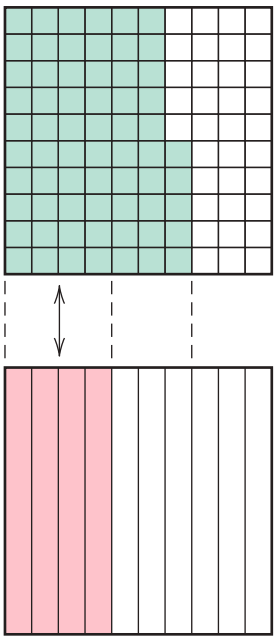
.2 + .8 = 1.0

1. The concept of addition of whole numbers, that is, *putting together* or *combining amounts,* is the same for addition of decimals. If the shaded amounts of Decimal Squares for .2 and .8 are combined, the total equals one whole square. Use your deck of Decimal Squares and answer parts a, b, and c. Write an addition equation for each pair of decimals.

***a.** Find three pairs of Decimal Squares for tenths (red squares) for which the sum of the decimals in each pair is 1.0.

b. Find three pairs of Decimal Squares for hundredths (green squares) for which the sum of the decimals in each pair is 1.0. Use decimals not equivalent to those used in part a.

c. Find three pairs of Decimal Squares for thousandths (yellow squares) for which the sum of the decimals in each pair is 1.0. Use decimals not equivalent to those used in parts a and b.



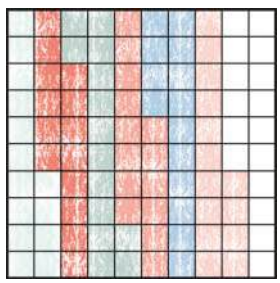
.65 – .4 = .25

2. The *comparison* concept for determining the difference of two whole numbers can also be used to find the difference of two decimals. By lining up the Decimal Squares for .65 and .4, as shown at the left, the shaded amounts can be compared to show the difference is .25. Find pairs of Decimal Squares from your deck that satisfy the following conditions and write a subtraction equation for each pair of decimals.

a. Two red Decimal Squares whose decimals have the greatest difference and two red Decimal Squares whose decimals have the smallest difference.

b. Two green Decimal Squares whose decimals have the greatest difference and two green Decimal Squares whose decimals have the smallest difference.

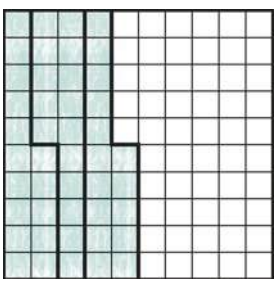
c. A red Decimal Square and a green Decimal Square whose decimals have the greatest difference.



6 × .14 = .84

3. The product 6 × .14 is illustrated here by using repeated addition and shading a blank hundredths square. Use Blank Decimal Squares and different colors of shading to illustrate each of the following products. Write a multiplication equation for each pair of decimals.

***a.** 5 × .05 **b.** 4 × .47 **c.** 3 × .650 **d.** 10 × .08



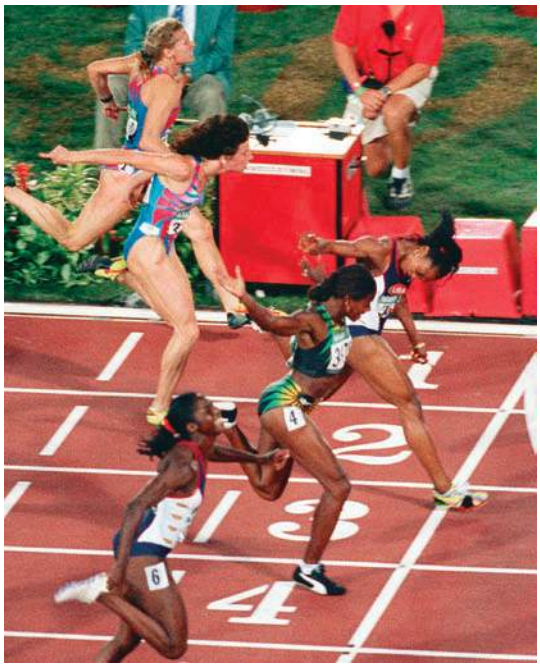
.45 ÷ 3 = .15

4. The quotient .45 ÷ 3 is illustrated at the left by shading a Blank Decimal Square for .45 and using the *sharing concept of* division to divide the shaded amount into 3 equal parts. Use Blank Decimal Squares to illustrate each of the following quotients. Write a division equation for each pair of decimals.

a. .8 ÷ 4 **b.** .50 ÷ 10 **c.** .400 ÷ 5 **d.** .60 ÷ 15

Section **6.2**

OPERATIONS WITH DECIMALS



Electronic timers for athletic competition measure time to hundredths and thousandths of a second. This photo shows the disputed finish of a 100-meter final as both the United States' Gail Devers and Jamaica's Merlene Ottey (in lanes 3 and 4, respectively) had times of 10.94 seconds at the Summer Olympics in Atlanta. Officials determined that Devers won the race by .005 second over Ottey. (Notice the right foot of Devers has crossed the finish line.)

PROBLEM OPENER

Helen Chen wants to seed her front lawn. Grass seed can be bought in 2-pound boxes that cost $7.95 or in 5-pound boxes that cost $15.95. She needs 14 pounds of seed. How many boxes of each size should she purchase to get the best buy?*

ADDITION

The concept of addition of decimals is the same as the concept of addition of whole numbers and fractions: It involves putting together, or combining, two amounts. Figure 6.15 on the next page shows a Decimal Square with 47 parts shaded out of 100, representing .47, and a Decimal Square with 36 parts shaded out of 100, representing .36. The total number of shaded parts is $47 + 36 = 83$, and since each of these parts is one-hundredth of a whole square, the total shaded amount represents .83.

This example with Decimal Squares indicates why addition of decimals is very similar to addition of whole numbers. We added whole numbers $(47 + 36)$ of parts, and then, taking into account that the small parts were hundredths, we located the decimal point.

The following equations compute this sum by using fractions. Notice that in going from the third to the fourth expression we compute the whole number sum $47 + 36$.

$$.47 + .36 = \frac{47}{100} + \frac{36}{100} = \frac{47 + 36}{100} = \frac{83}{100} = .83$$

*Inspired by C. R. Hirsch, ed., *Activities for Implementing Curricular Themes from the Agenda for Action*.

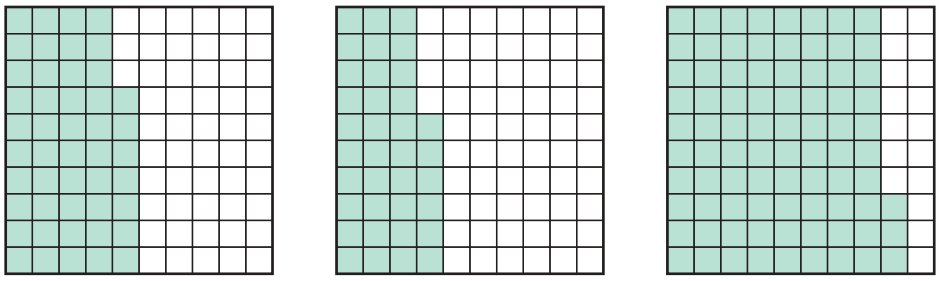


Figure 6.15

Pencil-and-Paper Algorithm In one pencil-and-paper algorithm for addition of decimals, the digits are aligned, tenths under tenths, hundredths under hundredths, etc., as shown in the following example. When the sum of the digits in any column is 10 or greater, regrouping is necessary. Since .47 can be thought of as 4 tenths and 7 hundredths and .36 as 3 tenths and 6 hundredths, the sum of 7 and 6 in the hundredths column is 13 hundredths.

$$\begin{array}{r} {\scriptstyle 1} \\ .47 \\ +\ .36 \\ \hline .83 \end{array}$$

Ten of the hundredths can be visualized as 1 tenth by recalling that in a Decimal Square, 10 hundredths fill one column ($\frac{1}{10}$ of a square). Also, we know that the fraction $\frac{10}{100}$ in lowest terms is $\frac{1}{10}$. So 1 tenth is regrouped to the tenths column, and 3 is recorded in the hundredths column, as shown above.

EXAMPLE A

Use the preceding pencil-and-paper algorithm to compute each sum, and show the numbers that are regrouped.

1. 62.47 + 114.86 **2.** 4.039 + 17.18 **3.** .267 + .5163

Solution

1.
$$\begin{array}{r} {\scriptstyle 1\ 1} \\ 62.47 \\ +\ 114.86 \\ \hline 177.33 \end{array}$$

2.
$$\begin{array}{r} {\scriptstyle 1\ 1} \\ 4.039 \\ +\ 17.180 \\ \hline 21.219 \end{array}$$

3.
$$\begin{array}{r} {\scriptstyle 1} \\ .2670 \\ +\ .5163 \\ \hline .7833 \end{array}$$

Notice in solution 2 of Example A that 17.18 was replaced by 17.180. This can be done because 18 hundredths is equal to 180 thousandths. Similarly, in solution 3, .267 was replaced by .2670.

SUBTRACTION

Subtraction of decimals, like subtraction of whole numbers and fractions, can be illustrated with the take-away concept, as shown in part a of Figure 6.16, or with the comparison concept, as shown in part b (see page 367). The red and yellow areas in part a represent .625, and the arrow shows 238 parts out of 1000 being taken away from 625 parts out of 1000. This leaves 387 parts out of 1000, which represents .387. In part b, we can compare the shaded amounts of the squares for .75 and .40 to see that the difference is 35 parts out of 100, or .35. The third concept of subtraction, *missing addend,* can be illustrated for .75 − .40 by beginning with the square for .40 in part b and counting up from 40 to 75 parts.

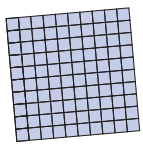
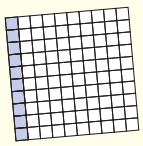
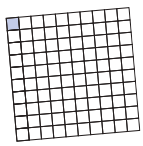
Explore 3-5

Math Lab
Adding and Subtracting Decimals Using Models

MAIN IDEA
Use models to add and subtract decimals.

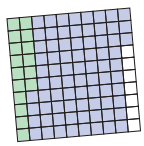
Math Online
glencoe.com
• Concepts In Motion

Decimal models can be used to add and subtract decimals.

Ones (1)	Tenths (0.1)	Hundredths (0.01)
		
One whole 10-by-10 grid represents 1 or 1.0.	Each row or column represents one tenth or 0.1.	Each square represents one hundredth or 0.01.

ACTIVITIES

1 **Find 0.16 + 0.77 using decimal models.**

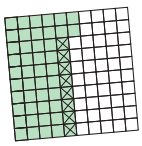


STEP 1 Shade 0.16 green.

STEP 2 Shade 0.77 blue. The sum is the total shaded area.

So, $0.16 + 0.77 = 0.93$.

2 **Find 0.52 − 0.08 using decimal models.**



STEP 1 Shade 0.52 green.

STEP 2 Use ×s to cross out 0.08 from the shaded area. The difference is the amount of shaded area with no ×s.

So, $0.52 - 0.08 = 0.44$.

CHECK Your Progress

Find each sum or difference using decimal models.

a. $0.14 + 0.67$ b. $0.35 + 0.42$ c. $0.03 + 0.07$

d. $0.75 - 0.36$ e. $0.68 - 0.27$ f. $0.88 - 0.49$

ANALYZE THE RESULTS

1. Explain how you can use grid paper to model $0.8 - 0.37$.
2. **MAKE A CONJECTURE** Write a rule you can use to add or subtract decimals without using models.

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Research Statement

Students who do not understand decimal notation frequently resort to memorizing procedural rules.

Bell, Swain, and Taylor; Sackur-Grisvard and Leonard

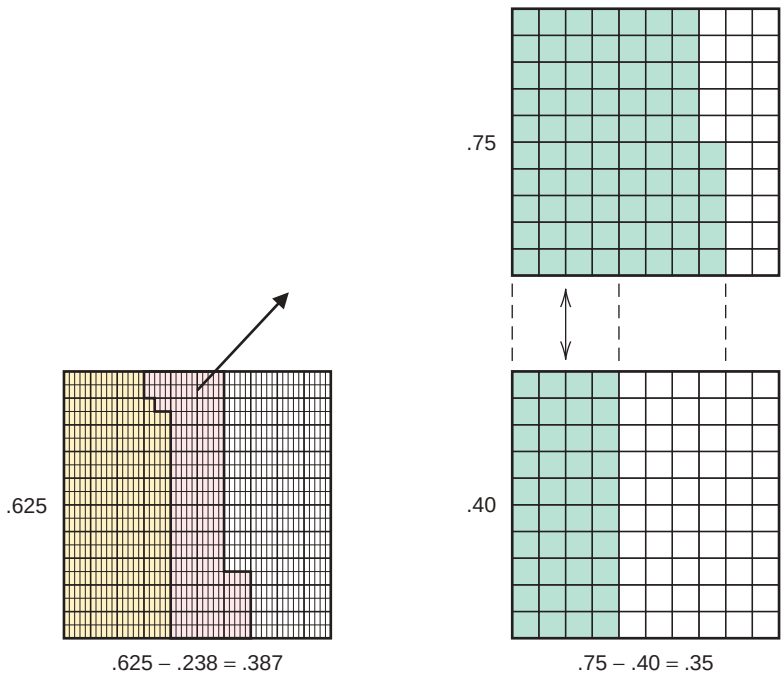


Figure 6.16

Pencil-and-Paper Algorithm The Decimal Squares provide a visual model for computing the difference of whole numbers of small parts. This provides a method of viewing subtraction of decimals as subtraction of whole numbers and helps to show this connection and the similarity of these operations.

In one pencil-and-paper algorithm for subtraction of decimals, the digits are aligned as they are for addition of decimals. Subtraction then takes place from right to left, with thousandths subtracted from thousandths, hundredths from hundredths, etc. When regrouping (borrowing) is necessary, it is done just as it is in subtracting whole numbers.

In the following example, .625 can be thought of as 6 tenths, 2 hundredths, and 5 thousandths and .238 as 2 tenths, 3 hundredths, and 8 thousandths. To subtract 8 thousandths from 5 thousandths, regrouping is needed. Since 1 hundredth equals 10 thousandths $\left(\frac{1}{100} = \frac{10}{1000}\right)$, 1 hundredth can be regrouped from the hundredths column to increase 5 thousandths to 15 thousandths. Then we can subtract 8 from 15 to obtain 7 in the thousandths column. A slash through the 2 indicates that it has been decreased by 1. Similarly, 6 in the tenths column is decreased by 1 to obtain 10 more hundredths for the hundredths column $\left(\frac{1}{10} = \frac{10}{100}\right)$. Then 3 is subtracted from 11 to obtain 8 in the hundredths column. Finally, 2 is subtracted from 5 to obtain 3 in the tenths column.

$$\begin{array}{r} {\scriptstyle 5\,1} \\ .6\not{2}5 \\ -\ .238 \\ \hline .387 \end{array}$$

The following equations show how this difference can be computed by using fractions. Notice that we subtract whole numbers in the third equation.

$$.625 - .238 = \frac{625}{1000} - \frac{238}{1000} = \frac{625 - 238}{1000} = \frac{387}{1000} = .387$$

EXAMPLE B Use the preceding pencil-and-paper algorithm to compute each difference, and show where regrouping is needed.

1. $46.32 - 18.47$ **2.** $.4074 - .356$ **3.** $15.06 - 2.743$

Solution

1.
$$\begin{array}{r} {\scriptstyle 3\,5\,2} \\ \not{46}.\not{3}2 \\ -\ 18.47 \\ \hline 27.85 \end{array}$$

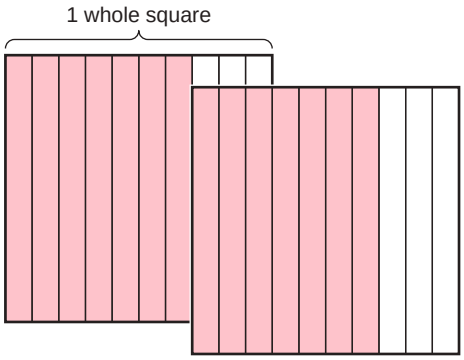
2.
$$\begin{array}{r} {\scriptstyle 3} \\ .4074 \\ -\ .3560 \\ \hline .0514 \end{array}$$

3.
$$\begin{array}{r} {\scriptstyle 4\ \ 5} \\ 1\not{5}.0\not{6}0 \\ -\ 2.743 \\ \hline 12.317 \end{array}$$

Notice that extra zeros were appended to some of the numbers in solutions 2 and 3 of Example B in order to perform the subtraction using the same number of decimal places.

MULTIPLICATION

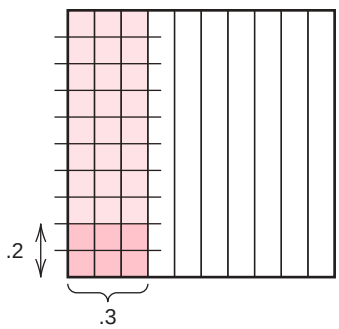
The product of a whole number and a decimal can be illustrated by repeated addition. Each of the Decimal Squares in Figure 6.17 has seven shaded parts. In all there is a total of $2 \times 7 = 14$ shaded parts. This is four more shaded parts than are contained in a whole square, so $2 \times .7 = 1.4$.



$2 \times .7 = 1.4$

Figure 6.17

The product of a decimal and a decimal, such as $.2 \times .3$, can be interpreted as .2 of .3. This is illustrated in Figure 6.18 by using a Decimal Square for .3 and taking .2 of its shaded part. To do this, we split the shaded part of the Decimal Square for .3 into 10 equal parts. The 6 darker parts of the square represent .2 of .3. Since each of the darker parts is 1 hundredth of a whole square, $.2 \times .3 = .06$.



.2 of .3 = .06

Figure 6.18

Pencil-and-Paper Algorithm The illustrations in Figures 6.17 and 6.18 show that computing products of decimals is closely related to computing products of whole numbers. To compute products involving decimals, we multiply the numbers as though they were whole numbers and then locate the decimal point in the product. The next example shows the

NCTM Standards

Multiplying and dividing fractions and decimals can be challenging for many students because of problems that are primarily conceptual rather than procedural. From their experience with whole numbers, many students appear to develop the belief that "multiplication makes bigger and division makes smaller." p. 218

product of a one-place decimal times a two-place decimal. The digits do not have to be positioned so that units are above units, tenths above tenths, etc., as they are for addition and subtraction of decimals. The number of decimal places in the answer is the total number of decimal places in the original two numbers.

$$\begin{array}{r} 27.48 \\ \times\quad 9.2 \\ \hline 5496 \\ 24732 \\ \hline 252.816 \end{array}$$

The following equations show why 9.2×27.48 can be computed by first computing 92×2748.

$$9.2 \times 27.48 = \frac{92}{10} \times \frac{2748}{100} = \frac{92 \times 2748}{10 \times 100} = \frac{252{,}816}{1000} = 252.816$$

EXAMPLE C

Use the pencil-and-paper algorithm for the multiplication of decimals to compute each product.

1. 3.7×2.5 **2.** $4.6 \times .35$ **3.** $1.8 \times .473$

Solution

1.
$$\begin{array}{r} 2.5 \\ \times\ 3.7 \\ \hline 175 \\ 75 \\ \hline 9.25 \end{array}$$

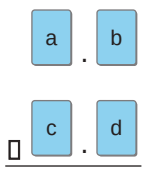
2.
$$\begin{array}{r} .35 \\ \times\ 4.6 \\ \hline 210 \\ 140 \\ \hline 1.610 \end{array}$$

3.
$$\begin{array}{r} .473 \\ \times\ \ 1.8 \\ \hline 3784 \\ 473 \\ \hline .8514 \end{array}$$

Laboratory Connection

Digit Draw

Ten cards marked 0 to 9 are placed in a container and selected one at a time without replacement. As each card is selected, each player must write the digit in one of the boxes shown below, with the object to form the largest product. Try this game and explore related questions in this investigation.



Mathematics Investigation, Chapter 6, Section 2
www.mhhe.com/bbn

The base-ten grid that was used for multiplying whole numbers in Chapter 3 can be used to model the products of decimals. The product 2.3×1.7 is illustrated in Figure 6.19 using a rectangle with dimensions 2.3×1.7. The two regions of this rectangle correspond to the two partial products 2×1.7 and $.3 \times 1.7$. The red region has the equivalent of 3 unit squares and 4 tenths of a unit square; the green region has 51 hundredths of a unit square.

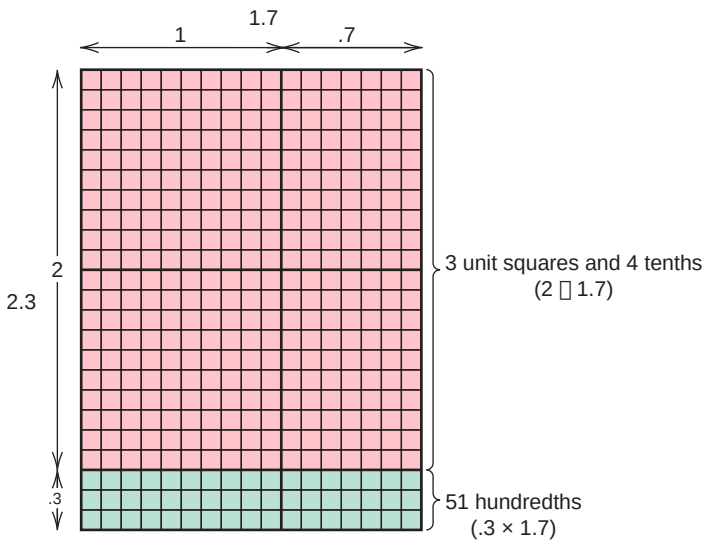


Figure 6.19

$$\begin{aligned} 2.3 \times 1.7 &= (2 + .3) \times 1.7 \\ &= (2 \times 1.7) + (.3 \times 1.7) \\ &= 3.4 + .51 \\ &= 3.91 \end{aligned}$$

NCTM Standards

NCTM's 5–8 Standard, *Number and Number Relationships* given in the *Curriculum and Evaluation Standards for School Mathematics* (p. 88), recommends the use of area models:

> Area models are especially helpful in visualizing numerical ideas from a geometric point of view. For example, area models can be used to show that $\frac{8}{12}$ is equivalent to $\frac{2}{3}$, that $1.2 \times 1.3 = 1.56$, and that 80% of 20 is 16. Later, students can extend area models to the study of algebra, probability, dimension analysis in measurement situations, and other more advanced subjects.

HISTORICAL HIGHLIGHT

One type of notation used in the sixteenth century called for the number of decimal places to be specified by a circled index to the right of the numeral. For example, 27.487 was represented as 27487 . . . ③, and 9.21 was 921 . . . ②. This notation is especially convenient for computing the product of two decimals. The whole numbers are multiplied, and then the numbers in circles are added to determine the location of the decimal point. Using our present notation, we would place the decimal point between the 3 and the 1 in the following product.

$$\begin{array}{rl} 27487 & \ldots ③ \\ \times\ 921 & \ldots ② \\ \hline 27487 & \\ 54974 & \\ 247383 & \\ \hline 25315527 & \ldots ⑤ \end{array}$$

Multiplying by Powers of 10 One way to illustrate multiplication of decimals by powers of 10 is to use Decimal Squares and repeated addition. Figure 6.20 on the next page shows squares for .1, .01, and .001 and corresponding squares whose shaded amounts are 10 times greater. Multiplying by higher powers of 10 can be similarly represented, for example, $100 \times .1 = 10$ and $100 \times .01 = 1$.

To illustrate $10 \times .165$ with Decimal Squares, we can replace each tenth by 1 whole square, each hundredth by 1 tenth of a square, and each thousandth by 1 hundredth of a square (see Figures 6.20 and 6.21 on the next page).

Another way to show the results of multiplying by powers of 10 is to replace the decimal by a sum of fractions and use the distributive property.

$$\begin{aligned} 10 \times .165 &= 10 \times \left(\frac{1}{10} + \frac{6}{100} + \frac{5}{1000}\right) \\ &= \frac{10}{10} + \frac{60}{100} + \frac{50}{1000} \\ &= 1 + \frac{6}{10} + \frac{5}{100} \\ &= 1.65 \end{aligned}$$

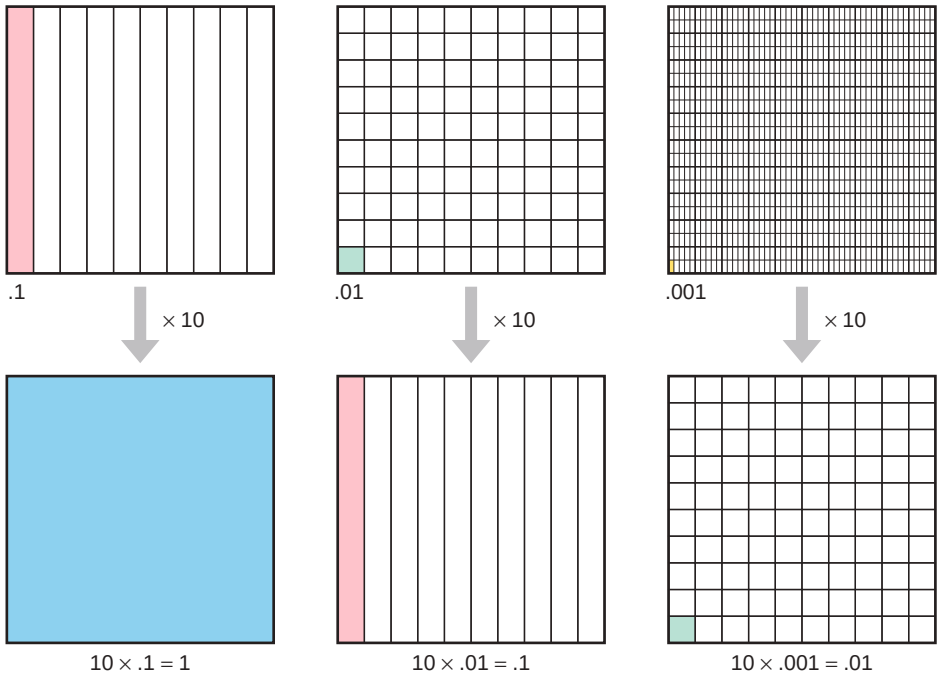


Figure 6.20

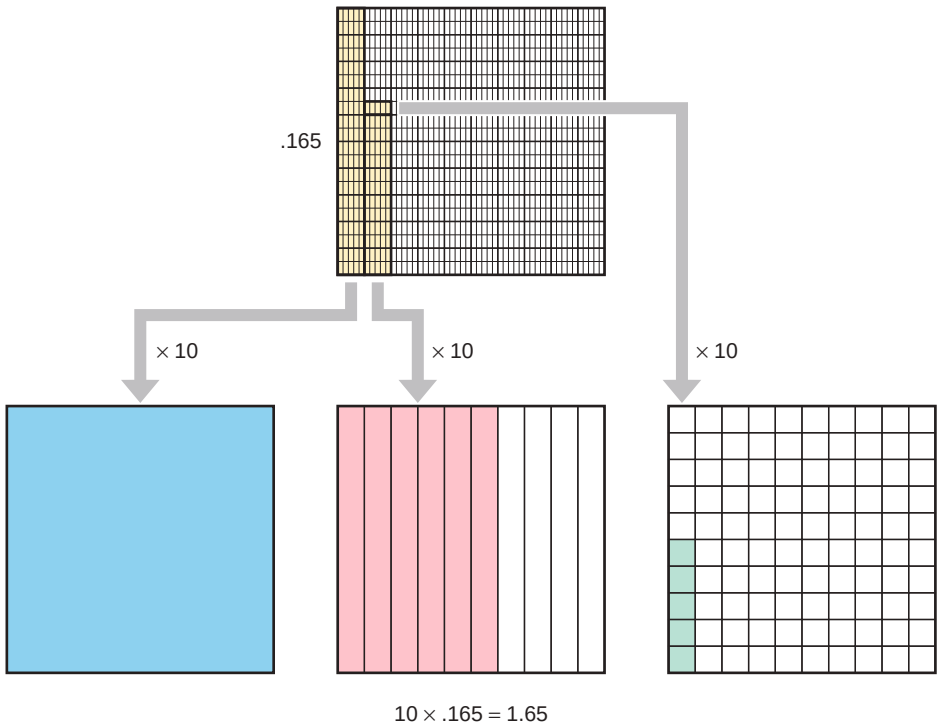


Figure 6.21

These examples suggest the following algorithm.

Multiplying by Powers of 10 To multiply a decimal by a power of 10, move the decimal point one place to the right for each power of 10.

EXAMPLE D

Compute each product mentally.

1. $100 \times .45$ **2.** 10×14.08 **3.** $1000 \times .32714$

Solution **1.** 45 **2.** 140.8 **3.** 327.14

DIVISION

The two concepts of division—*the measurement (subtractive) concept* and the *sharing (partitive) concept*—are both useful for illustrating division with decimals. The measurement concept involves repeatedly measuring off or subtracting one amount from another. For example, to compute $.90 \div .15$, determine how many times .15 can be subtracted from .90. The Decimal Square in Figure 6.22 has been marked off to show that the quotient is 6.

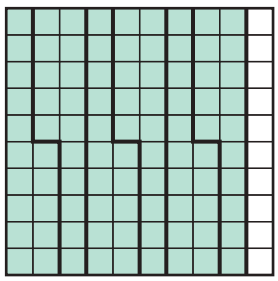


Figure 6.22 $.90 \div .15 = 6$

To illustrate the division of a decimal by a whole number, we can use the sharing concept. In this case, the divisor is the number of equal parts into which a set or region is divided. The shaded part of the Decimal Square in Figure 6.23 has been divided into four equal parts to illustrate $.80 \div 4$. Since each part has 20 hundredths, the quotient is .20.

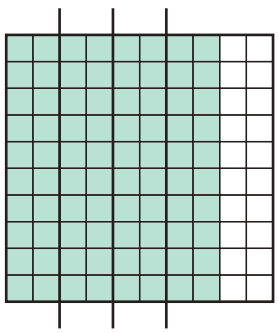


Figure 6.23 $.80 \div 4 = .20$

Pencil-and-Paper Algorithm The preceding examples illustrate the close relationship between division of whole numbers and division of decimals. Consider the illustration of dividing .80 by 4 in Figure 6.23. Since the Decimal Square has 80 parts shaded out of 100, we were able to think in terms of whole numbers and divide 80 by 4. Then since the quotient has 20 small squares, each one-hundredth of a whole square, the quotient $.80 \div 4$ is equal to .20. Similar steps are carried out to divide any decimal by a whole number: first divide, using the *long*

division algorithm for whole numbers, and then place the decimal point in the quotient directly above its location in the dividend. These steps are shown here for dividing .80 by 4.

$$\begin{array}{r} .20 \\ 4\overline{).80} \\ \underline{8} \\ 0 \\ \underline{0} \end{array}$$

In the long division algorithm for dividing with decimals, we never actually divide by a decimal. Before we divide, an adjustment is made so that the divisor is always a whole number. For example, the algorithm shown below indicates that we are to divide 1.504 by .32. Before dividing, however, we move the decimal points in .32 and 1.504 both two places to the right. This has the effect of changing the divisor and the dividend so that we are dividing 150.4 by 32.

$$.32\overline{)1.504}$$

changed to

$$\begin{array}{r} 4.7 \\ 32.\overline{)150.4} \\ \underline{128} \\ 224 \\ \underline{224} \end{array}$$

The rule for dividing by a decimal is to count the number of decimal places in the divisor and then move the decimal points in the divisor and the dividend that many places to the right. In the previous example, the decimal points in .32 and 1.504 were moved two places to the right because .32 has two decimal places. The justification for this process of shifting decimal points is illustrated in the following equations. In the first equation, we use the fact that the numerator and denominator of a fraction can be multiplied by the same nonzero number to produce an equal fraction.

$$\frac{1.504}{.32} = \frac{1.504 \times 10^2}{.32 \times 10^2} = \frac{150.4}{32}$$

These equations show that the answer to $1.504 \div .32$ is the same as that for $150.4 \div 32$. No further adjustment is needed as long as we shift the decimal points in both the divisor and the dividend by the same amount. Thus, division of a decimal by a decimal can always be carried out by dividing a decimal (or whole number) by a whole number.

EXAMPLE E

Use the long division algorithm to compute each quotient.

1. $106.82 \div 7$ **2.** $.498 \div .6 = 4.98 \div 6$ **3.** $34.44 \div 1.4 = 344.4 \div 14$

Solution

1.
$$\begin{array}{r} 15.26 \\ 7\overline{)106.82} \\ \underline{7} \\ 36 \\ \underline{35} \\ 18 \\ \underline{14} \\ 42 \\ \underline{42} \end{array}$$

2.
$$\begin{array}{r} .83 \\ .6\overline{).498} \\ \underline{48} \\ 18 \\ \underline{18} \end{array}$$

3.
$$\begin{array}{r} 24.6 \\ 1.4\overline{)34.44} \\ \underline{28} \\ 64 \\ \underline{56} \\ 84 \\ \underline{84} \end{array}$$

Dividing by Powers of 10 In Section 6.1 we saw that a whole number can be divided by a power of 10 by relocating a decimal point. This is also true for dividing decimals by powers of 10, as Figure 6.24 shows for .37 ÷ 10. Since 3 full columns (see first square) represent .3 or 300 thousandths, dividing by 10 results in 30 thousandths (see second square). Since 7 small hundredths squares (see first square) represent 70 thousandths, dividing by 10 results in 7 thousandths (see second square).

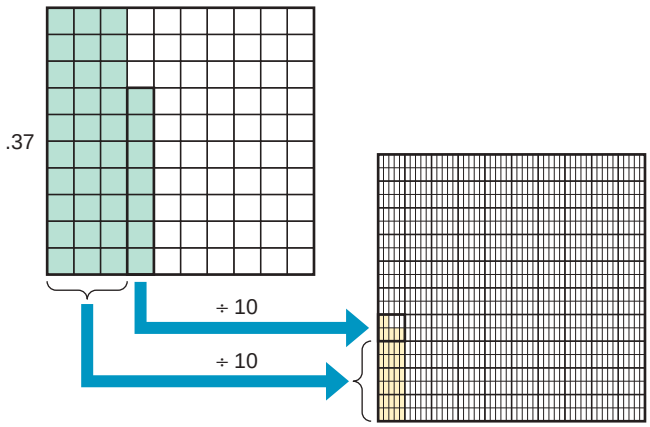


Figure 6.24

.37 ÷ 10 = .037

The effects of dividing by powers of 10 can be shown by writing the decimals in expanded form. Consider dividing .37 by 10 and 100.

$$.37 \div 10 = \frac{37}{100} \times \frac{1}{10} = \frac{37}{1000} = .037$$

$$.37 \div 100 = \frac{37}{100} \times \frac{1}{100} = \frac{37}{10{,}000} = .0037$$

These examples are special cases of the following algorithm.

Dividing by Powers of 10 To divide a decimal by a power of 10, move the decimal point one place to the left for each power of 10.

EXAMPLE F

Compute each quotient mentally.

1. .35 ÷ 10 **2.** 4.6 ÷ 100 **3.** .8 ÷ 1000

Solution **1.** .035 **2.** .046 **3.** .0008

ORDER OF OPERATIONS

When addition and subtraction are combined with multiplication and division, care must be taken regarding the order of operations on decimals. As in the case of whole numbers, multiplication and division are performed before addition and subtraction. Consider the following example of computing a tax. According to the instructions on the tax form, the tax on an income greater than $27,950 and less than $67,700 is $3892.50 plus .27 times the amount over $27,950. Therefore, the tax on $34,600 is

$$\$3892.50 + .27 \times \$6650$$

Technology Connection

On calculators that are designed to follow the *order of operations,* this tax can be computed by entering the numbers and operations into the calculator as they appear from left to right. On calculators without the *order of operations,* this tax can be obtained by computing $.27 \times 6650$ and then adding 3892.50.

EXAMPLE G

Determine the tax on the following incomes according to the instructions at the bottom of page 374.

1. \$42,672 **2.** \$51,320

Solution **1.** $\$3892.50 + .27 \times \$14{,}722 = \$3892.50 + \$3974.94 = \$7867.44$. **2.** $\$3892.50 + .27 \times \$23{,}370 = \$3892.50 + \$6309.90 = \$10{,}202.40$.

REPEATING DECIMALS

In Section 6.1 we saw that terminating decimals can be written as fractions whose numerators are integers and whose denominators are powers of 10.

$$.47 = \frac{47}{100} \qquad 3.802 = \frac{3802}{1000} \qquad 64.3 = \frac{643}{10}$$

A repeating decimal can also be written as a fraction whose numerator and denominator are integers. Before showing this on the next page, we first look at the repeating decimals for the fractions $\frac{1}{9}$, $\frac{1}{99}$, and $\frac{1}{999}$. Figure 6.25 shows the first three steps of a demonstration for finding the decimal for $\frac{1}{9}$. In step 1, we begin dividing 1 by 9 by dividing 10 tenths into 9 equal parts. There is a remainder of 1 tenth, which is regrouped to 10 hundredths so that the process of dividing by 9 can continue in step 2 and then similarly in step 3. Figure 6.25 shows that $\frac{1}{9} = .111$ with .001 remaining. This process can be continued to show that $\frac{1}{9} = .111\ldots$.

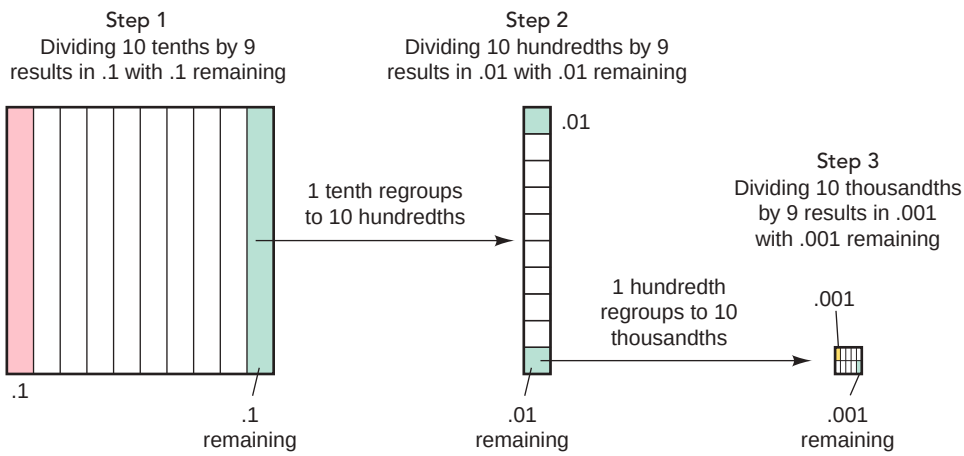


Figure 6.25

The demonstration in Figure 6.26 on the next page shows how to obtain the first few decimal places for $\frac{1}{99}$. In step 1, we begin dividing 1 by 99 by dividing 100 hundredths into 99 equal parts. There is a remainder of one hundredth, and this is regrouped to 100 ten-thousandths so that the process of dividing by 99 can continue in step 2. The first two steps show that $\frac{1}{99} = .0101$ $(.01 + .0001)$ with .0001 remaining. Continuing this process will show that $\frac{1}{99} = .010101\ldots$.

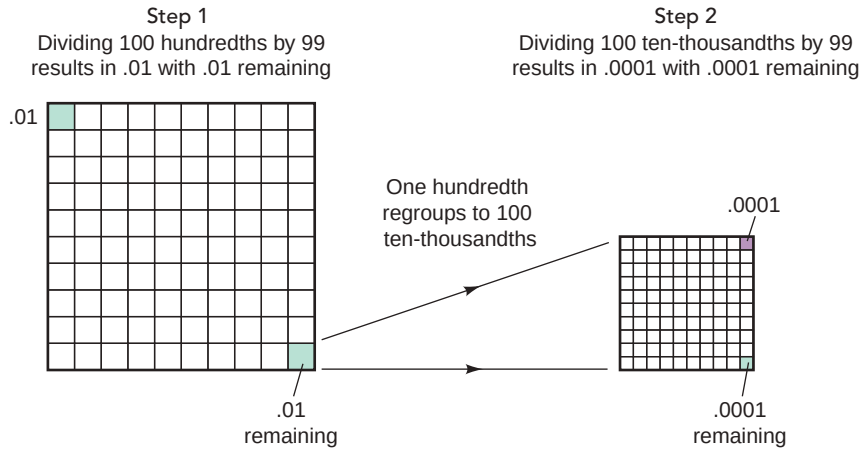


Figure 6.26

Similarly, by beginning with a Decimal Square for thousandths and dividing it into 999 equal parts, it can be shown that $\frac{1}{999} = .001$ with a remainder of .001; and if the process of regrouping and dividing by 999 were to be continued, we would see that $\frac{1}{999} = .001001\ldots$.

Now let's consider writing a repeating decimal as the quotient of two integers. The following equations show that the infinite repeating decimal .888 . . . is equal to $\frac{8}{9}$.

$$\begin{aligned} .888\ldots &= 8 \times .111\ldots \\ &= 8 \times \frac{1}{9} \\ &= \frac{8}{9} \end{aligned}$$

A similar process is used when the *repetend* has two digits, as here:

$$\begin{aligned} .373737\ldots &= 37 \times .010101\ldots \\ &= 37 \times \frac{1}{99} \\ &= \frac{37}{99} \end{aligned}$$

Any digits in the decimal that precede the repetend can be separated by multiplying by powers of 10. For example,

$$\begin{aligned} .2373737\ldots &= \frac{1}{10} \times 2.373737\ldots \\ &= \frac{1}{10} \times \left(2 + \frac{37}{99}\right) \\ &= \frac{2}{10} + \frac{37}{990} = \frac{235}{990} \end{aligned}$$

Technology Connection

Check the preceding examples by using a calculator to divide the numerators of the fractions by their denominators. Except for a possible rounded digit at the end of the decimal, you will obtain the original decimal in each case. (*Note:* Some calculators that are designed for fractions have a $\boxed{\text{F} \leftrightarrow \text{D}}$ key that will convert fractions to decimals, and decimals of three or fewer decimal places to fractions.*)

*CASIO fx-55 and Texas Instruments TI-15 are examples of such calculators.

EXAMPLE H

Replace each repeating decimal with a quotient of two integers. Check the results with a calculator.

1. $.\overline{17}$ **2.** $.\overline{7}$ **3.** $.\overline{238}$ **4.** $.\overline{18}$

Solution **1.** $\frac{17}{99}$ **2.** $\frac{7}{9}$ **3.** $\frac{238}{999}$ **4.** $\frac{2}{11}$

PROPERTIES OF RATIONAL NUMBERS

We have seen that a rational number can always be represented as the quotient of two integers $\frac{a}{b}$ or as a decimal. That is, there are two different types of numerals that represent rational numbers. In Section 5.3 we used the properties of addition and multiplication of integers to illustrate the properties of addition and multiplication of fractions. These properties are restated here for the rational numbers.

Closure Properties The set of rational numbers is closed under addition and multiplication. For any rational numbers a and b, $a + b$ and $a \times b$ are unique rational numbers.

Commutative Properties Addition and multiplication are commutative. For any rational numbers a and b, $a + b = b + a$ and $a \times b = b \times a$.

Associative Properties Addition and multiplication are associative. For any rational numbers a, b, and c, $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$.

Identity Properties The identity for addition is 0, and the identity for multiplication is 1. For any rational number b, there are unique identity elements 0 and 1 such that $0 + b = b$ and $1 \times b = b$.

Inverse Properties For every rational number, there is a unique inverse for addition; and for every nonzero rational number, there is a unique inverse for multiplication. In other words, for any rational number b, there is a unique rational number ^{-}b such that $b + {}^{-}b = 0$; and for any rational number $c \neq 0$, there is a unique rational number $\frac{1}{c}$ such that $c \times \frac{1}{c} = 1$.

Distributive Property Multiplication is distributive over addition. For any rational numbers a, b, and c, $a \times (b + c) = a \times b + a \times c$.

NCTM Standards

Students should also develop and adapt procedures for mental calculation and computational estimation with fractions, decimals, and integers. p. 220

MENTAL COMPUTATION

We have seen that computations with decimals can be carried out by first computing with whole numbers and then placing decimal points. This fact can be used to compute decimals mentally.

EXAMPLE I

Calculate each answer mentally by first computing with whole numbers and then placing decimal points.

1. $.4 \times .22$ **2.** $.35 + .55$ **3.** $5 \times .003$

4. $.345 - .2$ **5.** $.001 \times 62$ **6.** $.24 \div .04$

Solution **1.** $4 \times 22 = 88$, and since there is a total of three decimal places, $.4 \times .22 = .088$. **2.** $35 + 55 = 90$, so $.35 + .55 = .90$. **3.** $5 \times 3 = 15$, and since there is a total of three decimal places, $5 \times .003 = .015$. **4.** $345 - 200 = 145$, so $.345 - .2 = .145$. **5.** $1 \times 62 = 62$, and since there is a total of three decimal places, $.001 \times 62 = .062$. **6.** $24 \div 4 = 6$, and since both numbers have the same number of decimal places, $.24 \div .04 = 6$.

Substitutions and Add-Up for Mental Calculations The mental calculating techniques for whole numbers can also be used for mental calculations with decimals. For example, to compute $.54 - .38$, we can use the add-up method:

$$.38 + .02 = .40 \quad \text{and} \quad .40 + .14 = .54 \quad \text{so} \quad .38 + .16 = .54$$

Thus, $.54 - .38 = .16$.

EXAMPLE J

Calculate each answer mentally, using either the substitutions or add-up techniques.

1. $.37 + .28$ **2.** $.76 - .29$ **3.** $3 \times .98$ **4.** 4.3×102

Solution **1.** Substitution: $.37 + .28 = .37 + (.20 + .08) = .57 + .08 = .65$ **2.** Add up: $.29 + .01 = .30$ and $.30 + .46 = .76$, so $.29 + .47 = .76$, and $.76 - .29 = .47$ **3.** Substitution: $3 \times .98 = 3 \times (1 - .02) = 3 - .06 = 2.94$ **4.** Substitution: $4.3 \times 102 = 4.3 \times (100 + 2) = 430 + 8.6 = 438.6$

Equal Quotients for Mental Calculations Multiplying both numbers in a quotient of two decimals by the same number is sometimes a convenient method of obtaining a mental calculation, especially if the divisor is replaced by a power of 10.

EXAMPLE K

Compute each quotient by using the equal-quotients technique to replace the divisor by a power of 10.

1. $.21 \div 2.5$ **2.** $.34 \div 50$ **3.** $.16 \div 200$

Solution **1.** $.21 \div 2.5 = 4(.21) \div 4(2.5) = .84 \div 10 = .084$ **2.** $.34 \div 50 = 2(.34) \div 2(50) = .68 \div 100 = .0068$ **3.** $.16 \div 200 = \frac{1}{2}(.16) \div \frac{1}{2}(200) = .08 \div 100 = .0008$

Compatible Numbers for Mental Calculations Sometimes in computing products mentally it helps to recognize the decimal equivalents of a few simple fractions. Here are some that are useful:

$$.25 = \frac{1}{4} \quad .5 = \frac{1}{2} \quad .75 = \frac{3}{4} \quad .2 = \frac{1}{5} \quad .4 = \frac{2}{5} \quad .6 = \frac{3}{5} \quad .8 = \frac{4}{5} \quad .125 = \frac{1}{8}$$

EXAMPLE L

Compute each product by replacing the decimal by an equivalent fraction.

1. $.25 \times 800$ **2.** $.5 \times .6$ **3.** $.2 \times 30$

4. $.125 \times 24$ **5.** $.6 \times 45$ **6.** $.75 \times 12$

Solution **1.** $\frac{1}{4} \times 800 = 200$ **2.** $\frac{1}{2} \times .6 = .3$ **3.** $\frac{1}{5} \times 30 = 6$ **4.** $\frac{1}{8} \times 24 = 3$
5. $\frac{3}{\cancel{5}} \times \overset{9}{\cancel{45}} = 3 \times 9 = 27$ **6.** $\frac{3}{\cancel{4}} \times \overset{3}{\cancel{12}} = 3 \times 3 = 9$

ESTIMATION

There are times when estimations are as helpful as exact computations. The techniques of rounding, front-end estimation, and compatible numbers are illustrated in Examples M through P.

Rounding Rounding to obtain an estimation mentally will often save time. For example, to make a decision regarding a purchase, all we may need is a rough idea of the cost.

EXAMPLE M

Suppose you are interested in the total cost of a stereo system whose components are priced as follows: CD changer, $219.50; docking station, $179; pair of speakers, $284; and receiver, $335.89. Estimate the cost by rounding each amount to the nearest hundred dollars.

NCTM Standards

Estimation serves as an important companion to computation. It provides a tool for judging the reasonableness of calculator, mental, and paper-and-pencil computations. p. 155

Solution The exact sum and the estimation obtained by using numbers rounded to the hundreds place are shown below.

Exact Sum	Estimation
$ 219.50	$ 200
179.00	200
284.00	300
335.89	300
$1018.39	$1000

Notice that the estimation in Example M can be obtained quickly by adding the leading digits if the leading digit in $179 is rounded to 2 and the leading digit in $284 is rounded to 3.

The rectangular grid in Figure 6.27 shows the reasonableness of estimating 1.7×3.2 by rounding each number to the nearest whole number. The green region shows the increase due to rounding 1.7 to 2, and the red region shows the decrease caused by rounding 3.2 to 3. Notice the 6 unit squares in the grid that illustrate the estimated product of 2×3. The fact that the green region (the increase) is larger than the red region (the decrease) shows that the estimated product is larger than the actual product.

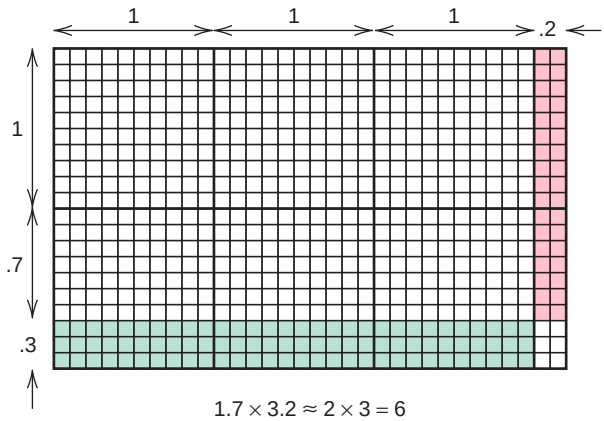


Figure 6.27

EXAMPLE N

Estimate each product mentally by rounding the decimals to the nearest whole numbers.

1. 4.6×8.21 **2.** 10.263×5.9

Solution **1.** $4.6 \times 8.21 \approx 5 \times 8 = 40$. **2.** $10.263 \times 5.9 \approx 10 \times 6 = 60$.

Front-End Estimation A quick and easy method of obtaining a rough estimation is to use only the leading nonzero digit. For example,

$$762 \times .26 \approx 700 \times .2 = 140$$

EXAMPLE O

Estimate each computation mentally by using the leading nonzero digit in each number.

1. $.328 + .511$ **2.** $.361 - .14$ **3.** $2.6 \div .53$ **4.** $3.8 \times .023$

Solution **1.** $.328 + .511 \approx .3 + .5 = .8$ **2.** $.361 - .14 \approx .3 - .1 = .2$ **3.** $2.6 \div .53 \approx 2 \div .5 = 4$ **4.** $3.8 \times .023 \approx 3 \times .02 = .06$

Compatible Numbers for Estimation Decimals can be replaced by compatible decimals or compatible fractions—numbers that are more convenient for estimations. For example, in the sum $3.71 + .24$, it is more convenient to use 3.7 in place of 3.71 and .3 in place of .24.

$$3.71 + .24 \approx 3.7 + .3 = 4$$

EXAMPLE P

Estimate each computation by replacing a decimal by a more compatible decimal or fraction.

1. $6 \div .26$ **2.** $1.43 - .5$ **3.** $.35 \times 268$

4. $2.87 + 5.15$ **5.** $.19 \times 45$ **6.** $27.7 - 1.8$

Solution **1.** $6 \div .26 \approx 6 \div \frac{1}{4} = 24$. **2.** $1.43 - .5 \approx 1.5 - .5 = 1$. **3.** $.35 \times 268 \approx \frac{1}{3} \times 270 = 90$. **4.** $2.87 + 5.15 \approx 2.85 + 5.15 = 8$. **5.** $.19 \times 45 \approx \frac{1}{5} \times 45 = 9$. **6.** $27.7 - 1.8 \approx 27.7 - 1.7 = 26$.

Whenever an exact answer is required, an estimation can serve as a guide for detecting large errors. One source of error is caused by misplacing a decimal point when a number is entered into a calculator. Suppose, in computing $.46 \times 34.28$, that you mistakenly enter 342.8. Then the calculator will show a product of

$$.46 \times 342.8 = 157.688$$

This error can be discovered by mental estimation: Replace .46 by $\frac{1}{2}$ and take one-half of 34.

$$.46 \times 34.28 \approx \frac{1}{2} \times 34 = 17$$

Since 17 is much smaller than 157.688, there is an indication of an error in the original computation.

PROBLEM-SOLVING APPLICATION

Occasionally a number is written using both decimal and fraction notation together, as in the next problem. Try to solve this problem. If you need help, read as much of the following information as you need.

Problem

Unleaded gas sells for $\$4.35\frac{9}{10}$ per gallon if you use a credit card and the gas pump meter is calibrated for this amount. A discount that lowers the price to $\$4.31\frac{9}{10}$ per gallon is offered if you choose to pay cash. If you hand the attendant a $20 bill and ask for $20 worth of gas, the gas pump amount that is calibrated for a credit card will register more than $20. What will be the dollar amount on the gas pump after the attendant has finished?*

Understanding the Problem These prices use hundredths of a dollar, $4.35 and $4.31, plus the fraction $\frac{9}{10}$ to indicate $\frac{9}{10}$ of a hundredth; $\frac{9}{10}$ of a hundredth is $\frac{9}{10} \times .01$, which is equal to .009. **Question 1:** What are the two prices per gallon, written as decimals to the thousandths place?

Devising a Plan Sometimes *forming a table* with a few calculations will suggest a plan for solving the problem. The following table shows the cost of the first few gallons of gas with cash payment or with credit card payment. For example, 3 gallons of gas will cost $3 \times \$4.319 = \$12.957$ with cash and $3 \times \$4.359 = \$13.077$ with a credit card.

Number of Gallons	Cash Cost	Credit Card Cost
1	$ 4.319	$ 4.359
2	$ 8.638	$ 8.719
3	$12.957	$13.077

Question 2: Since a cash payment of $12.957 will buy 3 gallons of gas ($12.957 \div 4.319 = 3$), how many gallons will a cash payment of $20 buy?

Carrying Out the Plan Dividing 20 by 4.319 shows that you should receive approximately 4.631 gallons of gas.

$$20 \div 4.319 \approx 4.631$$

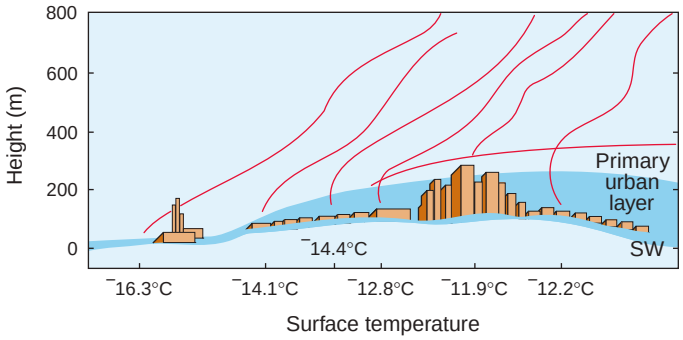
The cost of purchasing 4.631 gallons of gas with a credit card is $4.631 \times \$4.359$, which is the cost that will show on the gas pump. **Question 3:** What is this amount rounded to the nearest hundredth of a dollar?

Looking Back Another way to determine the total cost on the gas pump is to determine the extra cost for each gallon when a credit card is used ($\$4.359 - \$4.319 = \$.04$, or 4 cents) and multiply this difference by the number of gallons, 4.631. This yields an extra cost that, when added to $20, will be the dollar amount on the gas pump. **Question 4:** What is this extra cost, rounded to the nearest hundredth of a dollar (nearest cent)?

Answers to Questions 1–4 **1.** $4.359 and $4.319 **2.** Approximately 4.631 gallons **3.** $20.19 **4.** $.19 (19 cents)

*"Problems of the Month," *Mathematics Teacher* 81.

Exercises and Problems 6.2



The preceding sketch is from a study showing the influence of surface temperature on air currents. Use these Celsius temperatures in exercises 1 and 2.

1. a. What is the highest surface temperature on this graph?
b. What is the difference between the highest and lowest surface temperatures?

2. a. What is the lowest surface temperature on this graph?
b. What is the difference between the two lowest surface temperatures?

Use Decimal Squares to illustrate each of the computations in exercises 3 and 4. (Copy Blank Decimal Squares from the website or use the Virtual Decimal Squares.)

3. a. $.3 + .45 = .75$ **b.** $.350 - .2 = .15$
c. $.3 \times .4 = .12$ **d.** $10 \times .37 = 3.7$
e. $.45 \div .15 = 3$ **f.** $.30 \div 10 = .03$

4. a. $.7 + .35 = 1.05$ **b.** $.850 - .45 = .4$
c. $.1 \times .2 = .02$ **d.** $10 \times .25 = 2.5$
e. $.60 \div 10 = .06$ **f.** $.75 \div .05 = 15$

In the example shown below, a 1 is regrouped from the hundredths column to the tenths column. This can be explained by adding the fractions for 4 hundredths and 9 hundredths to get 13 hundredths (that equals 1 tenth and 3 hundredths).

$$\begin{array}{r} \frac{4}{100} \\ +\ \frac{9}{100} \\ \hline \frac{13}{100} \end{array} = \frac{10}{100} + \frac{3}{100} = \frac{1}{10} + \frac{3}{100}$$

$$\begin{array}{r} {\scriptstyle 1} \\ 58.347 \\ +\ 1.091 \\ \hline 59.438 \end{array}$$

In exercises 5 and 6, there is one column for which regrouping is needed. Mark this column and use equations to explain how the regrouping takes place.

5. a. $\begin{array}{r} 4.821 \\ +\ 61.73 \\ \hline \end{array}$ **b.** $\begin{array}{r} .046 \\ -\ .018 \\ \hline \end{array}$

6. a. $\begin{array}{r} 66.43 \\ -\ 41.72 \\ \hline \end{array}$ **b.** $\begin{array}{r} .367 \\ .015 \\ +\ .509 \\ \hline \end{array}$

Use a grid or a sketch to illustrate the products in exercises 7 and 8. Label the unit squares and parts of unit squares in the grid. (Copy the base-ten grid from the website.)

7. a. $1.7 \times 2.2 = 3.74$
b. $4.1 \times 2.7 = 11.07$

8. a. $2.5 \times 3.7 = 9.25$
b. $1.8 \times 4.6 = 8.28$

Use the pencil-and-paper algorithm in exercises 9 and 10 to compute each product or quotient. Indicate how the location of the decimal point in the product or quotient was found.

9. a. 3.2×7.8
b. $1.4146 \div .22$

10. a. $1.44 \div .3$
b. $.012 \times 9.3$

Mentally calculate each product or quotient in exercises 11 and 12. Explain your method.

11. a. $.01 \times 7.6$ **b.** $.001 \times 34$
c. $.03 \div 100$ **d.** $.04 \div 10$

12. **a.** $100 \times .65$ **b.** $.01 \times 362$
c. $.7 \div 10$ **d.** $7.2 \div 100$

Write each repeating decimal in exercises 13 and 14 as a fraction.

13. **a.** $.\overline{5}$ **b.** $.\overline{14}$ **c.** $.1\overline{5}$

14. **a.** $.\overline{217}$ **b.** $.4\overline{19}$ **c.** $.\overline{7}$

Find a decimal that is between each pair of decimals in exercises 15 and 16.

15. **a.** .6 and .7 **b.** .005 and .006

16. **a.** 5.16 and 5.17 **b.** 13.99 and 14

Mentally calculate each answer in exercises 17 and 18. Explain your method.

17. **a.** $.337 - .294$
b. $4.3 + .8$
c. 2.6×101
d. $12.9 \div 300$

18. **a.** $9 \times .6$
b. $.81 - .35$
c. $3.2 \div 25$
d. $17.3 + 8.9$

Mentally calculate each product in exercises 19 and 20 by replacing a decimal by an equivalent fraction. Show your replacement.

19. **a.** $.25 \times 48$
b. $.5 \times 40.8$
c. $5.5 \times .2$

20. **a.** $8 \times .125$
b. $.6 \times 555$
c. $.75 \times 40$

Obtain a front-end estimation for each sum or difference in exercises 21 and 22. Then obtain a second estimation by rounding to the leading digit.

21. **a.**
$$\begin{array}{r} \$26.31 \\ 47.66 \\ 21.18 \\ +\ 14.92 \\ \hline \end{array}$$

b.
$$\begin{array}{r} \$471.32 \\ -\ 113.81 \\ \hline \end{array}$$

22. **a.**
$$\begin{array}{r} \$346.32 \\ 260.40 \\ 118.63 \\ +\ 752.01 \\ \hline \end{array}$$

b.
$$\begin{array}{r} \$58.14 \\ -\ 16.71 \\ \hline \end{array}$$

Estimate each product in exercises 23 and 24 by rounding the decimals to the nearest whole number. Sketch a rectangular grid that illustrates the actual product. Then sketch and shade the regions that represent the increase due to rounding up and the decrease due to rounding down. Use these regions to predict whether the estimated product is less than or greater than the actual product. (Copy the base-ten grid from the website.)

23. **a.** 3.1×4.9 **b.** 5.3×1.6

24. **a.** 3.4×5.8 **b.** 6.5×2.1

Estimate each computation in exercises 25 and 26 by replacing a decimal by a compatible decimal or fraction.

25. **a.** $8 \div .48$
b. $11.63 + .4$
c. $.34 \times 120$
d. $.23 \times 81.6$

26. **a.** $.19 \times 80$
b. $.34 - .101$
c. $2 \div .49$
d. $6.85 + 10.17$

The following decimal estimation exercise was given to 13-year-olds as a part of the National Assessment of Educational Progress.* The numbers at the right in the table shown below indicate the percentages of students who selected each response on the left. Use this table in exercises 27 and 28.

Estimate the Answer to 3.04×5.3	Percentages
1.6	28
16	21
160	18
1600	23
I don't know	9

27. **a.** What percentage of the students selected either the incorrect response or "I don't know"?
b. What misunderstandings about decimal concepts might have caused students to select 1600?

28. **a.** What percentage of the students selected the correct response?
b. What misunderstandings of decimal concepts might have caused students to select 1.6?

Many types of errors can occur in computation, even when a student knows the basic operations with single-digit numbers. Determine which common types of errors were committed in exercises 29 and 30.

29. **a.**
$$\begin{array}{r} .4 \\ +.8 \\ \hline .12 \end{array}$$

b.
$$\begin{array}{r} 99.4 \\ -\ 27.86 \\ \hline 71.66 \end{array}$$

c.
$$\begin{array}{r} 21.8 \\ \times\ \ .4 \\ \hline 87.2 \end{array}$$

d.
$$\begin{array}{r} 9.62 \\ 4\overline{)38.6} \\ 36 \\ \hline 26 \\ 24 \\ \hline 2 \end{array}$$

*M. M. Lindquist, T. P. Carpenter, E. A. Silver, and W. Matthews, "The Third National Mathematics Assessment: Results and Implications for Elementary and Middle Schools," *The Arithmetic Teacher*.

30. a.
$$\begin{array}{r} 2.34 \\ \times\ .75 \\ \hline 1170 \\ 1638 \\ \hline 175.50 \end{array}$$

b.
$$\begin{array}{r} 2.7 \\ +\ .8 \\ \hline 2.15 \end{array}$$

c.
$$\begin{array}{r} 30.08 \\ -\ 7.32 \\ \hline 22.36 \end{array}$$

d.
$$\begin{array}{r} .62 \\ 7\overline{)4.214} \\ \underline{42} \\ 14 \\ \underline{14} \end{array}$$

Enter 273.5186 into a calculator. What single addition or subtraction can be performed on a calculator to change this number to each number in exercises 31 and 32?

31. a. 273.5193
b. 203.5180
c. 273.0156

32. a. 273.5086
b. 273.5196
c. 273.5786

Every decimal except 0 has a reciprocal. The product of a decimal and its reciprocal is 1. The reciprocal of 2.318 is $\frac{1}{2.318}$, which in decimal form to seven places is .4314064. Use a calculator to compute the reciprocals of the decimals in exercises 33 and 34, and multiply each number by its reciprocal. Record each reciprocal by rounding it to five decimal places.

33. a. 2.4 **b.** .48 **c.** .0046

34. a. 46.3 **b.** .087 **c.** 3.41

Assume in exercises 35 and 36 that a calculator is being used that has a constant function that repeatedly carries out the operation in step 2 by pressing [=]. Determine the numbers for the view screens in steps 4 and 5.

35. a.

Keystrokes	View Screen
1. 14.06	14.06
2. [×] ⁻1.6	–1.6
3. [=]	–22.496
4. [=]	
5. [=]	

b.

Keystrokes	View Screen
1. 472.8	472.8
2. [−] 16.3	16.3
3. [=]	456.5
4. [=]	
5. [=]	

36. a.

Keystrokes	View Screen
1. 192.38	192.38
2. [+] ⁻6.55	–6.55
3. [=]	185.83
4. [=]	
5. [=]	

b.

Keystrokes	View Screen
1. 160	160
2. [÷] 2.5	2.5
3. [=]	64
4. [=]	
5. [=]	

Determine whether each sequence in exercises 37 and 38 is arithmetic or geometric. Then find each common difference or common ratio, and write the next number for each sequence.

37. a. 4.2, 7.56, 13.608, 24.4944
b. 16.3, 15.6, 14.9, 14.2

38. a. 3.22, 8.82, 14.42, 20.02
b. 44.3, 13.29, 3.987, 1.1961

Reasoning and Problem Solving

39. Michelle pays a FICA/OASDI (Federal Insurance Contributions Act/Old Age Survivors and Disability Insurance) tax of .062 and a FICA/Medicare tax of .0145 on her annual salary of $61,150. What is the total amount she pays for these two taxes?

40. Brian's annual salary is $25,600 before any deductions. After setting aside some of his salary in a tax-deferred retirement plan and subtracting certain deductions, he pays taxes on an adjusted gross salary of $18,400.

a. Find his federal tax by multiplying .15 and $18,400.

b. Find his state tax by multiplying .029 by $18,400.

c. Find his FICA tax by multiplying .0765 by $18,400.

d. Subtract the sum of the taxes in parts a to c from $18,400 to determine Brian's "after-tax" amount.

41. Credit card companies have different policies and rates. One company's policies are as follows.

a. The monthly finance charge on the amount due is determined by the following rule: .0125 times the first $500; .0095 times the next $500; and .0083 times the amount over $1000. What is the finance charge on $1200?

b. The late charges are .05 times the amount past due. Compute the late charge on $75.25, rounding the answer to the nearest hundredth of a dollar.

Use the following table to answer exercises 42 and 43. This table shows the annual percentage changes in energy consumption for a large corporation over an 11-year period, where positive numbers represent increases and negative numbers represent decreases.

Annual Percentage Change in Energy Consumption

Year	Coal	Natural Gas	Petroleum
2001	4.7	4.6	4.1
2002	.4	4.5	.0
2003	.9	$^{-}$.4	$^{-}$1.9
2004	$^{-}$1.7	1.6	$^{-}$2.1
2005	.5	2.6	2.1
2006	2.9	3.4	.9
2007	.6	2.4	2.6
2008	.4	4.0	$^{-}$.3
2009	6.8	1.7	3.1
2010	2.9	.1	1.3
2011	.9	$^{-}$3.1	.5

42. **a.** Find a year in which the consumption of both coal and petroleum decreased. What happened to the demand for natural gas during this year?

b. Find 7 consecutive years in which there was an increase in the demand for natural gas. What is the total of these increases?

c. Find 2 consecutive years during which the demand for petroleum decreased. What is the total of these decreases?

43. **a.** Over the 11-year period covered by the table in the left column there was 1 year in which the energy consumption of petroleum and coal decreased and the consumption of natural gas increased. What year was this?

b. Find 7 consecutive years in which there was an increase in the demand for coal. What is the total of these increases?

c. In which year did the consumption of natural gas decrease the most and what happened to the consumption of coal and petroleum during this time?

The basic unit for measuring electricity is the kilowatt-hour. This is the amount of electric energy required to operate a 1000-watt appliance for 1 hour. For example, it takes 1 kilowatt-hour of electricity to light ten 100-watt bulbs for 1 hour. The table shown below lists the average number of kilowatt-hours required to operate each appliance for 1 month. If it costs $.12 for each kilowatt-hour, use this information to answer the questions in exercises 44 and 45.

Appliance	Kilowatt-Hours per Month	Cost per Month
Microwave oven	15.8	_____
Range with oven	97.6	_____
Refrigerator	134.7	_____
Tankless water heater	67.7	_____
Water heater	400.0	_____
Laptop computer	3.5	_____
Computer with wide-screen monitor	4.6	_____
Television	55.0	_____

44. **a.** What is the sum of the kilowatt-hours required for operating these eight appliances for 1 month?

b. What is the monthly cost of operating each of these appliances? Round your answers to the nearest penny.

c. Compute the sum of the monthly costs in part b for the eight appliances.

45. 

a. Compute the difference in the monthly costs of electricity for a tankless water heater and a (regular) water heater.

b. How much could a person save in 1 year by operating a laptop computer rather than a computer with a wide-screen monitor?

c. How much more expensive would it be to operate a range with an oven for 1 year than to operate a microwave oven for 1 year?

The greatest record-breaking spree in Olympic swimming competition occurred when world records were set in 22 out of 26 events. In one of these events, an East German, Petra Thumer, set a world record in the 400-meter freestyle, winning in 4:09.89 (4 minutes 9.89 seconds). Use this information in exercises 46 and 47.

46. **a.** Thumer's time was 1.87 seconds faster than the old record. What was the old record?

b. A world record in the 200-meter freestyle was set by another East German, Kornelia Ender, whose time was 1:59.26. If this rate of speed could be maintained, how long would it take to swim the 400-meter event? Compare this time with Thumer's time for the 400-meter event.

c. In the Tokyo Olympics, Don Schollander, of the United States, had a winning time of 4:12.2 in the 400-meter freestyle. How many seconds faster was Thumer's time for the 400-meter event?

47. **a.** Janet Evans of the United States set an Olympic swimming record in the 400-meter freestyle with a time of 4:03.85. How much faster was her time than Petra Thumer's Olympic time for this event?

b. In the summer Olympics in Barcelona, Spain, Dagmar Hase of Germany won the 400-meter swimming freestyle with a time of 4:07.18. How much faster was Janet Evan's time for this event?

48. 

Featured Strategy: Making a Table. During their summer vacation, Holly and Kathy traveled to Canada. Holly exchanged her U.S. money for Canadian money before leaving. For each 82 cents she received $1 in Canadian money. Kathy exchanged her money in Canada. For each U.S. dollar she received $1.20 in Canadian money. Who had the better rate of exchange?

a. Understanding the Problem. Let's answer a few easy questions to become more familiar with the problem. If Holly received $100 in Canadian money, what did it cost her in U.S. money? If Kathy exchanged $50 in U.S. money, how much did she receive in Canadian money?

b. Devising a Plan. It is tempting to conclude that Kathy had the better rate of exchange because it looks as if she "gained" 20 cents while Holly only "gained" 18 cents. Let's form a table to look at the cost of the first few dollars. Complete the next line of the table.

Holly		Kathy	
U.S.	Canada	U.S.	Canada
$ .82	$1	$1	$1.20
$1.64	$2	$2	$2.40

c. Carrying Out the Plan. Extend the table to determine how much it will cost each person in U.S. dollars to buy a gift in Canada for 6 Canadian dollars. Who had the better rate of exchange, Holly or Kathy?

d. Looking Back. The results in the table can be used to answer questions involving larger amounts of money. For example, how much less in U.S. money would it cost Holly than Kathy to buy an item for 48 Canadian dollars?

49. 

Misplacing a decimal point can result in a costly mistake, as described in this newspaper article.

Subtraction error to cost Rochester schools $3,000

By MARK C. BUDRIS
Rochester Bureau Chief

ROCHESTER - An arithmetic error may end up costing the School Department almost $3,000 next year.

School Board Chairman Roland Roberge said Thursday night the subtraction error during a comparison of milk bids led the board to accept a bid it thought was only $298 higher than a second. It was actually $2,986 higher.

"Well, the decimal point was put in the wrong place," Roberge told the board.

He said the error was in turning a half-cent difference in milk prices into a .05-cent difference, which was then multiplied out over the more than 650,000 cartons of milk used in a year by the School Department.

a. This article says that a .05-cent difference was used rather than a half-cent difference. What is the decimal for one-half cent?

b. If a bid for 650,000 cartons is .05 cent per carton higher than another bid, how many dollars greater is it?

c. If a bid for 650,000 cartons of milk is .5 cent per carton higher than another bid, how many dollars greater is it?

d. How much money did the school lose by misplacing the decimal point? (*Hint:* Use the answers from parts b and c.)

50. A pharmacist filled a prescription for a patient that called for an injection of .10 milliliter of insulin before the first meal of each day. The nurse's assistant at the hospital called the pharmacy to question the .10 amount of the injection, explaining that on several preceding occasions the prescription had called for .1 milliliter of insulin.

a. Is .1 equal to .10?

b. Explain how these two decimals could have different meanings.

Teaching Questions

1. One of your students made the following decimal calculations: $1.7 + 3.8 = 4.15$ and $4.3 + 21.8 = 25.11$. What does this student seem to be doing incorrectly and how can you use models to help him understand addition with decimals?

2. Suppose your students were counting together orally by tenths—one-tenth, two-tenths, . . . , nine-tenths and ten-tenths—and at the same time writing the decimals, .1, .2, ... If some of the students incorrectly wrote .10 for ten-tenths and .11 for eleven-tenths, how would you use decimal models so they can correct their notation?

3. Write an explanation with diagrams that would convince school students that when multiplying two positive decimals, both of which are less than 1, the product will be smaller than either decimal.

Classroom Connections

1. On page 366 the two activities from the **Elementary School Text** use 10-by-10 grids to illustrate addition and subtraction of decimals. (a) Use blank Decimal Squares to sketch and compute both $.35 + .7$ and $0.8 - 0.37$. (b) Answer the "Make a Conjecture" question on the school page by "Writing a rule you can use to add or subtract decimals without using models." Explain.

2. The **Grades 3–5 Standards—Number and Operations** (see inside front cover) under *Compute fluently*. . . says that strategies should be used to estimate computations with fractions and decimals that are relevant to students. Use the strategies for estimating with decimals in this section to create some examples that would be meaningful to students.

3. Explain how the research reported in the **Research** statement on page 367 is consistent with the results reported in the table preceding exercise 27 in **Exercises and Problems** of this section. Write a few questions that you would ask 13-year-olds to help them understand which estimation in the table is most appropriate.

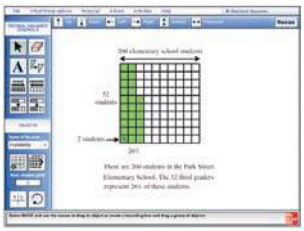
4. **Standards 2000** has several statements saying that mathematics should be a "sense making experience" for students. What is meant by this statement and how is the **Standards** quote on page 379 related to this type of experience?

5. Read the **Standards** quote on page 343 and then explain how you would help students see that decimals for numbers less than 1 are extensions of our base-ten numeration system for whole numbers.

MATH ACTIVITY 6.3

Percents with Decimal Squares

Virtual Manipulatives



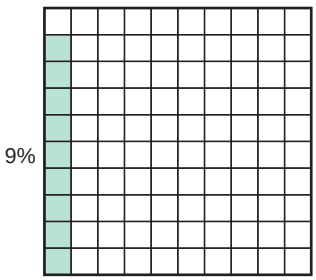
www.mhhe.com/bbn

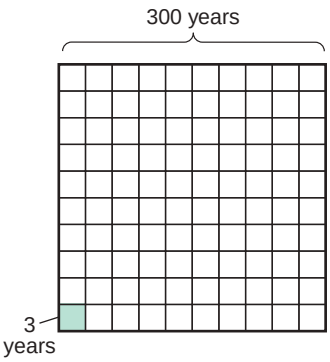
Purpose: Use 10-by-10 percent grids to visually solve percent problems.

Materials: Copies of Blank Decimal Squares from the website or use Virtual Manipulatives.

1. The 10 × 10 Decimal Square represents 1 unit, or 100 percent. By shading 9 out of 100 parts of the unit, shown below, 9 percent is represented. Percents greater than 100 are illustrated by shading more than one 10 × 10 square. Shade and label squares on the Blank Decimal Squares to illustrate the following percents.

a. 37 percent **b.** 2 percent **c.** $\frac{1}{2}$ percent **d.** 126 percent

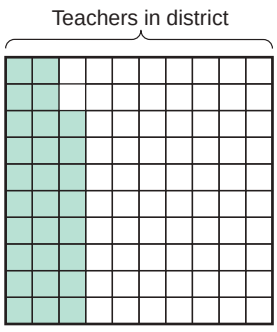




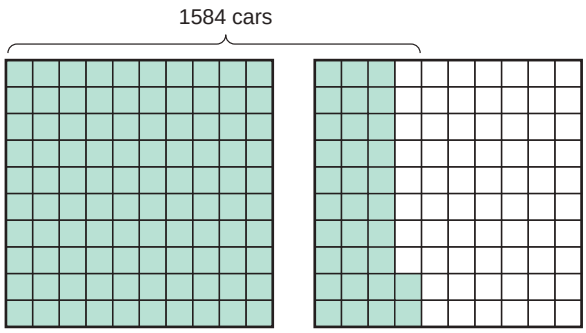
***2.** If the square at the left represents 300 years and this amount is evenly divided among the 100 smaller squares, then each small square represents 3 years. How many years are represented by 14 small squares (14 percent)? How many small squares are required to represent 24 years?

***3.** Shade a portion of a square on the Blank Decimal Squares to represent 45 percent. If a 10 × 10 square represents 180 skateboards, how many skateboards are represented by 45 small squares? Write the letter S in as many small squares as necessary to represent 27 skateboards. How many small squares should be labeled with S?

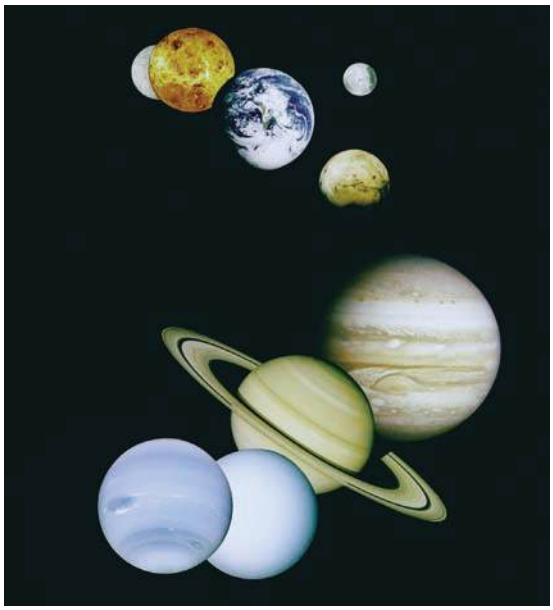
4. The 10 × 10 square at the left represents the number of teachers in a school district, and the 28 shaded squares represent 70 of the teachers. Label a square on the Blank Decimal Squares with this information. How many teachers are represented by 1 small square (1 percent)? How many teachers are in the school district?



5. One year a car dealer sold 1584 cars, which was 132 percent of the number sold the previous year. The shaded region of the squares shown below represents 1584 cars, which is 132 percent. Shade and label squares on the Blank Decimal Squares with this information, and determine the value of 1 small square (1 percent), 15 small squares (15 percent), and 100 small squares (100 percent).



Section 6.3 RATIO, PERCENT, AND SCIENTIFIC NOTATION



This is a montage of planetary images taken by spacecraft managed by the Jet Propulsion Laboratory in Pasadena, California. Included are (from top to bottom) images of Mercury, Venus, Earth (and moon), Mars, Jupiter, Saturn, Uranus, and Neptune. The inner planets (the first four) are roughly to scale to each other, and the outer planets (Jupiter, Saturn, Uranus, and Neptune) are roughly to scale to each other.

PROBLEM OPENER

A jar containing 140 marbles weighs 20 ounces, and the same jar containing only 100 marbles weighs 16 ounces. What is the weight of the jar?

One method of measuring large distances in our solar system is to compare each distance to the distance from Earth to the Sun. The distance from Earth to the Sun is called an **astronomical unit.** The distance from Jupiter to the Sun is 5.2 astronomical units, which means that Jupiter's distance from the Sun is 5.2 times Earth's distance from the Sun. Measuring with astronomical units involves the idea of *ratios,* which is introduced in this section.

HISTORICAL HIGHLIGHT



Isaac Newton, 1642–1727

England's Isaac Newton was born on Christmas in the year in which Galileo died. He was born prematurely and was so small and frail that his mother said he could have fit into a quart pot. Newton once told of how he performed his first scientific experiment as a young man. To determine the strength of the wind, he first broad-jumped with the wind and then broad-jumped against the wind. Comparing these distances with the extent of his broad jump on a calm day, he obtained the strength of the wind, expressed as so many feet strong. Newton is ranked by many as the greatest mathematician the world has produced. His *Principia,* which contains his laws of motion and describes the motions of the planets, is regarded as the greatest scientific work of all time. There are many testimonials to Newton's accomplishments, including the following lines by Alexander Pope:

Nature and Nature's laws lay hid in night:
God said, "Let Newton be," and all was light.*

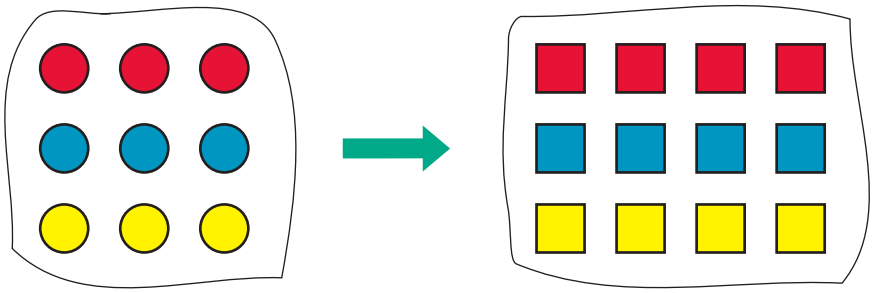
*H. W. Eves, *In Mathematical Circles* (Boston: Prindle, Weber, and Schmidt, 1969), 7–11.

NCTM Standards

In the middle grades students should encounter problems involving ratios (e.g., 3 adult chaperons for every 8 students) and rates (e.g., scoring a soccer goal on 3 of every 8 penalty kicks). p. 216

RATIOS

The **ratio** concept of a fraction was introduced in Section 5.2. A **ratio** is a pair of positive numbers that is used to compare two sets. Figure 6.28 shows that for every 3 chips there are 4 tiles. This ratio is written as 3:4 (read "3 to 4") or as the fraction $\frac{3}{4}$.



Ratio of 3 to 4

Figure 6.28

A ratio gives the relative sizes of two sets but not the actual numbers of objects in those sets. For example, the fact that the ratio of boys to girls in a certain classroom is 1 to 3 tells us that for every boy there are 3 girls, or that the number of boys is one-third the number of girls, but it does not tell us the number of boys or girls.

Ratio For any two positive numbers a and b, the **ratio** of a to b is the fraction $\frac{a}{b}$. This ratio is also written as $a:b$.

EXAMPLE A

In recent years, for every female inmate in the United States, there were 6 male inmates.*

1. What is the ratio of the number of male inmates to the number of female inmates?
2. What is the ratio of the number of female inmates to the number of male inmates?

Solution 1. 6 to 1 or $\frac{6}{1}$. 2. 1 to 6 or $\frac{1}{6}$.

Research Statement

There is ample evidence that students experience difficulty solving problems involving fractions and proportions, even simple ones.

Behr et al. Noelting; Vergnaud

PROPORTIONS

Comparing the relative sizes of large sets through the use of small numbers is a common use of ratios. In Figure 6.29 on the next page there are 8 teeth on the small gear and 40 teeth on the large gear. This is a ratio of 8 to 40 and, since $\frac{8}{40} = \frac{1}{5}$ in lowest terms, the ratio of teeth on the small gear to teeth on the large gear is 1 to 5 (1:5).

**Statistical Abstract of the United States,* 128th ed. (Washington, DC: Bureau of the Census, 2009), p. 208.

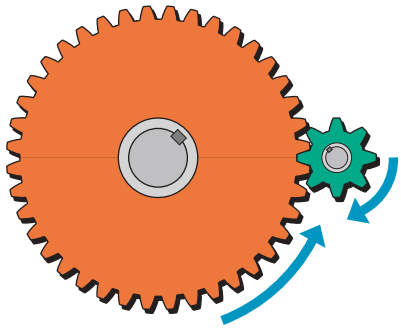


Figure 6.29

An equality of ratios is called a **proportion.** Each ratio gives rise to many pairs of equal ratios. For example, in a one-year period the ratio of trucks to cars involved in fatal crashes was 2 to 21. This means that for every 2 trucks there were 21 cars, for every 4 trucks there were 42 cars, etc., as shown in the following table. These are all equal ratios.

NCTM Standards

Proportionality is an important integrative thread that connects many of the mathematics topics studied in grades 6–8. p. 217

Truck Accidents	Car Accidents
2	21
4	42
6	63
8	84
.	.
.	.
.	.

Proportion For any two ratios $\frac{a}{b}$ and $\frac{c}{d}$, the equality

$$\frac{a}{b} = \frac{c}{d}$$

is called a **proportion.**

Proportions are useful in problem solving. Typically, three of the four numbers in a proportion are given and the fourth is to be found.

EXAMPLE B If the ratio of teachers to students in a school is 1 to 18 and there are 360 students, how many teachers are there?

Solution One method for obtaining the solution is to form a table showing equal ratios and continue this list until you reach 360 students.

Number of Teachers	Number of Students
1	18
2	36
3	54
4	72
.	.
.	.
.	.

Another method is to write a proportion as two equal fractions, with x representing the number of teachers.

$$\frac{1}{18} = \frac{x}{360}$$

We know that the numerator and denominator of $\frac{1}{18}$ must be multiplied by the same number to result in an equal fraction. Since the denominator of $\frac{1}{18}$ must be multiplied by 20 to get 360 ($360 \div 18 = 20$),

$$\frac{1 \times 20}{18 \times 20} = \frac{20}{360}$$

So the number of teachers is 20.

Historically, the rule of proportions was so valuable to merchants that it was called the *golden rule*. Often we know the price of some quantity of a given item and want to determine the price of a different amount as in the next example.

EXAMPLE C If 4.8 pounds of flour costs \$8.40, how much will 6 pounds cost?

Solution Using the ratio of pounds to cost produces the following proportion, with x representing the cost for 6 pounds.

$$\frac{4.8}{8.40} = \frac{6}{x}$$

By the rule for equality of fractions, we obtain

$$4.8 \times x = 8.40 \times 6$$
$$x = \frac{8.40 \times 6}{4.8}$$
$$x = 10.50$$

Thus, the cost of 6 pounds is \$10.50.

PERCENT

The word **percent** comes from the Latin *per centum,* meaning *out of 100*. Percent was first used in the fifteenth century for computing interest, profits, and losses. Currently it has much broader applications, as illustrated by the news clippings in Figure 6.30 on the next page.

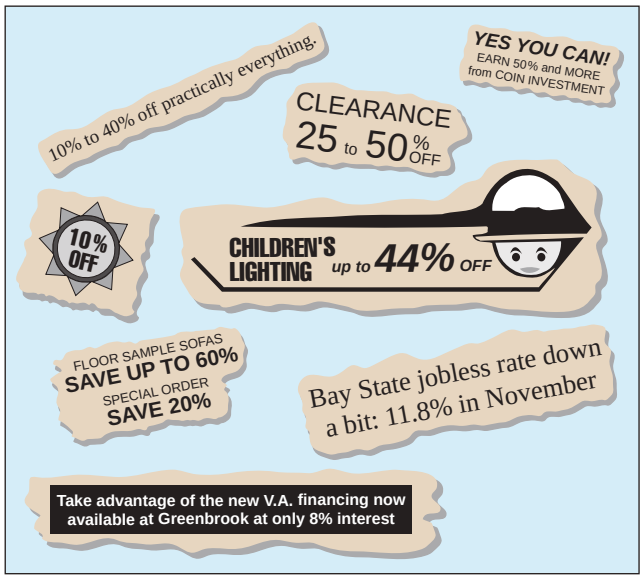


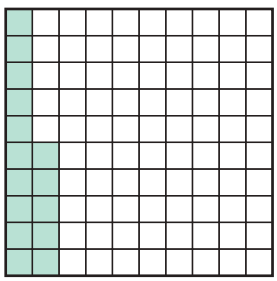
Figure 6.30

Research Statement

One reason percents may be difficult is concise linguistic form which leads students to manipulate numbers based on learned procedures rather than on underlying relationships.

Parker and Leinhardt

Percents are ways of representing fractions with denominators of 100. For example, 15 percent means $\frac{15}{100}$ and is written as 15%. Diagrams are one method of gaining an understanding of percents. A 10×10 grid with 100 equal parts is a common model in elementary school texts for illustrating percents (see Figure 6.31). Decimal Squares for tenths, hundredths, and thousandths will be used in this section to describe percents.



15%

Figure 6.31

EXAMPLE D

Describe a Decimal Square to represent each percent.

1. 90% **2.** 9% **3.** 35.5%

Solution **1.** 90 parts shaded out of 100, or 9 parts shaded out of 10. **2.** 9 parts shaded out of 100. **3.** 35.5 parts shaded out of 100, or 355 parts shaded out of 1000.

Notice the similarity between the percent symbol % and the numeral 100. This is helpful in remembering how to replace a percent by a fraction or a decimal. First, drop the percent symbol and write the percent as a fraction with a denominator of 100. Then, to obtain a decimal, divide the numerator by 100.

EXAMPLE E

Write each percent as a decimal. Then describe a Decimal Square that represents or approximately represents the decimal.

1. 42% **2.** 6.8% **3.** $21\frac{3}{4}\%$ **4.** 100%

Solution **1.** $42\% = \frac{42}{100} = .42$ (42 parts shaded out of 100). **2.** $6.8\% = \frac{6.8}{100} = .068$ (between 6 and 7 parts shaded out of 100, or 68 parts shaded out of 1000). **3.** $21\frac{3}{4}\% = \frac{21\frac{3}{4}}{100} = \frac{21.75}{100} = .2175$ (between 21 and 22 parts shaded out of 100, or between 217 and 218 parts shaded out of 1000). **4.** $100\% = \frac{100}{100} = 1$ (100 parts shaded out of 100).

Example E suggests a method for writing a percent as a decimal: *drop the percent symbol and divide by 100.* To write a decimal as a percent, reverse the process: Write the decimal first as a fraction with a denominator of 100 and then as a percent.

EXAMPLE F

Describe the Decimal Square for each decimal, and then write the decimal as a percent.

1. .07 **2.** .647 **3.** 3.25 **4.** .008

Solution **1.** 7 parts shaded out of 100: $.07 = \frac{7}{100} = 7\%$.

2. 647 parts shaded out of 1000: $.647 = \frac{647}{1000} = \frac{64.7}{100} = 64.7\%$.

3. 3 whole squares and 25 parts shaded out of 100: $3.25 = \frac{325}{100} = 325\%$.

4. 8 parts shaded out of 1000: $.008 = \frac{8}{1000} = \frac{.8}{100} = .8\%$.

To write a fraction as a percent, first write it as a decimal and then write the decimal as a fraction with a denominator of 100. For example, to write $\frac{1}{6}$ as a percent,

$$\frac{1}{6} = .1\overline{6} = \frac{16.\overline{6}}{100} = 16.\overline{6}\% \quad \text{or} \quad 16\frac{2}{3}\%$$

EXAMPLE G

Write each fraction as a percent.

1. $\frac{1}{5}$ **2.** $\frac{1}{8}$ **3.** $\frac{1}{3}$

Solution **1.** $\frac{1}{5} = .20 = \frac{20}{100} = 20\%$. **2.** $\frac{1}{8} = .125 = \frac{12.5}{100} = 12.5\%$.

3. $\frac{1}{3} = .33\overline{3} = \frac{33.\overline{3}}{100} = 33.\overline{3}\%$ or $33\frac{1}{3}\%$.

NCTM Standards

The importance of recognizing equivalent forms of numbers is noted in the *Curriculum and Evaluation Standards for School Mathematics* (p. 87):

> . . . the development of concepts for fractions, ratios, decimals, and percents and the ideas of multiple representations of these numbers need special attention and emphasis. The ability to generate, read, use, and appreciate multiple representations of the same quantity is a critical step in learning to understand and do mathematics.

CALCULATIONS WITH PERCENTS

Calculations with percents fall into three categories:

1. Given the *whole* and the *percent,* find the *part.*
2. Given the *whole* and the *part,* find the *percent.*
3. Given the *percent* and the *part,* find the *whole.*

Whole and Percent When the whole and the percent are given, the part can be found by multiplying the percent times the whole. For example, suppose 12 percent of the 250 teachers in a school have master's degrees. The word *of* is a clue that the percent is acting as a multiplier and that the number of teachers with master's degrees is found by multiplying $.12 \times 250$.

$$12\% \text{ of } 250 = .12 \times 250 = 30$$

Figure 6.32 illustrates the information in the preceding problem. The Decimal Square is 12 percent shaded, and the whole square represents 250 teachers. This suggests another way to solve the problem. If the whole square represents 250 teachers, then each small hundredths square represents $\frac{250}{100} = 2.5$ teachers. So the number of teachers represented by 12 small squares is $12 \times 2.5 = 30$.

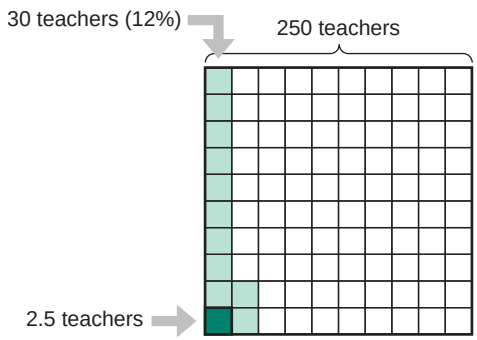


Figure 6.32

EXAMPLE H

A survey of football players revealed that 20 percent of 1180 players had knee injuries. How many players had knee injuries?

Solution $20\% \text{ of } 1180 = .20 \times 1180 = 236$. So, 236 players had knee injuries.

Research Statement

Several research studies have found that students have trouble solving problems involving percents.

Kouba et al.; Kouba, Zawojewski, and Strutchens; Risacher

Part and Whole When the part and the whole are given, the percent can be found by writing the fraction for the part of the whole and then writing this fraction as a percent. For example, if 8 of a radio station's top 40 songs for a given week are new songs, then $\frac{8}{40}$ of the songs are new. To represent $\frac{8}{40}$ as a percent, we can divide 8 by 40 to obtain a decimal and then replace the decimal by a percent.

$$8 \div 40 = .2 \qquad \text{and} \qquad .2 = \frac{20}{100} = 20\%$$

Figure 6.33 provides a visual approach to solving the preceding problem: 8 is what percent of 40? If we let the whole square represent 40, then since the square has 100 equal parts, each part represents $40 \div 100 = .4$. Thus, 2 small squares represent .8, 10 small squares represent 4, and 20 small squares represent 8. Since 20 squares out of 100 is 20 percent, 8 is 20 percent of 40.

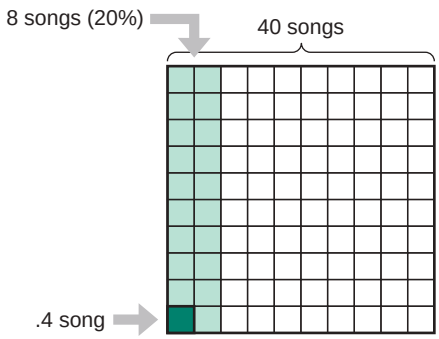


Figure 6.33

In some cases it is necessary to compare one number to a smaller one. For example, since 90 is 2 times 45, it is 200 percent of 45.

$$\frac{90}{45} = 2 = \frac{200}{100} = 200\%$$

EXAMPLE I

Determine the following percents.

1. 120 is what percent of 80?
2. 33 is what percent of 11?
3. 60 is what percent of 50?

Solution **1.** 120 is 150 percent of 80: $\frac{120}{80} = 1.5 = \frac{150}{100} = 150\%$. **2.** 33 is 300 percent of 11: $\frac{33}{11} = 3 = \frac{300}{100} = 300\%$. **3.** 60 is 120 percent of 50: $\frac{60}{50} = 1.2 = \frac{120}{100} = 120\%$.

EXAMPLE J

1. If \$880 of a \$2000 loan has been paid off, what percent of the loan has been paid off?
2. If a company's profits were 1.4 billion dollars in the year 2010 and 1.8 billion dollars in 2011, the 2011 profits were what percent of the 2010 profits?

Solution **1.** The fraction of the loan that has been paid off is $\frac{880}{2000}$, which equals .44. So, 44 percent of the loan has been paid off. **2.** 1.8 is approximately 1.29 times 1.4: $\frac{1.8}{1.4} \approx 1.29 = \frac{129}{100} = 129\%$.

NCTM Standards

As with fractions and decimals, conceptual difficulties [with percents] need to be carefully addressed in instruction. In particular, percents less than 1 percent and greater than 100 percent are often challenging . . . p. 217

Percent and Part When the percent and the part are given, the whole can be found by using a proportion. Suppose a down payment of \$14,400 (the part) is required for a home loan and this down payment is 18 percent (the part) of the loan. Then the amount of the loan (the whole) is the denominator x in the following proportion.

$$\frac{\text{Part}}{\text{Whole}} \quad \frac{18}{100} = \frac{14{,}400}{x}$$

Using the rule for equality of fractions, we can write this equation as

$$18 \times x = 14{,}400 \times 100$$
$$x = \frac{1{,}440{,}000}{18}$$
$$x = 80{,}000$$

So the amount of the loan is \$80,000.

Another method of solution for the preceding problem is illustrated in Figure 6.34. Here, 18 percent of the Decimal Square is shaded, and the shaded amount represents the \$14,400 down payment. If 18 of the small hundredths squares represent \$14,400, then each small square represents \$14,400 ÷ 18 = \$800. So the total of 100 squares represents 100 × \$800 = \$80,000.

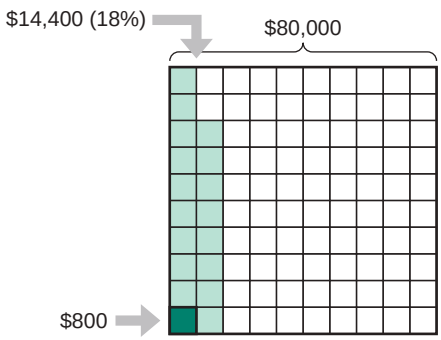


Figure 6.34

EXAMPLE K

At the turn of the twentieth century, Nebraska had 112 one-teacher schoolhouses. To the nearest percent, this number was 23 percent of the total number of one-teacher schoolhouses in the United States. How many one-teacher schoolhouses were there in the United States at this time?

Solution The ratio of 23 percent $\left(\frac{23}{100}\right)$ is equal to the ratio of 112 to the total number of one-teacher schoolhouses. If x equals the number of one-teacher schoolhouses, then

$$\frac{23}{100} = \frac{112}{x}$$

Using the rule for equality of fractions, we can rewrite the above equation as

$$23 \times x = 112 \times 100$$
$$x = \frac{11{,}200}{23}$$
$$x \approx 487.0$$

So, there were 487 one-teacher schoolhouses in the United States at the turn of the century.

Technology Connection

Palindromic Decimals

Notice that 6.7 + 7.6 = 14.3 and 14.3 + 3.41 = 17.71, a palindromic number. Try some other decimals to see if they will eventually go to palindromic numbers using this process of reversing and adding digits. The online 6.3 Mathematics Investigation will help you explore this and similar questions.

Mathematics Investigation
Chapter 6, Section 3
www.mhhe.com/bbn

The same procedure can be used for setting up a proportion when the percent is greater than 100. Suppose we know that after a physical exertion test a person's pulse rate is 144 beats per minute, and this is 180 percent of the person's resting pulse rate. The ratio of 180 percent $\left(\frac{180}{100}\right)$ is equal to the ratio of 144 to the resting pulse rate. If x equals the person's resting pulse rate, then,

$$\frac{180}{100} = \frac{144}{x}$$

Using the rule for equality of fractions, we find that

$$\begin{aligned} 180 \times x &= 144 \times 100 \\ x &= \frac{14{,}400}{180} \\ x &= 80 \end{aligned}$$

So the person's resting pulse rate is 80 beats per minute.

EXAMPLE L The school population for the new year in a certain town is 135 percent of the school population for the previous year. If the new population is 378, how many students did the school have the previous year?

Solution The ratio of 135 percent $\left(\frac{135}{100}\right)$ is equal to the ratio of the new population to the previous year's population. If x equals the previous year's population, then

$$\begin{aligned} \frac{135}{100} &= \frac{378}{x} \\ 135 \times x &= 378 \times 100 \\ x &= \frac{37{,}800}{135} \\ x &= 280 \end{aligned}$$

Thus, there were 280 students the previous year.



Technology Connection

Discounts A common occurrence of percents is found in computing discounts and sales taxes. In the case of discounts we want to subtract a certain percent of the original cost. For example, what is the cost of an object that is listed for \$36.40 with a 15 percent discount? Some calculators with percent keys are designed to compute this cost by using the keystrokes shown below. When 15 percent is entered in step 3, the view screen shows the amount to be subtracted from \$36.40. Pressing [=] in step 4 shows the discounted price. You may wish to try these keystrokes if you have a calculator with a percent key.

Keystrokes	View Screen
1. Enter 36.40	36.40
2. [−]	36.40
3. Enter 15 [%]	5.46
4. [=]	30.94

Sales Tax The cost of an object plus the sales tax is computed in a similar way with the minus sign in step 2 of the preceding example of a discount replaced by a plus sign. The cost of a \$30.94 item plus 6 percent sales tax is obtained by the following steps, where step 3 shows the amount of the added sales tax.

Keystrokes	View Screen
1. Enter 30.94	30.94
2. [+]	30.94
3. Enter 6 [%]	1.8564
4. [=]	32.7964

Calculators with a percent key that add a given percentage, as in the preceding example, make it easy to compute interest on saving accounts, credit cards, etc. Suppose that you owe $500 on a credit card that charges 1 percent interest compounded monthly. This means that at the end of each month you owe 1 percent of the unpaid balance. After the first month you will have a debt of $500 plus 1 percent of $500, which according to step 2 in the following example is $505. Step 3 shows the amount owed at the end of the second month, and each succeeding step shows the amount owed at the end of each month, if no payments or further charges are made to the account. This is an example of **compound interest** because interest is charged on unpaid interest.

Keystrokes	View Screen	
1. Enter 500	500.	
2. [+] 1 [%] [=]	505.	(end of first month)
3. [+] 1 [%] [=]	510.05	(end of second month)
4. [+] 1 [%] [=]	515.1505	(end of third month)

Notice that the balance of $505 at the end of the first month, which was obtained from $500 + (.01 \times 500)$, also can be computed by multiplying 1.01×500, as shown by the distributive property.

$$\begin{aligned} 500 + (.01 \times 500) &= (1 \times 500) + (.01 \times 500) \\ &= (1 + .01) \times 500 \\ &= 1.01 \times 500 \end{aligned}$$

This gives us another approach to determine each balance in the preceding example: Simply multiply each monthly balance by 1.01 (100% + 1%). This method can be used on any calculator. The following keystrokes produce the credit card balance at the end of 3 months.

500 [×] 1.01 [×] 1.01 [×] 1.01 [=] 515.1505

Another expression for the left side of the preceding equation is $500 \times (1.01)^3$. Using a calculator key for raising numbers to powers, such as $[y^x]$ or [^], the credit card balance after 3 months is

500 [×] 1.01 [^] 3 [=] 515.1505

NCTM Standards

Mental computation and estimation are also useful in many calculations involving percents. Because these methods often require flexibility in moving from one representation to another, they are useful in deepening students' understanding of rational numbers. p. 220

MENTAL CALCULATIONS WITH PERCENTS

The frequent occurrence of percents in everyday life has led people to adopt certain techniques for mental calculations. Two of these—*compatible numbers* and *substitutions*—are introduced here.

Compatible Numbers for Mental Calculations Certain percents are convenient for calculations. One of these is 10 percent, because multiplying by .10 is just a matter of moving a decimal point. For example, 10 percent of $16.50 = .10 \times 16.50 = 1.65$. Once we know 10 percent of a number, we can use that amount to determine other percents such as 5, 15, 20, 25, and 40 percent. The relationship of these percents to 10 percent is illustrated in Figure 6.35 on the next page.

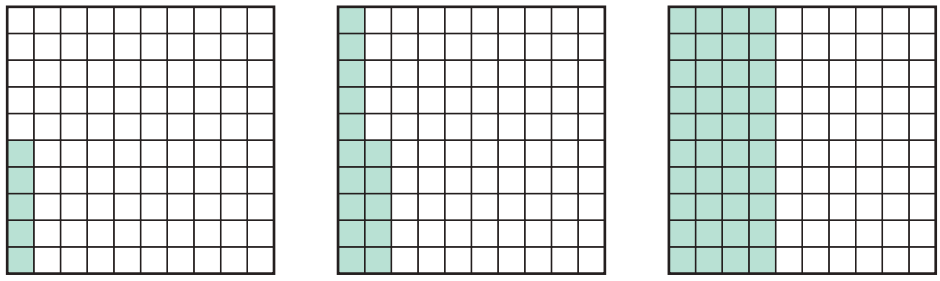


Figure 6.35

EXAMPLE M

A store is having a sale, and prices are being discounted 15, 20, and 25 percent. Calculate the amount of each discount mentally.

1. 15 percent of \$82 **2.** 20 percent of \$31.40 **3.** 25 percent of \$30

Solution **1.** Since 10% of \$82 = \$8.20 and 5% is one-half as much, 15% of \$82 = \$8.20 + \$4.10 = \$12.30. **2.** Since 10% of \$31.40 = \$3.14 and 20% is twice as much, 20% of \$31.40 = \$6.28. **3.** Since 10% of \$30 = \$3 and 25% = 10% + 10% + 5%, 25% of \$30 = \$3 + \$3 + \$1.50 = \$7.50.

NCTM Standards

By middle grades children should understand that numbers can be represented in various ways, so that they can see that $\frac{1}{4}$, 25%, and 0.25 are all different names for the same number. p. 33

For some computations it is convenient to replace a percent by a fraction. A few percents and their fractions are shown in Figure 6.36.

$$10\% = \frac{1}{10} \quad 12\frac{1}{2}\% = \frac{1}{8} \quad 20\% = \frac{1}{5} \quad 25\% = \frac{1}{4} \quad 33\frac{1}{3}\% = \frac{1}{3}$$

$$50\% = \frac{1}{2} \quad 66\frac{2}{3}\% = \frac{2}{3} \quad 75\% = \frac{3}{4} \quad 80\% = \frac{4}{5}$$

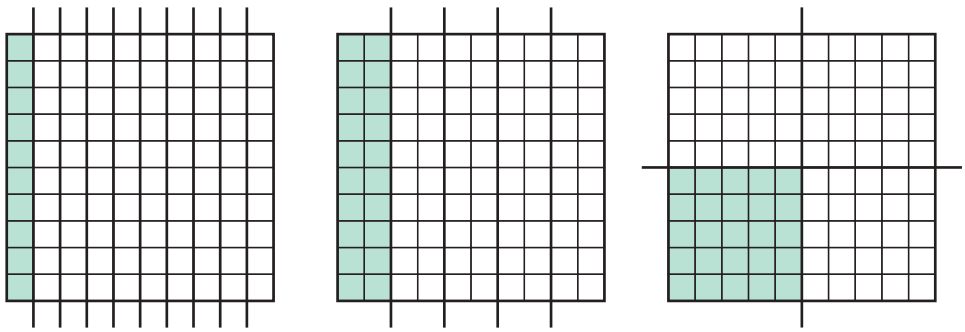


Figure 6.36

EXAMPLE N

Calculate each percent mentally by first replacing the percent by a fraction.

1. 25 percent of 88 **2.** 20 percent of 55 **3.** $33\frac{1}{3}$ percent of 45

4. 75 percent of 24

Solution **1.** 25 percent of 88 = $\frac{1}{4} \times 88 = 22$. **2.** 20 percent of 55 = $\frac{1}{5} \times 55 = 11$. **3.** $33\frac{1}{3}$ percent of 45 = $\frac{1}{3} \times 45 = 15$. **4.** 75 percent of 24 = $\frac{3}{4} \times 24 = 18$.

Substitutions for Mental Calculations The solution to a problem is sometimes easily obtained by replacing a percent by a sum or difference of two percents. For example, to find 90 percent of 140, we can use the fact that $90\% = 100\% - 10\%$.

$$90\% \text{ of } 140 = 100\% \text{ of } 140 - 10\% \text{ of } 140 = 140 - 14 = 126$$

EXAMPLE O

Calculate each percent mentally by replacing the percent by a sum or difference of two convenient percents.

1. 95 percent of 200 **2.** 110 percent of 430 **3.** 45 percent of 18

Solution **1.** 95% of 200 = 100% of 200 − 5% of 200 = 200 − 10 = 190 (*Note:* 10% of 200 = 20 so 5 percent of 200 is one-half of 20). **2.** 110% of 430 = 100% of 430 + 10% of 430 = 430 + 43 = 473. **3.** 45% of 18 = 50% of 18 − 5% of 18 = 9 − .9 = 8.1 (*Note:* 10% of 18 = 1.8 so 5% of 18 = .9).

ESTIMATION

Compatible Numbers for Estimation Sometimes it is convenient to replace a given percent by an approximation, which may be either another percent or a fraction. For example, percents such as 47, 52, and 48 percent, which are close to 50 percent, may be replaced by $\frac{1}{2}$; percents such as 34, 35, and 33 percent, which are close to $33\frac{1}{3}$ percent, are sometimes replaced by $\frac{1}{3}$; etc. At times, both numbers in a calculation are replaced by approximations to obtain compatible numbers. For example, 24 percent of \$18.75 could be replaced by $\frac{1}{4}$ of \$20, because $\frac{1}{4}$ and \$20 are compatible numbers that are approximately equal to the original numbers.

EXAMPLE P

Estimate each percent mentally by replacing one or both numbers by compatible numbers.

1. 34 percent of 62.4 **2.** 47 percent of \$87.62 **3.** 8 percent of 65

Solution Here are some possible solutions: **1.** $\frac{1}{3} \times 60 = 20$, or $\frac{1}{3} \times 63 = 21$. **2.** $\frac{1}{2}$ of \$88 = \$44, or $\frac{1}{2}$ of \$80 = \$40 (*Note:* Since 47 percent is increased to $\frac{1}{2}$, \$87.62 is decreased to \$80 for the second estimation). **3.** 10% of 65 = 6.5 or 10% of 60 = 6.

When a percent is written as a fraction, replacing the numerator and denominator by compatible numbers often provides a close estimation. For example, when answers to 47 out of 60 questions on a test are correct, $\frac{47}{60}$ is the fraction of correct answers. Here are two possibilities for expressing this fraction as a percentage:

$$\frac{47}{60} \approx \frac{48}{60} = \frac{8}{10} = 80\%$$

$$\frac{47}{60} \approx \frac{49}{63} = \frac{7}{9} = \frac{77}{99} \approx \frac{77}{100} = 77\%$$

EXAMPLE Q

Determine approximate percents by replacing the numerators and/or denominators by compatible numbers.

1. $\frac{16}{62}$ **2.** $\frac{300}{2490}$ **3.** $\frac{42}{87}$

Solution **1.** $\frac{16}{62} \approx \frac{15}{60} = \frac{1}{4} = 25\%$ or $\frac{16}{62} \approx \frac{16}{64} = \frac{2}{8} = \frac{1}{4} = 25\%$. **2.** $\frac{300}{2490} \approx \frac{300}{2500} = \frac{3}{25} = \frac{12}{100} = 12\%$. **3.** $\frac{42}{87} \approx \frac{1}{2} = 50\%$.

NCTM Standards

In the middle grades, students should . . . develop a sense of magnitude of very large numbers. For example, they should recognize and represent 2,300,000,000 as 2.3×10^9 in scientific notation and also as 2.3 billion. p. 217

SCIENTIFIC NOTATION

Very large and very small numbers can be written conveniently by using powers of 10. Consider the following example: Some computers can perform 400,000,000 calculations per second. Using a power of 10, we can write

$$400{,}000{,}000 = 4 \times 10^8$$

Decimals that are less than 1 can be written by using negative powers of 10. For example, the thickness of the average human hair, which is approximately .003 inch thick, can be written as

$$.003 = \frac{3}{1000} = \frac{3}{10^3} = 3 \times 10^{-3}$$

where $\frac{1}{10^3} = 10^{-3}$. In general, for any number $x \neq 0$ and any integer n,

$$\frac{1}{x^n} = x^{-n}$$

Any positive number can be written as the product of a number from 1 to 10 and a power of 10. For example, 2,770,000,000 can be written as 2.77×10^9. This method of writing numbers is called **scientific notation.** The number between 1 and 10 is called the **mantissa,** and the exponent of 10 is called the **characteristic.** In the preceding example, the mantissa is 2.77 and the characteristic is 9.

EXAMPLE R

The following table contains examples of numbers written in scientific notation. Fill in the two missing numbers in the table.

	Positional Numeration	Scientific Notation
Years since age of dinosaurs	150,000,000	1.5×10^8
Seconds of half-life of U-238	142,000,000,000,000,000	1.42×10^{17}
Wavelength of gamma ray (meters)	.0000000000003048	3.048×10^{-13}
Size of viruses (centimeters)	.000000914	_____
Orbital velocity of Earth (kilometers per hour)	_____	4.129×10^4

Solution $.000000914 = 9.14 \times 10^{-7}$; $4.129 \times 10^4 = 41{,}290$.

Numbers written in scientific notation are especially convenient for computing. The graph in Figure 6.37 on the next page shows increases in the world's population. It wasn't until 1825 that the population reached 1 billion (1×10^9); by 2010 it was 6.9 billion (6.9×10^9). Since

there is about 1.998×10^3 square yards of cultivated land per person, the total amount of cultivated land worldwide, in square yards, is

$$(6.9 \times 10^9) \times (1.998 \times 10^3)$$

Rearranging these numbers and using the rule for adding exponents, we can rewrite this product as

$$(6.9 \times 1.998) \times 10^{12}$$

Finally, we compute the product of the mantissas (6.9×1.998) and write the answer in scientific notation:

$$(6.9 \times 1.998) \times 10^{12} = 13.7862 \times 10^{12} = 1.37862 \times 10^{13}$$

So there is approximately 1.37862×10^{13}, or 13,786,200,000,000 square yards of cultivated land in the world. Notice in the preceding equation that 13.7862 is not between 1 and 10, so we divide by 10 to obtain the mantissa of 1.37862 and then increase the characteristic (the power of 10) from 12 to 13 to obtain an answer in scientific notation.

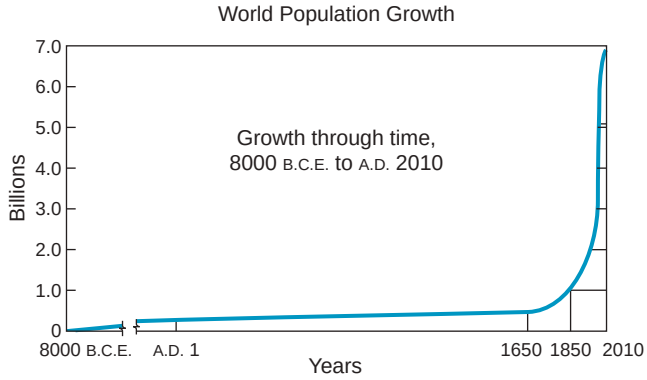


Figure 6.37

The preceding example illustrates the method of computing products of numbers in scientific notation: (1) Multiply the mantissas (numbers from 1 to 10); and (2) add the characteristics to obtain a new power of 10. Sometimes, to write an answer in scientific notation, it will be necessary to divide the mantissa by 10 and increase the characteristic by 1, as in the above example.

EXAMPLE 5

Compute each product and write the answer in scientific notation.

1. $(6.3 \times 10^4) \times (5.21 \times 10^3)$
2. $(1.55 \times 10^4) \times (8.7 \times 10^{-6})$

Solution **1.** $(6.3 \times 5.21) \times (10^4 \times 10^3) = 32.823 \times 10^7$, but since 32.823 is not between 1 and 10, a requirement for scientific notation, we replace it by 3.2823×10.

$$32.823 \times 10^7 = 3.2823 \times 10 \times 10^7 = 3.2823 \times 10^8$$

2. $(1.55 \times 8.7) \times (10^4 \times 10^{-6}) = 13.485 \times 10^{-2}$, but since 13.485 is not between 1 and 10, we replace it by 1.3485×10.

$$13.485 \times 10^{-2} = 1.3485 \times 10 \times 10^{-2} = 1.3485 \times 10^{-1}$$

Technology Connection

Calculators that operate with scientific notation will display the mantissa and the characteristic whenever a computation produces a number that is too large for the screen. The keystrokes for entering 4.87×10^{17} and the view screen are shown here.

Keystrokes	View Screen
4.87	4.87
×	4.87
10 ∧ 17	1. 17
=	4.87 17

To compute with numbers in scientific notation, the numbers and operations can be entered as they are written from left to right. The following keystrokes compute (4.87 × 10^{17}) × (9.2 × 10^{5}) and display the answer in scientific notation. Notice that parentheses are not needed as long as the calculator is programmed to follow the order of operations.

Keystrokes	View Screen
4.87 × 10 ∧ 17 × 9.2 × 10 ∧ 5 =	4.4804 23

Try the following product to see if your calculator uses scientific notation: 2,390,000 × 1,000,000. This product equals 2.39×10^{12}, where the mantissa is 2.39 and the characteristic is 12. Notice that the base of 10 does not appear in the calculator view screen of the first display in Figure 6.38, but it does in the second screen. Most calculators that display numbers in scientific notation show the mantissa and the characteristic but not the base 10.



Figure 6.38

If a number is too small for the view screen, it will be represented by a mantissa and a negative power of 10. Use your calculator to compute .0004 × .000006, which is .0000000024, or 2.4×10^{-9} in scientific notation. If this is computed on a calculator whose view screen has only eight places for digits and no scientific notation, it may show a product of 0, or an error message. The calculator in Figure 6.39 on the next page shows a mantissa of 2.4 and a characteristic of $^{-}9$ with the letter E indicating an exponent of $^{-}9$.

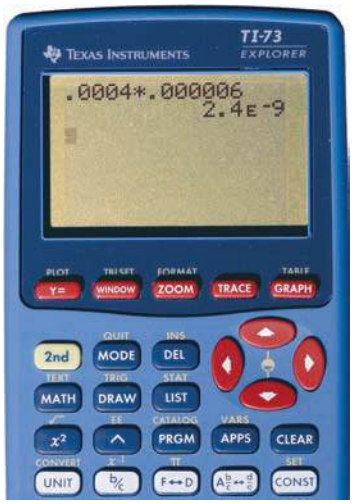


Figure 6.39

PROBLEM-SOLVING APPLICATION

Problem

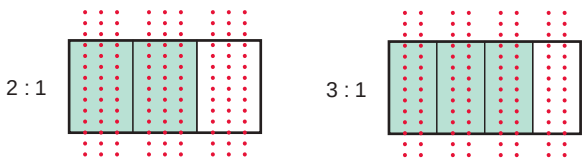
Two elementary school classes have equal numbers of students. The ratio of girls to boys is 3 to 1 in one class and 2 to 1 in the other. If the two classes are combined into one large class, what is the new ratio of girls to boys?

Understanding the Problem To obtain a better understanding of the ratios, let's select a particular number of students and compute the number of girls and boys. Suppose there are 24 students in each class. Then the class with the 3-to-1 ratio has 18 girls and 6 boys. **Question 1:** How many girls and how many boys are in a class of 24 students with the 2-to-1 ratio?

Devising a Plan One approach is to *make a drawing* representing the two classes and indicate their ratios. The following figures illustrate the girl-to-boy ratios in the two classes and show that each class is the same size. **Question 2:** Why can't we conclude from these figures that the ratio of girls to boys in the combined class is 5 to 2?



Carrying Out the Plan To obtain information from the sketches of the classes, we need to subdivide the parts so that each figure has parts of the same size. The smallest number of such parts is 12, as shown in the following figure. The combined class will have 24 equal parts. **Question 3:** What is the ratio of girls to boys in the combined class?



Looking Back In *Understanding the Problem,* we chose 24 students per class as a numerical example and from this established girl-to-boy ratios of 18 to 6 and 16 to 8. **Question 4:** Do these numbers produce the same ratio for the combined class as the ratio of 17 to 7 that was obtained from the sketches?

Answers to Questions 1–4 **1.** 16 girls and 8 boys. **2.** Because the parts of the sketches are different sizes. **3.** 17 to 7. **4.** Yes; the ratio of 34 to 14 is equal to the ratio of 17 to 7.

Exercises and Problems 6.3



1. The preceding public service advertisement points out the need for carpooling to reduce traffic.
- **a.** According to this advertisement, what fraction of the car seats are empty during rush hour?
- **b.** In a city the size of Los Angeles, there would be 9,000,000 empty seats during rush hour. How many seats would be filled?

Use a sketch to illustrate the given information and the solutions in exercises 2 and 3.

2. **a.** The ratio of apples to oranges in a gift box is 3 to 2, and there are 18 apples. How many oranges are there?
- **b.** If the ratio of cars to trucks in a parking lot is 7 to 2 and there are 26 trucks, how many cars are there?

3. **a.** The ratio of U.S. citizens to noncitizens among patent applicants during a given period was 11 to 3. If 407 patent applications were received from U.S. citizens, how many were received from noncitizens?
- **b.** Erik is twice as fast at typing as Trenton. If Trenton types 25 words every minute, how many words will Erik type in 3 minutes?

Answer each question in exercises 4 and 5 by assuming that the rate for the smaller quantity and the rate for the larger quantity remain the same.

4. **a.** Jessica purchased 2.3 pounds of chicken for $6.87. If Glen purchased 3.8 pounds of chicken, how much should he pay to the nearest cent for his purchase?
- **b.** Jason has $9.50, and sliced ham is $5.32 for 1.2 pounds. How many pounds of ham to the nearest tenth can he buy for $9.50?
- **c.** If Lakeside Farm cheese costs $1.19 for each $\frac{1}{4}$ pound, how many pounds of cheese to the nearest tenth can be purchased for $5?

5. **a.** If 1.5 pounds of fish cost $7.44, how much does 3.5 pounds cost?
- **b.** If 8 ounces of yarn cost $2.66, what is the cost for 20 ounces?
- **c.** If 10 pounds of nails cost $4.38, what is the cost to the nearest cent of 3.2 pounds of the same type of nail?

Write each percent as a decimal in exercises 6 and 7, and shade or describe a 10×10 Decimal Square to illustrate the percent. (Copy Blank Decimal Squares from the website or use Virtual Decimal Squares.)

6. **a.** 7% **b.** 18.2% **c.** $34\frac{1}{4}\%$

7. **a.** $37\frac{1}{2}\%$ **b.** 6.5% **c.** $28\frac{1}{3}\%$

Use or describe a Decimal Square for each decimal in exercises 8 and 9, and write the decimal as a percent. (Copy Blank Decimal Squares from the website or use Virtual Decimal Squares.)

8. **a.** .426 **b.** .003 **c.** .09

9. **a.** .60 **b.** .06 **c.** .256

Write each fraction in exercises 10 and 11 as a percent, rounded to the nearest tenth of a percent.

10. **a.** $\frac{7}{25}$ **b.** $\frac{1}{8}$ **c.** $\frac{5}{12}$

11. **a.** $\frac{4}{5}$ **b.** $\frac{5}{6}$ **c.** $\frac{7}{4}$

Determine each answer in exercises 12 and 13 to the nearest tenth. Use Decimal Squares and the visual methods outlined in Figures 6.32 through 6.34 for parts a and b. Label each diagram and explain your reasoning. (Copy Blank Decimal Squares from the website or use Virtual Decimal Squares.)

12. **a.** What percent of 20 is 14?
b. What is 12 percent of 60?
c. If 12 is 8 percent of some number, what is the number?
d. 36.25 is what percent of 14.5?

13. **a.** What is 27 percent of 160?
b. 40 is what percent of 200?
c. If 10 percent of a number is 4, what is the number?
d. What is 140 percent of 65?
e. 75 is what percent of 50?

Mentally calculate each percent in exercises 14 and 15, and explain your method.

14. **a.** 15 percent of \$42 **b.** 25 percent of 28
c. $33\frac{1}{3}$ percent of 15 **d.** 5 percent of \$42.60

15. **a.** 10 percent of \$128.50 **b.** 75 percent of 32
c. 90 percent of \$60 **d.** 110 percent of 80

Mentally estimate each percent in exercises 16 and 17 by replacing one or both numbers by compatible numbers. Show your replacements.

16. **a.** 51 percent of 78.3 **b.** 23 percent of 1182
c. 11 percent of \$19.99 **d.** 32 percent of \$612.40

17. **a.** 9 percent of \$30.75 **b.** 19 percent of 60
c. 4.9 percent of 128 **d.** 15 percent of 241

Mentally calculate approximate percents for the fractions in exercises 18 and 19 by replacing the numerators or denominators by compatible numbers. Show your replacements.

18. **a.** $\frac{14}{27}$ **b.** $\frac{9}{38}$ **c.** $\frac{7}{32}$

19. **a.** $\frac{2}{19}$ **b.** $\frac{408}{1210}$ **c.** $\frac{100}{982}$

Write the numbers in exercises 20 and 21 in scientific notation.

20. **a.** Size of a minute insect in inches is .013
b. Length of a day in seconds is 86,400

21. **a.** Number of years since Earth's formation is 4,600,000,000
b. Diameter of an atom in centimeters is .000000027

Write the numbers in exercises 22 and 23 in positional numeration.

22. **a.** Total number of possible bridge hands is 6.35×10^{11}
b. U.S. trade balance in June, 2010 was $-\$49.9 \times 10^{9}$

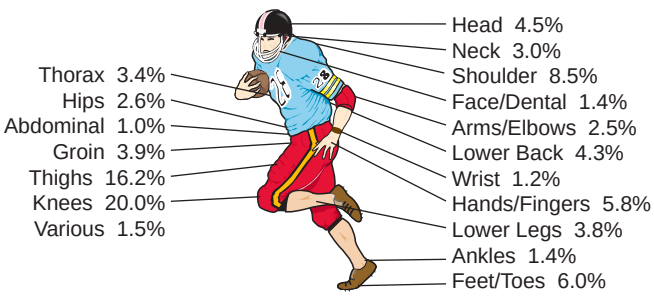
23. **a.** Wavelength of X-rays in inches is 1.2×10^{-9}
b. Approximate length of solar year in seconds is 3.15569×10^{7}

Write the answers for exercises 24 and 25 in scientific notation.

24. **a.** The velocity of a jet plane is 1.1×10^{3} miles per hour, and the escape velocity of a rocket from Earth is 22.7 times the plane's velocity. Find the rocket's velocity by computing $1.1 \times 10^{3} \times 22.7$.
b. Earth travels 6.21×10^{8} miles around the Sun each year in approximately 9×10^{3} hours. Compute $(6.21 \times 10^{8}) \div (9 \times 10^{3})$ to determine Earth's speed in miles per hour.

25. **a.** A light-year, the distance that light travels in 1 year, is 5.868×10^{13} miles. The Sun is 2.7×10^{4} light-years from the center of our galaxy. Find this distance in miles by computing $5.868 \times 10^{13} \times 2.7 \times 10^{4}$.
b. At one point in *Voyager I*'s journey to Jupiter, its radio waves traveled 4.619×10^{8} miles to reach Earth. These waves travel at a speed of 3.1×10^{5} miles per second. Compute $(4.619 \times 10^{8}) \div (3.1 \times 10^{5})$ to determine the number of seconds it took these signals to reach Earth.

The following figure shows the percentages of injuries to different parts of the body, as revealed in a study of 1180 injured professional football players. Determine the number of players with each of the types of injuries in exercises 26 and 27 rounded to the nearest whole number.



Professional Football Players' Injuries

26. **a.** Head injuries **b.** Shoulder injuries

27. **a.** Lower back injuries **b.** Injuries to the feet and toes

The table shows the numbers of students and teachers (to the nearest thousand) in grades K–12 in public schools in several states.* The student/teacher ratio for each state is the number of students divided by the number of teachers. Use this table in exercises 28 and 29.

State	Number of Teachers	Number of Students	Student/ Teacher Ratio
Alabama	59,100	744,600	_____
Florida	146,700	2,625,900	_____
Hawaii	10,900	180,300	_____
Iowa	34,700	479,300	_____
Maine	16,800	193,000	_____
Missouri	65,600	912,500	_____
Oregon	26,500	546,600	_____
Wyoming	6,400	84,900	_____

28. **a.** Compute the ratios to the nearest tenth for the first four states in the table.
b. Which of these states has the best (lowest) student/teacher ratio?
c. Which has the poorest student/teacher ratio?

29. **a.** Compute the ratios to the nearest tenth for the last four states in the table.
b. Which of these states has the best (lowest) student/teacher ratio?
c. Which has the poorest student/teacher ratio?

Reasoning and Problem Solving

30. Compute each percent to the nearest tenth of a percent.
a. A down payment of $200 is what percent of the cost of $1460?
b. A cost of $3.63 in the year 2012 is what percent of a 2010 cost of $2.75?
c. The school has collected $744, which is 62 percent of its goal. What is the total amount of the school's goal?
d. During a flu epidemic, 17 percent of a school's 283 students were absent on a particular day. How many students were absent?

31. Compute each percent to the nearest tenth of a percent and each dollar amount to the nearest hundredth of a dollar.
a. With a discount of 70 percent, the cost of a bracelet is $17.99. What is the price before the discount?
b. A wireless intercom system is marked down from $99.99 to $79.99. By what percent is the intercom discounted?
c. A teacher's salary in the year 2012 is 107.5 percent of her salary in 2011. If the salary in 2011 is $52,000, what is the salary in 2012?
d. If 6 of the 28 students in a class did not enroll in the school's insurance plan, what percentage did enroll in the plan?

32. Which package in each of the following pairs is the better buy?
a. Betty Crocker Complete Buttermilk Mix: small size, 40 ounces at $4.35, or large size, 56 ounces at $5.05
b. Bisquick Variety Baking Mix: small size, 20 ounces at $2.47, or large size, 32 ounces at $3.47
c. Plastic tape: small roll, 15.2 yards for 69 cents, or large roll, 23.6 yards for $1.60

33. One method for determining which of two packages is the better buy is to compute the price per unit of both packages. For example, each ounce of flour in the large package costs 299 cents ÷ 80 = 3.74 cents.



80 ounces for $2.99 32 ounces for $2.67

a. What is the cost per ounce to the nearest hundredth of a cent for the small package?
b. Which is the better buy?
c. If the large package provides enough flour for 10 batches of Drop Cookies, the smaller package will provide enough flour for how many batches?

34. Todd sells 3 newspapers every 10 minutes at his newsstand, and Shauna sells 4 newspapers every 12 minutes at her newsstand. If they work at these rates and combine their sales, how many newspapers will they sell in 30 minutes?

**Statistical Abstract of the United States,* 128th ed. (Washington, DC: Bureau of the Census, 2009), p. 153.

35. The cost of a \$9.85 item that is being discounted 12 percent can be determined by subtracting 12 percent of \$9.85 from \$9.85. The following equations show that the cost of this item can also be found by taking 88 percent of \$9.85. What number properties are used in the first two of these equations?

$$\begin{aligned} 9.85 - (.12 \times 9.85) &= (1 \times 9.85) - (.12 \times 9.85) \\ &= (1 - .12) \times 9.85 \\ &= .88 \times 9.85 \end{aligned}$$

Use one of these two methods to compute the discounted cost of each of the following items.

a. Portable DVD player, \$209.50 (15 percent off)
b. Backpacker sleeping bag, \$153.95 (20 percent off)
c. Snowshoes, \$86 (28 percent off)

36. The total cost of a \$15.70 item plus a 6 percent sales tax can be determined by adding 6 percent of \$15.70 to \$15.70. The total cost can also be found by multiplying 1.06 times \$15.70, as shown by the following equations. What number properties are used in the first two equations?

$$\begin{aligned} 15.70 + (.06 \times 15.70) &= (1 \times 15.70) + (.06 \times 15.70) \\ &= (1 + .06) \times 15.70 \\ &= 1.06 \times 15.70 \end{aligned}$$

Use one of these two methods to compute the cost plus the sales tax for each of the following items. Round each answer to the nearest hundredth of a dollar.

a. Fishing tackle outfit, \$48.60 (4 percent tax)
b. Ten-speed bike, \$189 (5 percent tax)
c. Cell phone, \$69.96 (6 percent tax)

37. Costs due to an annual inflation rate of 4 percent can be determined by repeatedly multiplying the cost of an item by 1.04. For example, if the cost of a \$50 item increases 4 percent, the new cost is $1.04 \times 50 = 52$. Use this rate of inflation to determine the following amounts to the nearest hundredth.

a. How much will the cost of a \$545 washing machine increase in 1 year due to inflation?
b. How much will the cost of a \$545 washing machine increase in 5 years if the rate of inflation remains at 4 percent?
c. Approximately how many years will pass before the cost of the washing machine in part a increases \$200 if the rate of inflation remains at 4 percent?
d. If a \$28,500 salary is increased 5 percent each year to keep ahead of inflation, what will this salary be after 3 years?

38. A consumer will be charged monthly interest at the rate of 1.5 percent (compounded monthly) on a loan of \$8000.

a. What is the interest on this loan for the first month?
b. If none of the loan is paid by the consumer, how much will be owed at the end of the first month?
c. How much will be owed by the consumer at the end of 3 months if no payments are made on the loan?
d. How much money will the consumer save if the loan of \$8000 for 3 months is obtained for a monthly interest rate of 1 percent rather than 1.5 percent?

39. In his will dated July 17, 1788, Benjamin Franklin stated that he wished "to be useful even after my death if possible," and to this end Franklin left 1000 pounds sterling (about \$4570) to be used to make loans to the inhabitants of Boston.

a. Franklin's will stipulated that not more than 60 pounds, about \$274, was to be loaned to apprentices at a 5 percent annual interest rate. What is the interest on this amount for 1 year?
b. The will also required that at the end of each year the borrower pay off 10 percent of the total amount owed. Add the interest from part a to \$274 to determine the total amount owed at the end of the first year. What is 10 percent of this amount?
c. Franklin predicted that the 1000 pounds he was leaving would grow to 131,000 pounds in 100 years if loaned at 5 percent interest and compounded yearly. This means that each year the 5 percent is computed on the total amount in the account, including the past interest. What will 1000 pounds grow to if interest is compounded yearly at 5 percent for 5 years?

40. Population density is a ratio. The ratio for each state is determined by dividing the state's population by its land area in square miles.

a. Calculate the 2009 population densities, using the information in the following table. Round each ratio to the nearest tenth.
b. Which of the four states had the greatest increase in population density from 2003 to 2009?

	California	New Jersey	Texas	Arkansas
2009 population	36,961,664	8,707,739	24,782,302	2,889,450
Square miles	155,959	7,417	261,797	52,068
2003 density	206.4	1134.2	74.2	47.4
2009 density	_____	_____	_____	_____

41. One astronomical unit is 93,003,000 miles, Earth's average distance from the Sun. The distance of the other planets from the Sun in astronomical units is their distance divided by 93,003,000. Determine the missing numbers in the following table. Compute each astronomical unit to the nearest tenth.

Planet	Scientific Notation	Positional Numeration	Astronomical Units
Mercury	3.6002×10^7	_____	_____
Venus	_____	67,273,000	_____
Earth	9.3003×10^7	_____	1
Mars	_____	141,709,000	_____
Jupiter	4.83881×10^8	_____	_____
Saturn	_____	887,151,000	_____
Uranus	1.784838×10^9	_____	_____
Neptune	_____	2,796,693,000	_____

42. Featured Strategy: Guessing and Checking. Suppose an item is on sale at a 20 percent discount but there is a 5 percent sales tax. Is the consumer better off if the discount is computed before the tax or if the tax is computed before the discount?

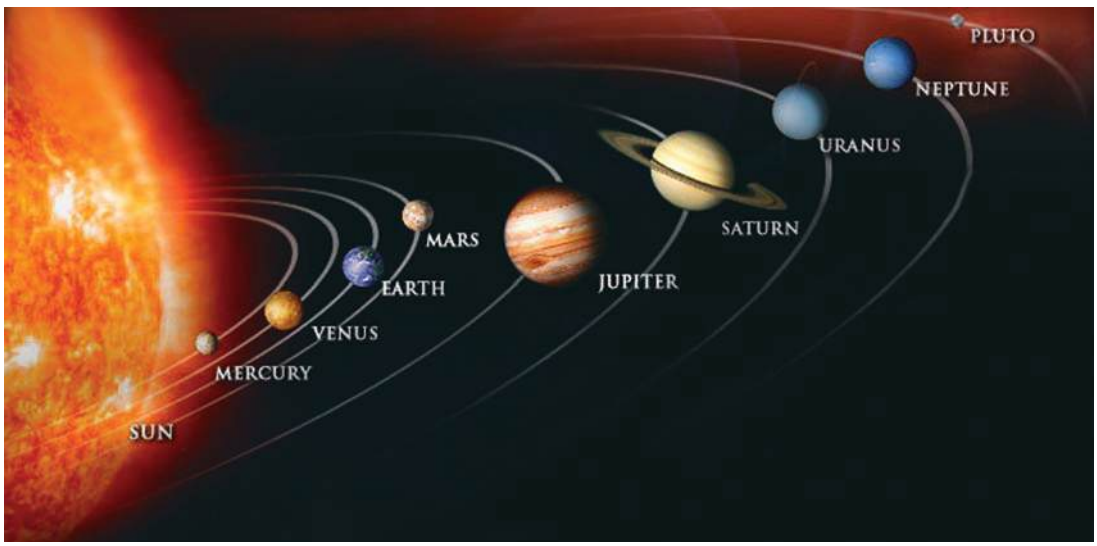
a. Understanding the Problem. If the discount is taken first, then the sales tax will be computed on an amount that is less than the original price. If the sales tax is computed first, then the discount will be taken on an amount that is more than the original price. Is one method better for the consumer than the other? Make an intuitive guess.

b. Devising a Plan. One approach to this problem is to *guess and check* by trying a few different prices, comparing the results, and using inductive reasoning. What happens when the two methods are used for an item that costs $25?

c. Carrying Out the Plan. Use the plan suggested in part b or your own plan to solve this problem.

d. Looking Back. It may have occurred to you to compute the final cost of the item by taking a discount of 15 percent (20 percent discount minus 5 percent tax). Will this method produce the correct result?

e. Looking Back Again. The methods described in the original problem result in the payment of different amounts of sales tax to the state. Which method would the owner of the business prefer, discount and then tax or tax and then discount?



43. After the first term, the top sequence shown here is a geometric sequence. Write the next two numbers in this sequence. Add 4 to each number in the top sequence, and divide the results by 10 to complete the lower sequence.

0 3 6 12 24 __ __
.4 .7 __ __ __ __ __

a. This famous sequence of numbers is the basis of Bode's law, which gives an amazingly close approximation of the distances from the first seven planets to the Sun in astronomical units. Using this sequence of numbers and inductive reasoning, astronomers predicted that there would be a planet between Mars and Jupiter (see the table on the next page). What was the predicted distance in astronomical units of this planet from the Sun? (Asteroids were eventually found between Mars and Jupiter, the biggest of which is Ceres, about 500 miles in diameter.)

b. In the 1770s when Bode's law was discovered, only the first five planets in the table on the next page had been discovered. Using Bode's law, astronomers found Uranus. What would its distance from the Sun

have been in astronomical units if it had conformed to Bode's law? Note: Since 2006 Pluto has not been classified as a planet. It is interesting that Pluto's distance to the Sun is not well predicted by Bode's law.

Planet	Distance from Sun in Astronomical Units	Distance Predicted by Bode's Law
Mercury	0.4	0.4
Venus	0.7	0.7
Earth	1.0	1.0
Mars	1.52	1.6
Ceres	2.77	—
Jupiter	5.2	5.2
Saturn	9.5	10.0
Uranus	19.2	—
Neptune	30.1	38.8
Pluto	39.5	77.2

44. In Lord Tennyson's poem "The Vision of Sin," there is a verse that reads

Every minute dies a man,
Every minute one is born.

In response to these lines, the English engineer Charles Babbage wrote a letter to Tennyson in which he noted that if this were true, the population of the world would be at a standstill. He suggested that the next edition of the poem should read

Every minute dies a man,
Every minute $1\frac{1}{16}$ is born.

a. Using Babbage's mixed number, what is the ratio of the number of people who are born to the number of people who die?

b. In recent years there has been a birth every 7 seconds and a death every 11 seconds. At these rates, what is the ratio of births to deaths?

c. What mixed number should be used in recent years in place of the mixed number suggested by Babbage?

Teaching Questions

1. Some students in a middle school class were having difficulty understanding the relationship between ratios and fractions. One student asked: "Since the ratio of the girls to boys in our class is 2 to 3, why isn't it correct to say $\frac{2}{3}$ of the class are girls?" How would you respond to this question using physical objects to help the students understand that $\frac{2}{3}$ is not correct?

2. Suppose two of your students were having a disagreement, with one saying that a percent could be greater than 100 and the other recalling that one of their parents said nothing could be greater than 100 percent. Explain with examples how you would help these students resolve this problem.

3. A class was discussing global warming and a newspaper statement saying that the temperature in a certain part of the artic region would increase .5 percent in the next 10 years. Some students thought this meant 50 percent. How would you help the students see the difference between .5 percent and 50 percent?

4. A student argued that because an object increases by 50 percent when it goes from \$4 to \$6, then when it drops from \$6 to \$4 it has decreased by 50 percent. Do you agree with this student? Explain.

Classroom Connections

1. In the **Grades 3–5 Standards—Number and Operations** (see inside front cover) under *Understand Numbers . . .* , read the expectation regarding fractions, decimals, and percents. Give a few examples using Decimal Squares to show equivalences of these three forms of rational numbers.

2. The **Standards** quote on page 396 mentions two types of percents that need to be carefully addressed in instruction. Write a problem for each type of percent and illustrate its solution using Decimal Squares.

3. The **Research** statement on page 395 reports that students have trouble solving problems involving percents. Percents are often taught by using three cases and learning a rule for each case. Explain how the approach in the one-page **Math Activity** at the beginning of this section is different than learning three rules and how this approach might be more helpful to students than just learning three rules.

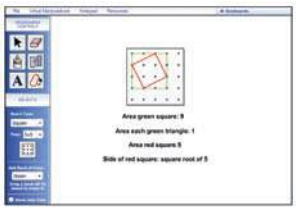
4. Read the **Standards** quote on page 402. In addition to learning the symbols and names for large numbers, how would you help students develop a "sense of magnitude"? Write a few specific suggestions as to how this can be accomplished.

5. The **Research** statement on page 390 speaks of the difficulty students have solving problems involving proportions. Two different methods for solving proportion problems are given on pages 391–392. Give two examples of proportion problems that are different from those in the text and give solutions using the methods of solution in this section.

MATH ACTIVITY 6.4

Irrational Numbers on Geoboards

Virtual Manipulatives

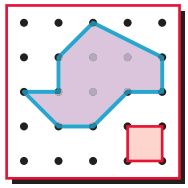


www.mhhe.com/bbn

Purpose: Use areas on rectangular geoboards to explore irrational number lengths.

Materials: Rectangular Geoboards (or copies from the website) or use Virtual Manipulatives.

1. The small shaded square on this geoboard has an area of **1 square unit.** Find the area of the second figure on the geoboard.

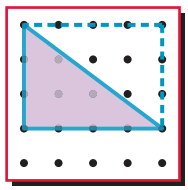


2. Form figures on geoboard paper that have the following areas. Use a geoboard square as in the preceding figure for the square unit. Label each figure with its area.

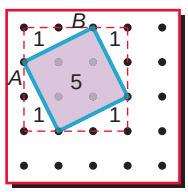
a. Area of $3\frac{1}{2}$ sq. units **b.** Area of 10 sq. units

c. Area of 7 sq. units **d.** Area of $14\frac{1}{2}$ sq. units

3. The triangle on the geoboard shown below has an area of 6 square units. One way to determine this area is to enclose the triangle in a rectangle and take one-half of the area of the rectangle. Form triangles on geoboard paper with areas of $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, and 3 square units. Label each triangle with its area.



***4.** Squares can be formed on the geoboard having areas of 1, 2, 4, 5, 8, 9, 10, and 16 square units. The square shown below has an area of 5 square units. Sketch the remaining seven squares on geoboard paper and label each with its area. (*Hint:* Areas of some figures can be easily found by enclosing the figure in a rectangle or square and subtracting the area of the region outside the figure, as shown below.)



***5.** The area of a square is the product of the length of one side and the length of the other. The square in the preceding figure has an area of 5 square units, so we need a number that, when multiplied by itself, yields 5. We call this number the **positive square root** of 5, written $\sqrt{5}$. Thus, line segment $\overline{AB}$ on the preceding geoboard has a length of $\sqrt{5}$ units. The positive square roots of all whole numbers that are not perfect squares are **irrational numbers.** This type of number is studied in Section 6.4. Label the side of each square in activity 4 with its length. Use your squares to show that $\sqrt{8} = 2\sqrt{2}$.

Section 6.4 IRRATIONAL AND REAL NUMBERS



An Egyptian painting from the Tomb of Mennah showing a scribe of the fields and the estate inspector under Pharaoh Thutmosis IV, dating from the fifteenth century B.C.E. The panels depict agricultural scenes and the needs of an advanced society. The top panel shows surveyors using rope to measure the land to determine the amount of the crop before it is cut.

PROBLEM OPENER

If the digits in the decimal .07007000700007 . . . continue, this pattern of increasing numbers of 0s followed by 7s (five 0s and a 7, six 0s and a 7, etc.), what will the 100th digit be?

The number line in Figure 6.40 shows the locations of a few positive rational numbers. Each rational number corresponds to a point on the number line, and between any two such numbers, no matter how close, there is always another rational number. It would seem that there is no room left for any new types of numbers.

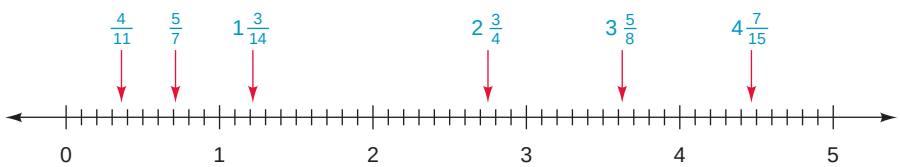


Figure 6.40

There are, however, points on the number line that correspond to numbers that are not rational. For example, there is no rational number that, when multiplied by itself, yields 2. The following equations show that such a number is close to, but greater than, 1.4. Try

these products on a calculator. Find a rational number that, when multiplied by itself, yields a value that is closer to 2 than the numbers shown below.

$$1.4 \times 1.4 = 1.96$$
$$1.41 \times 1.41 = 1.9881$$
$$1.414 \times 1.414 = 1.999396$$
$$1.4142 \times 1.4142 = 1.99996164$$

The purpose of this section is to introduce a new type of number whose decimals are nonrepeating. In the last of the preceding equations, 1.4142 contains the first few digits of a **nonrepeating decimal.** In such a number, there is no repeating pattern of digits as there is for a rational number. Such nonrepeating decimals are called **irrational numbers.** It is easy to create examples of this type of decimal. In the following numeral, each 7 is preceded by one more 0 than the previous 7: .07007000700007 . . . . Although there is a pattern here, there is no block of digits (repetend) that is repeated over and over, as in the case of some rational numbers. Therefore, this is an irrational number.

EXAMPLE A

Which of the following numbers are irrational?

1. .006006006 . . .
2. .060060006 . . .
3. .01001
4. .731731173111731111 . . .
5. .73737373 . . .
6. .21060606 . . .

Solution There is no block of repeating digits in either 2 or 4, so these are irrational numbers. The numbers in 1, 5, and 6 are repeating decimals, and 3 has a terminating decimal, so these are all rational numbers.

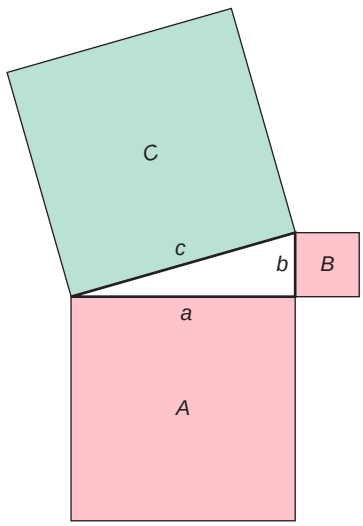
PYTHAGOREAN THEOREM

Research Statement

According to results from the 7th national mathematics assessment, students possess limited knowledge of the Pythagorean theorem and performance levels for even direct application of the theorem are quite low.

Martin and Strutchens

Numbers that are not rational were first recognized by the Pythagoreans, followers of the Greek mathematician Pythagoras who lived in the fifth century B.C.E. It is possible that the discovery of such numbers arose in connection with the Pythagorean theorem. This theorem concerns triangles with a right angle, that is, a 90° angle (see Figure 6.41). The two shorter sides of such a triangle are called **legs,** and the longest side, which is opposite the right angle, is called the **hypotenuse.** The theorem states that for any right triangle, the sum of the areas of the squares on the legs (square A and square B) is equal to the area of the square on the hypotenuse (square C).



Area A + area B = area C

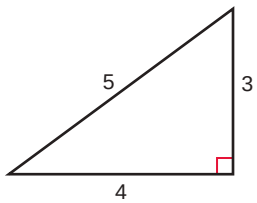
Figure 6.41

Since the area of a square is the square of the length of its side, in Figure 6.41 on the previous page, the area of square A is a^2, the area of square B is b^2, and the area of square C is c^2. The following theorem is one of the most familiar statements in all mathematics.

Pythagorean Theorem For any right triangle with legs of lengths a and b and hypotenuse of length c,

$$a^2 + b^2 = c^2$$

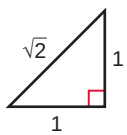
Figure 6.42 shows a right triangle with legs of lengths 3 and 4 and a hypotenuse of length 5. Notice that the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse.



$3^2 + 4^2 = 5^2$

Figure 6.42

Numbers that are not rational may have been discovered by using a right triangle whose legs both have a length of 1, as shown in Figure 6.43. In this case the sum of the squares of the lengths of the two legs is $1^2 + 1^2 = 2$, and the length of the hypotenuse is the number that, when multiplied by itself, equals 2. As we have noted, this number is irrational, and, as shown in Math Activity 6.4 on page 412, this number is $\sqrt{2}$. Thus, the hypotenuse of this triangle has a length that is an irrational number.



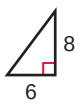
$1^2 + 1^2 = (\sqrt{2})^2$

Figure 6.43

EXAMPLE B

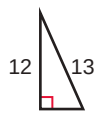
Use the Pythagorean theorem to find the missing length in each triangle.

1.



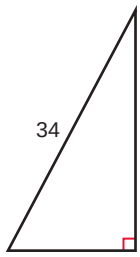
$6^2 + 8^2 = \square^2$

2.



$\square^2 + 12^2 = 13^2$

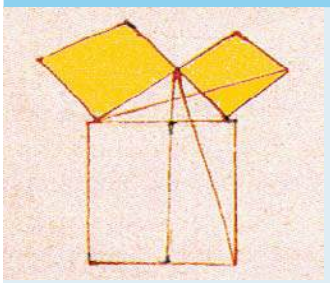
3.



$16^2 + \square^2 = 34^2$

Solution **1.** $6^2 + 8^2 = 100$. Since $10^2 = 100$, the missing length is 10. **2.** $13^2 - 12^2 = 169 - 144 = 25$. Since $5^2 = 25$, the missing length is 5. **3.** $34^2 - 16^2 = 1156 - 256 = 900$. Since $30^2 = 900$, the missing length is 30.

HISTORICAL HIGHLIGHT



Before the Pythagoreans discovered that some numbers were not rational, they believed that all practical and theoretical affairs of life could be explained by ratios of whole numbers, that is, positive rational numbers. The discovery of line segments whose lengths were not rational numbers caused a logical scandal that threatened to destroy the Pythagorean philosophy. According to one legend, the Pythagoreans attempted to keep the matter secret by taking the discoverer of such numbers, Hippacus, on a sea voyage from which he never returned. The diagram to the left was used by the Persian mathematician Nasr al-Din al-Tusi (1201–1274) to present a version of Euclid's proof of the Pythagorean theorem.

There are many proofs of the Pythagorean theorem. *The Pythagorean Proposition* is a book that describes 370 proofs of this theorem.* The proof suggested in Figure 6.44 was known by the Greeks and may have been the one given by Pythagoras. Part a has a small square of area a^2, a larger square of area b^2, and four right triangles. The total area of the figure in part a is $a^2 + b^2 + 4T$, where T is the area of each triangle. Part b has a square of area c^2 and four right triangles that are each the same size as those in part a. The total area of the figure in part b is $c^2 + 4T$. Since the large square in part a and the large square in part b both have sides of length $a + b$, they have the same area. Setting these areas equal to each other, we have

$$a^2 + b^2 + 4T = c^2 + 4T$$

and subtracting $4T$ from both sides leaves

$$a^2 + b^2 = c^2$$

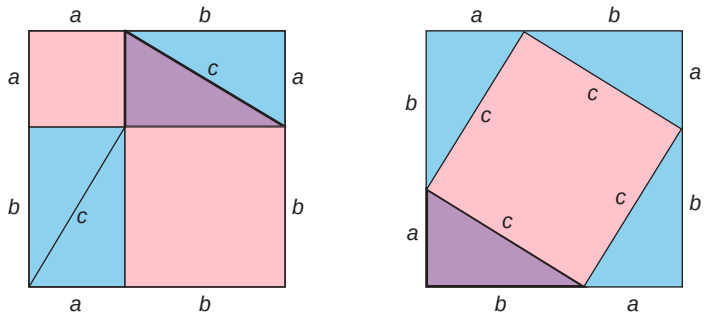


Figure 6.44

The converse of the Pythagorean theorem also holds. *If the sum of the squares of two sides of a triangle equals the square of the third side, then the triangle is a right triangle.* This means, if you used a rope with 30 knotted intervals of equal length and formed a triangle of sides 5, 12, and 13, as shown in Figure 6.45 on the next page, it would be a right triangle. This fact was undoubtedly known by the ancient Egyptians and used by their surveyors to form right triangles (see rope in the top panel of the photograph on page 413).

*E. S. Loomis, *The Pythagorean Proposition* (Washington, DC: National Council of Teachers of Mathematics, 1968).

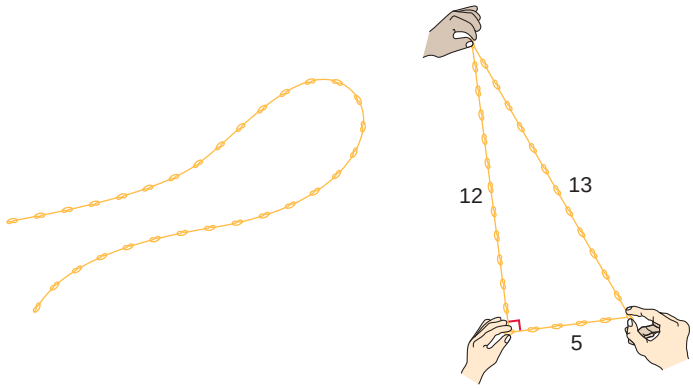


Figure 6.45

HISTORICAL HIGHLIGHT



Babylonian clay tablet with approximation of $\sqrt{2}$

The first proof of the Pythagorean theorem is thought to have been given by Pythagoras (ca. 540 B.C.E.). According to legend, when Pythagoras discovered this theorem, he was so overjoyed that he offered a sacrifice of oxen to the gods. The theorem had been used for centuries, however, by the Babylonians and Egyptians. It is illustrated on this 4000-year-old Babylonian tablet, which shows a square and its diagonals. The numbers on this tablet are in base-sixty numeration and show that the Babylonians had computed the value of $\sqrt{2}$ to several decimal places. The $\sqrt{2}$ is equal to 1 plus a decimal and this decimal in base sixty is $\frac{24}{60} + \frac{51}{60^2} + \frac{10}{60^3}$, which is written in Babylonian numerals on the tablet. Use a calculator to compute 1 + this decimal. What was the Babylonian value for $\sqrt{2}$? How does this compare with your calculator's value for $\sqrt{2}$?

NCTM Standards

In the middle grades, students should also add another pair to their repertoire of inverse operations—squaring and taking square roots. In grades 6–8, students frequently encounter squares and square roots when they use the Pythagorean relationship. p. 220

SQUARE ROOTS AND OTHER ROOTS

A **square root** of a non-negative number is defined as a number that, when multiplied by itself, yields the original number. For example, 3 is a square root of 9, since $3 \times 3 = 9$. However, $^{-}3$ is also a square root of 9, because $^{-}3 \times {}^{-}3 = 9$. Often we are concerned with only the positive square root of a number. For example, suppose the area of the square in Figure 6.46 is 64 square units. Since the area of a square is the product of the lengths of two of its sides, the length of one side of the square is the positive square root of 64, which is 8. In this example the negative square root $^{-}8$ has no meaning. The positive square root of a number is called the **principal square root.**

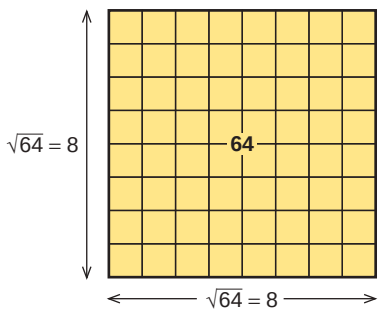


Figure 6.46

EXAMPLE C

Find the principal square root and the negative square root of each number.

1. 49 **2.** 20.25 **3.** .64 **4.** $\frac{1}{4}$

Solution **1.** 7, $^-7$ **2.** 4.5, $^-4.5$ **3.** .8, $^-.8$ **4.** $\frac{1}{2}, \frac{^-1}{2}$

The symbol for the principal square root of a number b is $\sqrt{b}$. The symbol $\sqrt{\ }$ is called the **radical sign** and was represented first by the letter r, then by ✓, and finally by $\sqrt{\ }$. The negative square root of b is written as $-\sqrt{b}$.

For any non-negative number $\boldsymbol{b}$,

$$\sqrt{\boldsymbol{b}} \times \sqrt{\boldsymbol{b}} = \boldsymbol{b}$$

EXAMPLE D

Evaluate the following expressions.

1. $\sqrt{14} \times \sqrt{14}$ **2.** $(\sqrt{6})^2$ **3.** $\sqrt{9} \times \sqrt{9}$

Solution **1.** 14 **2.** 6 **3.** 9

Technology Connection

The square roots of square numbers (1, 4, 9, 16, 25, etc.) are whole numbers. The square roots of all other whole numbers greater than zero ($\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, etc.) are irrational. These numbers all have nonrepeating decimals. If a square number is entered into a calculator and the square root key $\boxed{\sqrt{x}}$ is pressed, the display will show the principal square root of the number. If the square root of the number is irrational, the decimal that appears in the display will be an approximation.

EXAMPLE E

Classify each number as either rational or irrational. If it is rational, evaluate the square root; if it is irrational, find an approximation to the nearest tenth.

1. $\sqrt{81}$ **2.** $\sqrt{10}$ **3.** $\sqrt{30}$

4. $\sqrt{\frac{4}{9}}$ **5.** $\sqrt{18}$ **6.** $\sqrt{.16}$

Solution **1.** Rational, 9 **2.** Irrational, approximately 3.2 **3.** Irrational, approximately 5.5 **4.** Rational, $\frac{2}{3}$ **5.** Irrational, approximately 4.2 **6.** Rational, .4

Even though we cannot write the complete decimals for irrational numbers, these numbers should not be thought of as mysterious or illusive. They are the lengths of line segments, as illustrated by the triangles in Figure 6.47 on the next page. The legs of the first triangle each have a length of 1, so, by the Pythagorean theorem, the hypotenuse is $\sqrt{2}$. The legs of the middle triangle have lengths of 1 and $\sqrt{2}$, and the hypotenuse is $\sqrt{3}$. In the third triangle, legs of length $\sqrt{2}$ and $\sqrt{3}$ are used to obtain a hypotenuse of $\sqrt{5}$. Line segments of lengths $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, etc., can be constructed in a similar manner. The lengths of the hypotenuses of these triangles, $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$, are indicated on the number line below the triangles in Figure 6.47. Check the locations of these numbers on the number line by using the edge of a piece of paper to mark off the lengths $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ from the triangles.

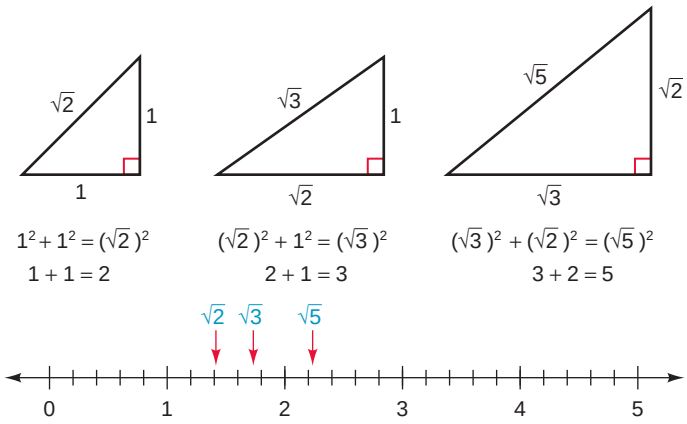


Figure 6.47

EXAMPLE F

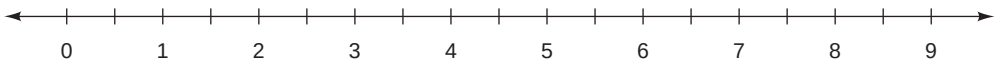
Find the length of each hypotenuse.



Solution **1.** $\sqrt{58}$ **2.** $\sqrt{29}$

EXAMPLE G

Mark the approximate location of the length of each hypotenuse from Example F on the following number line.



Solution **1.** To the nearest tenth, $\sqrt{58}$ is 7.6, which is 1 tenth beyond 7.5 on the number line. **2.** To the nearest tenth, $\sqrt{29}$ is 5.4, which is 1 tenth before 5.5 on the number line.

The **cube root** of a number n is written as $\sqrt[3]{n}$. This is the number s such that $s \times s \times s$ equals n. The cube roots of **perfect cubes,** 1, 8, 27, 64, etc. are whole numbers. The cube roots of all other whole numbers are irrational numbers. For example, the cube roots of 4, 10, and 35 are nonrepeating decimals. Their approximate locations are shown on the number line in Figure 6.48. Cube these decimals to see how close you get to 4, 10, and 35.

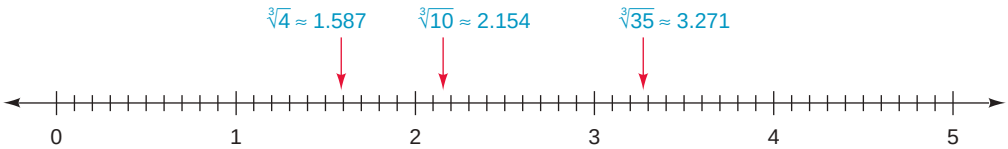


Figure 6.48

EXAMPLE H

Classify each number as either rational or irrational. If it is rational, find the cube root; if it is irrational, find an approximation to the nearest tenth.

1. $\sqrt[3]{30}$ **2.** $\sqrt[3]{125}$ **3.** $\sqrt[3]{100}$

Solution **1.** Irrational, approximately 3.1 **2.** Rational, 5 **3.** Irrational, approximately 4.6

In addition to square roots and cube roots, for positive numbers b, the fourth root of b is $\sqrt[4]{b}$ and $(\sqrt[4]{b})^4 = b$; the fifth root of b is $\sqrt[5]{b}$ and $(\sqrt[5]{b})^5 = b$; etc. In general, the ***n*th root** of a positive number b is written as $\sqrt[n]{b}$, or expressed in exponential form as $b^{\frac{1}{n}}$, and n is called the **index.** Notice that for square roots ($\sqrt{10}$, $\sqrt{3}$, etc.) the index 2 is not written. Furthermore, since it is possible to have odd roots of negative numbers, for example $\sqrt[3]{-8} = -2$ because $(-2)^3 = -8$, the following definition is stated in two parts.

nth Roots (1) For $b \geq 0$ and positive integer n, or (2) for $b < 0$ and odd positive integer n,

$$(\sqrt[n]{b})^n = b \qquad \text{and} \qquad \sqrt[n]{b} = b^{\frac{1}{n}}$$



Technology Connection

Some calculators have a key for finding roots (cube root, fourth root, etc.). One common notation for this key is $\boxed{\sqrt[x]{y}}$. Here are the keystrokes for finding the cube root of 12 using such a calculator. The number in the view screen in step 4 is only an approximation because $\sqrt[3]{12}$ is an irrational number.

Keystrokes	View Screen
1. Enter 12	12.
2. Press $\boxed{\sqrt[x]{y}}$	12.
3. Enter 3	3.
4. $\boxed{=}$	2.289428485

Your calculator may not have a key for roots ($\boxed{\sqrt[x]{y}}$), but it may have a key for raising numbers to powers, such as $\boxed{y^x}$, $\boxed{x^y}$, or $\boxed{\wedge}$. The second part of the definition above enables us to find the roots of numbers by using exponents. For example, since $\sqrt[3]{20} = 20^{\frac{1}{3}}$, the first few digits in the cube root of 20 are obtained by the next keystrokes.

$$20 \ \boxed{x^y} \ \left(\frac{1}{3}\right) \ \boxed{=} \ \boxed{2.714417617}$$

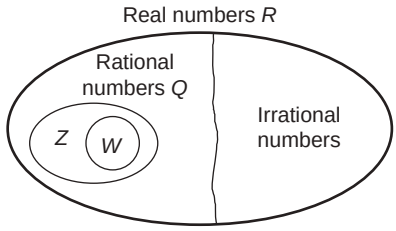
Similarly, square roots can be obtained by raising numbers to the $\frac{1}{2}$ power. The following keystrokes make use of the fact that $\sqrt{60} = 60^{\frac{1}{2}}$. Note that $\frac{1}{2}$ can be replaced by .5.



$$60 \ \boxed{x^y} \ \left(\frac{1}{2}\right) \ \boxed{=} \ \boxed{7.745966692}$$

REAL NUMBERS

The irrational numbers and the rational numbers together form the set of **real numbers.** Figure 6.49 on the next page shows the relationships among the familiar sets of numbers. The set of rational numbers and the set of irrational numbers are disjoint, and their union is the set of real numbers R. The rational numbers Q contain the integers $Z = \{0, \pm1, \pm2, \pm3, \ldots\}$, and the integers contain the whole numbers $W = \{0, 1, 2, 3, \ldots\}$. Viewed in another way, the sets W, Z, and Q form an increasing sequence of subsets of R. The set of whole numbers is contained in the set of integers, $W \subset Z$; the set of integers is contained in the set of rational numbers, $Z \subset Q$; and the set of rational numbers is contained in the set of real numbers, $Q \subset R$. (*Note*, Q is from the first letter of *Quotient* and Z is from the first letter of the German word for numbers, *Zahlen.*)



$W \subset Z \subset Q \subset R$

Figure 6.49

Each whole number is in all the sets W, Z, Q, and R. Other numbers are in only one, two, or three of these sets. For example, $\frac{1}{7}$ is in the set of rational numbers Q and the set of real numbers R, but not in the set of whole numbers W or integers Z.

EXAMPLE 1 Use the letters W, Z, Q, and R to indicate to which set(s) each number belongs.

1. $^{-}12$ **2.** $\sqrt{15}$ **3.** .23 **4.** 130

5. $\frac{3}{5}$ **6.** $.\overline{27}$ **7.** $\sqrt{10}$ **8.** $\sqrt{25}$

Solution **1.** Z, Q, R **2.** R **3.** Q, R **4.** W, Z, Q, R **5.** Q, R **6.** Q, R **7.** R **8.** W, Z, Q, R

PROPERTIES OF REAL NUMBERS

We have seen that the whole numbers, integers, rational numbers, and real numbers form an increasing sequence, with each set of numbers contained in the next. Although these number systems have several properties in common (the commutative, associative, and distributive properties), each number system was developed because it had number properties that the existing number systems did not have. For example, the whole numbers do not have inverses for addition (negative numbers), so the integers were developed; the integers do not have inverses for multiplication (reciprocals), so the rational numbers were developed. Similarly, the rational numbers lack a number property that the real numbers have.

The real numbers have the property of *completeness*. Intuitively, we can interpret **completeness** as meaning that all line segments can be measured. If we limit ourselves to the rational numbers, this is not true. For example, we have seen that there is no rational number corresponding to the length of the hypotenuse of a right triangle whose legs have lengths of 1 unit.

Expressed in a slightly different way, completeness of the real numbers means that there is a one-to-one correspondence between the real numbers and the points on a line. Because of this relationship, a line called the **real number line** is used as a model for the real numbers. Once a zero point has been labeled and a unit has been selected, each real number can be assigned to a point on the line. Each positive real number is assigned a point to the right of zero such that the real number is the distance from this point to the zero point. The negative of this number corresponds to a point that is the same distance to the left of zero. A few examples are shown in Figure 6.50.

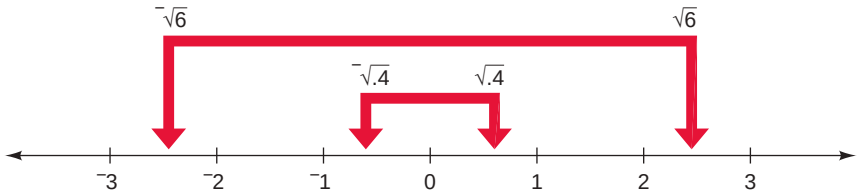


Figure 6.50

The following list contains the 12 properties of the real number system.

Closure Under Addition The sum of any two real numbers is another unique real number. That is, the set of real numbers is closed under addition.

Closure Under Multiplication The product of any two real numbers is another unique real number. That is, the set of real numbers is closed under multiplication.

Addition Is Commutative For any real numbers r and s, $r + s = s + r$.

Multiplication Is Commutative For any real numbers r and s, $r \times s = s \times r$.

Addition Is Associative For any real numbers r, s, and t, $(r + s) + t = r + (s + t)$.

Multiplication Is Associative For any real numbers r, s, and t, $(r \times s) \times t = r \times (s \times t)$.

Identity for Addition For any real number r, $0 + r = r$. Zero is called the **identity for addition,** and it is the only number with this property.

Identity for Multiplication For any real number r, $1 \times r = r$. The number 1 is called the **identity for multiplication,** and it is the only number with this property.

Inverses for Addition For any real number r, there is a unique real number ^{-}r, called its **opposite** or **inverse for addition,** such that $r + {}^{-}r = 0$.

Inverses for Multiplication For any nonzero real number r, there is a unique real number $\frac{1}{r}$, called its **reciprocal** or **inverse for multiplication,** such that $r \times \frac{1}{r} = 1$.

Multiplication Is Distributive Over Addition For any real numbers r, s, and t, $r \times (s + t) = r \times s + r \times t$.

Completeness Property All line segments can be measured with real numbers.

OPERATIONS WITH IRRATIONAL NUMBERS

At first it is difficult to imagine how to perform arithmetic operations with numbers that cannot be expressed exactly in decimal notation. One solution is to replace irrational numbers by rational approximations. The square in Figure 6.51 has an area of 2 square units and sides of length $\sqrt{2}$ units. Since $\sqrt{2} \approx 1.4$, the total length of the four sides is approximately 4×1.4, or 5.6 units. For many purposes this is sufficient accuracy.

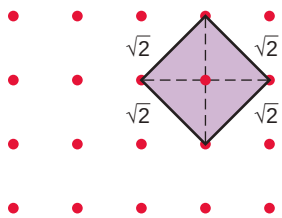


Figure 6.51

Another solution is to write products and sums by using irrational numbers. The total length of the sides of the square in Figure 6.51 can be written as $4\sqrt{2}$, which means 4 times $\sqrt{2}$.

Is $4\sqrt{2}$ a rational or an irrational number? We know by the property of closure for multiplication of real numbers that $4\sqrt{2}$ is a real number, so it is either rational or irrational. Let's suppose it is a rational number and denote it by r. That is,

$$r = 4\sqrt{2}$$

Multiplying both sides of this equation by $\frac{1}{4}$, we get

$$r \times \tfrac{1}{4} = \sqrt{2}$$

Now by the property of closure for multiplication of rational numbers $r \times \frac{1}{4}$ is a rational number. However, this cannot be true because $\sqrt{2}$ is an irrational number and the preceding equation would then have an irrational number equal to a rational number. Since the assumption that $4\sqrt{2}$ is rational leads to a contradiction, $4\sqrt{2}$ must be irrational. A similar argument can be used to prove that **the product of any nonzero rational number and an irrational number is an irrational number.** Thus, we can obtain an infinite number of irrational numbers by multiplying each nonzero rational number by $\sqrt{2}$. Let's consider another example of computing with irrational numbers. The total length of the sides of the triangle in Figure 6.52 can be written as $3 + \sqrt{5}$.

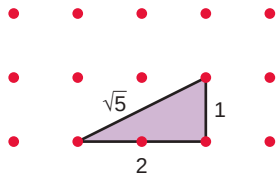


Figure 6.52

This is also an irrational number, as can be proved by an argument similar to the previous one. For, if we assume that $3 + \sqrt{5}$ is a rational number and denote it by s, then

$$s = 3 + \sqrt{5} \quad \text{and} \quad s - 3 = \sqrt{5}$$

But this equation contains a contradiction. Since s and $^{-}3$ are rational numbers, by the closure property for addition of rational numbers, their sum is also a rational number. But $\sqrt{5}$ is an irrational number. So, the assumption that $3 + \sqrt{5}$ is rational is false. In general, **the sum of any rational number and irrational number is an irrational number.** So once again, an infinite number of irrational numbers can be obtained by adding rational numbers to an irrational number.

EXAMPLE J

Classify each sum or product as rational or irrational, and if it is rational, evaluate the expression.

1. $3\sqrt{24}$ **2.** $5\sqrt{36}$ **3.** $14 + \sqrt{14}$ **4.** $\sqrt{81} + 18$

Solution **1.** Irrational **2.** Rational, 30 **3.** Irrational **4.** Rational, 27

The total length of the sides of the square in Figure 6.51 on page 422, and the triangle in Figure 6.52 can be approximated by using rational number approximations for $\sqrt{2}$ and $\sqrt{5}$. However, there are cases in which we can compute with irrational numbers and obtain rational numbers without replacing them by decimal approximations. For example, we know that $\sqrt{2} \times \sqrt{2} = 2$. This can also be seen by looking at the square in Figure 6.51. The area of the square is $\sqrt{2} \times \sqrt{2}$, and we can see that the area is 2 because it has been divided into four smaller half-squares, or triangles. Thus, in this example the product of two irrational numbers is a rational number.

Let's consider another example of a product of two irrational numbers. The rectangle in Figure 6.53 has a length of $\sqrt{18}$ because it is the hypotenuse of a right triangle whose legs each have a length of 3. The width of this rectangle is $\sqrt{2}$. Its area, according to the formula for the area of a rectangle, length times width, is $\sqrt{18} \times \sqrt{2}$. Using a second method, we can show this area to be 6 square units by dividing the rectangle into two squares and eight half-squares, or triangles (see shaded region). These two methods of finding area show that $\sqrt{18} \times \sqrt{2} = 6$. This example illustrates a case in which the product of two different irrational numbers is a rational number.

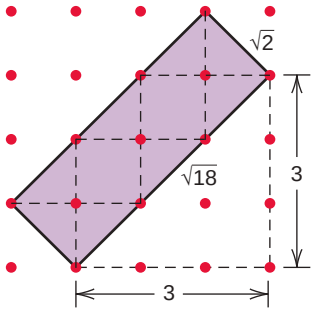


Figure 6.53

In the preceding example we saw that $\sqrt{18} \times \sqrt{2} = 6$. But since $6 = \sqrt{36} = \sqrt{18 \times 2}$, we see that $\sqrt{18} \times \sqrt{2} = \sqrt{18 \times 2}$. This illustrates that **the product of the square roots of two numbers is equal to the square root of the product of the two numbers.** This result is stated in the following theorem.

For positive numbers a and b,

$$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$$

EXAMPLE K

Compute each product and determine if it is rational or irrational.

1. $\sqrt{8} \times \sqrt{6}$ **2.** $\sqrt{12} \times \sqrt{3}$ **3.** $\sqrt{5} \times \sqrt{20}$ **4.** $\sqrt{6} \times \sqrt{10}$

Solution **1.** $\sqrt{48}$, irrational **2.** $\sqrt{36} = 6$, rational **3.** $\sqrt{100} = 10$, rational **4.** $\sqrt{60}$, irrational

In addition to enabling us to compute the products of square roots, the preceding theorem is useful for simplifying square roots. For example, $\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$. A square root is in **simplified form** if the number under the radical sign has no factor other than 1 that is a square number.

EXAMPLE L

Write each square root in simplified form.

1. $\sqrt{50}$ **2.** $\sqrt{54}$ **3.** $\sqrt{80}$

Solution **1.** $\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$. **2.** $\sqrt{54} = \sqrt{9 \times 6} = \sqrt{9} \times \sqrt{6} = 3\sqrt{6}$. **3.** $\sqrt{80} = \sqrt{16 \times 5} = \sqrt{16} \times \sqrt{5} = 4\sqrt{5}$.

Quotients of real numbers, such as $2 \div \sqrt{3}$, are often written as fractions, such as $\frac{2}{\sqrt{3}}$. When the denominator of a fraction contains a square root, cube root, etc., it is sometimes necessary to find an equal fraction that has a rational number for its denominator. The process of replacing a denominator that is irrational by a denominator that is rational is called **rationalizing the denominator.** The denominator of $\frac{2}{\sqrt{3}}$ can be rationalized by using the fundamental rule for equality of fractions to multiply the numerator and denominator by $\sqrt{3}$.

$$\frac{2}{\sqrt{3}} = \frac{2 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{2\sqrt{3}}{3}$$

EXAMPLE M

Rationalize the denominator of each fraction.

1. $\frac{1}{\sqrt{2}}$ 2. $\frac{-6}{\sqrt{5}}$ 3. $\frac{7}{\sqrt{7}}$

Solution 1. $\frac{1}{\sqrt{2}} = \frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{\sqrt{2}}{2}$. 2. $\frac{-6}{\sqrt{5}} = \frac{-6 \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}} = \frac{-6\sqrt{5}}{5}$.

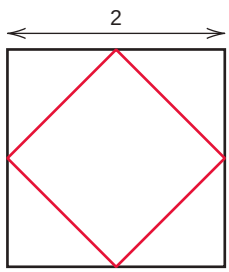
3. $\frac{7}{\sqrt{7}} = \frac{7 \times \sqrt{7}}{\sqrt{7} \times \sqrt{7}} = \frac{7\sqrt{7}}{7} = \sqrt{7}$.

PROBLEM-SOLVING APPLICATION

To solve the following problem, we use the Pythagorean theorem and the fact that the length of the sides of a square is the square root of its area. Try to solve this problem before you read the solution. You may find the strategies of *making a table* and *finding a pattern* to be useful in obtaining the solution.

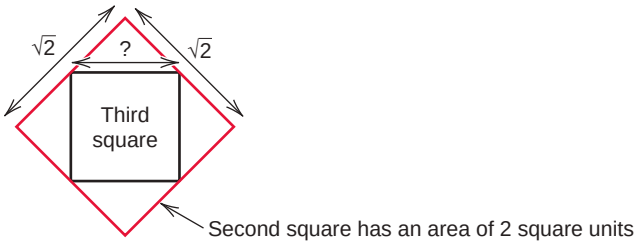
Problem

The inner square in the following sketch was obtained by connecting the midpoints of the sides of the outer square. If this process of forming smaller inner squares by connecting the midpoints of the sides of the preceding square is continued, what will be the dimensions of the ninth square?



Understanding the Problem The outer square is 2 units by 2 units and has an area of 4 square units. The second square is contained inside the first square and is smaller. **Question 1:** What is the length of the side of the second square, and what is the area of this square?

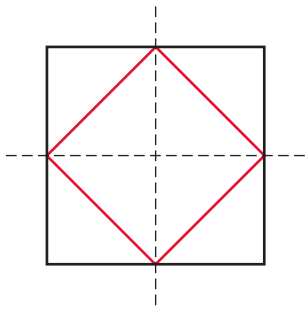
Devising a Plan One approach to solving the problem is to *form a table* listing the lengths of the sides and the areas of the first few squares. The second square has sides of length $\sqrt{2}$ units and an area of 2 square units (see the figure on top of the next page). **Question 2:** What is the length of the side of the third square, and what is its area?



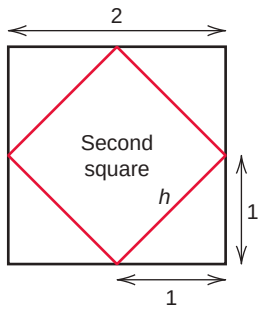
Carrying Out the Plan The following table lists the lengths of the sides and areas of the first four squares. Find a pattern and predict the area of the ninth square. **Question 3:** What is the length of the side of the ninth square?

	Square 1	Square 2	Square 3	Square 4	Square 5	Square 6
Length of side (units)	2 by 2	$\sqrt{2}$ by $\sqrt{2}$	1 by 1	$\frac{1}{\sqrt{2}}$ by $\frac{1}{\sqrt{2}}$		
Area (square units)	4	2	1	$\frac{1}{2}$		

Looking Back You may have noticed that the area of each square is one-half the area of the preceding square. Based on this pattern, the area of the ninth square is $\frac{1}{64}$ square units. Thus, the length of the side of the ninth square is $\sqrt{\frac{1}{64}} = \frac{1}{8}$ units. The fact that the area of each succeeding square decreases by one-half is suggested by the following figure. **Question 4:** How can the dashed lines be used to show that the inner square has one-half the area of the outer square?



Answers to Questions 1–4 **1.** The length h of the side of the second square (see the red square below) is the hypotenuse of a triangle whose sides have length 1. The length of the hypotenuse is $\sqrt{2}$, and the area of the second square is $\sqrt{2} \times \sqrt{2} = 2$.



$h^2 = 1^2 + 1^2$

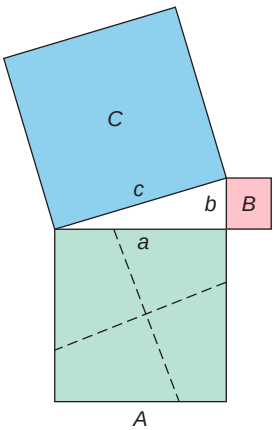
$h^2 = 2$

$h = \sqrt{2}$

Laboratory Connections

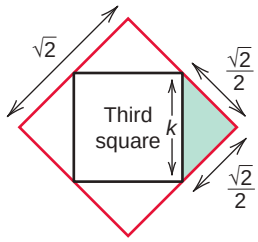
Pythagorean Theorem

This theorem can be illustrated visually in many ways. Some of these demonstrations involve cutting and fitting the two squares on the legs of a right triangle onto the square on the hypotenuse. Experiment with a right triangle and its squares to try demonstrating this theorem. Explore this and related questions using Geometer's Sketchpad® student modules available at the companion website.



Mathematics Investigation
Chapter 6, Section 4
www.mhhe.com/bbn

2. The length k of the sides of the third square is the hypotenuse of a triangle whose sides have length $\frac{\sqrt{2}}{2}$. The length of the hypotenuse is 1, and the area of the third square is 1.



$$k^2 = \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2$$
$$k^2 = \frac{2}{4} + \frac{2}{4} = 1$$
$$k = \sqrt{1} = 1$$

3. The area of the ninth square is $\frac{1}{64}$, and the length of its side is $\sqrt{\frac{1}{64}} = \frac{1}{8}$. **4.** The outer square region is formed by eight triangular regions of equal size, and the inner square region is formed by four of these triangular regions.

Exercises and Problems 6.4

There once was a student named Lew,
Who tried to compute the square root of 2.
When no pattern repeated
He gave up defeated,
Two million digits is all he would do.

$\sqrt{2}$ = 1.41421356241933916628197598871307959868348906509619318943243523661427981910045554616704325437650546094505945702825327193147647412885463 . . .

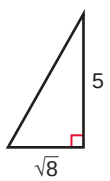
Determine which of the numbers in exercises 1 and 2 are irrational.

1. a. $\sqrt{49}$ **b.** .131131113 . . .
c. .113113113 . . . **d.** $\sqrt{14}$

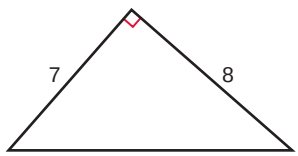
2. a. $\sqrt{50}$ **b.** 6.404004000 . . .
c. $2\sqrt{25}$ **d.** 151551555

Use the Pythagorean theorem to find the missing length for each of the right triangles in exercises 3 and 4. Write both the exact answer and the decimal approximation to one decimal place.

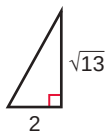
3. a.



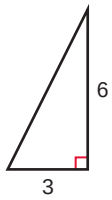
b.



4. a.



b.



Find the indicated root of each number in exercises 5 and 6.

5. a. $\sqrt{\frac{1}{16}}$ **b.** $\sqrt[3]{64}$ **c.** $\sqrt{9.61}$ **d.** $\sqrt[3]{-125}$

6. a. $\sqrt[3]{8000}$ **b.** $\sqrt{625}$ **c.** $\sqrt[3]{\frac{8}{27}}$ **d.** $\sqrt[3]{-64}$

Classify each number in exercises 7 and 8 as rational or irrational. If it is rational, find its root; if it is irrational, find an approximation to the nearest tenth.

7. a. $\sqrt{18}$ **b.** $\sqrt[3]{216}$ **c.** $\sqrt{\frac{1}{9}}$

8. a. $\sqrt[3]{30}$ **b.** $\sqrt{80}$ **c.** $\sqrt{121}$

Write each number in exercises 9 and 10 as a decimal to the nearest tenth, and mark its approximate location on a number line.

9. a. $\sqrt{7}$ **b.** $\sqrt[3]{30}$ **c.** $\sqrt{3}$

10. a. $\sqrt{15}$ **b.** $\sqrt{8}$ **c.** $\sqrt[3]{3.5}$

Any three whole numbers a, b, and c such that $a^2 + b^2 = c^2$ are called **Pythagorean triples.** We can generate Pythagorean triples a, b, and c, by substituting whole numbers for u and v in the following equations:

$$a = 2uv \quad b = u^2 - v^2 \quad c = u^2 + v^2$$

Use the given values of u and v in exercises 11 and 12 and the equations on the previous page to create Pythagorean triples. Check your answers by showing that $a^2 + b^2 = c^2$.

11. a. $u = 2, v = 1$
b. $u = 3, v = 2$
c. $u = 6, v = 5$

12. a. $u = 4, v = 3$
b. $u = 4, v = 2$
c. $u = 5, v = 3$
d. For $a = 2uv$, $b = u^2 - v^2$, and $c = u^2 + v^2$, use algebra to show $a^2 + b^2 = c^2$.

13. Form a table like the one shown below, and put checks in the appropriate columns to indicate the relevant set membership(s) for each number at the left. For example, $^-3$ is an integer, a rational number, and a real number.

	Whole Numbers	Integers	Rational Numbers	Real Numbers
$^-3$		✓	✓	✓
$\frac{1}{8}$				
$\sqrt{3}$				
π				
14				
$\frac{1.6}{4}$				
$.8\overline{2}$				

14. Form a table like the one in exercise 13, and write the following numbers in the leftmost column. Then put checks in the columns for each number to indicate relevant set membership(s). For example, see the checkmarks for $^-3$ in exercise 13.

$$\frac{^-1}{2} \quad \frac{1}{1.2} \quad \sqrt{1000} \quad .42\overline{7} \quad \sqrt{24} \quad 365 \quad \frac{17}{6}$$

Which of the sets in exercises 15 and 16 are closed with respect to the given operations? If a set is not closed under a given operation, provide an example to show this.

15. a. The set of whole numbers under subtraction
b. The set of nonzero rational numbers under division

16. a. The set of irrational numbers under multiplication
b. The set of integers under addition

State the property of the real numbers that is being used in each equality in exercises 17 and 18.

17. a. $\sqrt{3} \times \sqrt{6} = \sqrt{6} \times \sqrt{3}$
b. $(3 + \sqrt{2}) \times \sqrt{7} = 3\sqrt{7} + \sqrt{2} \times \sqrt{7}$
c. $(4 + \sqrt{8}) + 2\sqrt{5} = 2\sqrt{5} + (4 + \sqrt{8})$

18. a. $^-\sqrt{10} + (\sqrt{10} + 6) = (^-\sqrt{10} + \sqrt{10}) + 6$
b. $\sqrt{3} \times \left(\frac{1}{\sqrt{3}} \times \sqrt{3}\right) = \sqrt{3} \times 1$
c. $^-18 + (\sqrt{12} + {}^-\sqrt{12}) + 471 = {}^-18 + 0 + 471$

Classify each sum or product in exercises 19 and 20 as rational or irrational. If the expression is rational, evaluate it.

19. a. $\sqrt{2} \times \sqrt{20}$ **b.** $10 + \sqrt{8}$ **c.** $\sqrt{6} \times \sqrt{24}$

20. a. $4\sqrt{15}$ **b.** $\sqrt{25} + 11$ **c.** $\sqrt{7} \times \sqrt{28}$

Simplify the square roots in exercises 21 and 22 so that the smallest possible whole number is left under the square root symbol.

21. a. $\sqrt{45}$ **b.** $\sqrt{48}$ **c.** $\sqrt{60}$

22. a. $\sqrt{150}$ **b.** $\sqrt{1000}$ **c.** $\sqrt{288}$

Determine whether the equations in exercises 23 and 24 are true or false for positive numbers a and b; and if an equation is false, show a counterexample.

23. a. $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
b. $\sqrt{a} - \sqrt{b} = \sqrt{a - b}$

24. a. $\sqrt{a} + \sqrt{b} = \sqrt{a + b}$
b. $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

Rationalize the denominator of each fraction in exercises 25 and 26.

25. a. $\frac{4}{\sqrt{7}}$ **b.** $\frac{3}{2\sqrt{6}}$

26. a. $\frac{5}{\sqrt{5}}$ **b.** $\frac{^-1}{\sqrt{2}}$

Suppose a calculator determines square roots when a number is entered and $\boxed{\sqrt{x}}$ is pressed. Write the number in exercises 27 and 28 that is displayed for step 4. If you continue to press the square root key in these examples, eventually you will see the same number in every view screen. What is this number?

27.

Keystrokes	View Screen
1. Enter 2	2
2. $\boxed{\sqrt{x}}$	1.4142136
3. $\boxed{\sqrt{x}}$	1.1892071
4. $\boxed{\sqrt{x}}$	

28.

Keystrokes	View Screen
1. Enter .5	.5
2. $\sqrt{x}$	.70710678
3. $\sqrt{x}$	.84089642
4. $\sqrt{x}$	

If you use a calculator with a square root key, what number will eventually show in the view screen if you enter the types of numbers in exercises 29 and 30 and repeatedly press this key?

29. Enter a positive number less than 1.

30. Enter a positive number greater than 1.

A calculator with the key y^x (or $\wedge$) for raising numbers to powers is used in exercises 31 and 32 to evaluate the roots of numbers. Write each root that is being evaluated in the form $\sqrt[x]{y}$, and determine the number that will be displayed in a view screen with eight places for digits.

31. **a.** 81 y^x .25 =
b. 2.25 y^x (1 ÷ 2) =
c. 3.0625 y^x .5 =
d. ⁻2.197 y^x (1 ÷ 3) =

32. **a.** 2116 y^x .5 =
b. 2744 y^x (1 ÷ 3) =
c. 10.5625 y^x (1 ÷ 2) =
d. ⁻4.096 y^x (1 ÷ 3) =

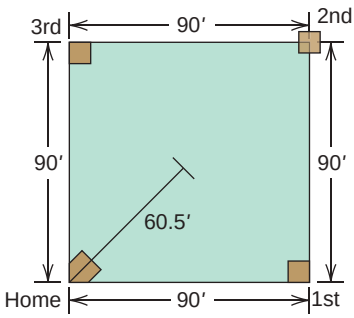
Reasoning and Problem Solving

33. Each day Ed walks past a rectangular athletic field on his way home from school. If the field is being used, Ed walks along two sides of the field. If the field is not being used, he cuts across from corner to corner. If he takes 300 steps along one edge of the field and 500 steps along the other edge, approximately how many steps can Ed save by walking from corner to corner?

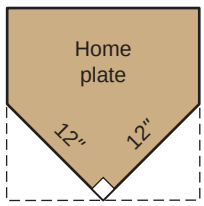
34. In a game for three students, one opens a book and multiplies the numbers of the facing pages. This student states the product, and the other two students race to see who can determine the page numbers. What page numbers yield the product 18,090?

35. A school's basketball hoop is mounted on a pipe that is cemented into the ground. The hoop is 10 feet above the ground. To stop it from swaying, some students put a brace from behind the hoop to a point on the ground that is 8 feet from the cement base. What is the length of the brace from the hoop to the ground (to the nearest foot)?

36. The infield diamond of a baseball field is a 90 × 90 foot square with the center of second base at a vertex of the square. The pitching mound is 60.5 feet from the corner of home plate. How far is the mound from the center of second base (to the nearest foot)?



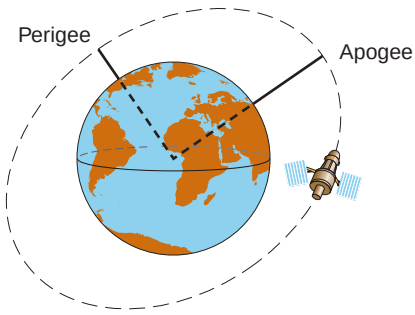
37. A home plate for a baseball field can be formed by making two 12-inch cuts from a square region, as shown in the following figure. What are the dimensions of the original square, to the nearest tenth of an inch?



38. A 30-foot ladder is leaning against a house, with the foot of the ladder 8 feet from the house. If the foot of the ladder is pulled 7 more feet from the house, how far down the side of the house will the ladder move (to the nearest foot)?

39. The time it takes a satellite to orbit Earth depends on its apogee and perigee. The **apogee** A of a satellite is its greatest distance from the center of the Earth and the **perigee** P is its smallest distance. The formula for the time in hours T required for one orbit is

$$T = \frac{(A + P)\sqrt{A + P}}{501{,}186}$$

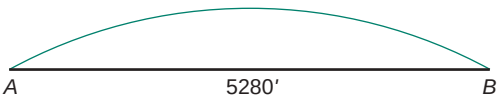


a. Suppose a satellite orbiting Earth has an apogee of 4300 miles and a perigee of 4100 miles. How long does it take to complete one orbit (to the nearest tenth of an hour)?

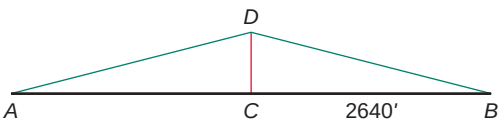
b. Satellites are often placed in circular orbits with apogees and perigees of approximately 26,300 miles. Why is this distance chosen?

40. Featured Strategy: Making a Drawing. Suppose a 1-mile-long metal bridge that was not built to allow for expansion nevertheless expands 2 feet and buckles upward at the center. How high will the center be pushed up?

a. Understanding the Problem. The first step in understanding the problem is to *make a drawing.* The distance along the line from A to B represents the bridge and the curve represents the expanded bridge. Since there are 5280 feet in a mile, what is the distance along the curve from A to B?



b. Devising a Plan. Let's approximate the shape of the buckled bridge by two right triangles, as shown in the next figure. The length from C to B is $5280 \div 2 = 2640$. Explain why the length from D to B is approximately 2641 feet.



c. Carrying Out the Plan. What is the distance from C to D?

d. Looking Back. If the bridge had been 2 miles long and had expanded 2 feet, how high would the bridge have buckled?

41. For over 2000 years, architects and artists have been fond of using a rectangle called the *golden rectangle.* The length of a golden rectangle divided by its width is an irrational number that, when rounded to six decimal places, is 1.618033. This irrational number is called the *golden ratio.*



Golden rectangle

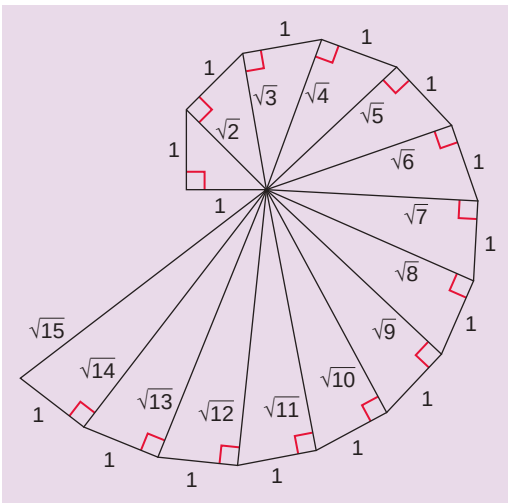
Fibonacci numbers are related to the golden ratio. The first 10 Fibonacci numbers are listed here.

1 1 2 3 5 8 13 21 34 55

a. Compute the ratios formed by dividing each Fibonacci number by the previous Fibonacci number.

b. Extend the sequence and find two consecutive Fibonacci numbers whose ratio to four decimal places equals the golden ratio to four decimal places.

42. The spiral of right triangles shown below somewhat resembles a cross section of the seashell of the chambered nautilus. It represents the square roots of consecutive whole numbers. The first triangle has two legs of unit length and a hypotenuse of $\sqrt{2}$. This hypotenuse becomes a leg of the next triangle, which has a hypotenuse of $\sqrt{3}$. Each triangle uses the hypotenuse of the preceding triangle as a leg.



a. Copy the rectangular grid from the website. Identify two grid lines to be horizontal and vertical axes, respectively. For each whole number n on the horizontal axis of the grid, plot a point approximately $\sqrt{n}$ units above the axis.

b. Connect the points on the grid. Use the graph to approximate $\sqrt{7.5}$. Could this graph be used to approximate the square root of any number greater than 1 and less than 12?

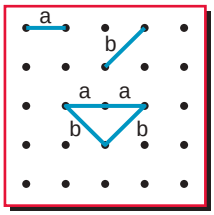
Teaching Questions

1. Some students wanted to know why "square of a number" and "square root of a number" both used the word "square." How would you answer their question?

2. A student who was writing a paper on the historical development of the number system got information from the website on the usefulness of whole numbers, fractions,

and integers and why they were developed, but could not understand why irrational numbers are important or useful. How would you help this student?

3. When forming and recording segments of different lengths on their geoboards, one group of students were disagreeing among themselves as to whether line segments **a** and **b** (see the geoboard) were the same length or not. (a) Explain a practical way for them to resolve the disagreement. (b) How can you use the triangle on this geoboard to explain which segment is longer?



4. Results from the Seventh Assessment of Educational Progress, as noted in the Research statement on page 414, show that students possess limited knowledge of the Pythagorean theorem and its applications. Describe a classroom activity that you would use to teach this theorem and one or more of its applications.

Classroom Connections

1. In **Grades 6–8 Standards—Geometry** (see inside back cover) under *Analyze characteristics . . .* , the third expectation requires that students create inductive and deductive arguments for the relationship between the lengths of the legs of a right triangle and the length of the hypotenuse. Write an inductive argument for this relationship and then show a deductive argument.

2. The awareness that some numbers are not rational occurred over 2000 years ago and is one of the great discoveries in the development of our number system. This discovery and the crises it caused is briefly described in the **Historical Highlight** on page 416. Write about some of the history and early details involving these new types of numbers.

3. The **Problem Opener** at the beginning of this section has a decimal whose digits form a pattern, but the explanation of irrational numbers on page 414 says that this decimal is not rational. Use the terms "core that repeats" and "core that grows" from the beginning of Exercises and Problems 1.2 in Chapter 1 to explain why the decimal in this problem opener is not a rational number.

4. Read the **Standards** quote on page 417. The terms "square number" and "square root" should suggest geometric images to students that will help them remember the meaning of these terms as well as the standard notation. Use a geometric model to illustrate the square of a number and its square root and discuss this inverse relationship.

CHAPTER 6 REVIEW

1. **Decimals**
   a. The word *decimal* comes from the Latin *decem,* meaning 10.
   b. The number of digits to the right of the decimal point is called the **number of decimal places.**
   c. The **place values** to the right of the decimal point are decreasing powers of 10.
   d. **Decimal Squares** and **number lines** are visual models for decimals.
   e. An **inequality** for decimals less than 1 can be determined by comparing their tenths digits. If these are equal, compare their hundredths digits, etc.

2. **Rational numbers**
   a. Any number that can be written in the form $\frac{a}{b}$, where $b \neq 0$ and a and b are integers, is called a **rational number.**
   b. Every rational number can be represented as either a **terminating** or a **repeating decimal.**
   c. Every terminating or repeating decimal can be written as a rational number in the form $\frac{a}{b}$.
   d. A rational number $\frac{a}{b}$ in lowest terms can be written as a terminating decimal if and only if b has only 2s or 5s in its prime factorization.
   e. The block of repeating digits in a repeating decimal is called the **repetend.**
   f. The rational numbers are **dense.** That is, between any two such numbers there is always another rational number.
   g. The operations of addition and multiplication on the set of rational numbers satisfy the 11 **number properties** stated in Section 6.2.

3. **Operations with decimals**
   a. **Decimal Squares** provide a visual model for decimal operations and can be used to show the similarity between these operations and the operations on whole numbers.
   b. The **pencil-and-paper algorithms** for decimals can be illustrated by computing with fractions.
   c. To compute the product of a decimal and a positive power of 10, move the decimal point one place to the right for each factor of 10.
   d. To divide a decimal by a positive power of 10, move the decimal point one place to the left for each power of 10.

4. **Mental calculations**
   a. Products and quotients of decimals can be calculated mentally by computing with whole numbers and then locating decimal points.
   b. **Compatible numbers** is the technique of using pairs of numbers that are especially convenient for mental calculation. In computing with decimals it is sometimes convenient to use equivalent fractions in place of the decimals.
   c. **Substitution** is the technique of replacing a decimal or a percent by a sum or difference of two decimals or percents.
   d. **Add-up** is the technique of obtaining the difference of two decimals by adding up from the smaller to the larger decimal.

5. **Estimation**
   a. **Compatible numbers** is the technique of computing estimations by replacing one or more numbers with approximate compatible numbers.
   b. **Rounding** is the technique of replacing one or both numbers in a computation with approximate numbers.
   c. **Front-end estimation** is the technique of using the leading nonzero digit to obtain a rough estimation.

6. **Ratio and percent**
   a. For any two positive numbers a and b, the **ratio** of a to b (a:b) is the fraction $\frac{a}{b}$.
   b. For two equal ratios $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a}{b} = \frac{c}{d}$ is called a **proportion.**
   c. The word *percent* is from the Latin *per centum*, meaning *out of 100.*
   d. There are three types of computations involving percents: computations using **whole and percent,** computations using **part and whole,** and computations using **percent and part.**

7. **Scientific notation**
   a. The method of writing a number as a product of a number from 1 to 10 and a power of 10 is called **scientific notation.**
   b. When a number is written in scientific notation, the part from 1 to 10 is called the **mantissa** and the exponent of 10 is called the **characteristic.**

8. **Real numbers**
   a. An infinite nonrepeating decimal is called an **irrational number.**
   b. The **principal square root** of a positive number b is denoted by $\sqrt{b}$ and is defined to be the positive number that, when multiplied by itself, yields b.
   c. For any positive number b and any positive whole number n, $\sqrt[n]{b}$ is called the ***n*th root** of b and defined by $(\sqrt[n]{b})^n = b$.

d. The rational numbers together with the irrational numbers form the set of **real numbers.**

e. In addition to the eleven **number properties** stated for the rational numbers, the real numbers have the property of **completeness:** all line segments can be measured with real numbers.

f. The sum or product of a nonzero rational number and an irrational number is an irrational number.

g. For any positive numbers a and b, $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$.

h. The process of replacing a denominator that is irrational by a denominator that is rational is called **rationalizing the denominator.**

CHAPTER 6 TEST

1. Describe Decimal Squares to explain each of the following:
a. $.4 > .27$ **b.** $.7 = .70$
c. $.225 < .35$ **d.** $.09 < .1$

2. Write each fraction as a decimal.
a. $\frac{3}{4}$ **b.** $\frac{7}{100}$ **c.** $\frac{2}{3}$ **d.** $\frac{7}{8}$ **e.** $\frac{4}{9}$ **f.** $\frac{6}{25}$

3. Write each decimal as a fraction.
a. .278 **b.** $.\overline{35}$ **c.** .03 **d.** $.7\overline{326}$

4. Round each decimal to the given place value.
a. .878 (hundredths) **b.** .449 (tenths)
c. .5096 (thousandths) **d.** $.\overline{6}$ (ten thousandths)

5. Perform each operation and describe Decimal Squares to illustrate each answer.
a. $.7 + .6 =$ **b.** $3 \times .4 =$
c. $.62 - .48 =$ **d.** $.80 \div .05 =$

6. Perform the following operations.
a. $.006 + .38 - .2$ **b.** $.62 \times .08$
c. $.14763 \div .21$ **d.** $47 + .8 \times 340$

7. Calculate each answer mentally and explain your method.
a. $100 \times .073$ **b.** $7 \times .6$
c. $4.9 \div 1000$ **d.** $.01 \times 372$
e. 15 percent of 260 **f.** 25 percent of 36

8. Estimate each answer by replacing the decimal or percent by a compatible fraction. Show your replacement.
a. $.49 \times 310$ **b.** $.24 \times 416$
c. 33 percent of 60 **d.** 76 percent of 40

9. Determine each answer to the nearest tenth.
a. What is 36 percent of 46?
b. 30 is what percent of 80?
c. 15 is 24 percent of what number?
d. What is 118 percent of 125?
e. 322 is what percent of 230?

10. Write each number in scientific notation.
a. 437.8 **b.** .000106

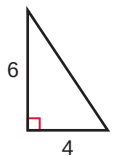
11. Classify each number as rational or irrational.
a. $\sqrt{60}$ **b.** $\sqrt[3]{27}$ **c.** $6\sqrt{8}$
d. $\sqrt{10} + 5$ **e.** $\sqrt{\frac{1}{4}}$ **f.** $\sqrt[3]{60}$

12. Approximate each number below to one decimal place.
a. $\sqrt{34}$ **b.** $\sqrt[3]{18}$

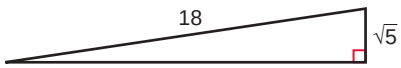
13. Determine whether each set is closed or not closed with respect to the given operation, and give a reason or show a counterexample.
a. The set of rational numbers under addition
b. The set of irrational numbers under multiplication
c. The set of irrational numbers under addition

14. Simplify each square root.
a. $\sqrt{405}$ **b.** $\sqrt{24}$

15. Find the missing length for each triangle.
a.



b.



16. Jon paid $187 for a coat that was on sale at 15 percent off. What was the original price of the coat?

17. A restaurant sells a 20-ounce glass of orange juice for $1.50 and a 16-ounce glass for $1.10. What size glass is the better buy?

18. A rectangular swimming pool has a length of 60 feet and a width of 30 feet. What is the distance from one corner of the pool to the opposite corner, to the nearest one-tenth foot?

19. A fuel company charges monthly finance fees of 1.2 percent for the first $500 due and .8 percent for any amount over $500. What is the monthly finance fee for an account with a balance of $650?

20. If the ratio of private school students to public school students in a city is 3 to 16 and there is a total of 18,601 students, how many students are in public schools?

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CHAPTER

7 Statistics

Spotlight on Teaching

Excerpts from NCTM's Standards for School Mathematics Prekindergarten through Grade 2*

Methods used by students in different grades to investigate the number of pockets in their clothing provide an example of students' growth in data investigations during the period through grade 2. Younger students might count pockets (Burns 1996). They could survey their classmates and gather data by listing names, asking how many pockets, and noting the number beside each name. Together the class could create one large graph to show the data about all the students by coloring a bar on the graph to represent the number of pockets for each student (see Figure 4.21). In the second grade, however, students might decide to count the number of classmates who have various numbers of pockets (see Figure 4.22).

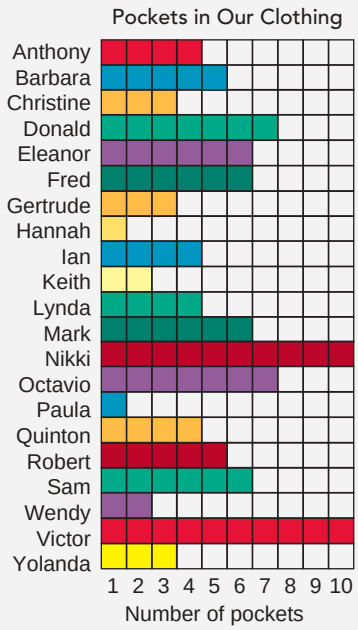


Figure 4.21
A bar graph illustrating the number of pockets in kindergarten students' clothes.

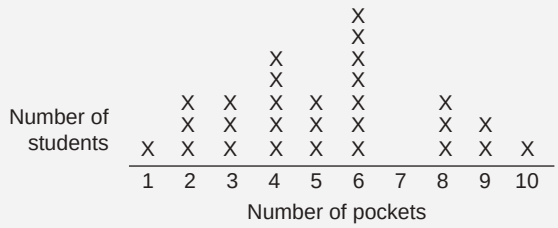


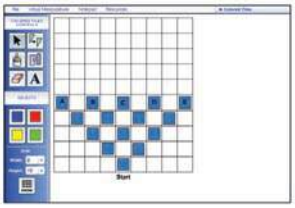
Figure 4.22
A line plot graph of the number of students in a second-grade class who have from one to ten pockets.

Principles and Standards for School Mathematics, pp. 110–111.

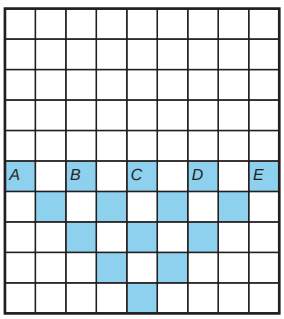
MATH ACTIVITY 7.1

Forming Bar Graphs with Color Tiles

Virtual Manipulatives



www.mhhe.com/bbn



Start

Purpose: Use an experiment to form color tile bar graphs and analyze a distribution.

Materials: Copies of the 2-Centimeter Grid from the website and 1-to-4 Spinner and Color Tiles in the Manipulative Kit or Virtual Manipulatives. The 1-to-4 Spinner from the Manipulative Kit can be used with a paper clip for the spinner, as shown on page 491.

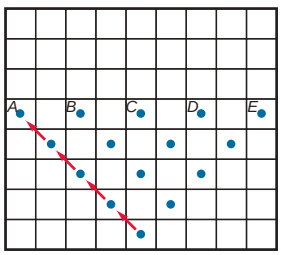
1. Use a 2-centimeter grid to make a copy of the grid shown at the left. Place a tile on the square marked *Start.* Spin the 1-to-4 spinner to obtain the color red or green. Move the tile right to a shaded square in the second row if the color is red (numbers 2 or 3) and left to a shaded square in the second row if the color is green (numbers 1 or 4). Continue spinning and moving the tile forward onto shaded squares (right for red and left for green) until it is at one of the shaded squares marked *A*, *B*, *C*, *D*, or *E.* Place the tile above the lettered square.

   a. Do you think that each lettered square has the same chance of receiving a tile? Assuming this process was carried out 32 times with 32 tiles, make a prediction as to how many tiles would land on each lettered square and write the predictions next to the letters.

   b. Using a new tile each time, place a tile on the Start square and carry out this process a total of 32 times. Placing each tile above the shaded square on which it lands forms a bar graph.

   c. From your results, which squares are most likely to have a tile land on them and which are least likely?

2. The diagram below shows that there is only one path from the Start square to the square lettered *A*.



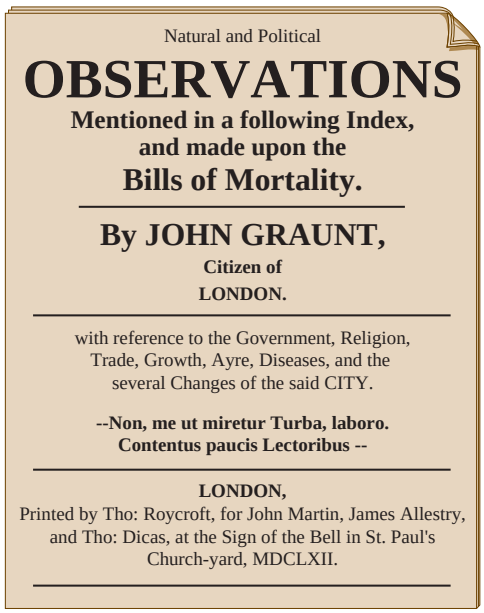
Start

   ***a.** Trace out and count the number of different paths to each lettered square. How many paths are there all together?

   b. For each lettered square, what percent of all possible paths lead to that square?

   c. If you completed the process of spinning and moving 128 tiles to the lettered squares, what is the number of tiles that might be expected to land on each lettered square?

Section 7.1 COLLECTING AND GRAPHING DATA



Natural and Political

OBSERVATIONS

Mentioned in a following Index, and made upon the Bills of Mortality.

By JOHN GRAUNT,

Citizen of LONDON.

with reference to the Government, Religion, Trade, Growth, Ayre, Diseases, and the several Changes of the said CITY.

--Non, me ut miretur Turba, laboro. Contentus paucis Lectoribus --

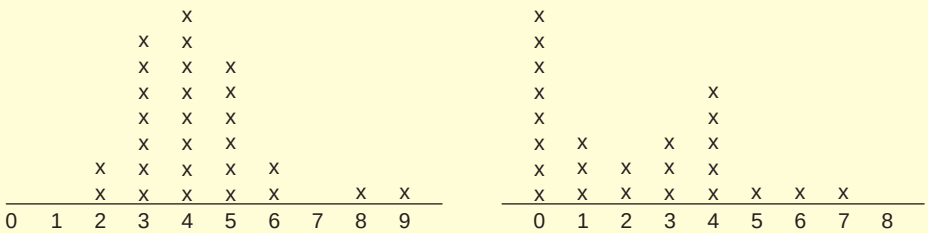
LONDON,

Printed by Tho: Roycroft, for John Martin, James Allestry, and Tho: Dicas, at the Sign of the Bell in St. Paul's Church-yard, MDCLXII.

John Graunt's seventeenth-century Bills of Mortality, one of the first publications containing statistical data.

PROBLEM OPENER

Two line plots are shown here with each x representing a fourth-grade student. One line plot shows the number of students having a given number of cavities, and the other shows the number of students having a given number of people in their families. Which plot contains the data on cavities? Explain how you know.*



Statistics had its beginning in the seventeenth century in the work of Englishman John Graunt (1620–1674). Born in London, Graunt developed human statistical and census methods that later provided the framework for modern demography. He used a publication called *Bills of Mortality,* which listed births, christenings, and deaths. Here are some of his conclusions: The number of male births exceeds the number of female births; there is a higher death rate in urban areas than in rural areas; and more men than women die violent deaths. Graunt used these statistics in his book *Natural and Political Observations of Mortality.* He analyzed the mortality rolls in early London as Charles II and other officials attempted to create a system to warn of the onset and spread of bubonic plague. Graunt summarized great amounts of information to make it understandable (*descriptive statistics*) and made conjectures about large populations based on small samples (*inferential statistics*).

*J. Zawojewski, "Polishing a Data Task: Seeking Better Assessment," *Teaching Children Mathematics.*

HISTORICAL HIGHLIGHT



Florence Nightingale, 1820–1910

Florence Nightingale exhibited a gift for mathematics from an early age, and later she became a true pioneer in the graphical representation of statistics. She is credited with developing a form of the pie chart now known as the polar area diagram, that is similar to a modern circular histogram. In 1854, she worked with wounded soldiers in Turkey during the Crimean War. During that time, she maintained records classifying the deaths of soldiers as the result of contagious illnesses, wounds, or other causes. Nightingale believed that using color emphasized the summary information, and she made diagrams of the nature and magnitude of the conditions of medical care for members of Parliament, who likely would not have understood traditional statistical reports. She used her statistical findings to support her campaign to improve sanitary conditions and to provide essential medical equipment in the hospital. Because of her efforts, the number of deaths from contagious diseases reduced dramatically.

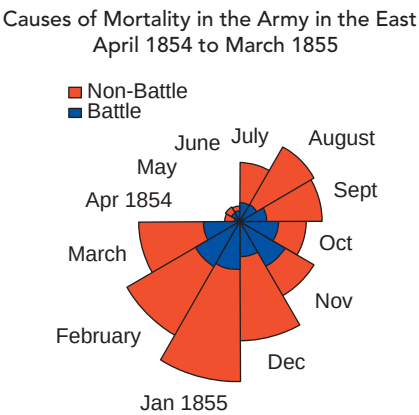


Figure 7.1

The polar pie graph (Nightingale Rose Diagram) used by Florence Nightingale is similar to the graph shown here. She used such graphs in "Notes on Matters Affecting the Health, Efficiency and Hospital Administration of the British Army," 1858.

Graphs provide quick visual summaries of information and methods of making predictions. Some of the more common graphs are introduced in this section.

NCTM Standards

By the end of the second grade, students should be able to organize and display their data through both graphical displays and numerical summaries. They should use counts, tallies, tables, bar graphs, and line plots. p. 109

BAR GRAPHS

The table in Figure 7.1 lists the responses of 40 teachers to a proposal to begin and end the school day one-half hour earlier. Teachers' responses are classified into one of three categories: favor (F); oppose (O); or no opinion (N).

Teacher	Category	Teacher	Category	Teacher	Category
1	F	15	O	28	O
2	F	16	F	29	F
3	O	17	N	30	O
4	N	18	O	31	F
5	F	19	F	32	F
6	O	20	N	33	N
7	O	21	F	34	F
8	F	22	F	35	O
9	F	23	O	36	F
10	F	24	F	37	N
11	O	25	N	38	F
12	O	26	O	39	O
13	N	27	N	40	O
14	F				

Research Statement

Fourth-grade students were successful at literal reading of bar graphs (over 95% success rate), they were less successful at interpreting (52% success rate) and predicting (less than 20% success rate).

Pereira-Mendoza and Mellor

The data from the table in Figure 7.1 on the previous page are summarized by the **bar graph** in Figure 7.2. The titles on the horizontal axis represent the three categories, and the scale on the vertical axis indicates the number of teachers for each category. Compare the graph to the table and notice that this graph provides a quick summary of the data.

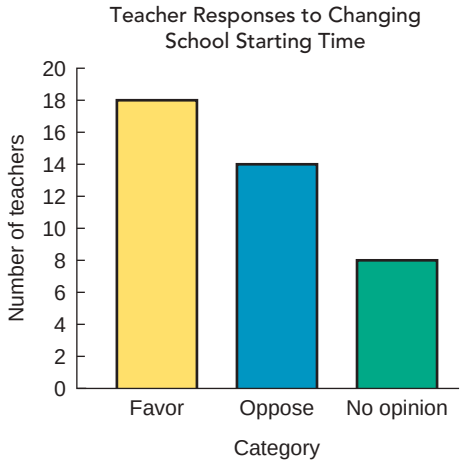


Figure 7.2

Some types of bar graphs have two bars for each category and are called **double-bar graphs,** while others have three bars for each category and are called **triple-bar graphs.** Figure 7.3 is a triple-bar graph that has four categories of age groups and compares the percentages of black children, Hispanic children, and white children who have not seen a physician in the past year.*

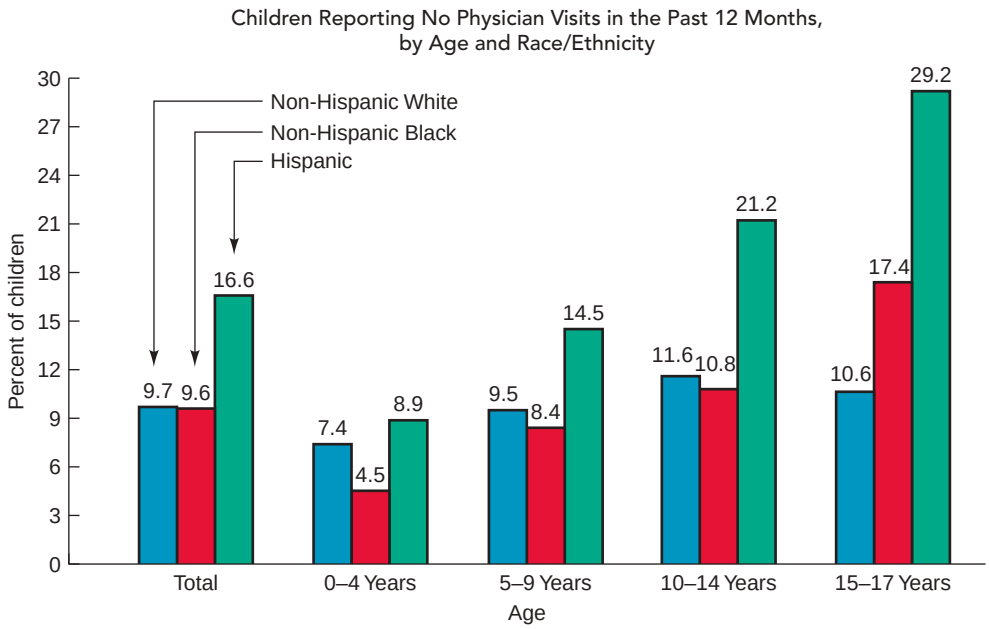


Figure 7.3

*U.S. Department of Health and Human Services, *Child Health USA 2008–2009* (Washington, DC: U.S. Government Printing Office, 2009), p. 60.

EXAMPLE A Use the information in the graph in Figure 7.3 on page 439.

1. What is the difference between the percent of Hispanic children and the percent of white children aged 15 to 17 who have not seen a physician in the past year?
2. What percent of black children aged 10 to 14 years visited a physician in the past year?
3. Was there a greater difference in the percents of black children and white children who did not visit a physician during the past year in the age group of 5 to 9 or the age group of 10 to 14?

Solution **1.** 18.6 percent **2.** 89.2 percent **3.** Age group of 5 to 9

PIE GRAPHS

A **pie graph** (circle graph) is another way to summarize data visually. A disk (pie) is used to represent the whole, and its pie slice–shaped sectors represent the parts in proportion to the whole. Consider, for example, the data from Figure 7.1 on page 438. A total of 40 responses are classified into three categories: 18 in favor, 14 opposed, and 8 with no opinion. These categories represent $\frac{18}{40}$, $\frac{14}{40}$, and $\frac{8}{40}$ of the total responses, respectively. To determine the central angles for the sectors of a pie graph, we multiply each of these fractions by 360°.

$$\frac{18}{40} \times 360° = 162° \qquad \frac{14}{40} \times 360° = 126° \qquad \frac{8}{40} \times 360° = 72°$$

The pie graph for this data is constructed by first drawing a circle and making three sectors, using the central angles, as in part a of Figure 7.4. Then each sector is labeled so that the viewer can easily interpret the results, as in part b.

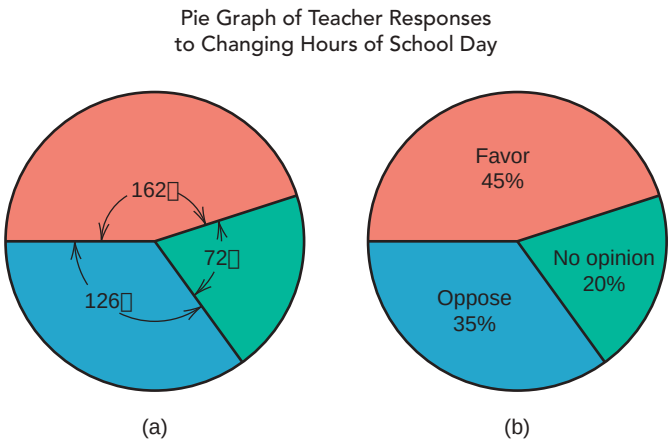


Figure 7.4

PICTOGRAPHS

A **pictograph** (see Figure 7.5 on the next page) is similar to a bar graph. The individual figures or icons that are used each represent the same value. For example, each stick figure in the pictograph represents 10,000 juveniles (persons under 18 years of age).* Notice how easily you can see increases and decreases in the numbers of juveniles for the given years.

**Statistical Abstracts of the United States,* 128th ed. (Washington, DC: Bureau of the Census, 2009), Table 315.

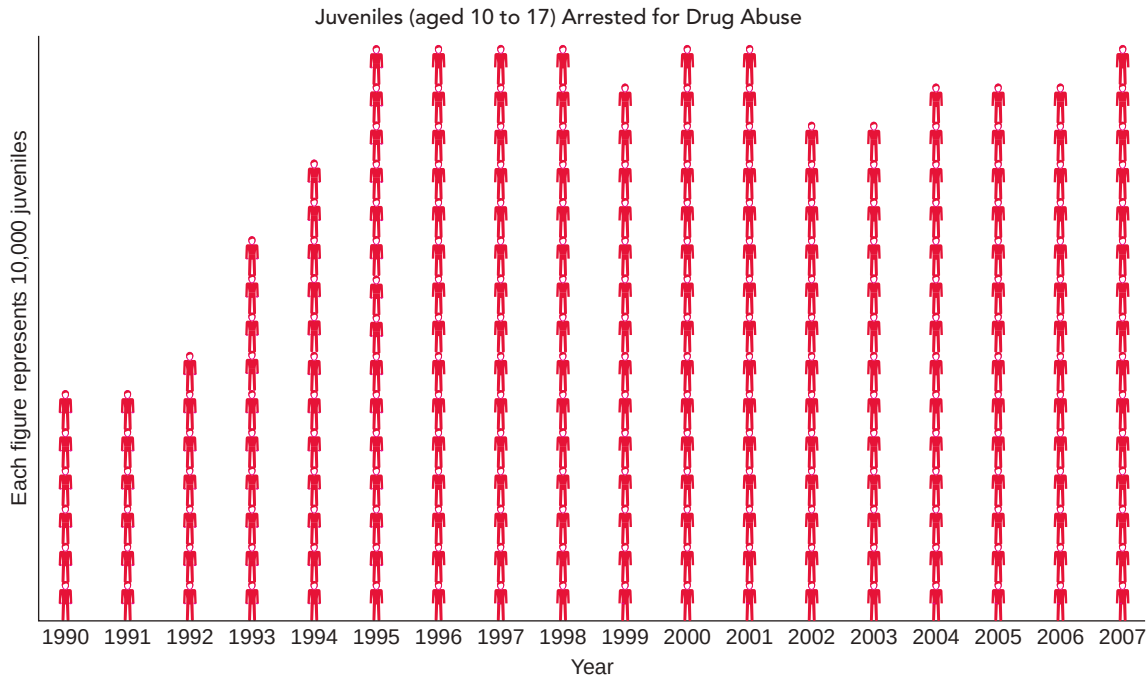


Figure 7.5

EXAMPLE B The number of juveniles arrested for drug abuse in each of the years given in Figure 7.5 is rounded to the nearest 10,000.

1. How many fewer juveniles, to the nearest 10,000, were arrested for drug abuse in 1990 than in 2000?
2. To the nearest 10,000, what was the total number of juveniles arrested for drug abuse in 1990 through 1995?
3. Were there more juveniles arrested in the 4-year period from 1995 through 1998 or the 4-year period from 2004 through 2007?

Solution **1.** 90,000 **2.** 560,000 **3.** More in the period from 1995 through 1998

LINE PLOTS

The table on the next page shows the countries that won one or more gold medals at the 2008 Summer Olympics in Beijing, China. Some information can be spotted quickly from the table, such as determining the countries that won large numbers of gold medals, but details such as comparing the numbers of countries that won one, two, or three gold medals are more time-consuming.

To assist in analyzing and viewing the data in the table, a line plot has been drawn in Figure 7.6 on the next page. A **line plot** is formed by drawing a line, marking categories and recording data by placing a mark such as a dot or an X above the line for each value of the data.

A line plot is easy to construct and interpret, and it gives a clear graphical picture of the data. Also certain features of the data become more apparent from a line plot than from a table. Such features include **gaps** (large spaces in the data) and **clusters** (isolated groups of data).

Countries That Won at Least One Gold Medal in the 2008 Summer Olympics

Argentina	2	Georgia	3	Norway	3
Australia	14	Germany	16	Panama	1
Azerbaijan	1	Great Britain	19	Poland	3
Belarus	4	Hungary	3	Portugal	1
Belgium	1	India	1	Romania	4
Brazil	3	Indonesia	1	Russia	23
Bulgaria	1	Iran	1	Slovakia	3
Cameroon	1	Italy	8	Slovenia	1
Canada	3	Jamaica	6	South Korea	13
China	51	Japan	9	Spain	5
Cuba	2	Kazakhstan	2	Switzerland	2
Czech Republic	3	Kenya	6	Thailand	2
Denmark	2	Latvia	1	Tunisia	1
Dominican Rep	1	Mexico	2	Turkey	1
Estonia	1	Mongolia	2	Ukraine	7
Ethiopia	4	Netherlands	7	United States	36
Finland	1	New Zealand	3	Uzbekistan	1
France	7	North Korea	2	Zimbabwe	1

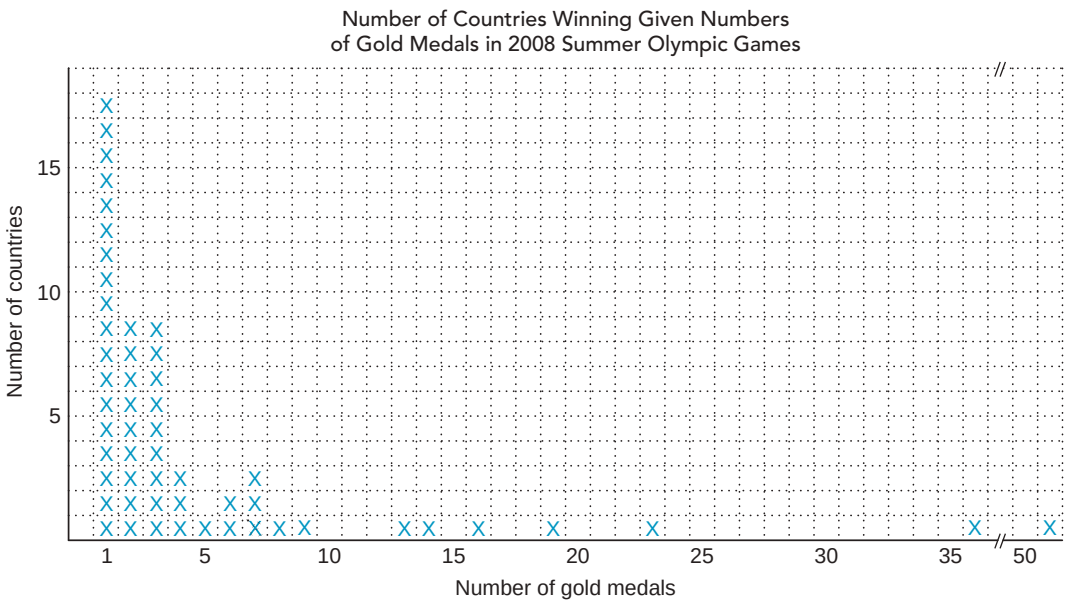


Figure 7.6

EXAMPLE C Use the line plot in Figure 7.6 (on the previous page) as you answer these questions.

1. There is one large cluster of data for the countries that won nine or fewer gold medals. How many countries are represented in this cluster?
2. The largest gap in the data occurs between which two numbers? How large is this gap?

Solution **1.** 47 **2.** Between 36 and 51. The size of the gap is 15.

STEM-AND-LEAF PLOTS

A **stem-and-leaf plot** is a quick numerical method of providing a visual summary of data where each data value is split into a **leaf** (usually the last digit) and a **stem** (the other digits). As the name indicates, this method suggests the stems of plants and their leaves. Consider the following test scores for a class of 26 students:

82 66 70 77 94 67 73 78 82 74 90 45 62
85 57 72 94 83 85 70 95 71 89 87 75 74

Since the scores in the preceding list range from the 40s to the 90s, the tens digits of 4, 5, 6, 7, 8, and 9 are chosen as the stems, and the unit digits of the numbers will represent the leaves (Figure 7.7). The first step in forming a stem-and-leaf plot is to list the stem values in increasing order in a column (see part a). Next, each leaf value is written in the row corresponding to that number's stem (part b). Here the leaf values have been recorded in the order in which they appear, but they could be listed in increasing order. For example, the leaves for stem 6 can be recorded as 2, 6, 7 rather than 6, 7, 2, as shown in Figure 7.7b. The stem-and-leaf plot shows at a glance the lowest and highest test scores and that the 70s interval has the greatest number of scores.

Stem	Leaf
4	
5	
6	
7	
8	
9	

(a)

Stem	Leaf
4	5
5	7
6	6 7 2
7	0 7 3 8 4 2 0 1 5 4
8	2 2 5 3 5 9 7
9	4 0 4 5

(b)

Figure 7.7

A stem-and-leaf plot shows where the data are concentrated and the extreme values. You may have noticed that this method of portraying data is like a bar graph turned on its side (rotate this page 90° counterclockwise). Although a stem-and-leaf plot is not as attractive as a bar graph, it has the advantage of showing all the original data. Furthermore, unlike a bar graph, it shows any gaps or clusters in the data.

A stem-and-leaf plot that compares two sets of data can be created by forming a central stem and plotting the leaves for the first set of data on one side of the stem and the leaves for the second set on the other side (see Figure 7.8 on the next page). This is called a **back-to-back stem-and-leaf plot.** Suppose the same class of students whose test scores are shown above obtains the following scores on a second test:

85 89 70 76 49 66 71 71 75 82 73 77 68
79 55 91 52 63 64 84 81 68 73 67 66 72

A back-to-back stem-and-leaf plot of the scores on both tests is shown in Figure 7.8. In this plot the leaves for both sets of scores have been arranged in order to aid in comparing the test scores. It appears that overall performance was better on the first test. For example, the first test has almost twice as many scores above 80 and one-half as many scores below 70 as the second test.

Second Test Leaf	Stem	First Test Leaf
9	4	5
5 2	5	7
8 8 7 6 6 4 3	6	2 6 7
9 7 6 5 3 3 2 1 1 0	7	0 0 1 2 3 4 4 5 7 8
9 5 4 2 1	8	2 2 3 5 5 7 9
1	9	0 4 4 5

Figure 7.8

HISTOGRAMS

When data fall naturally into a few categories, as in Figure 7.1 on page 438, they can be illustrated by bar graphs or pie graphs. However, data are often spread over a wide range with many different values. In this case it is convenient to group the data in intervals.

The following list shows the gestation periods in days for 42 species of animals.

Ass 365	Deer 201	Moose 240
Baboon 187	Dog 61	Mouse 21
Badger 60	Elk 250	Opossum 15
Bat 50	Fox 52	Pig 112
Black bear 219	Giraffe 425	Puma 90
Grizzly bear 225	Goat, domestic 151	Rabbit 37
Polar bear 240	Goat, mountain 184	Rhinoceros 498
Beaver 122	Gorilla 257	Sea lion 350
Buffalo 278	Guinea pig 68	Sheep 154
Camel 406	Horse 330	Squirrel 44
Cat 63	Kangaroo 42	Tiger 105
Chimpanzee 231	Leopard 98	Whale 365
Chipmunk 31	Lion 100	Wolf 63
Cow 284	Monkey 165	Zebra 365

Since there are many different gestation periods, we group them in intervals. The intervals should be nonoverlapping, and their number is arbitrary but usually a number from 5 to 15. One method of determining the length of each interval is to first compute the difference between the highest and lowest values, which is $498 - 15 = 483$. Then select the desired number of intervals and determine the width of the interval. If we select 10 as the number of intervals, then

$$483 \div 10 = 48.3$$

and we may choose 49 (because of its convenience) as the width of each interval (note $49 - 0 = 99 - 50 = 149 - 100 = 49$). Figure 7.9 lists the number of animals in each interval and is called a **frequency table.**

Frequency Table

Interval	0–49	50–99	100–149	150–199	200–249	250–299	300–349	350–399	400–449	450–499
Frequency	6	9	4	5	6	4	1	4	2	1

Figure 7.9

The graph for the grouped data in the frequency table is shown in Figure 7.10. This graph, which is similar to a bar graph, is called a **histogram.** A histogram is made up of adjoining bars that have the same width, and the bars are centered above the midpoints of the intervals or categories. The vertical axis shows the frequency of the data for each interval or category on the horizontal axis. We can see from this histogram that the greatest number of gestation periods occurs in the interval from 50 to 99 days, and there are only a few animals with gestation periods over 400 days.

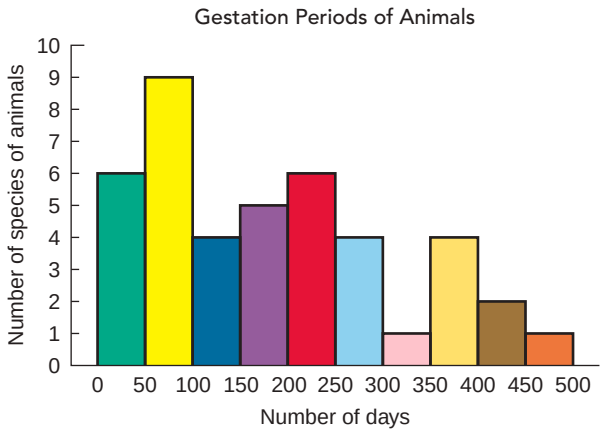


Figure 7.10

LINE GRAPHS

Another method of presenting data visually is the *line graph*. A **line graph** is a sequence of points connected by line segments and is often used to show changes over a period of time. For example, the line graph in Figure 7.11 shows the increase in the U.S. population from 1800 to 2020 at 20-year intervals with an estimate for 2020 of 332 million people.

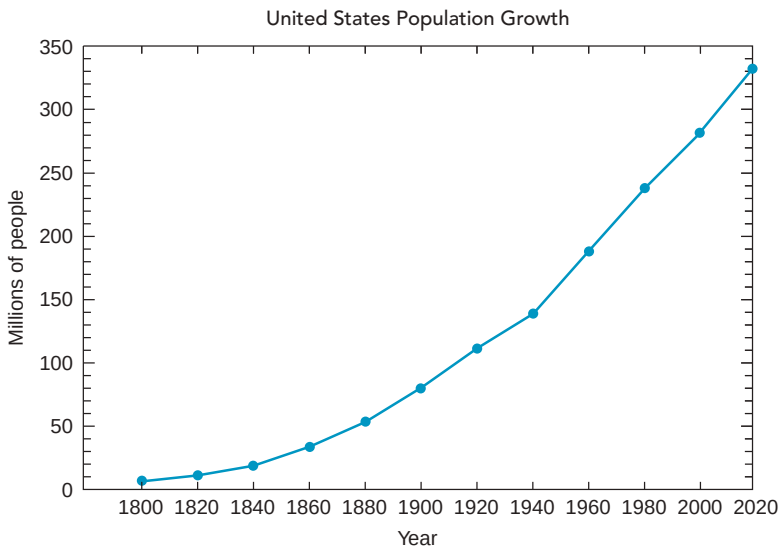


Figure 7.11

EXAMPLE D Use the line graph in Figure 7.11 to answer these questions.

1. What was the approximate population increase from 1880 to 1920?
2. Compare the population change for the period from 1800 to 1900 to the population change from 1980 to 2020. Which period had the greater increase in population?

Solution **1.** 55 million **2.** 1980 to 2020

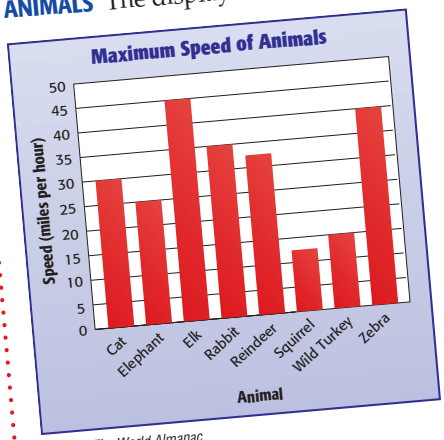
2-8 Selecting an Appropriate Display

MAIN IDEA
Select an appropriate display for a set of data.

Math Online
glencoe.com
- Extra Examples
- Personal Tutor
- Self-Check Quiz

GET READY for the Lesson

ANIMALS The displays show the maximum speed of eight animals.



Source: *The World Almanac*

Maximum Speed of Animals

Stem	Leaf
1	2 5
2	5
3	0 2 5
4	0 5

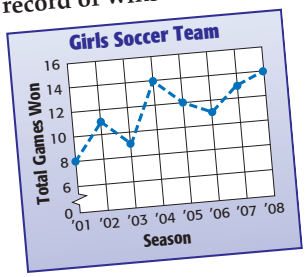
3|0 = 30 mi/h

1. Which display allows you to find a rabbit's maximum speed?
2. In which display is it easier to find the range of the data?

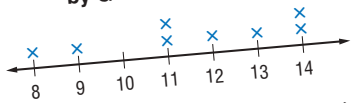
Data can often be displayed in several different ways. The display you choose depends on your data and what you want to show.

EXAMPLE 1 Choose Between Displays

SOCCER Which display allows you to see whether the team's record of wins has steadily improved since 2001?



Games Won Per Season by Girls Soccer Team



The line graph shows the change in games won from season to season, revealing some declines in the number of wins.

CHECK Your Progress

a. **SOCCER** Which of the above displays allows you to easily find the number of seasons in which the team won 11 or more games?

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SCATTER PLOTS

Consider the following table, which records the heights and corresponding shoe sizes of 30 fourth-grade to eighth-grade boys. It is difficult to see any patterns or relationships between the heights and shoe sizes from this information.

Height (inches)	59	71	57	72	64	60	64	62	66	63	74	60	67
Shoe size	6.5	11.5	4	10.5	9.5	5	7.5	8.5	9	7	11.5	4.5	8

Height (inches)	64	65	62	56	69	61	58	62	63	67	69	64	68	60	58	66	57
Shoe size	6.5	12	6	4.5	10	7	4	5	5.5	10	9	6	10.5	6.5	3.5	8.5	5

NCTM Standards

Students should see a range of examples in which plotting data suggests linear relationships, nonlinear relationships, and no apparent relationship at all. When a scatter plot suggests that a relationship exists, teachers should help students determine the nature of the relationship from the shape and direction of the plot. p. 253

The pairs of numbers in the table have been graphed in Figure 7.12, where the first coordinate (horizontal axis) of each point on the graph is a height and the second coordinate (vertical axis) of the point is the corresponding shoe size. Such a graph is called a **scatter plot.** The scatter plot enables us to see if there are any patterns or trends in the data. Although there are boys who have larger shoe sizes than some of the boys who are taller, in general it appears that taller boys have larger shoe sizes.

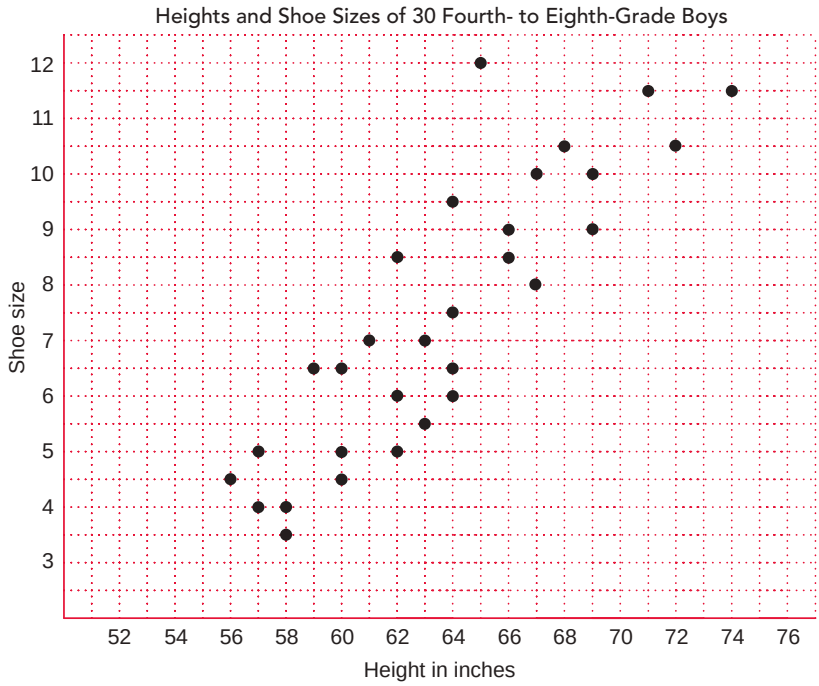


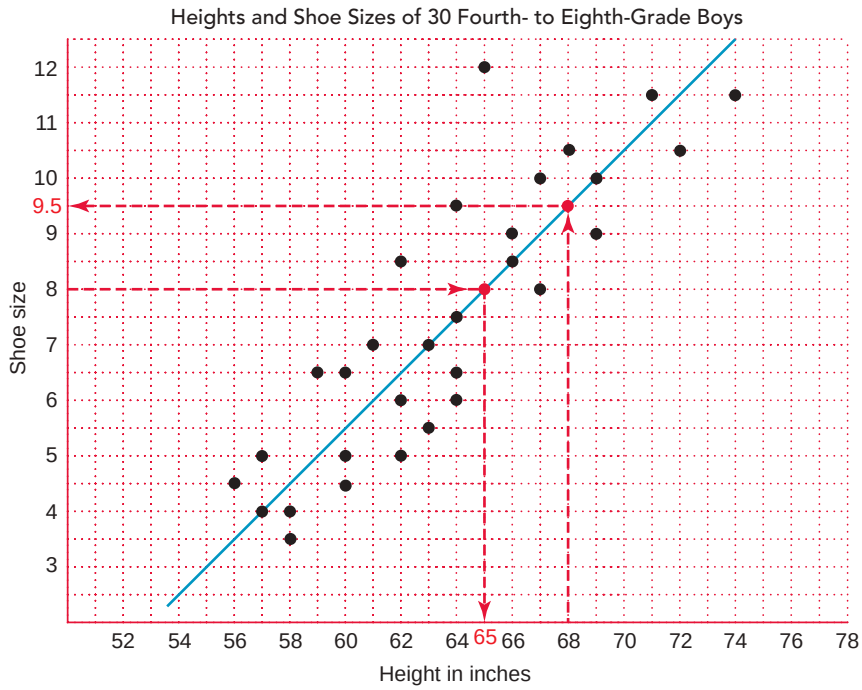
Figure 7.12

Trend Lines A straight line can be drawn from the lower left to the upper right that approximates the points of the graph in Figure 7.12. A line that approximates the location of the points of a scatter plot is called a **trend line.** One method of locating a trend line is to place a line so that it approximates the location of the points and there are about the same number of points of the graph above the line as below.

EXAMPLE E

1. Draw a trend line for the scatter plot in Figure 7.12.
2. Use your trend line to predict the shoe size for a boy of height 68 inches and the height of a boy with a shoe size of 8.

Solution **1.** While different people may select different locations for a trend line, these lines will be fairly close to the line shown on the scatter plot below. **2.** Approximately 9.5; approximately 65 inches.



Some scatter plots, such as the one in Figure 7.13a, may show *no correlation* between the data. Or, if the trend line goes from lower left to upper right, as for Figure 7.13b, there is a **positive correlation** and the slope of line is positive. If the trend line goes from upper left to lower right, as for Figure 7.13c, there is a **negative correlation** and the slope of line is negative (see pages 85–86 for a discussion of positive and negative slopes). When data are entered into a graphing calculator or computer, the value of a variable r will be computed that indicates the strength of the association between the data. This number is called a **correlation coefficient** and it varies from $^{-}1$ to 1 ($^{-}1 \le r \le 1$). If r is close to 0, there is **little or no correlation.** If r is close to 1, there is a **strong positive correlation;** and if r is close to $^{-}1$, there is a **strong negative correlation.**

The scatter plot in Example E shows a positive correlation between the heights of the boys and their shoe sizes: as heights increase, shoe sizes increase. In Example F on the next page, there is a negative correlation between the two types of data.

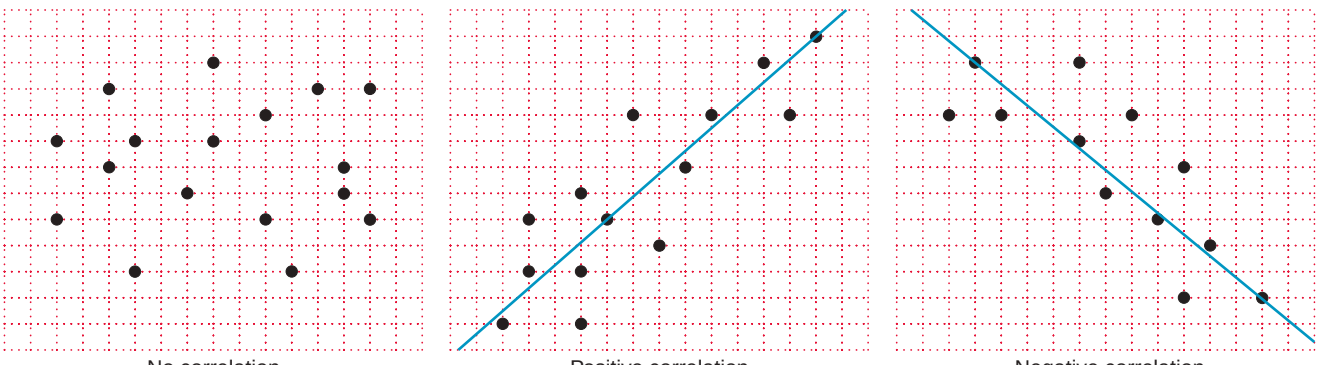


Figure 7.13

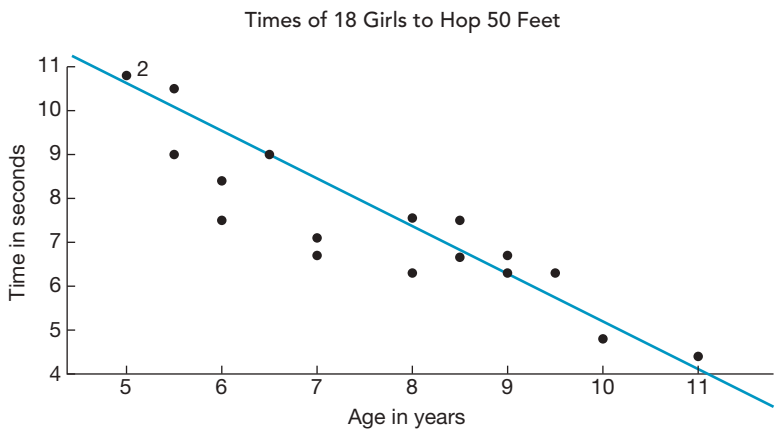
EXAMPLE F

The following table contains data on one aspect of child development—the time required for 18 girls to hop a given distance.* The age of each girl is rounded to the nearest half-year.

Age (years)	5	5	5.5	5.5	6	6	6.5	7	7	8	8	8.5	8.5	9	9	9.5	10	11
Time (seconds) to hop 50 feet	10.8	10.8	10.5	9.0	8.4	7.5	9.0	7.1	6.7	7.5	6.3	7.5	6.8	6.7	6.3	6.3	4.8	4.4

1. Form a scatter plot of these data. Mark intervals for ages on the horizontal axis and intervals for time on the vertical axis.
2. Locate a trend line.
3. Use your line to predict the time for a $7\frac{1}{2}$-year-old girl to hop 50 feet.
4. The negative (downward) slope of your line shows a correlation between the age of the girl and the time required to hop 50 feet. Describe this correlation.

Solution **1.** The "2" in the following scatter plot indicates that two children of the same age required the same time to hop 50 feet. **2.** Counting the point labeled "2" as two points, the trend line sketched below has 2 points on the line (marked by red dots), 8 points above the line, and 8 points below the line. **3.** Approximately 8 seconds **4.** The older a 5- to 11-year-old girl becomes, the less time is required to hop 50 feet.



CURVES OF BEST FIT

Technology Connection

Most graphing calculators and some computer software such as Excel and Minitab have graphing features that include scatter plots and trend lines or **curves of best fit.** A graphing calculator screen is shown in Figure 7.14 with the scatter plot and trend line for the data in Example F.

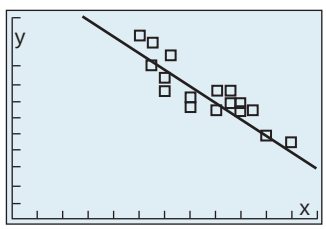


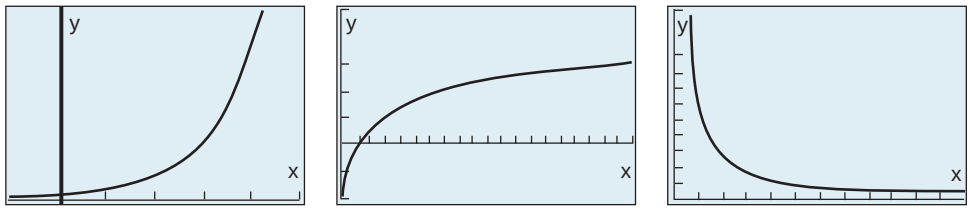
Figure 7.14

If you have a graphing calculator or a computer with suitable software, you may wish to enter the data from some of the preceding examples and obtain the trend lines. The

*Adapted from Kenneth S. Holt, *Child Development*, p. 143.

calculator or computer will print the equation of a trend line $y = ax + b$, where a is the slope of the line and b is the y intercept. You may find it interesting to use such equations to obtain new predictions. For example, the equation of the trend line in Figure 7.14 with the slope and y intercept rounded to the nearest hundredth is $y = {}^{-}.92x + 14.45$. Using this equation with $x = 7.5$, approximately how much time is required for a $7\frac{1}{2}$-year-old girl to hop 50 feet? Calculate this time and compare it to the time of 8 seconds obtained from the trend line in Example F.

Sometimes the *curve of best fit* for a scatter plot is not a straight line. Graphing calculators and some computer software have several types of curves of best fit. The equations for curves and trend lines are called **regression equations,** and such equations are algebraic models for approximating the location of points in a scatter plot. In addition to a straight line, the three types of curves shown in Figure 7.15 are common curves of best fit.



Exponential curve (a) | Logarithmic curve (b) | Power curve (c)

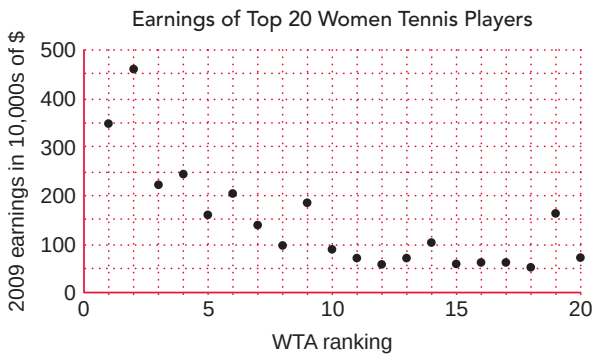
Figure 7.15

A curve of best fit often can be obtained by visualizing a curve that approximates the location of points on a graph, as in Example G.

EXAMPLE G

The scatter plot shows the end-of-year WTA rankings and earnings of the top 20 women tennis players in 2009.

1. Which type of curve from Figure 7.15 best fits the points of this graph?
2. Visualize this curve and use it to predict the earnings for the woman tennis player who ranked 22nd at the end of 2009.



Solution **1.** The power curve. **2.** About 60 × $10,000 or $600,000.

PROBLEM-SOLVING APPLICATION

A strong correlation between data does not necessarily imply that one type of measurement causes the other. Cigarette consumption and coronary heart disease (CHD) mortality rates (see the problem-solving application that follows) are an example of a strong association between data that has generated debate over cause and effect between these measurements.

Problem

The table for the following problem lists the number of deaths for each 100,000 adults due to coronary heart disease and the average number of cigarettes consumed per adult per year for 20 countries. Given this information, what is the number of deaths per 100,000 people due to coronary heart disease for a country that consumes 1000 cigarettes per adult per year?

Understanding the Problem The cigarette consumption and mortality rates that are given for 20 countries in the table below and on the following page do not have a cigarette consumption of 1000 per adult per year. The problem requires using the graph on page 452 to predict the mortality rate due to coronary heart disease for a country with a consumption of 1000 cigarettes per adult per year. **Question 1:** Which country in the table has a cigarette consumption that is closest to 1000?

Devising a Plan One approach is to use the data in the table for Finland and Germany, the two countries whose cigarette consumption is closest to 1000. **Question 2:** How can the data for these countries be used, and what approximation do you obtain by your method? Another approach is to use a scatter plot of the data in the table and a trend line to make a prediction.

Carrying Out the Plan The scatter plot and trend line on the next page are for the data in the table. **Question 3:** Given this trend line, what is the coronary heart disease mortality rate for a country that consumes an average of 1000 cigarettes per adult per year?

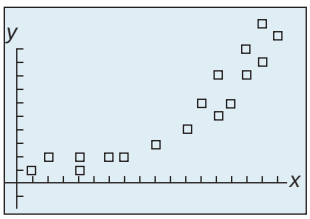
Country	Cigarette Consumption per Adult per Year	Mortality Rate per 100,000 People
Australia	1130	111
Austria	1684	110
Belgium	1763	65
Canada	897	95
Czech Republic	2368	149
Denmark	1495	106
Finland	956	144
France	876	40
Germany	1125	106
Hungary	1623	193
Ireland	1391	153

(continues)

Technology Connection

Trend Lines

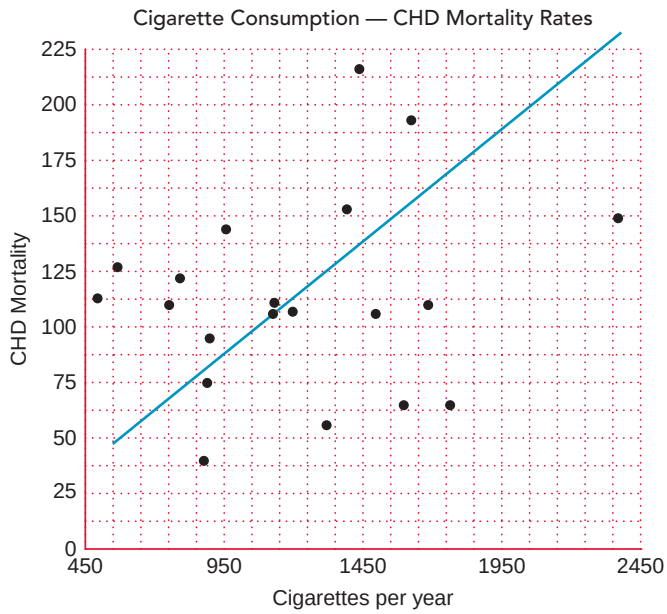
Try to predict the curve of best fit for the points plotted below. By entering the coordinates of the points for this graph, (1, 1), (2, 2), (4, 1), (4, 2), etc., into a graphing calculator and using the statistical menu, you can obtain the curve of best fit and its correlation coefficient. This is one of the graphs you will explore using your calculator in this investigation.



Mathematics Investigation
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(continued)

Country	Cigarette Consumption per Adult per Year	Mortality Rate per 100,000 People
Italy	1596	65
Netherlands	888	75
New Zealand	565	127
Norway	493	113
Portugal	1318	56
Sweden	751	110
Slovakia	1436	216
United Kingdom	790	122
United States	1196	107



Looking Back The trend line also enables predictions regarding the cigarette consumption for a country, if the coronary heart disease mortality rate is known. **Question 4:** What is the average number of cigarettes consumed per adult per year by a given country whose coronary heart disease mortality rate is 200 for every 100,000 people?

Answers to Questions 1–4 **1.** Finland. **2.** One possibility is 125, the mean of 144 and 106. Another possibility is 87.5, since 1000 is approximately 27 percent of the distance between 493 (minimum consumption) and 2368 (maximum consumption) and 87.5 is approximately 27 percent of the distance between 40 (minimum rate) and 216 (maximum rate). **3.** Approximately 100. **4.** Approximately 1750.

Sometimes the *looking-back* part of solving a problem involves using a different approach. Figure 7.16 shows a computer printout of the scatter plot for the data in the preceding table using Excel. Notice that the trend line is in about the same position as the one shown in the preceding scatter plot. The equation for this line, $y = .0182x + 90.943$, is displayed on the screen by the computer. Using this equation, we can obtain another prediction of the number of coronary heart disease mortalities for a country that consumes an average of 1000 cigarettes per adult per year:

$$y = .0182(1000) + 90.943 = 109.143$$

To the nearest whole number, this is 109 deaths for each 100,000 people. Compare this to the prediction we obtained from drawing the trend line in the preceding scatter plot.

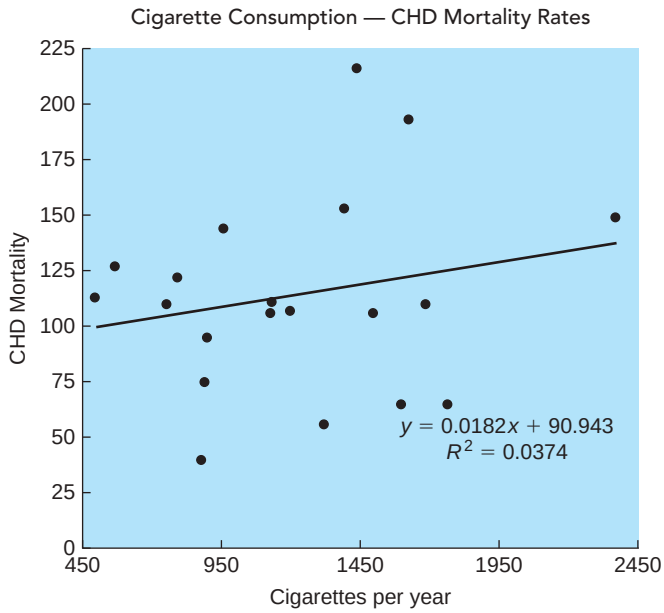


Figure 7.16

Summary Bar and pie graphs, pictographs, line plots, stem-and-leaf plots, and histograms provide visual descriptions for interpreting data that involve **one variable,** that is, data with one type of measurement. For example, the pictograph in Figure 7.5 on page 441 records the number of juveniles arrested for drug abuse, and there is one variable—*number of arrests.* Often we wish to compare two or more sets of data that involve one type of measurement (one variable) and double-bar or triple-bar graphs, and back-to-back stem-and-leaf plots are used for this purpose. As examples, in Figure 7.3 on page 439 a triple-bar graph compares data on three sets of children, and in Figure 7.8 on page 444 a back-to-back stem-and-leaf plot compares two sets of test scores.

A line graph and a scatter plot, on the other hand, provide a visual description of data that involve **two variables,** that is, data with two different types of measurement. As examples, the line graph in Figure 7.11 on page 445 plots population for given years and the two variables are *numbers of people* and *years,* and in Figure 7.12 on page 447 the heights and shoe sizes of boys were graphed on a scatter plot, and the two variables are *height* and *shoe size.*

Usually any one of several graphical methods can be chosen for one-variable sets of data, but there are some general guidelines. Bar graphs, pie graphs, and pictographs are best chosen when there are relatively small numbers of categories, such as 3 to 10. A histogram is often used for grouped data and 10 to 12 is a convenient number of groups.

A line plot is used for plotting intermediate numbers of data, such as 25 to 50. Stem-and-leaf plots accommodate a greater number of data, such as 20 to 100. To compare two sets of data with a back-to-back stem-and-leaf plot, there should be approximately the same number of values on both sides of the stem. Line plots and stem-and-leaf plots have an advantage over bar graphs and histograms in showing individual values of data, gaps (large spaces between data points), clusters (groupings of data points), and data points substantially larger or smaller than other values.

Exercises and Problems 7.1

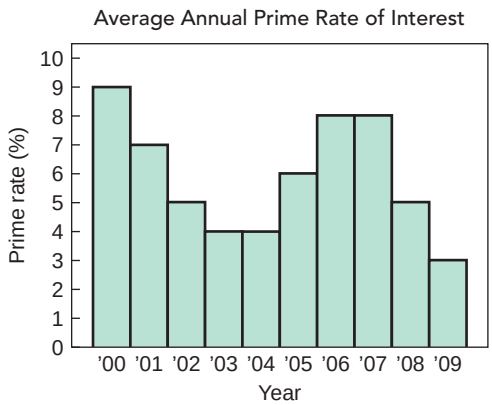


"We understand you tore the little tag off your mattress."

The numbers of troops (in thousands) in active service in 2008 in the four major branches of the service are as follows: Army, 544; Navy, 332; Air Force, 327; and Marine Corps, 199.* Use this information in exercises 1 and 2.

1. Draw a bar graph for the data.†
   - **a.** The number of people in the Army is how many times the number of people in the Marine Corps?
   - **b.** How many more people are in the Navy than in the Marine Corps?
2. Draw a pie graph for the data.
   - **a.** What is the measure of the central angle for each of the four regions of the pie graph?
   - **b.** What is the total number of people in the four branches of the service?

The following graph shows the average annual prime rate of interest (to the nearest whole percent) charged by banks for each of the years from 2000 to 2009.‡ Use this graph in exercises 3 and 4.



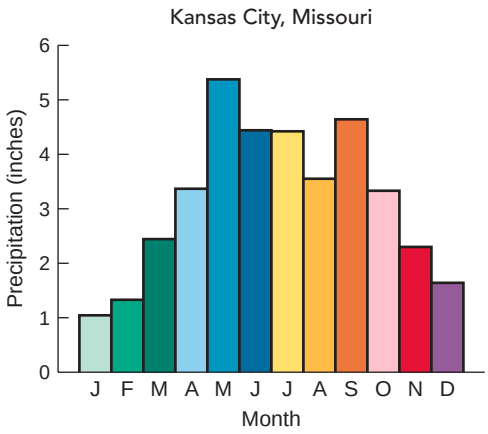
3. **a.** In which year was the prime rate the highest, and what was the rate?
   b. In which years did the prime rate increase, and how much was the increase?
4. **a.** In which year was the prime rate the lowest, and what was the rate?
   b. In which years did the prime rate decrease, and how much was the decrease?
5. A family's monthly budget is divided as follows: rent, 32 percent; food, 30 percent; utilities, 15 percent; insurance, 4 percent; medical, 5 percent; entertainment, 8 percent; other; 6 percent.
   - **a.** Draw a pie graph of the data.
   - **b.** What is the measure of the central angle (to the nearest degree) in each of the seven regions of the graph?

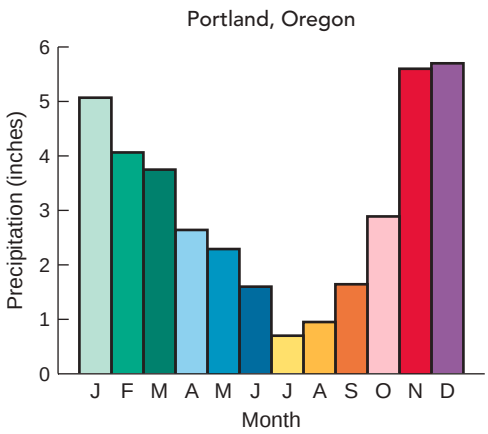
**Statistical Abstract of the United States,* 128th ed. (Washington, DC: U.S. Department of Defense, 2009), Table 498.

†Copy the rectangular grid from the website for the bar graphs and histograms on these pages.

‡*Statistical Abstract of the United States,* 128th ed. (Washington, DC: Bureau of the Census, 2009), Table 1160.

The following graphs show the average monthly amounts of precipitation for Kansas City, Missouri, and Portland, Oregon, over the course of a single year. Use these graphs in exercises 6 and 7.





6. a. In Portland, which two months had the greatest amounts of precipitation?

b. In Kansas City, which month had the least amount of precipitation?

7. a. Compare the amounts of precipitation during the summer months (June, July, and August). Which city had the most precipitation during the summer?

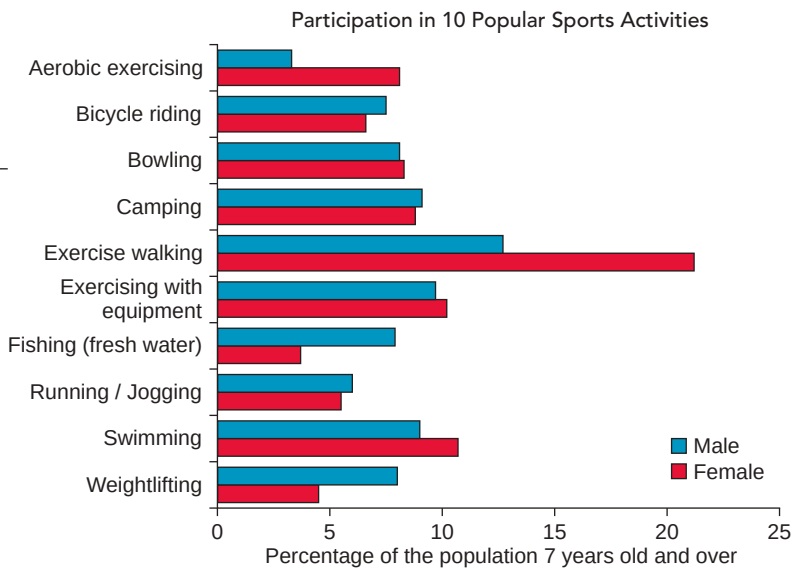
b. Rounding the amount of precipitaion for each month to the nearest whole number, determine the approximate amount of precipitation for each city for the year.

Sometimes the bars on a graph are placed horizontally rather than vertically. The following double-bar graph compares the percentages of participation of males and females over 7 years of age in the 10 most popular sports activities.* Use this graph in exercises 8 and 9.

8. a. In which sports activities do the females have the greater percent of participation?

b. In which sports activities is the percent of participation by females more than double the percent of participation by males?

c. In which sports activities is the participation by males about 1 percent greater than the participation by females?



**Statistical Abstract of the United States,* 128th ed. (Washington, DC: Bureau of the Census, 2009), Table 1212.

9. a. In which sports activities do the males have the greater percent of participation?
b. In which sports activities is the percent of participation by males more than double the percent of participation by females?
c. In which sports activity is the percent of participation by females almost 5 percent greater than the percent of participation by males?

The following triple-bar graph shows the birth rates among adolescent females by age and race.* Use this information in exercises 10 and 11.

10. a. Which ethnic groups have a difference of over 50 births (per 1000 females) between 18- to 19-year-olds and 15- to 17-year-olds?
b. What is the birth rate for all girls 10 to 14 years of age?
c. Which ethnic group has the smallest change in birth rates between 10- to 14-year-olds and 15- to 17-year-olds?

11. a. Which ethnic group has the greatest change in birth rates between 10- to 14-year-olds and 15- to 17-year-olds?
b. Which ethnic group has the least number of births for ages 10 to 19? What is this birth rate?
c. For which ethnic group is the change in birth rates between 15- to 17-year-olds and 18- to 19-year-olds about 70 births per 1000 females?

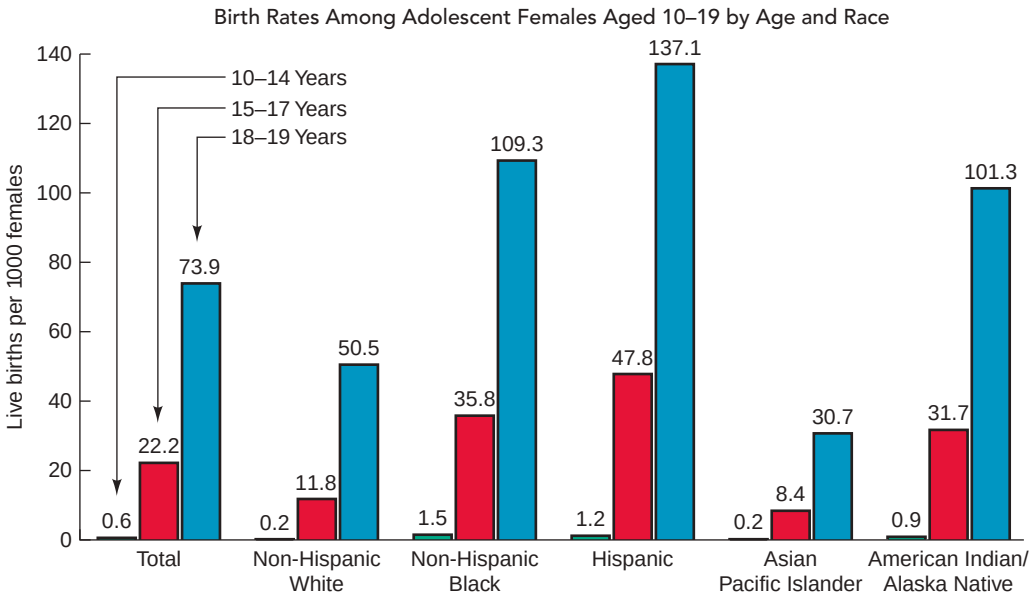
The following table shows the percentage of health-care coverage for children under 18 years of age and for children in poverty under 18 years of age.† Use this information in exercises 12 and 13.

	Children under Age 18	Children in Poverty under Age 18
No coverage	11.2%	17.6%
Public assistance	23.7%	70.1%
Private insurance	72.7%	17.2%

Note: Percentages add to more than 100 because some individuals receive coverage from more than one source.

12. a. Form a double-bar graph for the following three categories: no coverage, public assistance, and private insurance on the horizontal axis.
b. What is the difference between the percent of private insurance for all children under age 18 and the percent of private insurance for children in poverty under age 18?

13. a. Form a double-bar graph for the following three categories: coverage, no public assistance, and no private insurance on the horizontal axis.
b. What is the difference between the percent of no public assistance for children in poverty under 18 years of age and the percent of no public assistance for all children under age 18?

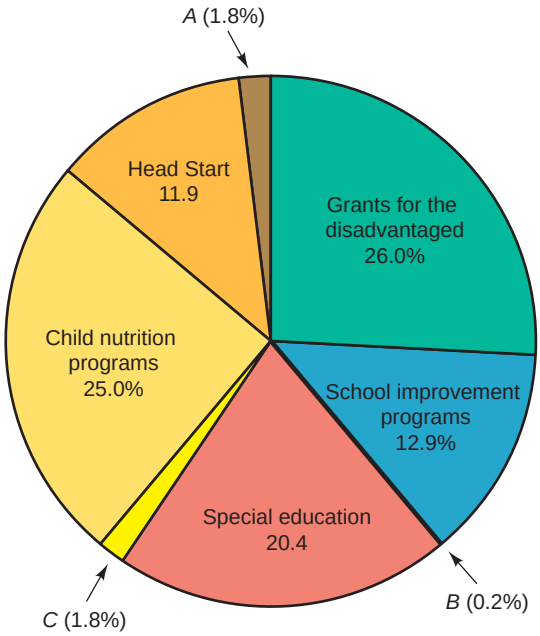


*U.S. Department of Health and Human Services, *Child Health USA 2008–2009* (Washington, DC: U.S. Government Printing Office, 2009), p. 40.

†U.S. Department of Health and Human Services, *Child Health USA 2008–2009* (Washington, DC: U.S. Government Printing Office, 2009), p. 54.

This pie graph represents the percentages of federal funds spent on a variety of programs for children. Use this graph in exercises 14 and 15.

A: Social security student benefits
B: Native American education
C: Overseas dependents' schools



14. a. What was the total percent spent on special and Native American education?
b. The amount of money spent on child nutrition programs was how many times to the nearest .1 the amount spent on Head Start?

15. a. What was the total percent spent on school improvement programs and overseas dependents' schools?
b. The amount of money spent on grants for the disadvantaged was how many times to the nearest .1 the amount spent on social security student benefits?

The following table shows the percentage to the nearest whole number of elementary schools, at a certain time, for various size categories.* Use this table in exercises 16 and 17.

Size of School (Number of Students)	Percentage of Schools
Under 100	6
100 to 199	9
200 to 299	13
300 to 399	16
400 to 499	16
500 to 599	13
600 to 699	9
700 to 799	6
800 to 899	7
900 to 999	0
1000 or more	5

16. a. Draw a pie graph of these data. Label the measure to the nearest degree of the central angle for each region of the graph.
b. What percentage of elementary schools has 500 or more students?

17. a. Draw a bar graph of these data.
b. What percentage of elementary schools has from 200 to less than 500 students?

18. The number of computers in public schools for student instruction is contained in the following table.†

School Level	Number of Computers
Elementary	5,612,000
Middle/Junior High	2,503,000
Senior High	4,067,000
K–12 other	733,000

a. Form a pictograph of the data in the table by choosing an icon and selecting the number of computers that the icon represents. Label the categories on the horizontal axis, and select an informative title for the graph.
b. Discuss the reasons for your choice of value for the icon. In general, what disadvantages result if the value of the icon is too large or too small?

**Statistical Abstract of the United States,* 128th ed. (Washington, DC: Bureau of the Census, 2009), Table 217.

†*Statistical Abstract of the United States,* 128th ed. (Washington, DC: Bureau of the Census, 2009), Table 254.

19. The following table shows the number of people, to the nearest 1000, in different age categories who were involved in automobile crashes.*

Age Levels	Number of People
16 to 19	10,034,000
20 to 24	17,173,000
25 to 34	35,712,000
35 to 44	40,322,000
45 to 54	40,937,000
55 to 64	30,355,000
65 to 74	17,246,000
75 and older	13,321,000

a. Form a pictograph of the data in the table by choosing an icon and selecting the number of people that the icon represents. Label the categories on the horizontal axis, and select an informative title for the graph.

b. What does the pictograph show about the total number of people aged 16 to 24 years who had crashes compared to the number of people aged 25 to 34 years?

20. The numbers of students in thousands in public schools, for a particular year, in grades K–8 are shown by states in the following table.

State	Students (1000)	State	Students (1000)
Alabama	536	Florida	1797
Alaska	95	Georgia	1075
Arizona	672	Hawaii	132
Arkansas	318	Idaho	171
California	4480	Illinois	1484
Colorado	529	Indiana	711
Connecticut	410	Iowa	330
Delaware	81	Kansas	322
District of Columbia	58	Kentucky	473
		Louisiana	537
Maine	144	Ohio	1287
Maryland	611	Oklahoma	446
Massachusetts	699	Oregon	382
Michigan	1223	Pennsylvania	1255
Minnesota	573	Rhode Island	113
Mississippi	362	South Carolina	500
Missouri	643	South Dakota	87
Montana	103	Tennessee	675
Nebraska	195	Texas	3016
Nevada	262	Utah	338
New Hampshire	144	Vermont	69
New Jersey	972	Virginia	826
New Mexico	225	Washington	696
New York	2017	West Virginia	200
North Carolina	956	Wisconsin	592
North Dakota	70	Wyoming	59

a. Form a line plot for the numbers of students in each state by marking off the horizontal axis in intervals of 100 students, that is, 0–99, 100–199, 200–299, etc. (*Note:* To accommodate all the intervals, you may find it convenient to place breaks in the horizontal axis.)

b. What percentage of the states to the nearest whole percent have less than 700,000 students?

c. What percentage of the states to the nearest whole percent have more than 2,000,000 students?

21. The following table shows the average annual salaries of elementary school teachers.†

Salaries (in thousands of dollars) of Elementary School Teachers

State	Salary	State	Salary
Alabama	$43.1	California	$63.6
Alaska	54.7	Colorado	45.5
Arizona	43.5	Connecticut	60.8
Arkansas	44.2	Delaware	54.9

(continues)

(continues)

**Statistical Abstract of the United States,* 128th ed. (Washington, DC: Bureau of the Census, 2009), Table 1077.

†*Statistical Abstract of the United States,* 128th ed. (Washington, DC: Bureau of the Census, 2009), Table 247.

(continued)

State	Salary	State	Salary
District of Columbia	$59.0	New Jersey	$58.7
Florida	45.3	New Mexico	42.6
Georgia	49.4	New York	59.6
Hawaii	51.9	North Carolina	46.1
Idaho	42.8	North Dakota	39.1
Illinois	55.8	Ohio	51.9
Indiana	49.1	Oklahoma	41.6
Iowa	43.3	Oregon	50.7
Kansas	43.4	Pennsylvania	55.0
Kentucky	43.4	Rhode Island	56.0
Louisiana	42.8	South Carolina	42.2
Maine	42.1	South Dakota	35.5
Maryland	56.7	Tennessee	43.2
Massachusetts	58.3	Texas	44.5
Michigan	55.5	Utah	41.2
Minnesota	49.7	Vermont	45.3
Mississippi	39.6	Virginia	45.3
Missouri	41.8	Washington	47.9
Montana	41.2	West Virginia	40.4
Nebraska	42.0	Wisconsin	48.0
Nevada	45.2	Wyoming	50.4
New Hampshire	46.5		

a. Form a line plot for the 51 teachers' salaries by using intervals of $1000 on the horizontal axis, that is, 35,000–35,999, 36,000–36,999, 37,000–37,999, etc.
b. What is the interval with the most salaries represented in the line plot?
c. The average salary of the elementary school teachers is $47,800. What percent of the salaries, to the nearest .1 percent, represented in the line plot is less than $47,800?

22. The following 40 scores are from a college mathematics test for elementary school teachers.

92, 75, 78, 90, 73, 67, 85, 80, 58, 87, 62, 74, 74, 76, 89, 95, 72, 86, 80, 57, 89, 97, 65, 77, 91, 83, 71, 75, 67, 68, 57, 86, 62, 65, 72, 75, 81, 72, 76, 69

a. Form a stem-and-leaf plot for these test scores.
b. How many scores are below 70?
c. What percent of the scores are scores greater than or equal to 80?

23. The life spans in years of the 38 U.S. presidents from George Washington to Gerald Ford are listed here.

67, 90, 83, 85, 73, 80, 78, 79, 68, 71, 53, 65, 74, 64, 77, 56, 66, 63, 70, 49, 57, 71, 67, 71, 58, 60, 72, 67, 57, 60, 90, 63, 88, 78, 46, 64, 81, 93

a. Form a stem-and-leaf plot of these data.
b. What percent, to the nearest .1 percent, of the 38 presidents lived 80 years or more?
c. What percent, to the nearest .1 percent, of the 38 presidents did not live 60 years?

24. The following test scores are for two classes that took the same test. (The highest possible score on the test was 60.)

Class 1 (24 scores): 34, 44, 53, 57, 19, 50, 41, 56, 38, 27, 56, 49, 39, 24, 41, 50, 45, 47, 35, 51, 40, 44, 48, 43

Class 2 (25 scores): 51, 40, 45, 28, 44, 56, 31, 33, 41, 34, 34, 39, 50, 36, 37, 32, 50, 22, 35, 43, 40, 50, 45, 33, 48

a. Form a back-to-back stem-and-leaf plot with one stem. Put the leaves for one class on the right side of the stem and the leaves for the other class on the left side. Record the leaves in increasing order.
b. Which class appears to have better performance? Support your answer.

25. The following data are the weights in kilograms of 53 third-graders.

19.3, 20.2, 22.3, 17.0, 23.8, 24.6, 20.5, 20.3, 21.8, 16.6, 23.4, 25.1, 20.1, 21.6, 22.5, 19.7, 19.0, 18.2, 20.6, 21.5, 27.7, 21.6, 21.0, 20.4, 18.2, 17.2, 20.0, 22.7, 23.1, 24.6, 18.1, 20.8, 24.6, 17.3, 19.9, 20.1, 22.0, 23.2, 18.6, 25.3, 19.7, 20.6, 21.4, 21.2, 23.0, 21.2, 19.8, 22.1, 23.0, 19.1, 25.0, 22.0, 24.2

a. Form a stem-and-leaf plot of these data, using 16 through 27 as stems and the tenths digits as the leaves. (*Note:* It is not necessary to write decimal points.)
b. Which stem value has the greatest number of leaves?
c. What are the highest and lowest weights?

26. The average annual per capita incomes by states are shown in the following table.*

Average Annual per Capita Income by States

State	Income
Alabama	$33,643
Alaska	43,321
Arizona	32,953
Arkansas	31,266
California	42,696
Colorado	42,377
Connecticut	56,248
Delaware	40,852
D.C.	64,991
Florida	39,070
Georgia	33,975
Hawaii	40,490
Idaho	32,133
Illinois	42,397
Indiana	34,103
Iowa	36,680
Kansas	37,978
Kentucky	31,826
Louisiana	36,271
Maine	35,381
Maryland	48,091
Massachusetts	50,735
Michigan	35,299
Minnesota	42,772
Mississippi	29,569
Missouri	35,228
Montana	34,256
Nebraska	37,730
Nevada	40,353
New Hampshire	42,830
New Jersey	50,919
New Mexico	$32,091
New York	48,076
North Carolina	34,439
North Dakota	39,321
Ohio	35,511
Oklahoma	36,899
Oregon	35,956
Pennsylvania	40,265
Rhode Island	41,008
South Carolina	31,884
South Dakota	37,375
Tennessee	34,330
Texas	38,575
Utah	30,291
Vermont	38,880
Virginia	42,876
Washington	42,356
West Virginia	30,831
Wisconsin	37,314
Wyoming	49,719

(continues)

a. Form a frequency table for the salary data, using the following intervals: 29,000–29,999, 30,000–30,999, etc.

b. Draw a histogram for the salary data for the intervals in part a. Use income in thousands of dollars for the horizontal scale.

c. Which interval has the greatest frequency of incomes?

d. In how many states was the per capita income greater than $32,000?

e. In how many states was the per capita income less than $36,000?

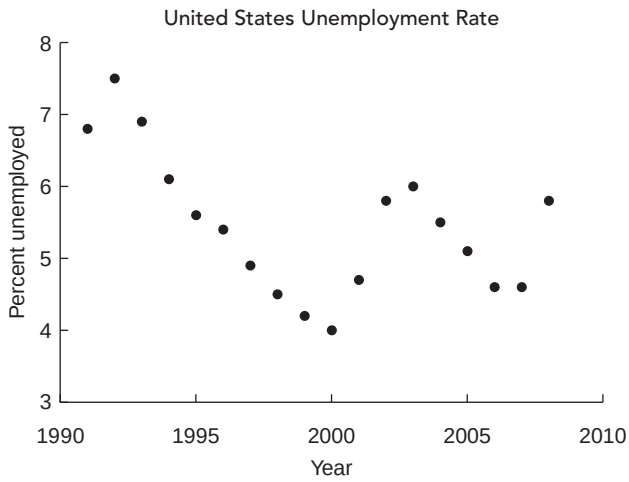
**Statistical Abstract of the United States,* 128th ed. (Washington, DC: Bureau of the Census, 2009), Table 665.

27. The following table records the amounts of snowfall (to the nearest inch) for selected cities in one year.

City	Snowfall	City	Snowfall
Juneau	97	St. Louis	20
Denver	60	Great Falls	58
Hartford	49	Omaha	30
Wilmington	21	Reno	24
Washington	17	Concord	64
Boise	21	Atlantic City	16
Chicago	38	Albany	64
Peoria	25	Buffalo	94
Indianapolis	23	New York	28
Des Moines	33	Bismarck	44
Wichita	16	Cincinnati	24
Louisville	17	Cleveland	56
Portland, ME	71	Pittsburgh	44
Baltimore	21	Providence	36
Boston	42	Salt Lake City	59
Detroit	41	Sault Ste. Marie	118
Burlington	78	Seattle-Tacoma	11
Duluth	81	Spokane	49
Minneapolis	50	Charleston	34

a. Form a frequency table for the snowfall data, using the following intervals: 0–15; 16–30; 31–45; 46–60; 61–75; 76–90; 91–105; 106–120.
b. Draw a histogram for the snowfall data for the intervals in part a.
c. Which interval contains the greatest number of cities?
d. How many cities had snowfalls of more than 60 inches?

The following scatter plot shows the changing unemployment rate in the United States from 1991 to 2009.* Use this information in exercises 28 and 29.

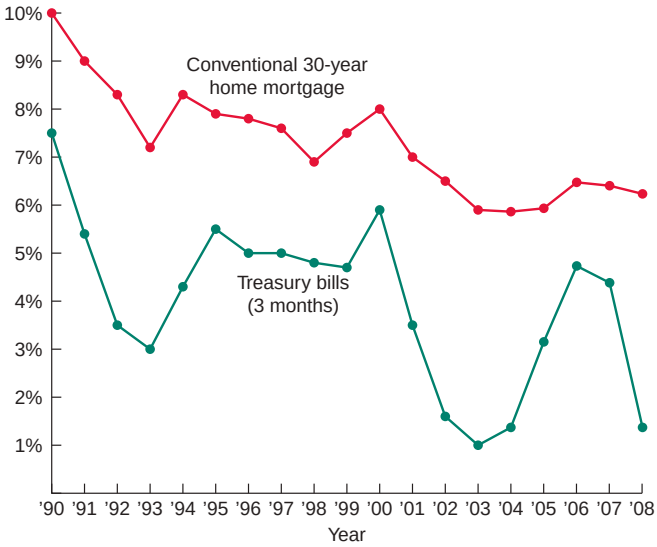


28. **a.** Was the unemployment rate increasing or decreasing from 1995 to 2000?
b. By what percent did the unemployment rate increase from 2000 to 2003?
c. In which year was the unemployment rate the lowest?
d. What was the longest time interval the unemployment rate was continuously dropping, and by what percent did it drop over this time?

29. **a.** Was the unemployment rate increasing or decreasing from 2003 to 2006?
b. By what percent did the unemployment rate decrease from 1994 to 2000?
c. In which year was the unemployment rate the highest?
d. What was the longest time interval the unemployment rate was continuously increasing and by what percent did it increase over this time?

**Statistical Abstract of the United States,* 128th ed. (Washington, DC: Bureau of the Census, 2009), Table 574.

The red line in the next figure shows the average annual mortgage rates for new homes from 1990 to 2008, and the green line shows the average annual interest rates on Treasury bills during the same period.* Use these line graphs to determine approximate answers in exercises 30 and 31.

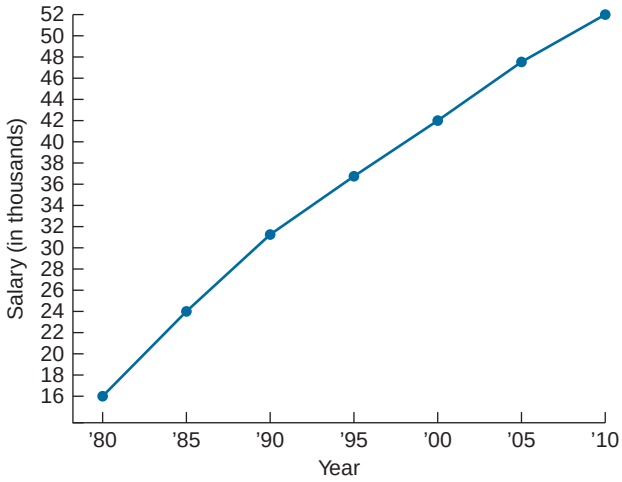


30. **a.** What was the highest annual mortgage rate from 1990 to 2008 and in what year did it occur?
b. What was the annual mortgage rate for 2002?
c. What was the lowest annual interest rate for Treasury bills from 1990 to 2008?
d. What was the greatest difference between the annual mortgage rate and the Treasury bill interest rate from 1990 to 2008, and what was the year?

31. **a.** In what 3 years were the mortgage rates below 6 percent?
b. What was the annual mortgage rate for 2006?
c. What was the highest annual interest rate for Treasury bills from 1990 to 2008?
d. What was the smallest difference between the annual mortgage rate and the Treasury bill interest rate from 1990 to 2008, and what was the year?

Statistical Abstract of the United States, 128th ed. (Washington, DC: Bureau of the Census, 2009), Table 1160.

32. This line graph shows the average salaries for public school teachers in five-year intervals from 1980 to 2010 (2010 is estimated).†



a. What is the projected average salary for public school teachers in 2015 if the increase from 2010 to 2015 is the same as in the preceding five-year period?
b. Which 5-year period had the greater increase in salaries? (*Hint:* Look at slopes of segments.)
c. What is the projected total increase in salaries from 1980 to 2015?

The percentages of public elementary schools with Internet access grew rapidly from 1994 to 2002 and are as follows: 1994, 30 percent; 1995, 46 percent; 1996, 61 percent; 1997, 75 percent; 1998, 88 percent; 1999, 95 percent; 2000, 98 percent; and 2002, 99 percent.‡ Use this information in exercises 33 and 34.

33. **a.** Draw a line graph with the years from 1994 to 2002 represented on the horizontal axis.
b. During which two-year period did the percentage of elementary schools with Internet access approximately double?

34. **a.** Draw a bar graph with the years from 1994 to 2002 represented on the horizontal axis.
b. During which group of years did the percentage of elementary schools with Internet access approximately triple?

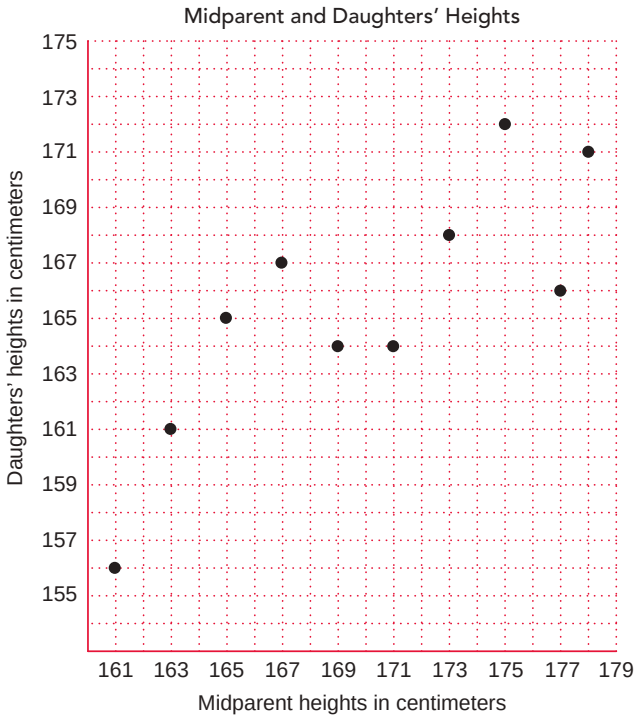
†*Statistical Abstract of the United States,* 128th ed. (Washington, DC: Bureau of the Census, 2009), Table 247.

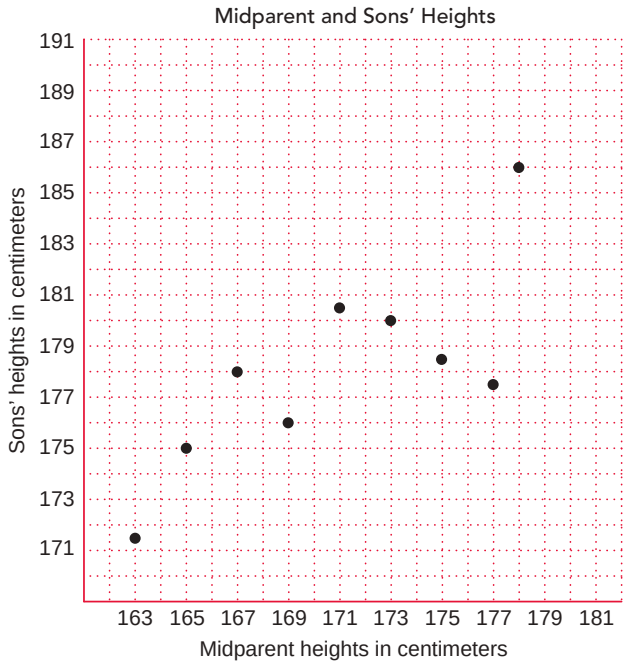
‡*Statistical Abstract of the United States,* 124th ed. (Washington, DC: Bureau of the Census, 2004), p. 159.

Inheritance factors in physical growth have been studied to compare the mother's height to the daughter's and son's heights and the father's height to the daughter's and son's heights. Some researchers have found that the **midparent height,** which is the number halfway between the height of each parent, is more closely related to the heights of their children. The scatter plots in exercises 35 and 36 compare midparent heights to the daughters' heights and midparent heights to the sons' heights.*

35. Using the Midparent and Daughters' Heights graph:
- **a.** Locate a trend line for the scatter plot. Briefly explain your method of determining this line.
- **b.** Use your line to predict the heights of daughters for midparent heights of 160 and 174 centimeters.
- **c.** Use your line to predict the midparent heights for daughters' heights of 163 and 170 centimeters.

36. Using the Midparent and Sons' Heights graph:
- **a.** Locate a trend line for the scatter plot. Briefly explain your method of determining this line.
- **b.** Use your line to predict the sons' heights for the midparent heights of 170 and 180 centimeters.
- **c.** Use your line to predict the midparent heights for the sons' heights of 179 and 182 centimeters.





*W. M. Krogman, *Child Growth,* p. 157.

The following table contains the percentages of 18 to 24 year olds who have not completed high school and are not enrolled in school.*

Percentages of 18 to 24 Year Old High School Dropouts

Year	White (Non-Hispanic)	Black (Non-Hispanic)	Hispanic Origin
1995	13.6	14.4	34.7
1996	12.5	16	34.5
1997	12.4	16.7	30.6
1998	13.7	17.1	34.4
1999	12.8	16	33.9
2000	12.2	15.3	32.3
2001	13.4	13.8	31.7
2002	12.2	14.6	30.1
2003	5.8	10.7	24
2004	5.8	11.0	23
2005	5.8	10.9	22.5
2006	5.8	10.7	22.1

37. a. What was the decrease in black non-Hispanic school dropouts from 1995 to 2006?

b. The decrease of Hispanic dropouts from 1995 to 2006 was how many times the decrease of black non-Hispanic dropouts over the same time period (to the nearest whole number percent)?

c. Form a scatter plot to compare the Hispanic dropouts to the black non-Hispanic dropouts by forming intervals from 22 to 35 percent on the vertical axis for Hispanic and from 10 to 18 percent on the horizontal axis for black non-Hispanic. Is there a positive or negative correlation?

d. Locate a trend line for your scatter plot, and use it to predict the black non-Hispanic dropout percent for an Hispanic dropout rate of 28 percent.

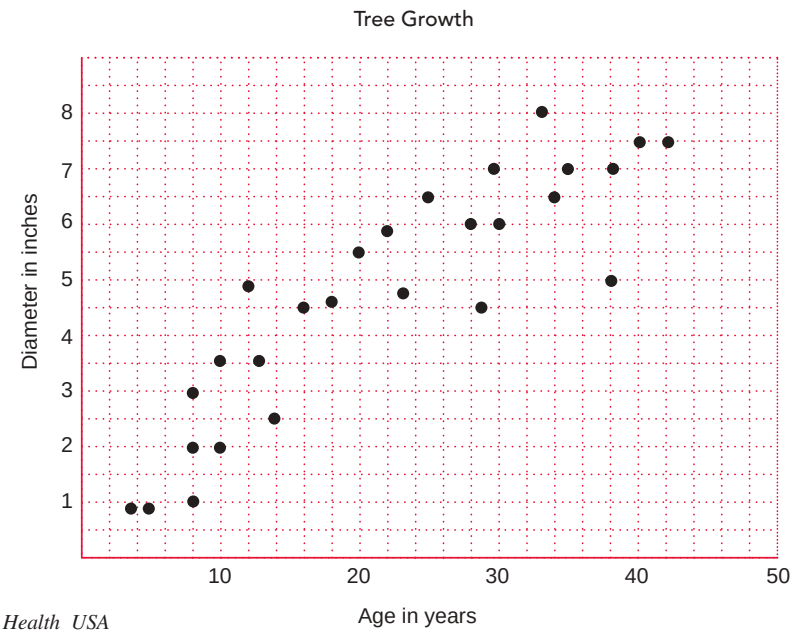
38. a. What was the decrease in white non-Hispanic dropouts from 1995 to 2006?

b. The decrease of black non-Hispanic dropouts from 1995 to 2006 was how many times the decrease of white non-Hispanic dropouts (to the nearest whole number percent)?

c. Form a scatter plot to compare the black non-Hispanic dropouts to the white non-Hispanic dropouts by forming intervals from 10 to 18 percent on the horizontal axis for black non-Hispanic dropouts and from 5 to 14 percent on the vertical axis for white non-Hispanic dropouts. Is there a positive or negative correlation?

d. Locate a trend line for your scatter plot, and use it to predict the white non-Hispanic dropout percentage to the nearest percent for a black non-Hispanic dropout rate of 17 percent.

39. This scatter plot shows the ages of 27 trees and their corresponding diameters.†



*U.S. Department of Health and Human Services, *Child Health USA 2008–2009* (Washington, DC: U.S. Government Printing Office, 2009), p. 15.

†NCTM *Data Analysis and Statistics across the Curriculum*, p. 43.

a. What are the greatest diameter and the oldest age of the trees represented in this graph?
b. Is there a positive or negative correlation?
c. Locate a trend line and use it to predict the diameters of a 26-year-old tree and a 32-year-old tree.
d. Use your trend line to predict the approximate age of a tree, if its diameter is 9 inches.

Reasoning and Problem Solving

40. Featured Strategy: Drawing a Graph. Two ardent baseball fans were comparing the numbers of home runs hit by the American and National Leagues' home run leaders and posed the following question: Is there a correlation from year to year between the numbers of these runs? That is, in general, if the number of home runs hit by one league's home run leader for a given year is low (or high), will the number of home runs hit by the other league's home run leader be low (or high)? Use the tables on pages 484 and 485 in Section 7.2.

a. Understanding the Problem. Consider the two years with the smallest numbers of home runs by the leaders and the two years with the largest numbers of home runs by the leaders. What were these years and numbers?
b. Devising a Plan. One possibility for considering a correlation between the data is to form a scatter plot. To form a scatter plot, first mark off axes for numbers of home runs by each league's home run leaders. For each league, what is the difference between the smallest number of home runs by the leaders and the greatest number?
c. Carrying Out the Plan. Form a scatter plot by plotting the number of home runs hit by each pair of leaders for each year to see if there appears to be a correlation between the data. If so, is the correlation positive or negative? Use your graph to determine the year with the greatest difference in the number of home runs hit by each league's home run leaders.
d. Looking Back. Draw a trend line and use your line to predict the number of home runs by the National League's home run leader for a given year if the number of home runs by the American League's leader is 54.

41. A company's record of amounts invested in advertisements and the corresponding amounts of sales produced are listed by the following pairs of numbers. The first number is the amount for advertisements to the nearest tenth of a million dollars, and the second number is the amount of sales to the nearest million dollars. (3.8, 17), (1.4, 3), (2.8, 7), (4.9, 26), (2.3, 5), (1.8, 3), (3.3, 10), (5.3, 39), (4.4, 23), (5.1, 31), (2.6, 6), (1.2, 2)

a. Form a scatter plot with the amounts for advertisements on the horizontal axis and the corresponding amounts for sales on the vertical axis.
b. Which type of curve from Figure 7.15 on page 450 best fits the points of the scatter plot?
c. Sketch the curve of best fit from part b and use your curve to predict the amount of sales for $3.6 million in advertisements.
d. Use your sketch in part c to predict the amount invested in advertisements, if the total resulting sales was $20 million.

42. For a math project, one middle school student recorded the number of words that her friend Amy was able to memorize in different amounts of time. In the following pairs of numbers, the first number is the amount of time in minutes, and the second is the number of words memorized for the given time: (.5, 5), (1, 9), (1.5, 11), (2, 12), (2.5, 13), (3, 14), (3.5, 15), (4, 15), (4.5, 13), (5.5, 16), (5.5, 18), (6, 17), (7, 18), (8, 18)

a. Form a scatter plot for this data with the times on the horizontal axis and the corresponding numbers of words on the vertical axis.
b. Which type of curve from Figure 7.15 on page 450 best fits the points of the scatter plot?
c. Sketch the type of curve from part b to approximate the location of points on the scatter plot and use your curve to predict the number of words that Amy could memorize in 9 minutes.
d. Use your curve in part c to predict the time period if Amy memorized 10 words.

Teaching Questions

1. The Spotlight on Teaching at the beginning of this chapter shows a bar graph and a line plot involving 21 students. Suppose this were your class. Explain how you would have your class transfer the bar graph data to a line plot. Then describe the different information each graph displays.

2. The scatter plot on page 447 compares the heights of boys to their shoe sizes. Use this data to form stem-and-leaf plots for these two types of measurements. How would you label each plot and explain what information each plot contains?

3. What is the purpose of the "zigzag" line on the vertical axis of the *Girls Soccer Team* graph on the Elementary School Text example on page 446? Explain how you would introduce this idea to a fourth-grade class by listing a series of questions that would lead them to understand and be able to scale similar data.

4. In an elementary school class that was graphing data obtained from student surveys, a student wanted to know if a histogram was the same as a bar graph. How would you answer this question?

Classroom Connections

1. Eight different ways of graphing data are presented in Section 7.1. Refer to the *Data Analysis and Probability* **Standards for PreK–2, Grades 3–5, and 6–8** (see inside front and back covers) to record the different graphical representations that are recommended and their grade levels.

2. The **Research** Statement on page 439 notes two of the difficulties fourth-graders have with bar graphs. Read this statement, and then use the bar graph on that page as an example to write a question involving the interpretation of the data and another question involving predictions.

3. The **Standards** quote on page 447 mentions three types of relationships that can exist for data when forming a scatter plot. Create an example for each of these types of relationships that would be meaningful to school students.

4. The **PreK–2 Standards, Data Analysis and Probability,** states that students should be able to represent data using concrete objects as well as pictures and graphs. Describe three examples of data that students at this level can collect and represent using concrete objects.

5. On page 446 the example from the **Elementary School Text** illustrates how the same data can be displayed in different ways. For the Speed of Animals displays:
   a. What information can you obtain from the bar graph that you cannot get from the stem-and-leaf plot?
   b. What information you can easily obtain from the stem-and-leaf that you cannot get from the bar graph? Explain.

The graphs for questions 6 and 7 are from a middle school textbook section on *Misleading Statistics* (Grade 7, Chapter 8, page 444).

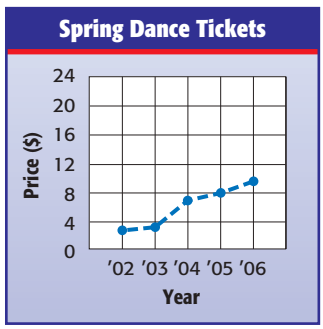
1. According to the size of the hockey players, how many times more points does Mark Messier appear to have than Jari Kurri? Explain.



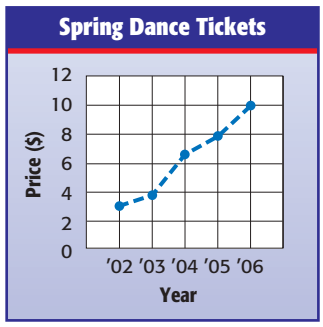
Source: *ESPN Sports Almanac*

6. a. About how many more points did Mark Messier score than Jari Kurri? Once you have determined this, give at least three questions you could ask your students to guide them as they answer the question posed in the schoolbook: "According to the size of the hockey players, how many times more points does Mark Messier appear to have than Jari Kurri? Explain."
   b. The schoolbook also asks the students if the graph is representative of the players' points. How can the graph be changed so that it is a better representation of the players' playoff points?

Graph A



Graph B

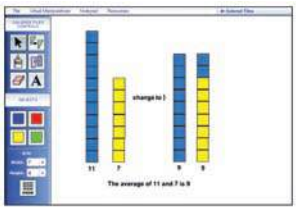


7. The schoolbook asks the question: "Do Graph A and Graph B show the same data? If so, explain how they differ."
   a. Give several typical responses you might expect from your students as they answer this question.
   b. Describe how you would address the question "which graph is better" in your own classroom.

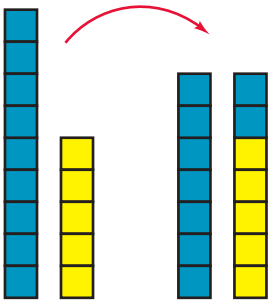
MATH ACTIVITY 7.2

Averages with Columns of Tiles

Virtual Manipulatives



www.mhhe.com/bbn



The average of 9 and 5 is 7

Purpose: Level off columns of color tiles to develop the concept of average (mean).

Materials: Color Tiles in the Manipulative Kit or Virtual Manipulatives.

When two or more columns of tiles like those below at the left are "leveled off" so they have the same height but the number of columns does not change, the common height is called the **average** of the original heights.*

***1.** Use the color tiles to build a column of height 9 and another column of height 15. Other than moving one tile at a time, find at least two methods for leveling the tiles so that there are two columns of the same height.

a. Using the heights of the columns 9 and 15, and the operations of arithmetic, express each of your leveling-off methods with a number expression.

b. Using the results from part a, write two rules for determining the average of two whole numbers x and y, where x is greater than or equal to y.

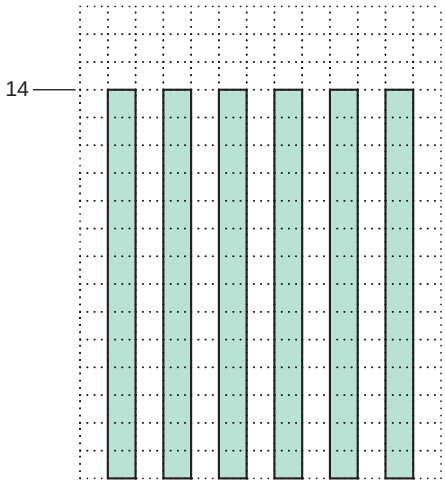
c. Express your two rules in part b as algebraic expressions and show these expressions are equal.

2. Use the color tiles to solve each of the following problems. Draw a diagram and explain your reasoning.

a. The average height of two columns is 10 and the difference of their heights is 3 times the height of the smaller. What are their heights?

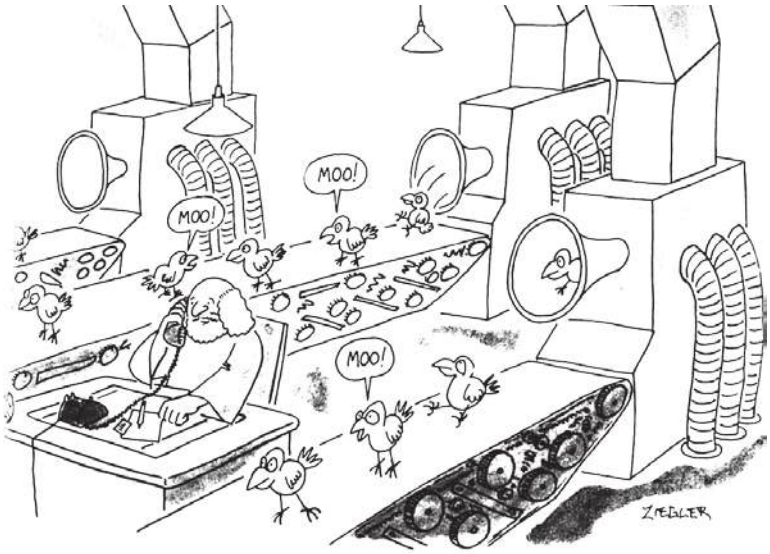
b. There are a total of 16 tiles in two columns and one of the columns is $1\frac{2}{3}$ times the height of the other. What is the height of each column?

3. The six columns at the right have been leveled off to represent the average score for one student on six 20-point quizzes. Her average is 14, but to obtain a satisfactory grade, an average of 15 is needed. The teacher gives two options to raise the average: (1) Throw out the lowest quiz score and base the average on the five remaining quiz scores or (2) take a seventh 20-point quiz to raise the average. Which of these options should she choose if the lowest score on the six quizzes was 9 points? Explain your reasoning.



*A. B. Bennett, E. Maier, and L. T. Nelson, "Visualizing Number Concepts," *Math and the Mind's Eye*.

Section 7.2 DESCRIBING AND ANALYZING DATA



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"Hello? Beasts of the Field? This is Lou, over in Birds of the Air. Anything funny going on at your end?"

PROBLEM OPENER

A large metropolitan police department made a check of the clothing worn by pedestrians killed in traffic at night. About $\frac{4}{5}$ of the victims were wearing dark clothes, and $\frac{1}{5}$ were wearing light-colored garments. Explain why this study does not necessarily show that pedestrians are less likely to encounter traffic mishaps at night if they wear something light-colored.

The employees of Animated Animal Company keep records of the number of toy birds each machine produces and the number of breakdowns of each machine. Figure 7.17 shows that machine III outproduced machines I and II, and machine II had the most problems. This is an example of numerical information or data, which can also be called **statistics.**

Weekly Record of Birds Produced

	M	T	W	Th	F	Breakdowns
Machine I	165	158	98	125	260	13
Machine II	117	82	46	6	30	24
Machine III	182	243	196	305	261	4

Figure 7.17

The word **statistics** also means the science of collecting and interpreting data. There are two broad areas of this science: *descriptive statistics* and *inferential statistics.* **Descriptive statistics** is the science of describing data. The average number of toy birds produced each day by machine I is an example of a descriptive statistic. **Inferential statistics** is the

science of interpreting data in order to make predictions. Sampling is an important part of inferential statistics. For example, if the manager of the Animated Animal Company randomly selects 100 birds from the assembly line and finds that 3 are defective, he can estimate that the number of bad birds in a batch of 5000 will be 150. How? Methods of describing and analyzing data will be introduced in this section.

MEASURES OF CENTRAL TENDENCY

An important source of data, especially for children, comes from conducting **surveys** (gathering of a sample of data or opinions), and this should begin in the early grades. Questions of interest to children, such as the amount of time spent watching television or the most popular soda pop, can lead to the design of a survey of their class, families, friends, etc., for gathering data.

Once data have been collected, they need to be organized and described so that the results can be understood and communicated. For example, if a student records the amount of time spent watching television each day for a 10-day period, these 10 numbers can be replaced by a "typical number" or "central number" to describe the amount of time in general that students watched television. There are three types of numbers that approximate the center of a set of data: the *mean* (also called the *average*), the *median,* and the *mode.* Such numbers are called **measures of central tendency.**

NCTM Standards

Students need to understand that the mean "evens out" or "balances" a set of data and that the median identifies the "middle" of the data set. They should compare the utility of the mean and median as measures of center for different data sets. p. 251

Mean The columns of centimeter tiles in Figure 7.18a represent the heights in centimeters of seven children by showing the tops of the columns above the 121-centimeter level. One way to represent all the heights by a single number is to "level off" the columns. If 3 tiles are taken from each of the three tallest columns and are distributed among the four shortest, as shown in part b, the columns (heights) level off at 126 centimeters.

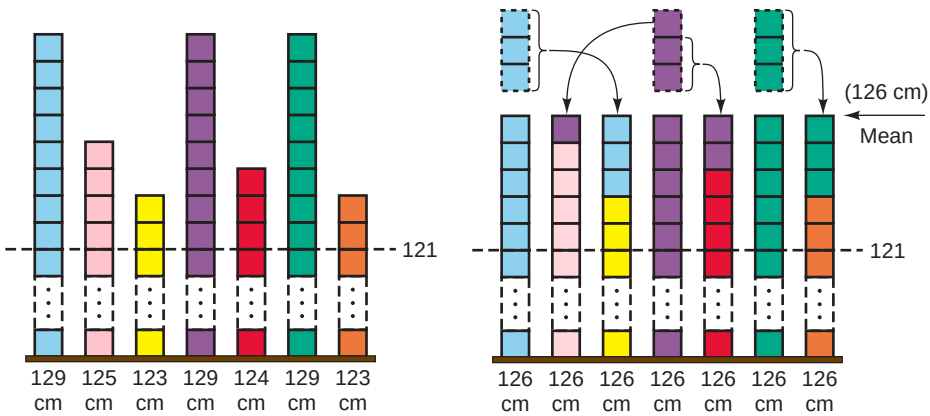


Figure 7.18

Perhaps you can see by looking at Figure 7.18 why the same result is obtained by adding the seven numbers

$$123 + 123 + 124 + 125 + 129 + 129 + 129 = 882$$

and dividing the sum by 7

$$882 \div 7 = 126$$

This type of "center" number is called the *mean* and is the number we often refer to as the *numerical average* or, simply, the *average.*

Selected Major Earthquakes from 1983 to 2010*

Date	Place	Deaths	Magnitude
1983 Mar. 31	Southern Colombia	250	5.5
1983 May 26	Honshu, Japan	81	7.7
1983 Oct. 30	Eastern Turkey	1,300	7.1
1985 Mar. 3	Chile	146	7.8
1985 Sept. 19–21	Mexico City	4,200	8.1
1987 Mar. 5–6	Ecuador	4,000	7.3
1988 Dec. 7	Northwestern Armenia	55,000	6.8
1990 June 21	Northwestern Iran	40,000	7.7
1992 Mar. 13	Eastern Turkey	4,000	6.2
1992 Dec. 12	Flores, Indonesia	2,500	7.5
1993 July 12	Hokkaido, Japan	200	7.8
1993 Sept. 29	Latur, India	9,784	6.3
1994 June 2	Jawa, Indonesia	250	7.2
1994 June 6	Colombia	295	6.4
1995 Jan. 16	Kobe, Japan	5,542	6.8
1995 May 27	Sakhalin Island	1,989	7.5
1995 Oct. 1	Turkey	100	6.2
1996 Feb. 3	Yunnan, China	322	6.5
1997 Feb. 28	Armenia-Azerbaijan	1,100	6.1
1997 May 10	Northern Iran	1,567	7.3
1998 Feb. 4	Afghanistan	2,323	6.1
1998 May 30	Afghanistan	5,000	6.9
1999 Jan. 25	Colombia	1,883	6.0
1999 Aug. 17	Turkey	17,000	7.6
1999 Sept. 20	Taiwan	2,400	7.7
2000 June 4	Sumatra, Indonesia	103	8.0
2001 Jan. 13	El Salvador	852	7.7
2001 Jan. 26	Gujarat, India	20,103	7.7
2002 Mar. 25	Afghanistan	1,000	6.1
2003 May 21	Northern Algeria	2,266	6.8
2003 Dec. 26	Southeastern Iran	31,000	6.6
2004 Feb. 24	Strait of Gibraltar	631	6.4
2004 Dec. 26	Sumatra, Indonesia	283,106	9.0
2005 Mar. 28	Sumatra, Indonesia	1,313	8.7
2005 Oct. 8	Kashmir, Pakistan	86,000	7.6
2006 May 26	Java, Indonesia	6,749	6.3
2006 July 17	West Java, Indonesia	730	7.7
2007 Aug. 15	Chincha Alta, Peru	514	8.0
2008 May 12	Sichuan, China	87,587	7.9
2009 April 6	L'Aquila, Italy	295	6.3
2009 Sept. 29	Samoa Islands	192	8.1
2009 Sept. 30	Sumatra, Indonesia	1,117	7.5
2010 Jan. 13	Léogâne, Haiti	222,570	7.0
2010 Feb. 27	SW of Santiago, Chile	507	8.8

*National Earthquake Information Center of the United States Geological Survey.

Figure 7.19

Mean The **mean** of a set of data is the sum of all measurements divided by the total number of measurements.

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}$$

where x_1, x_2, etc. are n measurements and $\bar{x}$ (read "x bar") denotes the mean.

The table in Figure 7.19 on the previous page lists magnitudes of selected major earthquakes from 1983 to 2010 and the number of deaths caused by each quake. The death tolls vary from a low of 81 to a high of 283,106. Let's calculate the mean for these data. The sum of the number of deaths from the 44 earthquakes is 907,867. The mean to the nearest whole number is $\frac{907{,}867}{44}$, which equals 20,633 deaths to the nearest whole number per earthquake. Notice that this *central number* is higher than all but seven of the numbers in the list, which indicates that the mean may not be the best number for describing the "typical" number of deaths for these earthquakes.

Median We have seen from the earthquake data that the mean is not always a *representative* central number. The next type of central number can also be informative and is illustrated in Figure 7.20. Consider the heights of the seven children that are represented in Figure 7.20a by showing the tops of the columns above the 121-centimeter level. In part b these columns have been placed in increasing order. There are three columns shorter than and three columns taller than the column in the fourth position of part b. Thus, it seems reasonable to select the height of the center column, the column in the fourth position (125 centimeters) as a representative height. This type of central number, the middle number, is called the *median.* Notice that this number is different from the mean for this set of data, which is 126 centimeters.

Research Statement

The 7th national mathematics assessment found that when given a choice about which statistic to use, students tended to select the mean over the median regardless of the distribution of data.

Zawojewski and Shaughnessy

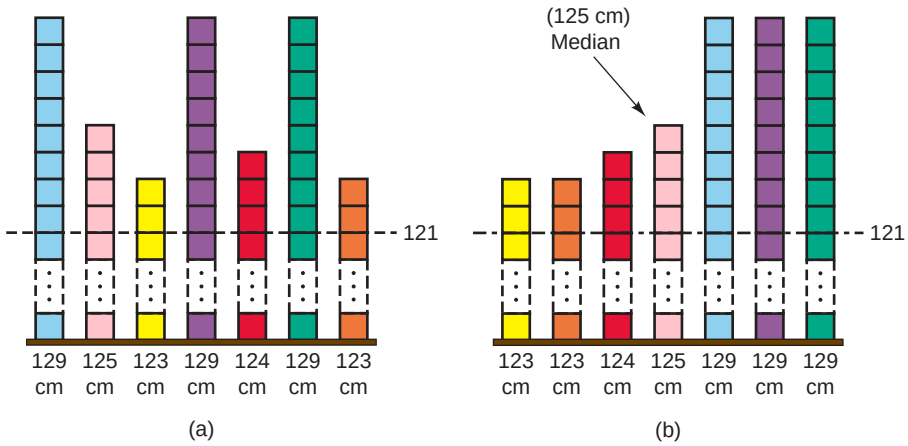


Figure 7.20

Median

1. The **median** of a set of data with an odd number of measurements is the middle number when the measurements are listed from smallest to largest (or largest to smallest).
2. The **median** of a set of data with an even number of measurements is the mean of the two middle numbers when the measurements are listed from smallest to largest (or largest to smallest).

The numbers of earthquake deaths from Figure 7.19 on page 470 are listed here from smallest to largest. Since the number of measurements is even, we must compute the mean of the two middle numbers. The median for this set of data is 1725, the mean of the two circled numbers. Twenty-two of the numbers are less than 1725, and 22 of the numbers are greater than 1725. Notice for this data, the median is significantly smaller than the mean.

81, 100, 103, 146, 192, 200, 250, 250, 295, 295, 322, 507, 514, 631, 730, 852, 1,000, 1,100, 1,117, 1,300, 1,313, (1,567), (1,883), 1,989, 2,266, 2,323, 2,400, 2,500, 4,000, 4,000, 4,200, 5,000, 5,542, 6,749, 9,784, 17,000, 20,103, 31,000, 40,000, 55,000, 86,000, 87,587, 222,570, 283,106

Mode The median seems to be a more representative central number for the earthquake data than the mean, but there are no death tolls that actually equal the median. Let's consider the third type of central measure. Observe that three of the children whose heights are represented by tiles in Figure 7.21 are 129 centimeters tall. This height might be considered the most representative as a central measure because it occurs most frequently. This type of central measure is called the *mode.* Notice that this height is different from the mean (126 centimeters) and the median (125 centimeters).

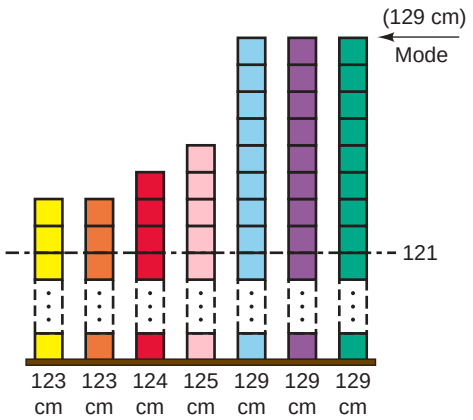


Figure 7.21

Mode The **mode** of a set of data is the measurement that occurs most often.

When several numbers occur most frequently, a set of data will have more than one mode. If there are two modes the data is called **bimodal;** if there are three modes, it is called **trimodal;** and if there are four or more modes we say the data is **multimodal.** The earthquake data in the table on page 470 is *trimodal* with modes 250, 295, and 4000.

EXAMPLE A

Determine the mean, median, and mode of the earthquake magnitudes in Figure 7.19 on page 470.

Solution The mean is approximately $7.2 \left(\frac{316.3}{444}\right)$. To determine the median, we must first list the earthquake magnitudes in increasing order.

5.5, 6.0, 6.1, 6.1, 6.1, 6.2, 6.2, 6.3, 6.3, 6.3, 6.4, 6.4, 6.5, 6.6, 6.8, 6.8, 6.8, 6.9, 7.0, 7.1, 7.2, (7.3), (7.3), 7.5, 7.5, 7.5, 7.6, 7.6, 7.7, 7.7, 7.7, 7.7, 7.7, 7.7, 7.8, 7.8, 7.9, 8.0, 8.0, 8.1, 8.1, 8.7, 8.8, 9.0

The median is 7.3, the mean of the two circled numbers. The mode is 7.7, the measure that occurs 6 times.

2-6 Mean

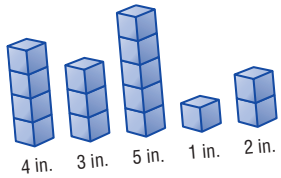
MAIN IDEA
Find the mean of a data set.

New Vocabulary
average
mean
outlier

Math Online
glencoe.com
- Extra Examples
- Personal Tutor
- Self-Check Quiz

MINI Lab

In five days, it snowed 4 inches, 3 inches, 5 inches, 1 inch, and 2 inches.



- Make a stack of centimeter cubes to represent the snowfall for each day, as shown at the right.
- Move the cubes until each stack has the same number of cubes.

1. On average, how many inches did it snow per day in five days? Explain your reasoning.
2. Suppose on the sixth day it snowed 9 inches. If you moved the cubes again, how many cubes would be in each stack?

When analyzing data, it is helpful to use a single number to describe the whole set. In the Mini Lab above, a good choice would be the number 3, the mean or **average** number of cubes in each stack that results from equally distributing all the cubes. The mean can be interpreted as a balancing point for a set of data. The mean of a set of data can also be calculated.

Key Concept

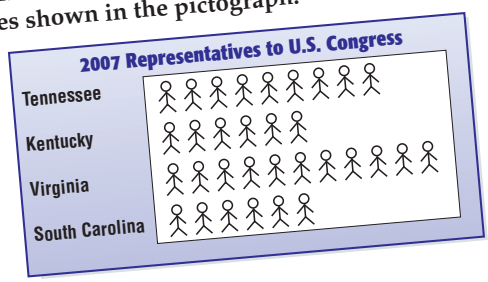
Mean

Words The **mean** of a set of data is the sum of the data divided by the number of pieces of data.

Example Data set: 4, 3, 5, 1, 2 → mean: $\frac{4+3+5+1+2}{5} = \frac{15}{5}$ or 3

EXAMPLE 1 Find the Mean

CIVICS Find the mean number of Representatives for the four states shown in the pictograph.



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The mean, median, and mode each have their advantages, depending on the data and type of information desired. In some cases one of these central numbers is clearly more representative of a set of data than another.

Values of data that are substantially larger or smaller than the other values such as 222,570 and 283,106 in the list of earthquake deaths can pull the mean above (or below) the median. The median often has an advantage over the mean in describing data because very large or very small data values usually have less effect on the median.

EXAMPLE B

Determine the mean, median, and mode of the following salaries for people in a small company. Which salaries might be considered atypical? What is the best measure of central tendency?

One president	$210,000
One vice-president	120,000
One salesperson	40,000
One supervisor	32,000
One machine operator	28,000
Five mill workers (each earning)	25,000
Six apprentice workers (each earning)	22,000

Solution The sum of the 16 salaries is

$$\$210{,}000 + \$120{,}000 + \$40{,}000 + \$32{,}000 + \$28{,}000 + 5(\$25{,}000) + 6(\$22{,}000) = \$687{,}000$$

So the mean is $\frac{\$687{,}000}{16} = \$42{,}937.50$. The median is the mean of the eighth and ninth salaries when the salaries are considered in increasing order. Since the eighth and ninth salaries are both $25,000, the median is

$$\frac{\$25{,}000 + \$25{,}000}{2} = \$25{,}000$$

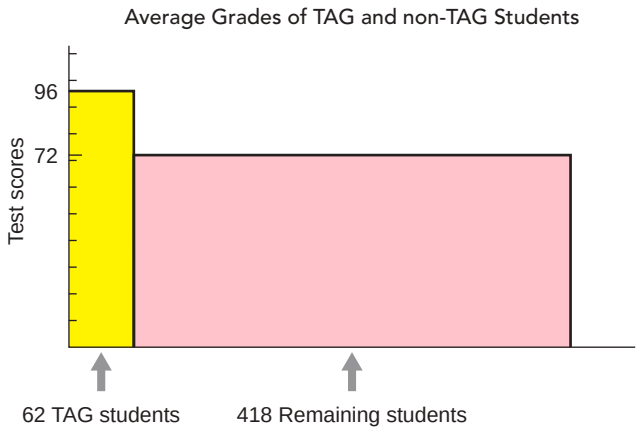
The mode is $22,000, since this salary occurs most frequently. Both the median and the mode are more representative of the majority of salaries than the mean. The mean of $42,937.50 is greater than 13 of the 16 salaries because it was strongly affected by the atypical salaries of $120,000, and $210,000.

PROBLEM-SOLVING APPLICATION

Problem

An elementary school principal was interested in computing the mean of the verbal reasoning scores on a differential aptitude test taken by all the students in the school. The mean test score for the 62 students in the Talented and Gifted (TAG) program was 96, and the mean of the scores for the remaining 418 students was 72. What was the mean of the test scores for all the students?

Understanding the Problem *Drawing a graph* is one way of visualizing the given information. We can think of each of the 418 students as having a score of 72 and each of the 62 TAG students as having a score of 96. The mean for the total group of 480 students can be visualized by "evening off" both columns. **Question 1:** Will the mean be closer to 72 or to 96?



Devising a Plan The graph shown above suggests a plan. The difference between the height of the region representing the TAG students and the height of the region for the remaining students is $96 - 72 = 24$. Thus, if the column for the TAG students is cut down to a height of 72, the 62×24 additional test score points can be "spread" across the entire top of the new graph. To determine the increase in the mean of 72, we can divide the 1488 extra points by the total number of students (480). **Question 2:** What is this increase?

Carrying Out the Plan The increase in the mean is $1488 \div 480 = 3.1$. So, if the extra points for the TAG students' scores are evenly spread across the top of the graph, the new mean will be $72 + 3.1 = 75.1$. You might have been tempted to obtain the new mean by finding the mean of 96 and 72: $\frac{96 + 72}{2} = 84$. **Question 3:** How does the graph help to show that this is not reasonable for the new mean?

Looking Back Another approach to solving the original problem is to find the total of all the scores for the 480 students and then divide by 480. **Question 4:** How can the sum of the scores for all 480 students be obtained if we know that the mean for 62 students is 96 and the mean for 418 students is 72?

Answers to Questions 1–4 **1.** Closer to 72 **2.** 3.1 **3.** A horizontal line drawn on the graph at a height of 84 (halfway between 72 and 96) makes it evident that the part of the TAG students' bar above 84 does not have enough area to increase the total graph to a height of 84. **4.** The sum of the scores for the total 480 students is $62(96) + 418(72) = 36{,}048$. So the new mean of the test scores for all the students is $\frac{36{,}048}{480} = 75.1$.

BOX-AND-WHISKER PLOTS

An application of the median is found in the formation of a visual diagram called a **box-and-whisker plot,** or more briefly, a **box plot.** For this type of plot, the data are divided into four parts with approximately the same amount of data in each part. First, the median is used to divide the data into a lower part and an upper part, and then each of these parts is separated into two parts by their medians.

Let's illustrate this process with a set of data. First, order the data from the smallest value to largest, and mark the location of the median, as shown on page 476. The "lower half" has the numbers 1, 3, 3, 7, 8, 8, 11, and its median is 7. The median of the "lower half" is called the **lower quartile** (Q_1). The "upper half" has the numbers 15, 19,

21, 21, 21, 26, 29, and its median is 21. The median of the "upper half" is called the **upper quartile** (Q_3).

1 3 3 7 8 8 11 14 15 19 21 21 21 26 29

lower quartile — median — upper quartile

The next sample shows 20 scores from a science test that have been listed in increasing order. The median test score is 70 and the lower and upper quartiles are 66 and 80, respectively.

57 58 62 63 66 66 67 67 68 70 70 72 73 75 80 80 81 83 85 99

lower quartile — median — upper quartile

The box-and-whisker plot for this set of data is shown in Figure 7.22. The diagram is drawn on or above a number line and the median is marked by the vertical line in the rectangle, and the lower quartile Q_1 and upper quartile Q_3 are marked by the left edge and right edge of the rectangle. Each of the four parts of the box-and-whisker plot represent approximately 25 percent of the data. The rectangle is the **box** of the plot and illustrates the middle 50 percent of the data. The lines extending from the ends of the rectangle to the smallest number (57) and to the largest number (99) are the **whiskers** and they illustrate the lower 25 percent and upper 25 percent of the data. The box plot shows that the 25 percent of the test scores in the interval from 70 to 80 are more than twice as widely spread as the 25 percent of the scores in the interval from 66 to 70. Also the length of the whiskers provides information about how close the smallest and greatest measurements are to the quartiles: The smallest test score (57) is much closer to the lower quartile (66) than the greatest score (99) is to the upper quartile (80), so the upper 25 percent of the test scores are more widely spread than the lower 25 percent.

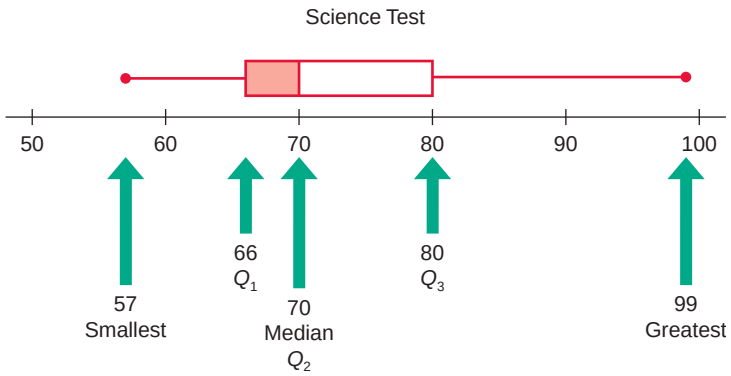


Figure 7.22

EXAMPLE C

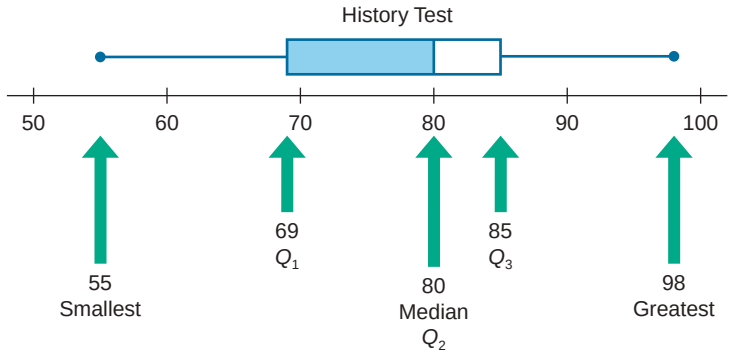
Form a box-and-whisker plot for the following 20 scores on a history test:

55 59 64 64 68 70 73 75 76 79 81 81 82 84 85 85 87 92 95 98

1. Make a few observations about this plot.
2. Compare the box-and-whisker plot of scores on the history test to the plot of scores on the science test (Figure 7.22). Write a few observations.

NCTM Standards

Box plots do not convey as much specific information about the data set, such as where clusters occur, as histograms do. But box plots can provide effective comparisons between two data sets because they make descriptive characteristics such as median and interquartile range readily apparent. p. 251



Solution **1.** The rectangle in the box-and-whisker plot for the history test shows that 25 percent of the scores are in the interval from 69 to 80 and 25 percent are in the interval from 80 to 85, which is approximately one-half as long. Also, 50 percent of the scores are above 80, and 25 percent are above 85. **2.** A comparison of the box-and-whisker plots for the history and science tests shows that, overall, performance on the history test was better. Although the ranges of both tests are approximately the same ($98 - 55 = 43$ compared to $99 - 57 = 42$), the median and quartiles for the history test are greater than the median and corresponding quartiles for the science test. For example, the median for the history test (80) equals the upper quartile for the science test. That is, 50 percent of the scores on the history test are above 80, whereas only 25 percent of the scores on the science test are above 80. Also, the lower quartile (69) for the history test is approximately equal to the median (70) for the science test, which means that there are approximately twice as many scores below 70 on the science test as on the history test.

Interquartile Range We have seen that the box in the box-and-whisker plot has special significance, as it represents approximately 50 percent of the data. The length of the box that is the difference between the upper quartile and the lower quartile is called the **interquartile range.** For example, the box plot in Example C shows that the interquartile range for the history test scores is $85 - 69 = 16$.

The interquartile range is used for determining which values of the data, if any, are significantly larger or smaller than the other data values. If a value of the data is more than 1.5 times the interquartile range above the upper quartile or below the lower quartile, the value is considered to be an **outlier.**

Lower Quartile The median of the lower half of a set of data is the **lower quartile.**

Upper Quartile The median of the upper half of a set of data is the **upper quartile.**

Interquartile Range The difference between the upper quartile and the lower quartile is the **interquartile range.**

Outlier A data value more than 1.5 times the interquartile range above the upper quartile or below the lower quartile is an **outlier.**

Let's consider Example B once again from page 474. The salaries from this example are shown here in thousands of dollars (that is, 22 represents 22,000, etc.), and the median and quartiles have been marked.

22 22 22 22 22 22 25 25 25 25 25 28 32 40 120 210

lower quartile | median | upper quartile

When looking at the data in Example B, you may feel that \$210,000 is an outlier (substantially larger or smaller than the other values), but you may be uncertain about \$120,000 or \$40,000. Let's use the previous test to determine if these numbers are outliers.

Interquartile range:	$30{,}000 - 22{,}000 = 8000$
$1.5 \times$ interquartile range:	$1.5 \times 8000 = 12{,}000$
Upper quartile + 12,000:	$30{,}000 + 12{,}000 = 42{,}000$
Lower quartile − 12,000:	$22{,}000 - 12{,}000 = 10{,}000$

Since \$120,000 and \$210,000 are greater than \$42,000, these salaries are outliers, and since \$40,000 is not greater than \$42,000, this salary is not an outlier. However, there are no salaries less than \$10,000, so there are no outliers at the lower end of the data.

EXAMPLE D

What are the outliers, if any, for the science test scores that are represented by the box plot in Figure 7.22 on page 476?

Solution The interquartile range is $80 - 66 = 14$. Since there are no scores above $80 + 1.5(14) = 101$ and no test scores below $66 - 1.5(14) = 45$, there are no outliers.

MEASURES OF VARIABILITY

The mean, median, mode, and quartiles are single numbers used to describe a set of data; yet, even when they are all used, they do not present the whole picture. You will find something quite interesting about the sets of data in the next example that will indicate the need for further methods of describing data.

EXAMPLE E

1. Compute the mean, median, mode, and quartiles for sets *A* and *B*.
2. In which set are the data more spread out about the mean?

Set *A:*	13	14	15	19	20	20	28	29	30	32	33
Set *B:*	1	4	15	16	17	20	20	21	30	31	78

Solution **1.** Both sets have the same mean, median, mode, and quartiles: mean, 23; median, 20; mode, 20; lower quartile, 15; and upper quartile, 30. **2.** The data in set *B* are more spread out about the mean.

Example E shows that it is possible for the mean, median, mode, and quartiles of one set of data to equal those of another, while the data in one of these sets are more spread out than the data in the other. To help describe a set of data, we need to measure the amount of spread among the data. A **measure of variability** is a number that describes the spread or dispersion in a set of data. We have seen that the interquartile range is one indicator of the spread in the middle 50 percent of the data. Another common method of measuring variation uses the *range*.

Range The range is the difference between the greatest and least values in a set of data.

The range of set A in Example E is 20 and the range of set B is 77. Even when two sets of data have the same range, there may be differences in the way the data are centered or spread about the mean. Consider the data and graphs in Figure 7.23. The range of the data in both part a and part b is 4 (the greatest value is 5, and the least value is 1). However, the data represented in part a are more concentrated about the mean than the data represented in part b.

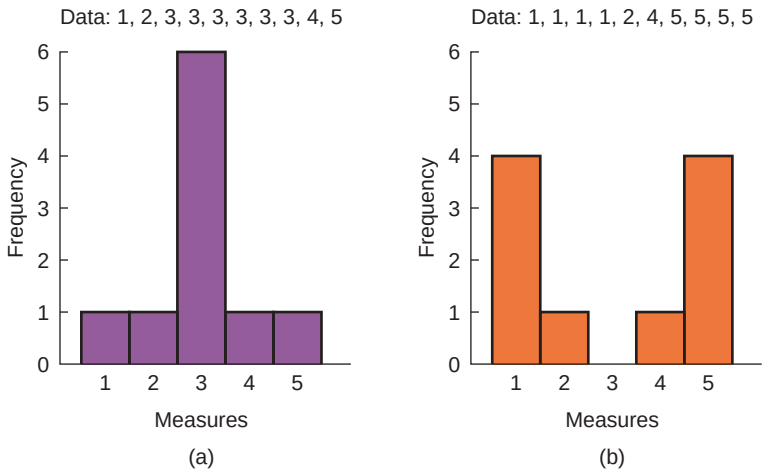


Figure 7.23

The two sets of data in Figure 7.23 show the need for a measure of variability that is more sensitive than the range. The next measure of variability is called the *standard deviation,* which essentially measures the average distance of the data from the mean.

Standard Deviation The following are steps in determining the **standard deviation** of a set of data.

1. Determine the mean.
2. Find the difference between each value of data and the mean.
3. Square the differences.
4. Determine the mean of the squared differences.
5. Compute the square root of this mean to find the standard deviation.

$$\text{Standard deviation } \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ \ldots \ + (x_n - \bar{x})^2}{n}}$$

where x_1, x_2, etc. are $\boldsymbol{n}$ values of data and $\boldsymbol{\bar{x}}$ is the mean.

The tables in Figure 7.24 illustrate the steps for computing the standard deviation for each of the sets of data in Figure 7.23. They show that the standard deviation for set A is 1, and the standard deviation for set B is approximately 1.8 (about twice as large). The larger standard deviation for set B confirms our visual interpretation of the graphs in Figure 7.23. The data graphed in part b are more widely spread about the mean than the data in part a. In general, the more varied (spread out) the data, the greater the standard deviation; and the less varied the data, the smaller the standard deviation (closer to zero).

Set *A* (mean = 3)
1, 2, 3, 3, 3, 3, 3, 4, 5

Measure x	Difference from Mean $x - \bar{x}$	Square of Difference $(x - \bar{x})^2$
1	$1 - 3 = -2$	4
2	$2 - 3 = -1$	1
3	$3 - 3 = 0$	0
3	$3 - 3 = 0$	0
3	$3 - 3 = 0$	0
3	$3 - 3 = 0$	0
3	$3 - 3 = 0$	0
3	$3 - 3 = 0$	0
4	$4 - 3 = 1$	1
5	$5 - 3 = 2$	4
		Total 10

Mean of squared differences $= \frac{10}{10} = 1$

Standard deviation $= \sqrt{1} = 1$

Set *B* (mean = 3)
1, 1, 1, 1, 2, 4, 5, 5, 5, 5

Measure x	Difference from Mean $x - \bar{x}$	Square of Difference $(x - \bar{x})^2$
1	$1 - 3 = -2$	4
1	$1 - 3 = -2$	4
1	$1 - 3 = -2$	4
1	$1 - 3 = -2$	4
2	$2 - 3 = -1$	1
4	$4 - 3 = 1$	1
5	$5 - 3 = 2$	4
5	$5 - 3 = 2$	4
5	$5 - 3 = 2$	4
5	$5 - 3 = 2$	4
		Total 34

Mean of squared differences $= \frac{34}{10} = 3.4$

Standard deviation $= \sqrt{3.4} \approx 1.8$

Figure 7.24

EXAMPLE F

Two sets of data and their means are given below. Inspect the sets of data and predict which set has the smaller standard deviation. Compute the standard deviation for both sets of data.

Set *A*: 18 19 20 20 26 28 30 mean = 23
Set *B*: 0 1 10 20 20 50 60 mean = 23

Solution The numbers in set *A* are fairly close together and less spread out than the numbers in set *B*. So the numbers in set *A* should have the smaller standard deviation.

Measure x	Difference from Mean $x - \bar{x}$	Square of Difference $(x - \bar{x})^2$
30	7	49
28	5	25
26	3	9
20	-3	9
20	-3	9
19	-4	16
18	-5	25
		Total 142

Mean of squared differences $= \frac{142}{7} \approx 20.3$

Standard deviation $\approx \sqrt{20.3} \approx 4.5$

Measure x	Difference from Mean $x - \bar{x}$	Square of Difference $(x - \bar{x})^2$
60	37	1369
50	27	729
20	-3	9
20	-3	9
10	-13	169
1	-22	484
0	-23	529
		Total 3298

Mean of squared differences $= \frac{3298}{7} \approx 471.1$

Standard deviation $\approx \sqrt{471.1} \approx 21.7$

These standard deviations (4.5 compared to 21.7) show that the measures in set *A* are less spread out than the measures in set *B*.

Standard deviations determine intervals about the mean. For set *A* in Example F, 1 standard deviation above the mean is 23 + 4.5, or 27.5, and 1 standard deviation below the mean is 23 − 4.5, or 18.5 (Figure 7.25). The interval within ±1 standard deviation of the mean is the interval from 18.5 to 27.5. The interval within ±2 standard deviations of the mean is from 14 to 32. Notice the use of the lowercase Greek letter sigma (σ) in Figure 7.25 to represent the standard deviation.

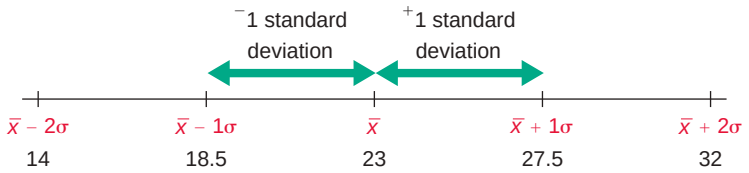


Figure 7.25

Distribution of Data At least 75 percent of the measurements in any set of data will lie within 2 standard deviations of the mean (see Figure 7.26).

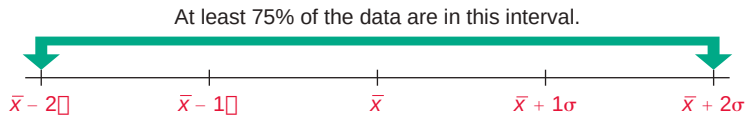


Figure 7.26

The percentage of data within 2 standard deviations of the mean is usually much higher than 75. In fact, if only 75 percent of the data in a given set are within 2 standard deviations of the mean, the data have a rather unique distribution (see Computer Investigation 7.2 on the website). For sets of data that reflect real-life situations, 90 percent or more of the data are usually within 2 standard deviations of the mean (see the section on *normal distributions,* page 497).

Technology Connection

Calculators

Calculators and computers that are programmed with statistics functions compute the mean, standard deviation, and other statistical measures for data that are entered. Figure 7.27 on the next page shows a calculator's view screen for the data in set *A* of Example F.* The variable *n* on this screen shows that seven numbers were entered, and the first line has the mean of these numbers. Also Σx and Σx^2 are the sum of the seven numbers and the sum of the squares of the numbers, respectively. Notice that the median, lower quartile Q_1, and upper quartile Q_3 are also given. The standard deviation computed in Example F was approximately 4.5, and this is denoted on the view screen by σx. The symbol Sx also represents a standard deviation, but for a sample. This standard deviation is larger than σx because in

*This view screen is from Texas Instruments TI-83 calculator.

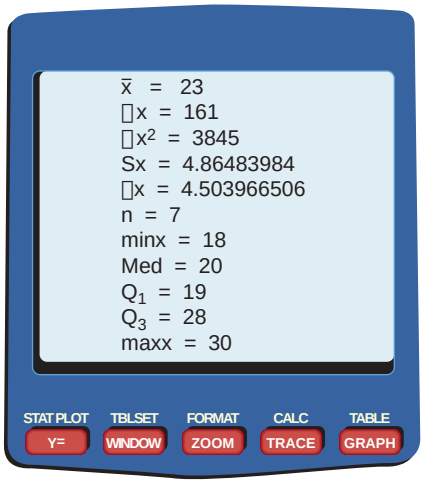


Figure 7.27

the formula for standard deviation on page 479, $n - 1$ is used for the denominator rather than n. Using the smaller denominator $n - 1$, rather than n, produces a larger standard deviation that is better suited for making inferences from samples.

Summary The mean, median, and mode are single numbers that describe the central tendency of a set of data. Often the median is a more reliable measure than the mean because a few extremely large or small values of data will have less effect on the median than on the mean. The median is used in forming box-and-whisker plots for visual illustrations of data. This type of plot makes it easy to focus attention on the median, quartiles, and the largest and smallest values of data. In particular, the box illustrates the "middle" 50 percent of the data, and the length of the box is the interquartile range, which is one measure of the spread of the data. The interquartile range provides a method of determining if there are outliers in the data. The standard deviation and the range are also methods of measuring the amount of spread of the data.

One major advantage of a box plot is that it does not become more cluttered with large amounts of data, as illustrated in Example G.

EXAMPLE G

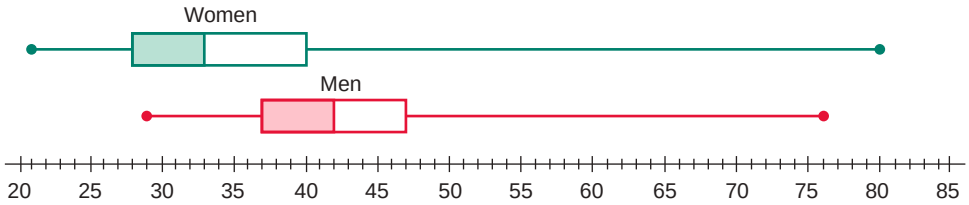
The ages of the women and men who have won Oscars for best actress and actor from 1928 to 2009 are shown below. The ages have been listed in increasing order.

Women:	21	22	24	24	24	24	24	25	25	25	26	26	26	26	26
	26	26	27	27	27	28	28	28	28	29	29	29	29	29	30
	30	30	30	30	31	31	31	32	33	33	33	33	33	34	34
	34	34	34	34	35	35	35	35	35	36	37	37	38	38	38
	39	40	41	41	41	41	41	42	43	44	45	45	48	49	49
	60	60	61	61	62	74	80								
Men:	29	30	31	32	32	32	33	33	34	34	35	35	35	35	36
	36	36	37	37	37	37	38	38	38	38	38	38	38	39	39
	39	40	40	40	40	41	41	41	41	42	42	42	43	43	43
	43	43	44	44	44	45	45	46	46	46	47	47	47	47	48
	48	48	49	49	49	49	51	51	52	52	53	54	55	56	56
	56	59	60	60	61	62	76								

1. What are the quartiles and median for each set of data?
2. Form a box-and-whisker plot to compare these sets of data. List some observations.
3. Use the interquartile range to determine if there are any outliers. If so, what are they?

Solution **1.** Women: quartiles, 28 and 40; median, 33. Men: quartiles, 37 and 48; median, 42.

2.



Observations: The median of the men's ages is greater than the upper quartile of the women's ages, which means that more than 75% of the women were younger than the median age of the men; and the median of the women's ages is below the lower quartile of the men's ages, which means that more than 50% of the women were younger than the lower quartile age of the men. In general, the men have higher ages, although the women have the greatest age. The boxes show that the middle 50 percent of each set of data is slightly more variable (spread out) for the women than for the men.

3. The interquartile range for the women is $40 - 28 = 12$, and $1.5(12) + 40 = 58$. So 60, 61, 62, 74, and 80 are outliers for the women's data. The interquartile range for the men is $48 - 37 = 11$, and $1.5(11) + 48 = 64.5$. So 76 is an outlier for the men's data.

Technology Connection

Standard Deviation

Suppose a make-up test you gave to six students had these scores: 70, 63, 47, 58, 81, and 62. What would happen to the mean and the standard deviation of these scores if you increased each of the six scores by the same amount? The online 7.2 Mathematics Investigation will help you quickly gather data to explore this and other similar questions.

Mathematics Investigation
Chapter 7, Section 2
www.mhhe.com/bbn

Technology Connection

The box plot in Example G shows that the range (greatest value minus the smallest value) is greater for the women's data (59) than for the men's (47). The boxes also indicate that the women's data are more variable. The standard deviation will provide another measure of this variability. A calculator with a statistics mode is convenient for finding the standard deviation as well as the mean, median, mode, and quartiles.* To the nearest .1, the means for the women's and men's data are 35.5 and 43.5 and the standard deviations are 11.4 and 8.6, respectively. The greater standard deviation for the women's data shows that these data are more spread out than the data for the men. Notice that the means for these sets of data are both greater than the medians.

*Also, the Mathematics Investigation, Standard Deviation, on the website, computes the mean, median, mode, quartiles, and standard deviation for any data entered, and it prints the percentage of data within ± 1, ± 2, and ± 3 standard deviations of the mean.

Exercises and Problems 7.2

Calculate the mean, median, and mode for each set of data in exercises 1 and 2.

1. a. 4, 7, 6, 2, 4, 5 **b.** 0, 1, 5, 0, 2, 0, 3, 1

2. a. 4, $^{-}3$, 2, 8, $^{-}2$, 0 **b.** 78, 85, 83, 71, 62, 83, 77

Which measure of central tendency—mean, median, or mode—is best for describing the instances in exercises 3 and 4?

3. a. The typical size of bicycles (by tire size) sold by a bicycle shop
b. The typical size of dresses sold in a store
c. The typical cost of homes in a community

4. a. The typical size of hats sold in a store
b. The typical heights of players on a basketball team
c. The typical age of seven people in a family if six of them are under 40 and one is 96 years old

The table below lists the number of nuclear power reactors operating in 21 countries and their average capacity in megawatts (1 million watts).* Use this table in exercises 5 and 6.

Country	Number of Reactors	Average Megawatt Capacity
Argentina	2	935
Belgium	7	5,943
Bulgaria	2	1,906
Canada	18	12,679
China	11	8,587
France	58	63,236
Germany	17	20,339
Great Britain	19	11,035
Hungary	4	1,880
India	19	4,183
Japan	54	47,102
Mexico	2	1,310
Netherlands	1	485
Russia	32	22,811
South Africa	2	1,842
South Korea	20	17,716
Spain	8	7,448
Sweden	10	9,399
Switzerland	5	3,252
Ukraine	15	13,168
United States	104	101,119

*World Nuclear Association (world-nuclear.org).

5. a. What is the median capacity in megawatts of the reactors in these 21 countries?
b. What is the average number of reactors in these 21 countries (to the nearest whole number)? Explain why the mean is a misleading measure of central tendency in this example.

6. a. What is the average overall capacity in megawatts in these 21 countries? Explain why the mean is a misleading measure of central tendency in this example.
b. What is the median for the numbers of reactors in these 21 countries?

The home run leaders in the National and American Leagues from 1990–2009 are listed here and on the next page. Use these data in exercises 7 and 8.

Home Run Leaders

Year	American League	Home Runs
1990	Cecil Fielder, Detroit	51
1991	Cecil Fielder, Detroit	
	Jose Canseco, Oakland	44
1992	Juan Gonzalez, Texas	43
1993	Juan Gonzalez, Texas	46
1994	Ken Griffey, Jr., Seattle	40
1995	Albert Belle, Cleveland	50
1996	Mark McGwire, Oakland	52
1997	Ken Griffey, Jr., Seattle	56
1998	Ken Griffey, Jr., Seattle	56
1999	Ken Griffey, Jr., Seattle	48
2000	Troy Glaus, Anaheim	47
2001	Alex Rodriguez, Texas	52
2002	Alex Rodriguez, Texas	57
2003	Alex Rodriguez, Texas	47
2004	Manny Ramirez, Boston	43
2005	Alex Rodriguez, Texas	48
2006	David Ortiz, Boston	54
2007	Alex Rodriguez, New York	54
2008	Miguel Cabrera, Detroit	37
2009	Mark Teixeira, New York	39

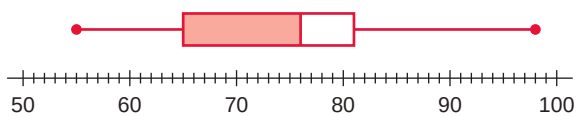
Home Run Leaders

Year	National League	Home Runs
1990	Ryne Sandberg, Chicago	40
1991	Howard Johnson, New York	38
1992	Fred McGriff, San Diego	35
1993	Barry Bonds, San Francisco	46
1994	Matt Williams, San Francisco	43
1995	Dante Bichette, Colorado	40
1996	Andres Galarrago, Colorado	47
1997	Larry Walker, Colorado	49
1998	Mark McGwire, St. Louis	70
1999	Mark McGwire, St. Louis	65
2000	Sammy Sosa, Chicago	50
2001	Barry Bonds, San Francisco	73
2002	Sammy Sosa, Chicago	49
2003	Jim Thome, Philadelphia	47
2004	Adrian Beltre, Los Angeles	48
2005	Andruw Jones, Atlanta	51
2006	Ryan Howard, Philadelphia	58
2007	Prince Fielder, Milwaukee	50
2008	Ryan Howard, Philadelphia	48
2009	Albert Pujols, St. Louis	47

7. **a.** Compute the mean, median, and mode for the numbers of home runs in the National League.
b. Compute the mean, median, and mode for the numbers of home runs in the American League.
c. In how many different years did the American League home run leaders hit more home runs than those of the National League?
d. Which league's home run leaders, if either, have the better record? Support your conclusion.

8. Form a box-and-whisker plot for the numbers of home runs hit by the American and for the National League home run leaders from 1990–2009. Based on observations of the plot, which league's home run leaders, if either, have the better record? Support your conclusion.

The following box-and-whisker plot is for 40 test scores. Use this plot in exercises 9 and 10.



9. **a.** What is the lower quartile?
b. Approximately how many test scores are between 65 and 76?
c. Approximately how many test scores are above 65?
d. What is the interquartile range?

10. **a.** What are the lowest and highest scores?
b. What is the median score?
c. What is the upper quartile?
d. Approximately how many of the test scores are below 65?

Draw a box-and-whisker plot for the data in exercises 11 and 12.

11. 52, 61, 67, 75, 79, 81, 82, 83, 90, 93, 96
a. What is the range of these data?
b. What observations can you make from the plot?

12. 30, 162, 201, 149, 157, 214, 227, 154, 153, 179, 147, 226, 188, 230, 174, 223
a. What is the range of these data?
b. What observations can you make from the plot?

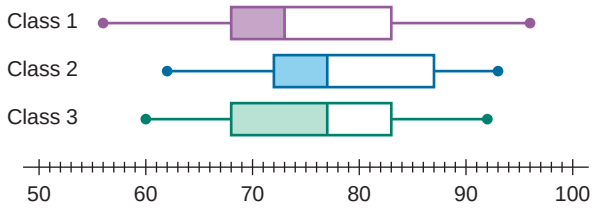
13. This list of the 25 largest states by population shows the percent (to the nearest .1) of students completing high school.*

State	Percent	State	Percent
Alabama	80.4	Missouri	85.6
Arizona	83.5	New Jersey	87
California	80.2	New York	84.1
Colorado	88.9	North Carolina	83
Florida	84.9	Ohio	87.1
Georgia	82.9	Pennsylvania	86.8
Illinois	85.7	South Carolina	82.1
Indiana	85.8	Tennessee	81.4
Louisiana	79.9	Texas	79.1
Maryland	87.4	Virginia	85.9
Massachusetts	88.4	Washington	89.3
Michigan	87.4	Wisconsin	89
Minnesota	91		

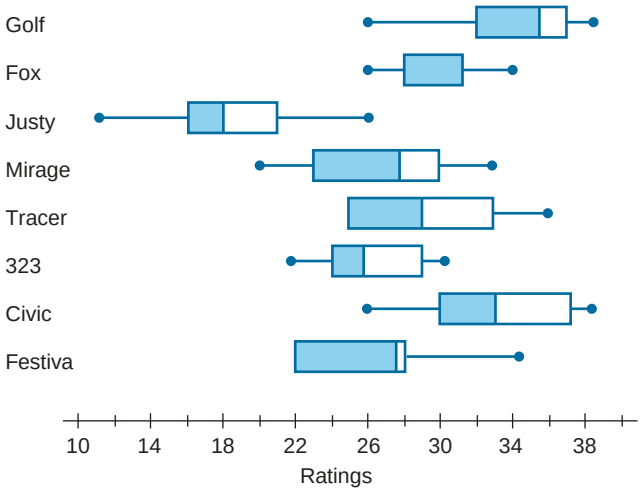
a. Draw a box-and-whisker plot of these data.
b. What is the median and what information does it provide?
c. What is the upper quartile, and what information does it provide?
d. What is the lower quartile, and what information does it provide?

**Statistical Abstract of the United States,* 128th ed. (Washington, DC: Bureau of the Census, 2009), Table 228.

14. The box-and-whisker plots shown below illustrate the test scores of three classes that took the same test. Which class performed the best and which performed the worst? Support your conclusion.



The box plots below are the results of the ratings of eight different kinds of cars for a particular year.* Each car was rated in 11 categories and received a number from 1 to 40 for each category. The 11 numbers for each car are the data for the box plots. Use these plots in exercises 15 and 16.



15. a. One measure of variability within the rating of each car is the size of the interquartile range, with greater size implying greater variability. Which car has the most variability in its ratings?
b. Which car's ratings have the highest upper quartile?
c. Which car's ratings have the lowest median?
d. Which car has the best ratings? Support your choice.

16. a. If the smallest interquartile range indicates the least variability, which car's ratings are the least variable?
b. Which car's ratings have the highest lower quartile?
c. Which car's ratings have the highest median?
d. The Justy has the poorest ratings. Which car has the second-poorest ratings? Support your conclusion.

17. The following list contains the percentage ratings of the top all-time television programs for four decades.† The ratings are the percent of television-owning households tuned into a program.

CBS	42.8	42.9	43.8	44.0	44.1	44.2	
	45.3	45.8	46.3	46.4	47.2	48.5	
	49.1	53.3	60.2				
ABC	43.1	43.2	43.4	43.4	43.8	44.1	44.8
	45.7	45.9	45.9	46.0	46.4	51.1	
NBC	42.7	43.5	44.4	44.4	44.5	45.0	45.1
	45.5	45.5	46.0	46.6	47.1	47.4	47.7
	48.3	48.6					

a. Form a box plot for each set of data, and use the plot to determine the network with the best ratings and the network with the worst ratings.
b. Use the interquartile range to determine if there are any ratings which are outliers. If there are any outliers, write a statement about their meaning.

18. The grades of eight students on a 10-point test were 1, 3, 5, 5, 7, 8, 9, and 10.
a. Compute the mean of these test scores.
b. Compute the standard deviation to the nearest .01.
c. Another class took the same test and had the same mean. What can be said about the two sets of scores if the second class had a standard deviation of 2?

19. The following two sets of data both have a mean of 6.

Set *A:*	0,	2,	4,	6,	8,	10,	12
Set *B:*	3,	4,	5,	6,	7,	8,	9

a. Predict which set of data will have the smaller standard deviation.
b. Calculate the standard deviation for both sets.
c. Do the standard deviations in part b support your prediction in part a? Explain.

Use the following list of resting pulse rates of 55 people in exercises 20 and 21. The mean of these rates is 72, and the standard deviation is approximately 9.2.

51	56	56	57	57	61	62	62	62	63	64	65
65	65	66	67	67	68	68	69	69	70	70	70
70	70	70	72	73	73	74	74	74	74	75	75
76	76	76	77	78	79	79	80	80	80		
81	82	84	84	86	86	89	91	92			

20. a. Determine the pulse rates that are within 1 standard deviation of the mean. What percentage of the total to the nearest .1 percent do these rates represent?
b. What percentage of the pulse rates to the nearest .1 percent are within 2 standard deviations of the mean?

*C. H. Hirch, ed., NCTM. *Data Analysis and Statistics,* Addenda Series, p. 25.

†*The World Almanac and Book of Facts.*

21. a. What percentage of the pulse rates to the nearest .1 percent are more than 2 standard deviations above the mean?

b. What percentage of the pulse rates to the nearest .1 percent are more than 2 standard deviations above or 2 standard deviations below the mean?

22. People often talk about "the good old days," but how good were they? The following table contains the amounts of time a person had to work in 1925 compared to the amount(s) of time in 2005 to buy the items listed.

	You Would Work in	
To Buy	1925	2005
New car	$41\frac{1}{2}$ wks	30 wks
Year in college	31 wks	20 wks
Gas range	138 hrs	54 hrs
Washing machine	$120\frac{1}{2}$ hrs	50 hrs
Sewing machine	$101\frac{1}{2}$ hrs	38 hrs
Woman's skirt	$6\frac{1}{2}$ hrs	2 hrs
Dozen oranges	$\frac{3}{4}$ hrs	$\frac{1}{4}$ hrs
Dozen eggs	53 min	12 min
Pound of coffee	48 min	10 min
Pound of butter	50 min	8 min

a. Divide each amount of work time in 1925 by the corresponding amount of work time in 2005. Each 1925 work time is how many times the corresponding 2005 value? Compute your answer to the nearest .1.

b. Compute the mean of your answers in part a. On the average, the work time needed in 1925 to buy the items listed is how many times the work time needed in 2005 to the nearest tenth?

The 50 states of the United States and the District of Columbia are classified in four regions in the table on page 488.* Four categories of data for a particular year are given in this table and are used in subsequent exercises.

Use the data from the table on page 488 involving the average amount of money spent by state per student in exercises 23 and 24.

23. a. What are the quartiles and medians of the data for the states of the south and west?

b. Form box plots for the data for these two regions, and list some observations.

c. Compare the upper quartile for the south and the median for the west. Which is greater, and what does this mean?

d. Which of these two regions has the greater interquartile range? What does this show?

*From *Statistical Abstract of the United States.*

24. a. What are the quartiles and medians of the data for the states of the midwest and northeast, mid-Atlantic?

b. Form box plots for the data for these two regions, and list some observations.

c. Compare the median for the northeast, mid-Atlantic region to the greatest value of data for the midwest. What does this show?

d. Compare the median of the midwest to the lower quartile for the northeast, mid-Atlantic region. What does this show, and what is its meaning?

Use the data from the table on page 488 involving the percent of the population (for a particular year) that is not composed of high school graduates in exercises 25 and 26.

25. a. What are the quartiles and medians of the data for the states of the west and midwest?

b. Form box plots for the data for these two regions.

c. Compare the box plots and state some observations.

d. Use the interquartile range for both sets of data to determine if there are any outliers.

26. a. What are the quartiles and medians of the data for the states of the south and northeast, mid-Atlantic?

b. Form box plots for the data for these two regions.

c. Compare the box plots and state some observations.

d. Use the interquartile range for both sets of data to determine if there are any outliers.

The data from the table on page 488 involving the average (mean) personal income per capita by states are in thousands of dollars for a particular year. For example, 28.5 represents $28,500. These incomes are rounded to the nearest $100. Use these data in exercises 27, 28, and 29.

27. a. Form box plots for the data for the states of the west and south.

b. What are the interquartile ranges for these sets of data. What do they show?

c. Compare the upper 50 percent of the incomes from the west to the upper 50 percent of the incomes from the south. What do the box plots show?

d. Which is greater, the median income for the south or the lower quartile income for the west?

e. Draw some conclusions from the box plots.

28. a. Form box plots for the data for the states of the midwest and northeast, mid-Atlantic.

b. What are the interquartile ranges for these sets of data. What do they show?

c. Compare the greatest income for the midwest with the incomes of the northeast, mid-Atlantic. What do the box plots show?

d. What do the upper quartile of the incomes of the midwest and the lower quartile of the incomes of the

Four Regions of U.S. Data

	Personal Income per Capita (in thousands of dollars)	Percentage of Population Not High School Graduates (to nearest percent)	Average Amount of Money Spent for Education per Student (in thousands of dollars)	Percentage of Population Below Poverty Level (to tenth of a percent)
Northeast, Mid-Atlantic				
Maine	28.8	13	10.0	11.3
New Hampshire	34.7	8	8.7	5.6
Vermont	30.7	11	11.5	9.9
Massachusetts	39.8	13	11.3	9.6
Rhode Island	31.9	19	10.5	10.3
Connecticut	43.2	12	12.0	7.8
New York	36.6	16	11.9	14.0
New Jersey	40.4	14	11.6	7.8
Pennsylvania	32.0	14	9.0	19.2
Delaware	32.8	11	10.9	8.1
Maryland	37.3	12	8.7	7.3
District of Columbia	48.3	14	15.1	16.8
West Virginia	24.4	21	9.6	16.0
Midwest				
Ohio	29.9	13	9.6	10.1
Indiana	28.8	14	8.9	8.7
Illinois	33.7	12	10.3	11.2
Michigan	30.4	12	9.0	10.3
Wisconsin	30.9	11	9.6	8.6
Minnesota	34.4	8	9.3	6.5
Iowa	29.0	10	7.4	8.3
Missouri	29.3	12	7.7	9.6
North Dakota	29.2	10	6.9	11.9
South Dakota	29.2	11	7.3	10.2
Nebraska	30.8	9	7.7	9.5
Kansas	29.9	11	8.7	9.4
South				
Virginia	33.7	12	6.9	8.7
North Carolina	28.2	19	7.2	13.1
South Carolina	26.1	19	7.9	13.5
Georgia	29.4	15	9.0	12.1
Florida	30.4	15	6.7	12.1
Kentucky	26.3	17	8.0	13.1
Tennessee	28.5	19	6.5	14.2
Alabama	26.3	20	5.6	14.6
Mississippi	23.4	19	6.2	17.6
Arkansas	24.3	19	6.1	18.0
Louisiana	26.1	20	7.6	17.0
Oklahoma	26.7	14	6.6	14.7
Texas	29.4	23	7.7	15.3
West				
Montana	25.9	10	8.1	13.7
Idaho	25.9	12	6.8	11.8
Wyoming	32.8	9	10.7	9.5
Colorado	34.3	11	8.0	9.4
New Mexico	25.5	18	8.2	17.8
Arizona	26.8	16	5.6	13.3
Utah	25.0	11	5.3	9.3
Nevada	31.3	14	6.7	8.3
Washington	33.3	11	7.6	10.8
Oregon	29.3	13	8.2	11.2
California	33.7	19	7.5	12.8
Alaska	33.6	9	11.2	8.3
Hawaii	30.9	11	8.1	10.6

northeast, mid-Atlantic show about the average personal income in these two regions?

e. The smallest data value for the northeast, mid-Atlantic is less than the smallest data value for the midwest. Is the data value for the northeast, mid-Atlantic an outlier?

29. a. Form box plots for the data for the states of the west and midwest.

b. The interquartile range of incomes for the states of the west is about how many times the interquartile range of incomes for the states of the midwest?

c. Compare the lower 25 percent of the incomes for the west to the lower 25 percent of the incomes of the midwest. What do the box plots show?

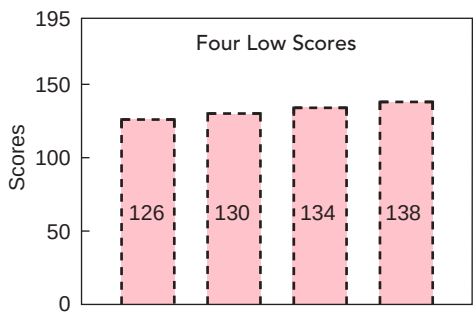
d. State some conclusions from the box plots.

Reasoning and Problem Solving

30. Featured Strategy: Drawing a Graph. Richard bowled 25 games, and the mean of his scores was 195. His four lowest scores were 126, 130, 134, and 138. If he throws out these four low scores, what is the new mean?

a. Understanding the Problem. Throwing out four scores that are below the mean of 195 will result in a greater mean. How many games will be used to determine the new mean?

b. Devising a Plan. Sometimes *drawing a graph* of the given information will suggest a plan. A mean of 195 for 25 games can be pictured as 25 games, each with a score of 195. This is shown in the graph below, where the heights of the four bars represent the four low scores. What amounts would need to be added to the low scores to bring them up to 195? Devise a plan to determine the new mean when these low scores are thrown out.



c. Carrying Out the Plan. Use your plan to determine the new mean for the 21 games.

d. Looking Back. You may have used the approach of determining the total score for the 25 games. What is the total? Then the four low scores can be subtracted to obtain a new total. Show how to determine the new mean by using this approach.

31. An ice cream shop owner hires a clerk and informs him that the mean weight of scoops of ice cream should be 45 grams. The clerk weighs out seven ice cream scoops with the following weights in grams: 43, 46, 44, 41, 44, 45, and 39. How many grams of ice cream should be added to the smallest of these scoops to obtain an average weight of 45 grams for the seven scoops?

32. The mean for the numbers of earthquake deaths (see Figure 7.19 on page 470) is 20,633. This mean is high because of the unusually high number of deaths from the 2004 earthquake in Indonesia and the 2010 earthquake in Haiti. What would the mean (to the nearest whole number) of the 44 numbers of earthquake deaths be if the 283,106 Indonesian and 222,570 Haiti deaths were each replaced by the mean of the remaining 42 numbers?

33. In Lee Middle School, the ratio of the number of fifth-graders to the number of sixth-graders is 4 to 3, and there are 18 more fifth-graders. If the ratio of the number of sixth-graders to the number of seventh-graders is 2 to 3, what is the mean number of students in the three grades?

34. Demetra has taken nine quizzes in an environmental conservation course, and the mean of her scores is 81. The course syllabus states that two of the lowest quiz scores will be thrown out. If her lowest scores are 57 and 62, what is the mean to the nearest whole number of her seven remaining scores?

35. A scientist is testing a new sweetener on 10 mice. In a period of 3 days the mice gain the following weights in grams: 4.9, 6.3, 5.1, 6.1, 5.8, 6.2, 5.7, 6.3, 6.0, and 5.6. To compare this experiment to the results of previous tests, the scientist needs to determine the percentage of these gains in weight that is within ± 1 standard deviation of the mean of the 10 weights.

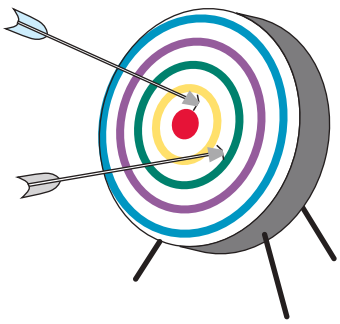
a. What is the mean of these weights?

b. What is the standard deviation to the nearest .01 of these weights?

c. What percentage of these weights is within ± 1 standard deviation of the mean?

36. Ian and Meredith each took eight quizzes in a biology course, one quiz for each of eight chapters of a text. If they both had a mean of 71 for their eight scores and Ian's standard deviation was 11 while Meredith's standard deviation was 4, what conclusion can you draw regarding their knowledge of the material of the eight chapters?

37. Peter and Sally each shoot 12 arrows at a target that is circular with a diameter of 72 inches. The mean of the distances that Peter's arrows strike from the center of the target is 20 inches, and the standard deviation of his distances is 3 inches. The mean of the distances that Sally's arrows strike from the center of the target is 16 inches, and the standard deviation of her distances is 4 inches. What statements can be made about the region of the target that at least 75 percent of Peter's arrows strike and the region of the target that at least 75 percent of Sally's arrows strike?



Teaching Questions

1. After building and leveling off stacks of cubes and doing paper-and-pencil computations to determine the arithmetic mean of a set of numbers, one of your students tells you that he still does not understand what the mean "really" represents. Devise two examples that you believe will help this student understand the concept of mean.

2. When asked to find the mean of the numbers 17, 43, 22, and 38, Ainsley turned in the computations illustrated below to show the mean was 30. Explain her method and illustrate it to find the mean of a set of different numbers.

17	43	22	38
20	40	22	38
20	30	32	38
28	30	32	30
30	30	30	30

3. Assume that you are teaching a middle school class and the method in the school text for finding the mean of two numbers is to add the two numbers and divide by 2. Suppose one of your students discovered the following method for finding the mean of two numbers and wanted to know if it would always work: subtract the smaller from the larger; divide the difference by 2; and add the result to the smaller number. Show by using diagrams that the text's method and the student's method both produce the mean. Then use algebra to show that both of these methods are equivalent.

4. The **Elementary School Text** example on page 473 approaches average as a leveling off of cubes and also gives an example of data displayed in a pictograph. (a) How can the leveling of cubes technique be used to find the mean number of Representatives in the pictograph? (b) Explain how you can extend this activity if the cubes (or pictograph) cannot be leveled evenly to the same whole number height. For example, suppose you started with stacks of cubes with heights 8, 5, 6, and 11.

Classroom Connections

1. The mean, median, and mode are called measures of central tendency and are three different ways to use a single or "typical" number to give general information about a data set. According to the **Research** statement on page 471, which of these three measures do students tend to use regardless of the distribution of data?

2. Read the **Spotlight on Teaching** at the beginning of Chapter 7 and examine the bar graph. Write a few questions that you believe would be appropriate to ask young children about the graph to introduce them to statistical thinking. Then, list additional questions that would be appropriate to ask young children to help them understand the line plot shown in this spotlight.

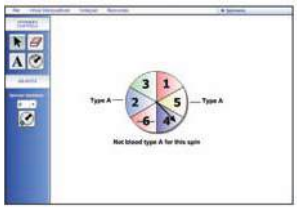
3. Read the **Standards** quote on page 469. Explain how the examples in Figures 7.18 and 7.20 on page 471 help to comply with the recommendations in this quote.

4. The one-page **Math Activity** and the **Elementary School Text** page in this section illustrate a visual method for obtaining average by "leveling-off" columns. If the bars in the bar graph in the **Spotlight on Teaching** at the beginning of this chapter were leveled off, what would the leveled off bars represent? Explain how you would interpret the leveling off of the Xs in the line plot on the same page.

MATH ACTIVITY 7.3

Simulations in Statistics

Virtual Manipulatives

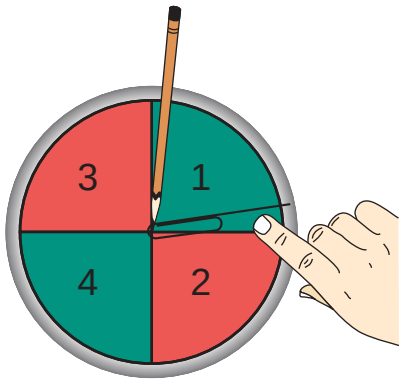


www.mhhe.com/bbn

Purpose: Use spinners to generate random numbers and explore simulation problems.

Materials: 1-to-4 and 1-to-6 Spinners in the Manipulative Kit or Virtual Manipulatives. For the cardstock spinners, bend a paper clip and hold a pencil at the center of the spinner, as shown below.

1. A certain brand of nonfat yogurt contains one of the letters A, B, C, or D on the inside of the cover. A person collecting one of each type of cover can return them to the store and receive one free container of yogurt. Assuming that each letter has the same chance of being selected, how many yogurts might a person expect to buy on average before getting one of each type of cover? Make a guess.



Rather than buy yogurts, we can devise an experiment (called a **simulation**) to model the situation. Since there are four types of covers, we can use the 1-to-4 spinner, letting 1, 2, 3, and 4 represent the A, B, C, and D covers, respectively. Spin the 1-to-4 spinner, and record the total number of spins needed to get each of the numbers at least once. Repeat this experiment 20 times, and record the number of spins needed to get all four numbers for each experiment.

a. What is the smallest number of your spins needed to get all four numbers?

b. What is the greatest number of your spins to get all four numbers?

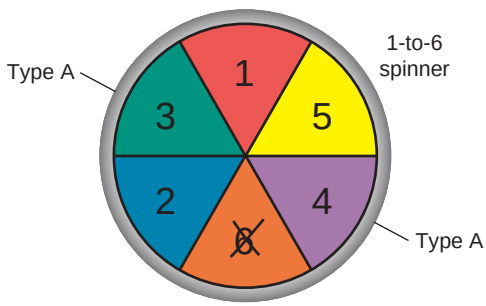
c. What is the average of the numbers of spins to get all four numbers for your 20 experiments?

d. On the basis of your experiments, write a statement about the number of yogurts a person might normally be expected to buy before getting all four types of covers.

***e.** Results tend to be more accurate for larger numbers of experiments. Pool your data from part c with several other students and compute the average of all of your answers.

2. Approximately two out of every five people have type A blood. How many people on average will need to be randomly selected to obtain three people with type A blood?

a. Model the situation by using the 1-to-6 spinner. Cross off one number (for example, the number 6) and designate two of the remaining five numbers as representing people with type A blood. Record the total number of spins needed to get three people with type A blood.



b. Repeat this experiment 20 times, and record the number of spins needed to get three people with type A blood for each experiment. What is the average of the number of spins to get three people with type A blood for your 20 experiments?

Section 7.3 SAMPLING, PREDICTIONS, AND SIMULATIONS



"That's the worst set of opinions I've heard in my entire life"

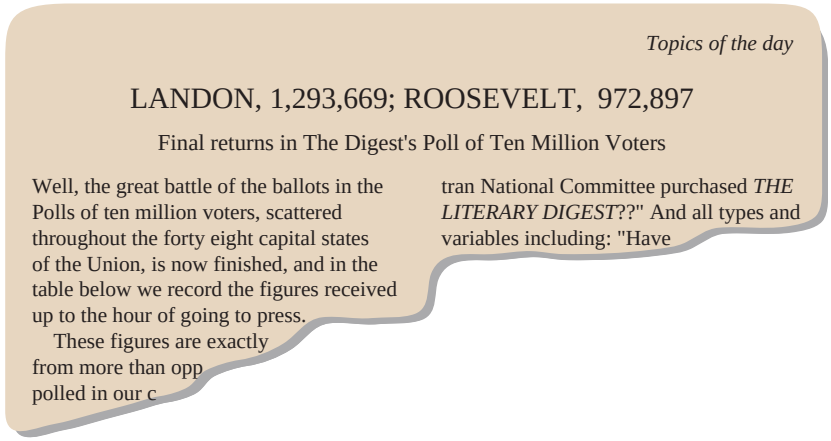
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PROBLEM OPENER

In one survey of trout populations, biologists caught and marked 232 trout from a lake. Three months later the biologists selected a second sample of 329 trout from the lake, and 16 were found to be marked. Assuming that the 232 marked trout intermingled freely with unmarked trout during the 3-month period, estimate the number of trout in the lake.

Making predictions from samples is an important part of *inferential statistics.* Because of the many possibilities for errors, strict procedures must be followed in gathering data. The sample must be large enough, and it must be a representative cross section of the whole.

The need for scientific sampling techniques was dramatically illustrated in the 1936 Presidential election. *Literary Digest,* which had been conducting surveys of elections since 1920, sent questionnaires to 10 million voters (see Figure 7.28 on the next page). Its sample was obtained from telephone directories and lists of automobile owners. By choosing the sample this way, the magazine's editors selected people with above-average incomes (that is, voters who could afford what were relative luxuries at that time: phones and cars) rather than voters from a range of income levels. Based on its sample, the *Digest* predicted that Alfred Landon would win. Instead, the election was a landslide victory for Franklin D. Roosevelt, who had much popular support among middle- and low-income voters.



Topics of the day

LANDON, 1,293,669; ROOSEVELT, 972,897

Final returns in The Digest's Poll of Ten Million Voters

Well, the great battle of the ballots in the Polls of ten million voters, scattered throughout the forty eight capital states of the Union, is now finished, and in the table below we record the figures received up to the hour of going to press.

These figures are exactly from more than opp polled in our c

tran National Committee purchased *THE LITERARY DIGEST*??" And all types and variables including: "Have

Figure 7.28

NCTM Standards

The concept of sample is difficult for young students. Most of their data gathering is for full populations such as their own class. p. 113

SAMPLING

A **sample** is a collection of people or objects chosen to represent a larger collection of people or objects, called the **population.** For example, when a national poll of 1873 people is used to determine the popularity of a television program, the 1873 people form the sample and all television watchers in the country are the population.

Random Sampling If a sample is obtained in such a way that every element in the population has the same chance of being selected, it is called a **random sample** and the process is called **random sampling.**

Randomness is difficult to achieve. Repeatedly tossing a coin may appear to be a random method of making yes-or-no decisions, but imbalances in the coin's weight and tossing it to approximately the same height each time are two factors that could cause a biased result. Similarly, dice and spinners produce fairly random sequences of numbers, but these also have slight biases due to their physical imperfections.

One method of obtaining a random sample is to use a table of **random digits.** Many types of calculators are programmed to generate random digits.* The list of random digits in Figure 7.29 is from a computer printout. The digits are printed in pairs and groups of 10 for ease of reading and counting.

40 09 18 94 06	62 89 97 10 02	58 63 02 91 44	79 03 55 47 69	14 11 42 33 99
33 19 98 40 42	13 73 63 72 59	26 06 08 92 65	63 08 82 45 85	14 45 81 65 21
69 49 02 58 44	45 45 19 69 33	51 68 97 99 05	77 54 22 70 97	59 06 64 21 68
17 49 43 65 45	04 95 82 76 31	85 53 15 21 70	59 17 27 54 67	07 76 13 95 00
43 13 78 80 55	90 80 88 19 13	13 89 11 00 60	41 86 23 07 60	22 77 93 30 83

Figure 7.29

Example A on the next page illustrates the use of random digits for obtaining a random sample. First a number is assigned to each element of the population. Then, to ensure randomness, an arbitrary starting place is selected in a table of random digits.

*The TI-73 Explorer, 83 and 83/84 Plus and the CASIO fx-55 calculators have function keys for producing lists of random digits.

EXAMPLE A

How can a table of random digits be used to select 10 questions randomly from a list of 50 questions?

Solution Number the questions from 1 to 50. Then arbitrarily select a pair of numbers from the table in Figure 7.29 on the previous page, say 26, the 11th pair in the second row. Beginning with this number and moving to the right along the row (and down to additional rows as needed), list the first 10 different numbers that are less than or equal to 50. The numbers of the 10 questions to be selected are 26, 6, 8, 45, 14, 21, 49, 2, 44, and 19.

. . . 26 06 08 92 65 63 08 82 45 85 14 45 81 65 21
69 49 02 58 44 45 45 19 . . .

The list of digits in Figure 7.29 may also be used to obtain random single-digit numbers or random numbers with three or more digits.

EXAMPLE B

How can a list of random digits be used to select a random sample of 65 items from a list of items numbered 1 to 650?

Solution One way is to start with any digit in a table of random digits and list consecutive groups of three digits until you have found 65 numbers between 1 and 650. The numerals 001, 002, etc., represent 1, 2, etc., and any triples of numbers from the table that are greater than 650 are discarded. If we use this method with the table of random digits in Figure 7.29 and begin with the first line, 400 is the first number. Then 918, 940, 662, 899, and 710 are discarded because they are greater than 650. The next acceptable number is 025, which represents 25. Using this process with a sufficiently large table of random digits will produce a random sample of 65 items.

Stratified Sampling In **stratified sampling,** a population is divided into groups. The number sampled from each group is then determined by the ratio of the size of the group to the size of the total population.

EXAMPLE C

A city council wants to sample the opinion of the city's adult population of 80,000 people on a plan to build a public swimming pool. The population is divided into three groups—high-income, middle-income, and low-income—and 1500 people are to be sampled. If 16,000 people are low-income, 56,000 are middle-income, and 8000 are high-income, determine the size of the sample for each income group.

Research Statement

The 7th national mathematics assessment found that proportional reasoning was a source of difficulty for students in reasoning about data, graphs, and chance.

Zawojewski and Shaughnessy

Solution Since $\frac{16{,}000}{80{,}000} = .2$ and $.2 \times 1500 = 300$, 300 people will be sampled from the low-income group. Since $\frac{56{,}000}{80{,}000} = .7$ and $.7 \times 1500 = 1050$, 1050 people will be sampled from the middle-income group. Finally, since $\frac{8000}{80{,}000} = .1$ and $.1 \times 1500 = 150$, 150 people will be sampled from the high-income group.

SKEWED AND SYMMETRIC DISTRIBUTIONS

The graph of a set of data provides a visual way of illustrating the **distribution** of the data, that is, how the data are clustered together, isolated from each other, or spread out.

For example, the bar graph in Figure 7.30 shows that about 97 families have no children, about 105 families have 1 child, etc. The most common number of children per family (the mode) is 2. Graphs that show the data piled up at one end of the scale and tapering off toward the other end are called **skewed.** The direction of skewness is determined by the longer "tail" of the distribution. The graph in Figure 7.30 is said to be **skewed to the right (positively skewed).**

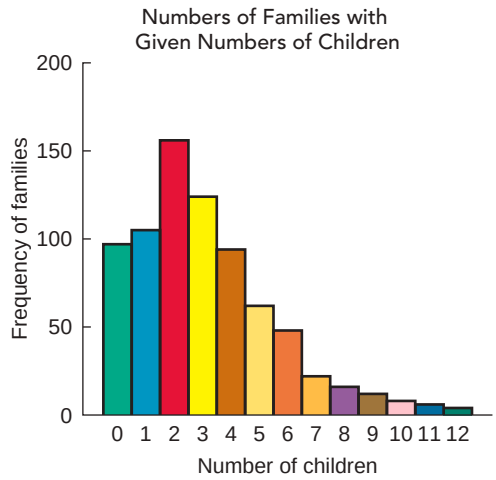


Figure 7.30

Similarly, a graph may have data piled up at the right with the "tail" extending to the left. This type of graph is said to be **skewed to the left (negatively skewed).** Such a graph is illustrated in Figure 7.31. It shows the numbers of teachers in a large school district who drive cars built in the years from 1995 to 2010; the greatest number of teachers have cars that were built in recent years. The mode in this example is the year 2006.

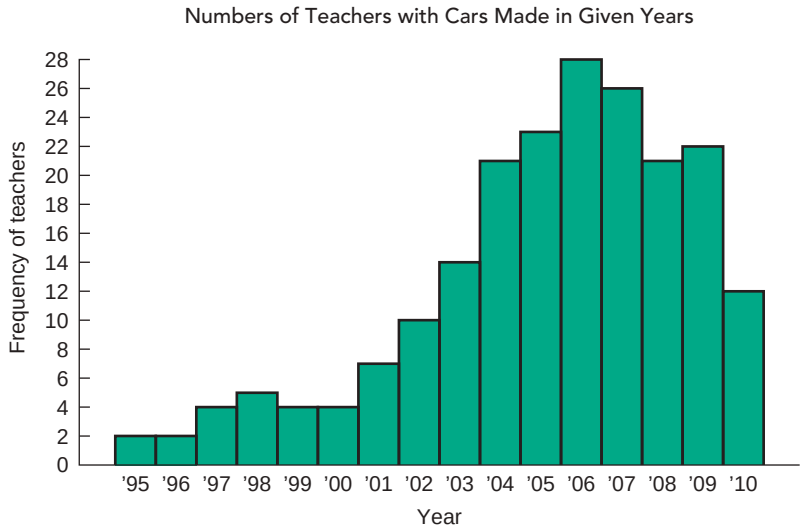


Figure 7.31

A distribution of data in which measurements at equal distances from the center of the distribution occur with the same frequency is said to be **symmetric.** A symmetric distribution and two skewed distributions are shown in Figure 7.32. The graph shows the relative positions of the mean, median, and mode for these distributions. Notice that in a symmetric distribution the mean, median, and mode are all equal.

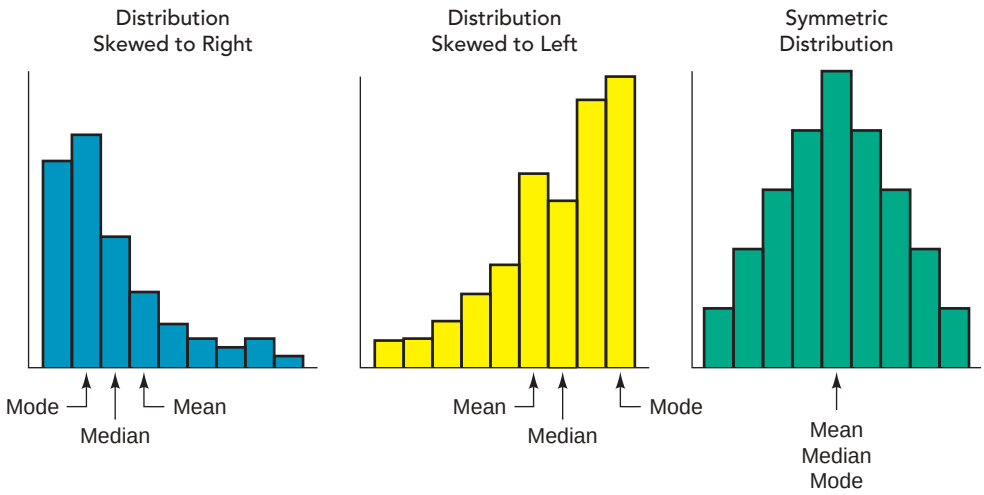


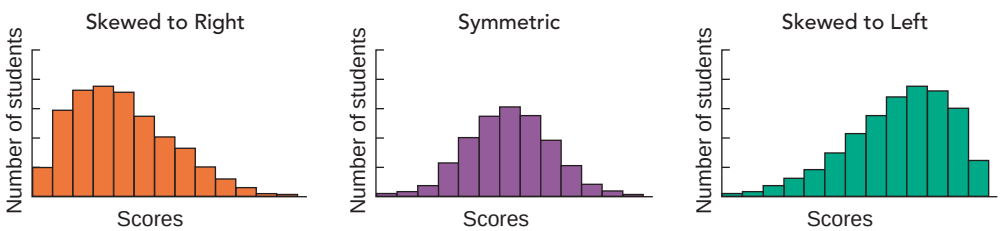
Figure 7.32

EXAMPLE D

School test results sometimes produce skewed graphs, especially if the test is too difficult or too easy for the students. Determine the types of distributions of test scores that will occur in each of the following cases.

1. A test designed for fifth-graders is given to second-graders.
2. A test designed for fifth-graders is given to fifth-graders.
3. A test designed for fifth-graders is given to eighth-graders.

Solution **1.** The majority of scores will be low, and the distribution will be skewed to the right, as shown in (a) below. **2.** The distribution of scores will be more or less symmetric, as illustrated in (b). **3.** The majority of scores will be high, and the distribution will be skewed to the left, as in (c).



NORMAL DISTRIBUTIONS

As the number of values in a set of data increases and the width of the intervals for the grouped data (width of bars) on the histogram becomes smaller, the shape of the top of the histogram approaches a smooth curve. Thus, in graphing large sets of data, it is customary to approximate a histogram (see Figure 7.33a) by a smooth curve (see Figure 7.33b).

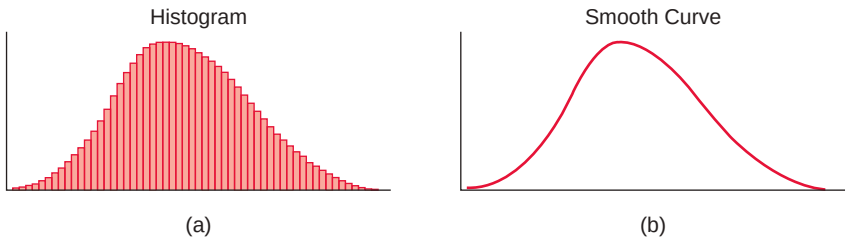


Figure 7.33

A smooth symmetric bell-shaped curve, such as the curve shown in Figure 7.34, is called a **normal curve,** and the distribution of its data is called a **normal distribution.** Normal distributions have certain important properties. About 68 percent of the values are within 1 standard deviation of the mean; about 95 percent are within 2 standard deviations of the mean; and about 99.7 percent fall within 3 standard deviations of the mean. The remaining percent is evenly divided above and below 3 standard deviations. These approximate percents hold for any normal distribution, regardless of the mean or the size of the standard deviation.

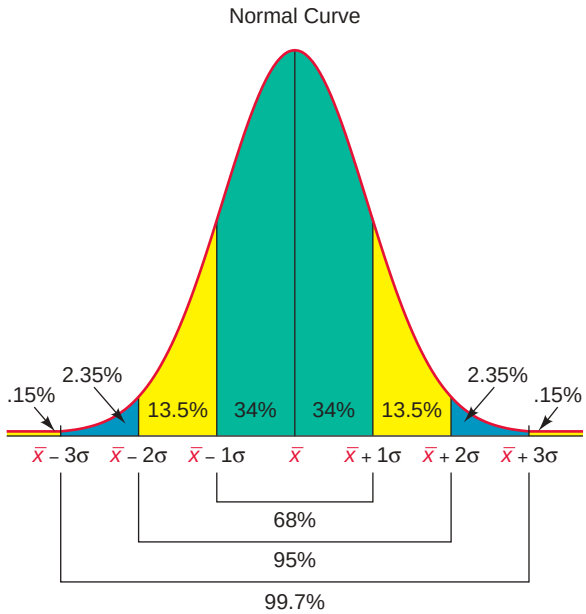


Figure 7.34

The shapes of normal curves vary, as shown in Figure 7.35 on the next page. The standard deviation of the data determines the shape of the curve. The smaller the standard deviation is, the less spread out the data and the taller and thinner the curve, as in Figure 7.35a. The larger the standard deviation is, the more spread out the data and the lower and flatter the curve, as in Figure 7.35c. The standard deviations of the three sets of data for the normal curves in Figure 7.35 increase from part a to part c.

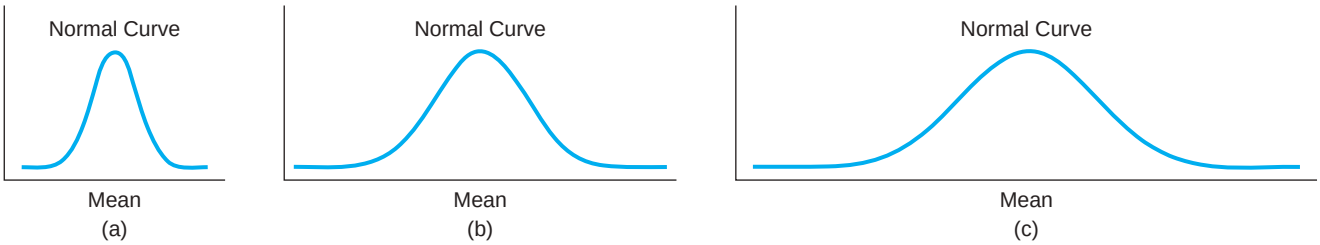
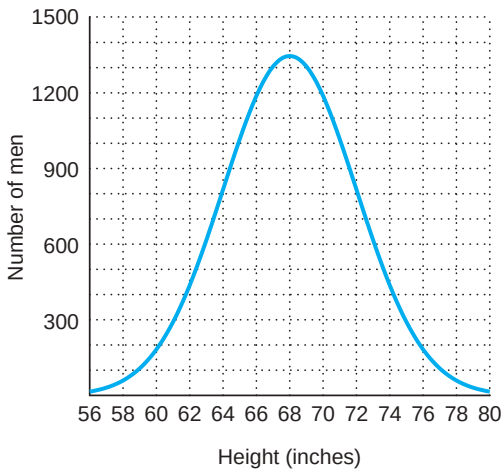


Figure 7.35

The mean and standard deviation of a normal distribution are used to provide information about the distribution of data, as shown in Examples E and F.

EXAMPLE E

The following graph, showing the distribution of the heights of 8585 men, is an approximation to a nearly normal distribution. The mean is approximately 68 inches (5 feet 8 inches), and the standard deviation is approximately 3 inches.



1. How many of these men are between 5 feet 5 inches and 5 feet 11 inches tall?
2. How many men are between 5 feet 2 inches and 6 feet 2 inches tall?
3. How many men are less than 5 feet 2 inches tall?

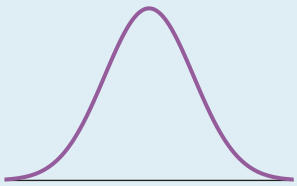
Solution **1.** One standard deviation above and below the mean includes the heights from 5 feet 5 inches to 5 feet 11 inches, and this interval contains 68 percent of the data. Since $.68 \times 8585 \approx 5838$, there are approximately 5838 men with heights in this interval. **2.** Two standard deviations above and below the mean includes the heights from 5 feet 2 inches to 6 feet 2 inches, and this interval contains 95 percent of the data. Since $.95 \times 8585 \approx 8156$, there are approximately 8156 men in this interval. **3.** More than 2 standard deviations above and below the mean corresponds to heights of less than 5 feet 2 inches and more than 6 feet 2 inches; these intervals together contain 5 percent of the data, so each contains approximately 2.5 percent of the data. Since $.025 \times 8585 \approx 215$, there are approximately 215 men who are less than 5 feet 2 inches tall.

EXAMPLE F

The National Academy of Sciences has suggested a standard for public water that allows no more than 100 milligrams of sodium per liter of water. The records of a certain city's water treatment plant for a 200-day period are normally distributed and show that the mean number of milligrams per liter is 94.0 with a standard deviation of 3.0. For how many days of this period were the sodium levels above 100 milligrams?

Solution Two standard deviations above the mean of 94 is 100. Since approximately 2.5 percent of the measurements of a normal distribution is above 2 standard deviations, and $.025 \times 200 = 5$, the sodium level was above 100 milligrams per liter on five days.

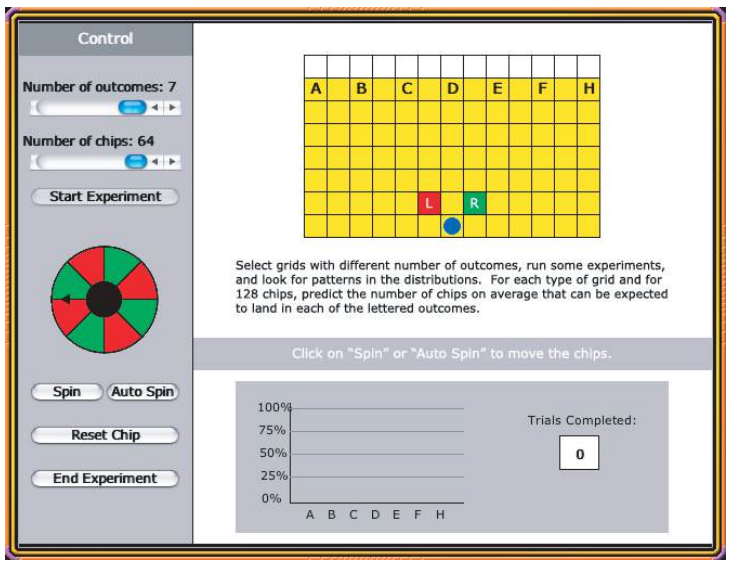
HISTORICAL HIGHLIGHT



The word *normal* is used to indicate that a normal distribution is very common in nature. About 1833, Belgian scientist L. A. J. Quetelet collected large amounts of data on human measurements: height, weight, length of limbs, intelligence, etc. He found that all measurements of mental and physical characteristics of humans tended to be normally distributed. That is, the majority of people have measurements that are close to the mean (average), and measurements farther from the mean occur less frequently. Quetelet was convinced that nature aims at creating the perfect person but misses the mark and thus, creates deviations on both sides of the ideal.

Technology Connection

If the blue chip shown at the bottom of this grid moves randomly up to the right (green square) or left (red square), in which lettered square will it most likely end? This applet will help you find patterns in different size grids in order to help you predict the final square on which the blue chip will most likely land. It is interesting that such random movements, when applied many times, produce dependable patterns.



Distributions Applet, Chapter 7, Section 3
www.mhhe.com/bbn

MEASURES OF RELATIVE STANDING

We often wish to determine the relative standing of one measurement in a given set of data, that is, to compare one value with the distribution of all values. This is especially important in analyzing test results. The mean is one common **measure of relative standing.** If the mean of some test scores is 70 and a student has a score of 85, then we know the student has done better than average. However, this information does not tell us how many students scored higher than 70 or whether 85 was the highest score on the test.

Percentiles One popular method of stating a person's relative performance on a test is to give the percentage of people who did not score as high. For example, a person who scores higher than 80 percent of the people taking a test is said to be in the *80th percentile.*

> ***p*th percentile** The ***p*th percentile** of a set of data is a number that is greater than ***p*** percent of the data and less than (100 − ***p***) percent of the data.

Percentiles range from a low of 1 to a high of 99 percent; the 50th percentile is the median. It is customary on standardized tests to establish percentiles for large samples of people. When you take such a test, your score is compared to those of the sample. A percentile score of 65 means that you did better than 65 percent of the sample group. The table and bar graph in Figure 7.36 show a student's performance on a differential aptitude test. Nine categories are listed in the table at the top of this form: verbal reasoning, numerical ability, VR + NA (verbal reasoning and numerical ability together), abstract reasoning, etc. The raw score in each category represents the number of questions that the student answered correctly. The student's percentile score is obtained by comparing these raw scores with the

	Verbal Reasoning	Numerical Ability	VR + NA	Abstract Reasoning	Clerical Sp & Acc	Mechanical Reasoning	Space Relations	Language Usage: Spelling	Language Usage: Grammar
Raw Score	33	31	64	36	41	48	28	56	32
Percentile	90	95	97	80	40	75	70	55	85

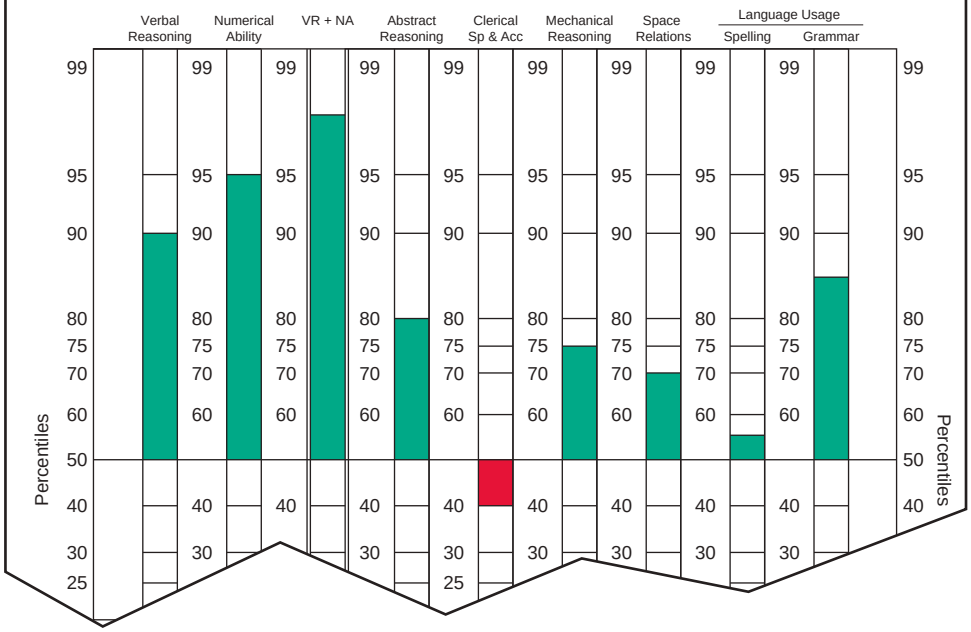


Figure 7.36

scores from a sample of thousands of other students. The bar graph is a visual representation of the percentile scores. A horizontal line at the 50th percentile makes it easier to spot scores above and below the median.

EXAMPLE G

1. The verbal reasoning score shown in Figure 7.36 on the previous page is at the 90th percentile. What does this mean?
2. In which of the nine categories is the student's performance below the median?
3. What percentage of people in the sample group had better spelling scores than this student?

Solution **1.** This student's verbal reasoning score is greater than the verbal reasoning scores of 90 percent of the people in the sample group. **2.** In the clerical category. **3.** 45 percent.

The 50th percentile (median) splits any set of data into two parts: the lower part and the upper part. The median of the lower part is the 25th percentile or lower quartile (see Figure 7.37), and the median of the upper part is the 75th percentile or upper quartile.

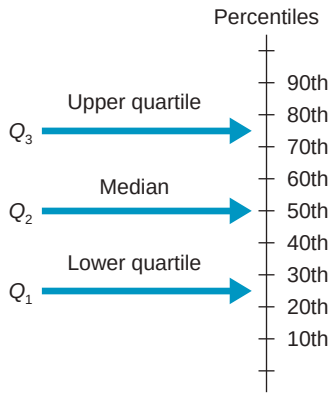


Figure 7.37

***z* Scores** Percentiles are a method of stating a person's relative standing on a test compared to that of others on the same test. But suppose you wish to compare performances on two different tests. One popular method of determining relative standing is to compute the number of standard deviations that a person's test score is from the mean.

EXAMPLE H

John scored 572 on the mathematics part of the Scholastic Aptitude Test (SAT). The mean score for this test was 460, and the standard deviation was 112. Bev scored 28 on the American College Test (ACT), and this test had a mean of 18 and a standard deviation of 5. Who had the better performance?

Solution John's score of 572 is 1 standard deviation above the mean ($460 + 112 = 572$). Bev's score of 28 is 2 standard deviations above the mean. Even though Bev's score of 28 on the ACT appears much lower than John's score of 572 on the SAT, Bev's performance was better.

The number of standard deviations a measurement is from the mean is called the ***z* score.** A z score can be defined for every measure in a set of data and can be computed as follows:

z Score The ***z* score** for a measurement x is denoted by

$$z = \frac{x - \bar{x}}{\sigma}$$

where $\bar{x}$ is the mean and σ is the standard deviation for the set of data.

Notice that in Example H on the previous page, John has a z score of 1 (his score is 1 standard deviation above the mean), since

$$z = \frac{572 - 460}{112} = 1$$

and Bev has a z score of 2 (her score is 2 standard deviations above the mean), since

$$z = \frac{28 - 18}{5} = 2$$

Figure 7.38 shows a few z scores and their relationship to the standard deviation and mean. Notice that if a measurement has a z score of 0, the measurement equals the mean.

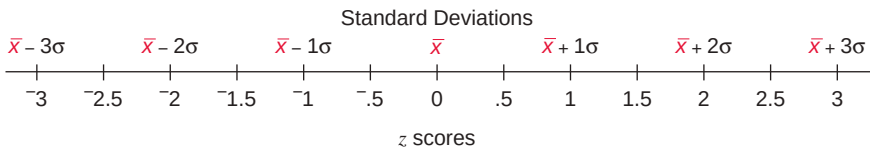


Figure 7.38

EXAMPLE I

Three students each took a different test, and these three tests had different means and standard deviations. The results are listed below. Which student had the best relative performance and which had the worst?

Student 1 scored 82 on test 1. The mean on this test was 78.5, and the standard deviation was 2.3.
Student 2 scored 55 on test 2. The mean on this test was 48.2, and the standard deviation was 4.3.
Student 3 scored 392 on test 3. The mean on this test was 460, and the standard deviation was 85.

Solution Student 1: $z = \frac{82 - 78.5}{2.3} \approx 1.52$

Student 2: $z = \frac{55 - 48.2}{4.3} \approx 1.58$

Student 3: $z = \frac{392 - 460}{85} = {}^{-}.8$

Student 2 had the best relative performance with a z score of 1.58, and student 3 had the worst performance. Notice that student 3 scored below the mean, so the z score is a negative number.

Using *z* Scores for Predictions Almost all z scores are between $^{-}3$ and $^{+}3$, and most z scores are between $^{-}2$ and $^{+}2$. Thus, since few measurements are more than 2 standard deviations from the mean, we will define such measurements to be **rare events.**

Rare Event If a measurement for a set of data is more than 2 standard deviations from the mean, it is considered to be a **rare event.** That is, if its z score is less than $^{-}2$ or more than $^{+}2$, it is considered to be a rare event.

This definition can be applied to make predictions.

EXAMPLE J

In a certain school system, the mean annual salary for male teachers with more than 20 years of teaching experience was \$47,320 with a standard deviation of \$2540. A female teacher with more than 20 years of teaching experience and an annual salary of \$42,000 filed a grievance procedure, claiming that her salary was low due to sex discrimination. Is there evidence to support her claim?

Solution The z score for the female teacher's salary is

$$\frac{42{,}000 - 47{,}320}{2540} \approx {}^{-}2.1$$

Since the z score is below $^{-}2$, the female teacher's salary is considered a rare event; that is, it is evidence of discrimination.

SIMULATIONS

There are many statistical problems that are of interest to children but are beyond their abilities to solve theoretically. Example K contains a type of problem that might appeal to elementary schoolchildren. Such problems are related to sampling and can be solved by conducting experiments.

EXAMPLE K

Each package of a certain brand of cereal contains one of seven cards about superheroes. The students in an elementary school class wanted to know how many boxes of cereal they could expect to buy before getting the entire set.*

Solution The elementary school class solved this problem by writing the names of the seven superheroes on slips of paper and then performing the following experiment: **1.** The seven slips of paper were put in a bag. **2.** A slip was drawn at random from the bag, the superhero's name was tallied, and the slip of paper was returned to the bag. **3.** Step 2 was repeated until the name of each superhero had been drawn at least once. **4.** The total number of draws was recorded. This total represents the number of boxes needed in this experiment to obtain an entire set of superhero cards.

This experiment was repeated 20 times. Here are the numbers of boxes obtained in the 20 experiments:

20, 14, 27, 18, 17, 15, 19, 19, 19, 20, 16, 11, 15, 21, 15, 22, 20, 28, 12, 26

The average (mean) number of "boxes" for this experiment before one gets the entire set of cards is the sum of these numbers divided by 20:

$$\frac{374}{20} = 18.7 \approx 19$$

The students could see that they might have to go through more than 19 boxes to get all seven cards if they were unlucky; or they might collect all seven cards after buying fewer than 19 boxes, if they were lucky.

*Ann E. Watkins, "Monte Carlo Simulations: Probability the Easy Way," in the NCTM yearbook, *Teaching Statistics and Probability,* pp. 203–209.

Finding an answer to the question in Example K by purchasing boxes of cereal would be expensive. Representing cereal box prizes by writing names on slips of paper and performing experiments is an example of a *simulation.* In general, a **simulation** is a procedure in which experiments that closely resemble the given situation are conducted repeatedly. This method relies on identifying a model, such as rolling dice, tossing a coin, or using a table of random digits, that can be used to *simulate* an event and then performing experiments using the model.

NCTM Standards

If simulations are used, teachers need to help students understand what the simulation data represent and how they relate to the problem situation, . . . p. 254

PROBLEM-SOLVING APPLICATION

Using a simulation is a powerful problem-solving technique.

Problem

If people are selected randomly, how many must be selected (on average) to find two who have a birthday in the same month?

Understanding the Problem It might be necessary to select several people to find two with a birthday in the same month. **Question 1:** What is the minimum number that must be chosen before this will happen?

Devising a Plan Conducting a simulation will be more convenient than interviewing large numbers of people. One method of simulation involves writing the whole numbers from 1 to 12 on 12 slips of paper to represent each of the 12 months and placing them in a box. A slip is then randomly selected, its number is recorded, and the slip is returned to the box. An experiment consists of selecting numbers one at a time until the same number is obtained twice. After this experiment has been carried out several times, we can determine the mean of the numbers of selections. **Question 2:** Why must each slip of paper be returned to the box after it is selected?

Technology Connection

Dice Roll Simulation

How many times on the average do you think three dice would have to be rolled to obtain a sum greater than or equal to 15? The online 7.3 Mathematics Investigation repeatedly simulates a roll of 2 to 5 dice until a desired sum is obtained. Explore this and related questions in this investigation.

Mathematics Investigation
Chapter 7, Section 3
www.mhhe.com/bbn

Carrying Out the Plan The following 19 groups of numbers were obtained by selecting slips of paper from a box until the same number was chosen twice for each group. Determine the mean (average) size of these groups of numbers. **Question 3:** According to this experiment, how many people must be interviewed (on average) to find two with a birthday in the same month?

7, 2, 2 1, 9, 10, 3, 5, 6, 9 7, 2, 2 10, 11, 8, 10 11, 9, 2, 3, 7, 9
9, 4, 9 7, 11, 9, 6, 4, 6 5, 9, 9 7, 12, 6, 11, 12 9, 12, 3, 11, 1, 4, 1
12, 2, 8, 6, 6 7, 2, 3, 9, 11, 9 12, 9, 3, 7, 5, 3 7, 2, 11, 4, 10, 5, 12, 9, 7
1, 4, 12, 2, 12 5, 9, 8, 3, 8 11, 11 6, 2, 12, 7, 1, 1 5, 2, 12, 10, 12

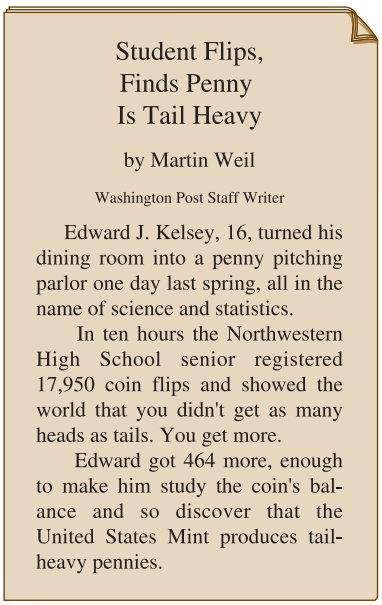
Looking Back Another solution to this problem can be found by using a table of random numbers for the simulation (see page 505). One way to use such a table is to select one of the numbers arbitrarily as a beginning point and then, moving from left to right, to record pairs of numbers that are greater than 0 and less than or equal to 12, with 01, 02, etc., being counted as 1, 2, etc. Solve the original problem by carrying out a simulation, using the

following list of random digits. **Question 4:** What number do you obtain as a solution to the problem by using this simulation?

89 81 80 69 77 09 86 76 77 71 21 52 23 86 53 95 20 94 29 48 33 37 58 33 93
24 30 87 37 31 80 37 25 47 06 72 78 11 30 08 88 84 78 78 46 51 14 96 58 12
77 02 18 48 54 50 76 36 05 12 33 77 59 58 76 17 68 58 89 84 38 35 42 17 55
58 01 63 92 45 47 24 54 42 80 55 53 09 95 46 98 94 67 27 15 52 56 08 82 56
24 39 68 08 01 15 72 23 88 37 38 00 36 94 14 47 88 90 44 74 28 27 01 71 16
05 61 62 60 18 72 01 75 51 88 52 95 13 39 81 75 76 66 02 76 29 69 77 96 77
62 23 95 43 71 34 38 09 45 82 85 62 72 58 62 74 51 95 87 44 45 01 77 67 26
43 07 96 21 98 68 25 01 17 11 59 32 39 70 13 21 43 81 57 55 86 59 28 45 34
95 78 66 81 10 85 54 62 86 27 44 89 51 18 75 48 62 29 43 54 44 46 13 32 13
55 00 90 00 42 27 01 23 24 10 49 21 46 26 14 82 31 94 54 39 55 07 81 32 57
81 57 86 88 83 81 54 91 42 82 82 14 44 13 30 27 84 31 77 21 88 67 72 04 36
99 94 94 09 62 81 41 09 62 30 95 13 69 92 15 18 76 02 78 22 15 86 90 86 72

Answers to Questions 1–4 **1.** Two **2.** There should be an equal chance of any number being chosen in any selection from the box. **3.** The following numbers represent the sizes of the groups: 3, 7, 3, 4, 6, 3, 6, 3, 5, 7, 5, 6, 6, 9, 5, 5, 2, 6, 5. The mean of these numbers is $\frac{96}{19} \approx 5.05$. Thus, on average approximately five people must be chosen before two are found with a birthday in the same month. **4.** Starting at the beginning of the list at 89, we obtain the following groups: [09, 06, 11, 08, 12, 02, 05, 12], [01, 09, 08, 08], [01, 01], [05, 01, 02, 09, 01], [07, 01, 11, 10, 01], [10, 07, 04, 09, 09]. The mean size of these six groups is approximately 4.8.

Exercises and Problems 7.3



Student Flips, Finds Penny Is Tail Heavy

by Martin Weil

Washington Post Staff Writer

Edward J. Kelsey, 16, turned his dining room into a penny pitching parlor one day last spring, all in the name of science and statistics.

In ten hours the Northwestern High School senior registered 17,950 coin flips and showed the world that you didn't get as many heads as tails. You get more.

Edward got 464 more, enough to make him study the coin's balance and so discover that the United States Mint produces tail-heavy pennies.

As this article from the *Washington Post* reports, a student recorded 17,950 coin flips and got 464 more heads than tails. He concluded that the U.S. Mint produces tail-heavy coins. For many repeated experiments of 17,950 tosses of a fair coin, we can expect an approximately normal distribution with a mean of 8975 heads and a standard deviation of 67. Use this information in exercises 1 and 2.

1. a. The area under a normal curve within ± 1 standard deviations of the mean is 68 percent of the total area under the curve. Therefore, 68 percent of the time, the number of heads should be between what two numbers?

b. The area under a normal curve within ± 2 standard deviations of the mean is approximately 95 percent of the total area under the curve. Therefore, approximately 95 percent of the time, the number of heads should be between what two numbers?

2. a. The area under a normal curve that is less than $^{+}3$ standard deviations above the mean is approximately 99.85 percent of the total area under the curve. Therefore, 99.85 percent of the time, the number of heads should be less than what number?

b. Edward Kelsey got 9207 heads. Was he justified in concluding that the coins are tail-heavy?

Describe a method for using a table of random digits in exercises 3 and 4 to obtain the random samples. Then use the table of random digits above to obtain the sample.

3. a. The names of 2 people from a list of 9

b. Ten test questions from a total of 60

4. a. Five people from a group of 30
b. The health records of 8 children from a class of 25

5. An elementary school has the following numbers of students in grades K to 4: grade K, 50; grade 1, 80; grade 2, 90; grade 3, 80; and grade 4, 100. If a stratified sample of 80 children is to be chosen, how many children will be chosen from each grade?

6. There are 18 girls and 12 boys in a class. If stratified sampling is used to select 10 students, how many girls will be selected?

Describe the distribution of scores (skewed to the right, symmetric, skewed to the left) for the tests in exercises 7 and 8. Explain your answers.

7. a. A test designed for third-graders and given to first-graders
b. A test designed for sixth-graders and given to eighth-graders

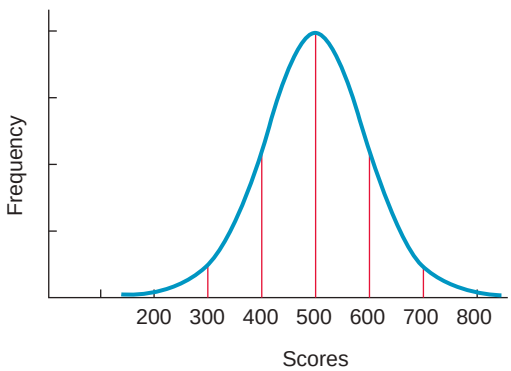
8. a. A test designed for fourth-graders and given to second-graders
b. A test designed for second-graders and given to second-graders

Would you expect the distribution of the sets of data in exercises 9 and 10 to be skewed to the right, symmetric, or skewed to the left? Explain your answers. (*Hint:* Sketch a graph with some typical values.)

9. a. Amounts of time students study in a 24-hour period before an exam
b. Widths of the hand spans of fifth-graders
c. Sneaker sizes of professional basketball players

10. a. Heights of college students
b. Test scores of third-graders on a pretest on fractions at the beginning of the school year
c. Weights of newborn babies

The normal curve here shows a distribution of college entrance exam scores that has a mean of 500 and a standard deviation of 100. Use this curve in exercises 11 and 12.



11. a. What percent of students scored between 400 and 600?
b. What percent of students scored above 600?
c. What percent of students scored below 300?

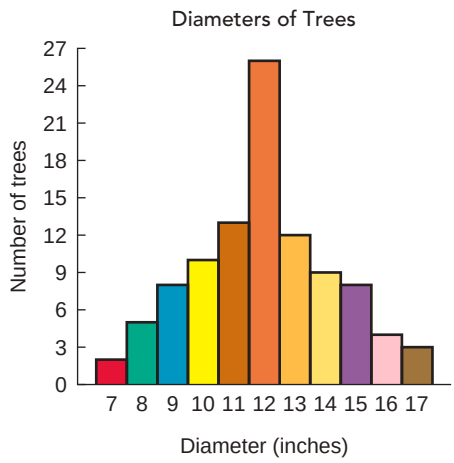
12. a. What percent of students scored between 300 and 600?
b. What percent of students scored above 700?
c. What percent of students scored above 800?

One method of grading tests that uses a normal curve gives students letter grades depending on the standard-deviation interval above or below the mean that contains their score: a grade of F for below -2 standard deviations; D for -2 to -1 standard deviation; C for -1 to $+1$; B for $+1$ to $+2$; and A for above $+2$ standard deviations. In exercises 13 and 14, use this grading system and the fact that on a test given to 50 students the mean score was 78 and the standard deviation was 6.

13. a. How many students received an F?
b. How many students received a grade of B?
c. How many students received a D?

14. a. How many students received a C?
b. How many students received a grade below C?
c. How many students received an A?

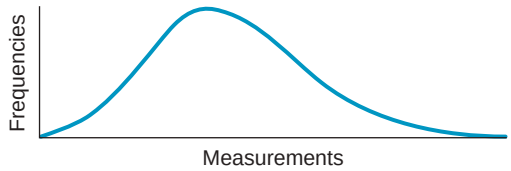
This bar graph shows the measures of the diameters (to the nearest inch) of 100 trees of the same species. The mean diameter is 12 inches, and the standard deviation is approximately 2 inches. Use this information in exercises 15 and 16.



15. a. What type of distribution does this graph illustrate?
b. Make a frequency table showing the numbers of trees of each diameter.
c. What percentage of the diameters are within 1 standard deviation of the mean?

16. a. What percent of the trees have diameters within 2 standard deviations of the mean?
b. What percent of the trees have diameters greater than 2 standard deviations above the mean?
c. What can be said about the difference between the mean diameter of the trees and the diameter of a tree that is more than 2 standard deviations below the mean?

Trace the graph shown here on a sheet of paper, and mark the approximate locations of the 10th, 25th, 50th, 75th, and 99th percentiles on the horizontal axis assuming the area between the curve and the x-axis is 100% of the data. Use this graph in exercises 17 and 18.



17. a. What percentage of the measurements is less than or equal to the 30th percentile?
b. What percentage of the measurements is greater than or equal to the 90th percentile?

18. a. What percentage of the measurements is between the 30th and 90th percentiles?
b. What percentage of the measurements is between the median and the 80th percentile?

The following score form shows a fourth-grade student's scores on six of the subtests of the Stanford Achievement Test. The top row of numbers listed for each subtest shows the number of questions answered correctly out of the total number of questions for that subtest. The first number in the second row of each column shows the national percentile; and the first number in the third row of each column shows the local percentile. For example, this student was in the 90th percentile nationally on the vocabulary test and in the 69th percentile locally on the vocabulary test. Use this score form in exercises 19, 20, and 21.

Stanford ACHIEVEMENT TEST

		SCORE TYPE	MATH COMP
	GR 4 NORMS GR 4.8	RS/NO POSS	29/44
	LEVEL INTER 1 FORM E	NAT'L PR-S	54 - 5
	STUDENT N0 400000044	LOCAL PR-S	68 - 6
	OTHER INFO	GRADE EQUIV	5.5
AGE 9–6	TEST DATE 5/10/99		

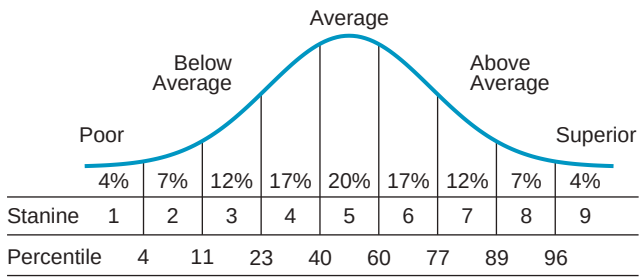
READING COMP	VOCAB-ULARY	MATH APPL	SPELLING	LANGUAGE
57/60	32/36	32/40	37/40	40/53
96 - 9	90 -	77 -	86 -	63 -
78 - 7	69 -	62 -	83 -	45 -
	8.4	6.4	8.9	5.4

19. a. Nationally, what percentage of the students scored below this student in mathematics comprehension?
b. Locally, what percentage of the students scored below this student in mathematics comprehension?
c. This student scored higher in reading comprehension than what percentage of the national group?
d. Which of the local percentile scores is not lower than the corresponding national percentile score?
e. If a local percentile is lower than a national percentile, it means that the local level of achievement is higher than the national level of achievement. Explain why.

20. a. Nationally, what percentage of the students scored below this student in mathematics application?
b. Locally, what percentage of the students scored below this student in language?
c. This student scored higher in spelling than what percentage of the national group?
d. The last row of each column shows the *grade equivalent*—the grade level of the student's achievement. What is the grade level of this student's achievement in vocabulary?
e. If a local percentile is higher than a national percentile, it means that the local level of achievement is lower than the national level of achievement. Explain why.

21. a. Nationally, what percentage of the students scored below this student in spelling?
b. Locally, what percentage of the students scored below this student in reading comprehension?
c. This student scored higher in vocabulary than what percentage of the national group?
d. What percentage of the questions on the mathematics application subtest did the student answer correctly?

Rather than being divided into 100 parts, as in the case of percentiles, a distribution is sometimes divided into 9 parts called **stanines,** a contraction of the term *standard nine.* Stanines are whole numbers numbered from a low of 1 to a high of 9, with 5 representing average performance. The stanines that correspond to the percentile intervals are shown in the following graph. Use this graph in exercises 22 and 23.



22. **a.** Determine the stanine for the 70th percentile.
b. The Stanford Achievement Test chart in exercises 19 through 21 records a stanine score beside each percentile. For example, under Math Comp, the 54th percentile corresponds to a stanine score of 5. Determine the student's four national stanine scores missing from that chart.

23. **a.** Determine the stanine for the 44th percentile.
b. The Stanford Achievement Test chart in exercises 19 through 21 records a stanine score beside each percentile. For example, the chart shows a stanine score of 7 for the local percentile of 78 under Reading Comp. Determine the student's four local stanine scores which are missing from the chart.

Determine a z score to the nearest .01 for each of the test scores in exercises 24 and 25 and determine the better score.

24. Test *A:* mean 74.3; standard deviation 3.6; test score 81.
Test *B:* mean 720; standard deviation 146; test score 840.

25. Test *A:* mean 3.1; standard deviation 2.1; test score 2.8.
Test *B:* mean 6.2; standard deviation 1.7; test score 5.3.

Use this computer-generated set of random numbers in exercises 26 and 27.

61	44	34	03	09	05	64	20	54	24
41	17	26	81	06	85	19	76	44	59
73	73	97	24	18	38	25	89	37	20
65	69	66	39	80	13	97	76	63	34
08	60	20	66	68	42	99	28	71	47
72	47	40	14	34	38	57	30	80	89

26. Explain how to simulate the flipping of a coin by using the table of random numbers. Use your method and record the first 10 "coin tosses."

27. Devise a way to use these random numbers to simulate the rolling of a die. Use your method and record the first 10 "rolls" of the die.

Reasoning and Problem Solving

28. A student scored 650 on the mathematics part of the Scholastic Aptitude Test in a year in which the mean for that subtest was 455 and the standard deviation was 112. When the same student took a university entrance exam for engineers, he scored 140 on a mathematics test that had a mean of 128 and a standard deviation of 9.5.
a. What was his z score for each test?
b. On which of the two tests was his performance stronger, relative to the performance of the other students taking the test?

29. A student scored 31 on the mathematics section of the American College Test. The mean score for the ACT was 17.4, and the standard deviation was 7.8. The same student scored 582 on the Scholastic Aptitude Test. The SAT had a mean of 458 and a standard deviation of 117.
a. What was her z score for each test?
b. On which of the two tests was her performance stronger relative to the performance of the other students taking the tests?

Objects that are manufactured to certain specifications tend to vary slightly from their specified measurements. In answering the questions in exercises 30 and 31, assume the measurements are normally distributed.

30. A certain type of lightbulb has a mean life of 2400 hours with a standard deviation of 200 hours. What percentage of these bulbs can be expected to burn longer than 2600 hours?

31. A brand of crockpots has a mean high temperature of 260°F and a standard deviation of 3°F. If a crockpot's highest temperature is below 254°F or above 266°F, it is considered defective. What percentage of these pots can be expected to be defective?

32. A DC-10 shuttle flight between two cities has a total seating of 312, a mean of 286 passengers, and a standard deviation of 13 passengers. What percentage of the time is this flight able to meet the demand for seats? (Assume that the numbers of passengers for this flight are normally distributed.)

33. A certain university's Wide Area Telephone Service (WATS) line can handle as many as 20 calls per minute. The average number of calls per minute during peak periods is 16 with a standard deviation of 4 calls. What percentage of the time will the WATS line be overloaded during peak periods? (Assume that the numbers of phone calls are normally distributed during the peak period.)

Use z scores and the definition of *rare event* to support your conclusions in exercises 34 through 36.

34. The guarantee on the box of a product called Rechargeable Lamp Light states that in case of power failure, the lamp automatically provides light for an average of 90 minutes. If the standard deviation of the times that these lamps burn is 9.5 minutes, what might you suspect about the guarantee if you purchased one of these lamps and it burned for only 70 minutes?

35. A company that manufactures an exercise belt claims that at least 15 minutes' use of the machine per day for 60 days will result in an average loss of 3.3 inches in

the distance around the abdomen. Assume that the standard deviation of these measurements is 1.4 inches. Suppose that a person is randomly selected from those who used this machine for 15 minutes per day for a 60-day period, and this person lost only .3 inch in distance around the abdomen. What might be predicted about the company's claim?

36. An automobile manufacturer claims that its midsize car will accelerate from 0 to 60 miles per hour in 8 seconds, and it is known that the standard deviation is 1.3 seconds. If you randomly select one of these cars from a dealer's lot and find that it requires 10 seconds to accelerate from 0 to 60 miles per hour, are you justified in being suspicious of the manufacturer's claim?

PS Use a simulation to solve exercises 37 through 39. Describe your use of this method.

37. A manufacturer puts one of five different randomly selected colored markers in each box of Crackerjacks. What is the average number of boxes that must be purchased to obtain all five different markers?

38. Pepe and Anna are playing a penny-tossing game. The player who can toss 10 heads in the fewest tosses wins the game. How many tosses of a fair coin on average are required to obtain 10 heads?

39. A newly married couple would like to have a child of each sex. Assuming that the chances of having a boy and a girl are equally likely, what is the average number of children the couple must have in order to have at least 1 boy and 1 girl?

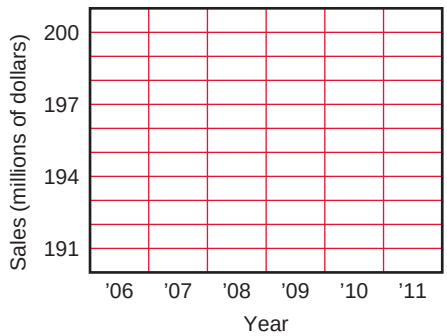
40. Featured Strategy: Using a Simulation. PS A cloakroom attendant receives 9 hats from 9 men and gets the hats mixed up. If the hats are returned at random and simultaneously, what is the average number of hats that will go to the correct owners?

- **a. Understanding the Problem.** There are 9 men, and each man has exactly 1 hat. Is it possible that each hat might be returned to its owner?
- **b. Devising a Plan.** One way to approach the problem is to conduct experiments with 9 men and 9 hats, using some random method of returning the hats. Another plan is to design a simulation. How might such a simulation be designed?
- **c. Carrying Out the Plan.** The simulation must be carried out several times to determine the number of "hats" (on average) that will be returned correctly. What is this number?
- **d. Looking Back.** Suppose that instead of 9 hats and 9 men, there are fewer men, each having 1 hat. Will the average number of hats that are returned correctly increase?

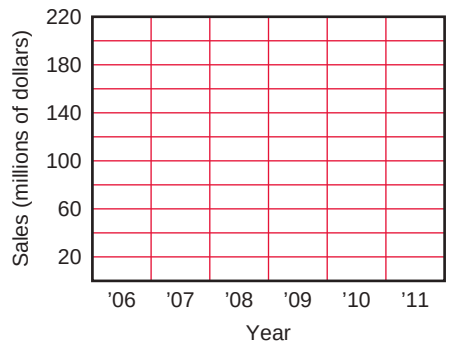
41. PS Graphs of distributions are sometimes intended to be misleading. The scale used on the vertical axis will determine whether the graph of the distribution of sales shown in the table below will be skewed. Plot bar graphs of these sales on copies of the following grids. What impression do these two graphs give to the reader? Which graph better illustrates the true increase in sales? Explain why. (Copy the rectangular grid from the website.)

Year	Sales
2006	$191,000,000
2007	191,500,000
2008	193,000,000
2009	195,000,000
2010	198,000,000
2011	200,500,000

(i)



(ii)



42. PS The Montagnais-Naskapi, a northeastern Native American tribe, bake the shoulder blade of the caribou to get guidance on decisions concerning the well-being of their tribe. They determine the direction of the next hunt from the direction of the cracks that appear in the animal's shoulder blade as it is baked. This method of determining direction is a fairly random device that avoids human bias. It suggests that some practices in magic need to be reassessed.*

*O. K. Moore, "Divination—A New Perspective," 121–128.

a. Explain how a table of random numbers can be used to randomly determine directions of 0° to 360° for hunting.

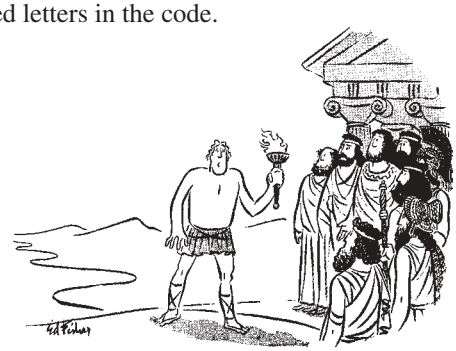
b. How can a table of random numbers be used to randomly determine both directions and distances in miles to be traveled for the hunt?

c. Use your method in part b and the table of random digits on page 505 to determine the direction and distance of your first hunt.

43. **Cryptology** is the science of coding and decoding secret messages. Decipher the following code, a statement by the nineteenth-century mathematician Pierre Laplace.

"CA CW GZJHGBHYRZ AOHA H
WVCZQVZ MOCVO YZFHQ MCAO
AOZ VLQWCXZGHACLQ LK
FHJZW LK VOHQVZ WOLERX YZ
ZRZSHAZX AL AOZ GHQB LK AOZ
JLWA CJILGAHQA WEYPZVAW LK
OEJHQ BQLMRZXFZ."

Hint: Make a *frequency distribution* (table) showing the number of times each letter occurs. The four most often used letters in English are e, t, a, and o, in that order. Substitute these letters in that order for the first, second, third, and fourth most frequently used letters in the code. The letters h, n, i, and s also occur with high frequency in our language. Substitute these letters for the fifth, sixth, seventh, and eighth most frequently used letters in the code.



"I forgot the message!"

Teaching Questions

1. After discussing the importance of randomness and random numbers in statistics, one of your students claims that she can list random digits off the "top of her head." Describe a set of criteria for determining randomness of a set of numbers. Write down a list of about 50 single-digit numbers that you feel are random and test them for randomness using your criteria. Then test your criteria on the list of random digits listed on page 505. Record your results.

2. The often quoted phrase, "There are three kinds of lies: lies, damned lies, and statistics," is attributed to the English Prime Minister, Benjamin Disraeli (1804–1881). Collect some examples of statistical information that you believe are misleading. Describe some ways you could help middle school students learn to detect misleading data and where you would have your students search for such material.

3. As a class project, suppose that your students decided to study the number of people talking on a cell phone while driving a car. Outline the steps you believe that the students should go through, starting with formulating their study question to drawing conclusions from their results.

4. A middle school teacher was introducing the blood-type problem described in question number 2 in the one-page Math Activity at the beginning of this section. When she asked for suggestions from the class, John said, "Put 5 cards in a container, two marked with an A and the other 3 blank. Draw and replace the cards until you get 3 cards marked A. The number of draws is the number of people you need in your sample." Explain how you would respond to John.

Classroom Connections

1. Explain what you think is meant by the **Standards** statement on page 504, and how it applies to the simulation in Example K.

2. Read the **Standards** statement on page 493. How would you introduce the concept of sampling a population rather than gathering data from a full population? Describe a data collection question involving a large population that you think would be of interest to fifth-graders. Explain how you can help students see the importance of sampling a population.

3. The **Research** Statement on page 494 states that proportional reasoning causes students difficulty in statistical thinking. Explain what you think the statement is referring to, and give some examples where proportional reasoning is important in deriving information from graphs and data.

4. The idea of random number and randomness is important in statistics. Look through the **Data Analysis and Probability** sections of the **Standards** expectations (see inside front and back covers) to see if there is any mention of the word random. If so, at what level does it occur and if not, explain at what grade level you think it should occur. Give an example of how you would introduce the notion of randomness to students.

CHAPTER 7 REVIEW

1. **Statistics**
   a. **Statistics** can refer to numerical information, called **data,** or to the science of collecting data and making inferences.
   b. **Descriptive statistics** is the science of collecting, describing, and analyzing data.
   c. **Inferential statistics** is the science of interpreting data in order to make predictions.
   d. A **survey** is the gathering of a sample of data or opinions.

2. **Measures of central tendency**
   a. A **measure of central tendency** is a number that approximates the center of a set of data.
   b. There are three measures of central tendency: **mean** (numerical average), **mode,** and **median.**
   c. The **mean** of n numbers is the sum of the numbers divided by n.
   d. The **median** of a set of numbers is the middle number when the numbers are placed in increasing order, or if no middle number exists, the mean of the two middle numbers.
   e. The **mode** of a set of numbers is the number that occurs most often.
   f. If there are two modes, the data is called **bimodal,** and if there are three modes, it is called **trimodal,** and if there are four or more modes the data is called **multimodal.**

3. **Charts and graphs**
   a. A **bar graph** provides an illustration of data and is used when there are a small number of distinct categories. A **double-bar graph** compares two sets of data and a **triple-bar graph** compares three sets of data.
   b. A **pie graph** is used for illustrating essentially the same information as bar graphs. Pie graphs, bar graphs, and pictographs are best chosen when there are relatively few categories, such as 3 to 10.
   c. A **pictograph** is similar to a bar graph, but copies of an icon (figure) that has a given numerical value are used to show the amount of data in each of several categories.
   d. A **line plot** is similar to a bar graph, but tally marks are used for each value of data, and the graph is best used for plotting intermediate numbers of data, such as 25 to 50.
   e. A **stem-and-leaf plot** is a visual method of listing up to 100 or so values of data where each data value is split into a **leaf** (usually the last digit) and a **stem** (the other digits). When it is used to compare two sets of data, a **back-to-back stem and leaf plot** should have approximately the same number of values on both sides of the stem.
   f. A **histogram** is similar to a bar graph. It has adjoining bars of equal width along one axis and the other axis contains the frequency of the data for each interval or category.
   g. A **line graph** is a sequence of points connected by line segments that is often used to show changes over a period of time.
   h. A **scatter plot** is a graph for comparing two sets of data by graphing coordinates.
   i. A **trend line** approximates the location of the points of a scatter plot. Such a line is used for making predictions about a value in one set of data when given the corresponding value in the other set of data.
   j. A **curve of best fit** may be a line or a curve that approximates the location of the points of a scatter plot. Such curves can be visually drawn, or their equations **(regression equations)** can be obtained from graphing calculators or computers.
   k. A **correlation coefficient** is a measure from $^{-}1$ to $^{+}1$ that indicates how well the curve of best fit approximates the points of the scatter plot. The closer the measure is to $^{-}1$ or $^{+}1$, the better the fit.
   l. If the correlation coefficient is zero, there is **no correlation** between the two sets of data; if it is close to 1, there is a strong **positive correlation;** and if it is close to $^{-}1$, there is a strong **negative correlation.**
   m. A **box-and-whisker plot** (or simply **box plot**) shows the data divided into four parts by the median and quartiles. Several sets of data can be compared with each having its own box plot, and the sizes of the sets of data may vary.

4. **Measures of variability**
   a. A **measure of variability** is a number that describes the spread or variation in a set of data.
   b. The **range** is the difference between the greatest and least measures in a set of data.
   c. The median of the lower half of a set of data is the **lower quartile** (Q_1) and the median of the upper half of a set of data is the **upper quartile** (Q_3).
   d. The **interquartile range** is the difference between the upper quartile and the lower quartile.
   e. If a value of the data is more than 1.5 times the interquartile range above the upper quartile or below the lower quartile, the value is considered to be an **outlier.**

f. The **standard deviation** is computed by subtracting the mean from each measurement, squaring each difference, finding the mean of the squared differences, and obtaining the square root of this mean.
g. For any set of data, at least 75 percent of the measurements will lie within 2 standard deviations of the mean.

5. **Measures of relative standing**
   a. A **measure of relative standing** is a number that determines the relative position of a measurement in a set of data.
   b. The ***p*th percentile** of a set of data is a number that is greater than p percent of the data and less than $(100 - p)$ percent of the data.
   c. The **25th percentile** is the lower quartile, the **50th percentile** is the median, and the **75th percentile** is the upper quartile.
   d. A z **score** is a number that can be calculated for any measurement in a set of data. It determines the number of standard deviations that a measurement is above or below the mean.
   e. If the z score of a value from a set of data is less than $^{-}2$ or greater than $^{+}2$, the value is called a **rare event.**

6. **Sampling and Simulations**
   a. A **sample** is a collection of people or objects chosen to represent a larger collection of people or objects, called the **population.**
   b. A **random sample** is a sample for which every element of the population has the same chance of being selected and the process is called **random sampling.**
   c. In **stratified sampling** the population is divided into groups, and the number sampled from each group is proportional to the size of the group.
   d. A **simulation** is a procedure in which experiments that closely resemble the given situation are conducted repeatedly.

7. **Distributions**
   a. **Distribution** refers to how the measurements of a set of data are clustered together, isolated from each other, or spread out.
   b. **Gaps** are large spaces in the data and **clusters** are isolated groups of data.
   c. If the data are concentrated at the right side of a graph with the "tail" extending to the left, the direction is **skewed to the left.**
   d. If the data are concentrated at the left side of the graph with the "tail" extending to the right, the distribution is **skewed to the right.**
   e. A distribution in which measurements at equal distances from the center of the distribution occur with the same frequency is called **symmetric.**
   f. A smooth symmetric bell-shaped curve is called a **normal curve,** and the distribution of its data is called a **normal distribution.**

CHAPTER 7 TEST

The following table shows sources of public school revenues in percent for each state for 2007.* Use these data to answer questions 1 through 6.

Public School Revenues, by Source, by State (in percent)

	Federal	State	Local
Hawaii	8.6	89.8	1.6
Vermont	7.7	86.8	5.6
Minnesota	7.5	73.3	19.2
New Mexico	14.2	71.3	14.4
Idaho	10.1	67.2	22.7
North Carolina	10.4	64.7	24.9
Delaware	9.3	64.0	26.7
Alaska	12.6	63.5	23.9
Washington	8.5	62.1	29.4
West Virginia	12.6	59.7	27.8
California	11.2	58.9	29.9
Michigan	8.1	58.7	33.3
Kansas	6.9	57.6	35.5
Arkansas	11.2	56.2	32.5
Mississippi	15.3	53.9	30.8
Utah	11.0	53.5	35.5
Indiana	8.0	53.2	38.8
Arizona	8.0	51.8	40.2
Wisconsin	5.7	51.6	42.6
Oklahoma	12.4	51.4	36.2
Alabama	8.9	51.0	40.2
Oregon	9.5	50.7	39.8
Kentucky	13.0	50.2	36.8
Wyoming	7.5	48.8	43.6
Montana	12.3	47.7	40.0
Massachusetts	5.4	47.0	47.6
Iowa	6.6	46.3	47.1
Tennessee	11.1	45.8	43.1
New York	7.3	45.7	47.1
Georgia	8.4	44.4	47.1
South Carolina	9.9	43.6	46.6
Colorado	7.5	42.6	49.8
Louisiana	17.3	42.6	40.1
Ohio	7.7	42.0	50.3
New Hampshire	6.4	41.4	52.2
Virginia	6.5	41.4	52.1
Maine	10.0	39.9	50.1
Maryland	6.8	39.8	53.4
Florida	8.9	39.2	51.9
Connecticut	6.4	38.6	55.0
Texas	10.4	37.9	51.7
Nebraska	8.1	37.3	54.6
New Jersey	3.3	36.6	60.1
Rhode Island	3.2	36.2	60.6
Pennsylvania	7.3	36.2	56.5
North Dakota	13.9	33.5	52.5
South Dakota	15.5	32.8	51.5
Missouri	9.4	32.5	58.1
Nevada	7.4	32.4	60.2
Illinois	8.6	27.5	63.9

**Statistical Abstract of the United States*, 128th ed. (Washington, DC: Bureau of the Census, 2009), Table 251.

1. a. Which state had the greatest percent of revenue from the state?
b. Which state had the smallest percent of revenue from the state?
c. What is the range of percents of revenue supplied by the states?
d. The states are listed in decreasing order by percent of revenue from state sources. What is the median of these percents?

2. a. Draw a pie graph showing the three sources of revenue for California. Label the size of the central angle, to the nearest degree, for each part of the graph.
b. Draw a bar graph showing the three sources of revenue to the nearest percent for Iowa. Label the axes of the graph.

3. Form a stem-and-leaf plot of the percent of revenue from the federal government. Use the tens and units digits for the stems and the tenths for the leaves.

4. Form a histogram for the percents of local revenue, using intervals on the horizontal axis of 0 to 9.95 percent, 10 to 19.95 percent, 20 to 29.95 percent, . . . , 90 to 99.95 percent.
a. What is the frequency of states in the interval from 20 to 29.95 percent?
b. What is the frequency of states in the interval from 50 to 59.95 percent?
c. What interval contains the most measurements?

5. a. What is the mean of the percents to the nearest tenth of local revenue for the five states with the greatest percents of local revenue?
b. What is the mean of the percents to the nearest tenth of local revenue for the five states with the smallest percents of local revenue?

6. Form a scatter plot for the first 25 states (Hawaii to Montana) in the table by plotting the percent of revenue from each state (horizontal axis) compared to the percent of revenue from local sources (vertical axis). Draw a trend line and use this line to predict the amount of revenue from the state if the revenue from local sources is 35 percent and to predict the amount of revenue from local sources if the revenue from the state is 75 percent. Is there a correlation between the two types of revenue?

7. **a.** Which of the following sets of data has the greatest mean?
b. Which has the greatest range?
c. Which has the greatest standard deviation?

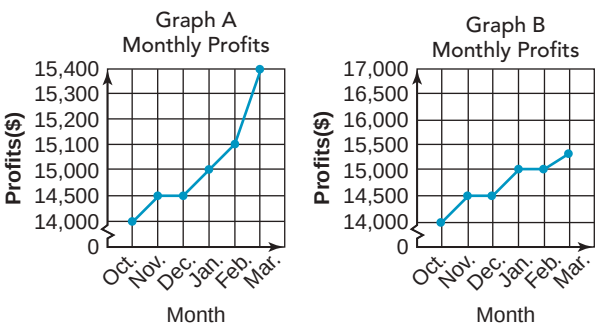
Set *A:* 1, 5, 10, 15, 20

Set *B:* 21, 22, 23, 24, 25

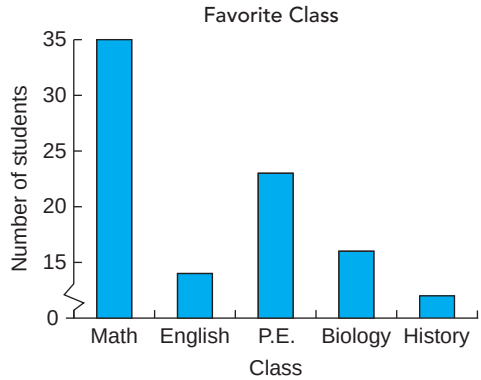
8. The following table gives the length, in minutes, of rides at a small amusement park. The park advertises that the average length of their rides is longer than 10 minutes. Explain why this is misleading.

Ride	Length (minutes)
Ferris Wheel	6
Wild Coaster	7
Swing Around	6
Park Tram Tour	25
Teacup Tilt	8
Bumper Cars	10

9. The following graphs are in an elementary school text section on misleading statistics.* Which graph suggests the company is extremely profitable? How would you help your students explain whether or not this is a valid conclusion?



10. A professor surveyed 100 students regarding their favorite class. The following graph might indicate that about three times as many students preferred their math class to their biology class. Explain why this graph is misleading. In reality, how many more students preferred math to biology?



11. Chinese athletes won 51 gold medals during the 2008 Summer Olympics in Beijing, China (see the table on page 442). Is this an outlier? The United States won a total of 36 gold medals; is this medal count an outlier?

12. Draw a box-and-whisker plot for the following data. Label the three quartiles and the smallest and largest values. Compute the interquartile range and determine if it is less than or greater than one-half the range of the data.

62, 63, 66, 66, 70, 72, 73, 77, 84, 86, 92, 95, 97

13. Mary obtained a mathematics score of 520 on the SAT; the SAT scores had a mean of 435 and a standard deviation of 105. Her mathematics score on the PSAT (Preliminary Scholastic Aptitude Test) was 56; the PSAT scores had a mean of 44 and a standard deviation of 9.5
a. Determine Mary's z score for the SAT (to the nearest .1).
b. Determine Mary's z score for the PSAT (to the nearest .1).
c. On which test was her mathematics performance stronger?

14. A fourth-grade class is given a mathematics pretest at the beginning of the school year and a posttest at the end of the school year.
a. Is the distribution of scores on the pretest most likely to be skewed to the left, skewed to the right, or normal?
b. Describe the most likely distribution for the posttest.

*(Math Connects Grade 7 Chapter 8, page 445.)

15. A few of the results from a Stanford Achievement Test taken by a fourth-grader are shown below.

Total Reading	Total Listening	Total Language	Total Math	Basic Bat Tot
107/120	68/76	77/93	81/118	333/407
93 - 8	93 - 8	76 - 6	64 - 6	77 - 7
90 - 8	77 - 7	65 - 6	60 - 6	69 - 6
10.0	9.2	6.5	5.5	8.0

a. The first row of numbers under each subtest shows the number of questions answered correctly out of the total number of questions on that subtest. What percent to the nearest .1 of the questions on Total Language did this student answer correctly?

b. The first number in the second row of each column is the student's percentile score relative to the national group. What percent of the national group of students scored below this student in Total Reading?

c. The first number in the third row of each column is the student's percentile score relative to the local group. This student is at the 60th percentile in Total Math. What does this mean?

d. The numbers in the fourth row are the grade equivalents indicated by the test results. What do these numbers indicate about this fourth-grader?

16. A district mathematics test for all third-graders had a normal distribution with a mean of 74 and a standard deviation of 11.

a. What percent of the third-graders tested scored within ± 1 standard deviation of the mean?

b. What percent of the students scored between 52 and 96?

17. Students in two fifth grades were given the same English test. One class of 26 students had a mean of 68, and the second class of 22 students had a mean of 73. What is the mean for the total number of students in both classes to the nearest .1?

18. The mean of three test scores is 74. What must the score on a fourth test be to raise the mean of the four tests to 78?

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CHAPTER

8 Probability

Spotlight on Teaching

Excerpts from NCTM's Standards for School Mathematics Grades 6–8*

Teachers should give middle-grade students numerous opportunities to engage in probabilistic thinking about simple situations from which students can develop notions of chance. They should use appropriate terminology in their discussions of chance and use probability to make predictions and test conjectures. For example, a teacher might give students the following problem:

> Suppose you have a box containing 100 slips of paper numbered from 1 through 100. If you select one slip of paper at random, what is the probability that the number is a multiple of 5? A multiple of 8? Is not a multiple of 5? Is a multiple of both 5 and 8?

Teachers can help students relate probability to their work with data analysis and to proportionality as they reason from relative-frequency histograms. For example, referring to the data displayed in Figure 6.27, a teacher might pose a question like, How likely is it that the next time you throw a one-clip paper airplane, it goes at least 27 feet? No more than 21 feet?

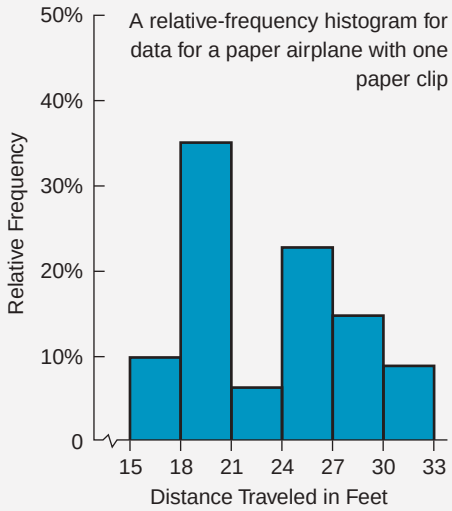


Figure 6.27

* *Principles and Standards for School Mathematics*, pp. 250, 253.

MATH ACTIVITY 8.1

Experimental Probabilities from Simulations

Virtual Manipulatives



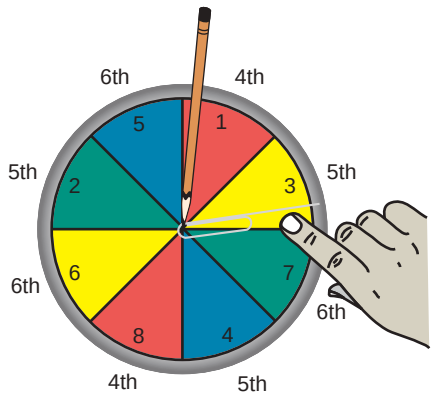
www.mhhe.com/bbn

Purpose: Use simulations and explore experimental probabilities.

Materials: 1-to-8 Spinner in the Manipulative Kit or Virtual Manipulatives. Bend a paper clip and hold a pencil at the center of the spinner, as shown below, if using the cardstock spinner.

1. A school will send two fourth-graders, three fifth-graders, and three sixth-graders to attend a one-day session of the state's House of Representatives. Two students will be picked at random to have lunch with the governor, and the others will have lunch with the legislators. Do you think it is likely or not so likely that at least one sixth-grader will be picked to have lunch with the governor?

a. One approach to answering this question is to represent the eight students and their grade levels as numbers on the 1-to-8 spinner. The first spin determines one student. Then spinning again until a different number is obtained determines a second student. Repeat this experiment of randomly selecting two students 20 times, recording the grade level of each student.



***b.** Based on your data, how would you rate the chances that at least one sixth-grader will be selected? Write a sentence or two to support your conclusion.

c. Describe what could be done to increase your confidence about your conclusion.



2. As we saw in Section 7.3, repeatedly carrying out experiments that closely resemble a given situation is called a *simulation.* The next problem can also be solved by a simulation:

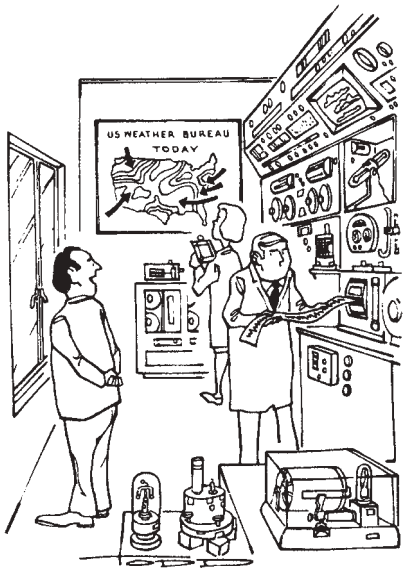
In New Hampshire, cars have inspection stickers indicating the month of the inspection on their windshields, as shown at the left. If there are approximately the same number of cars due for inspection each month, what is the likelihood that out of six randomly selected cars, all six will have different inspection months?

a. Use the 1-to-12 spinner, and let each number represent a different month. Spin the spinner 6 times to simulate selecting cars at random and record the numbers. Repeat this experiment 15 times, and determine the percentage of experiments with six different inspection months. This percentage is called an *experimental probability.*

b. On the basis of your experimental probability, what is the likelihood (unlikely, about 50:50, or likely) that when six cars are selected, they will all have different inspection months?

c. Explain how the accuracy of your experimental probability can be improved.

Section 8.1 SINGLE-STAGE EXPERIMENTS



"Looks like it might be a nice day tomorrow!"

PROBLEM OPENER

The numbers 3, 4, 5, and 6 are written on four cards. If two numbers are randomly selected and the first is used for the numerator of a fraction and the second is used for the denominator of the fraction, what is the probability that the fraction is greater than 1 and less than $1\frac{1}{2}$?



Probability, a relatively new branch of mathematics, emerged in Italy and France during the sixteenth and seventeenth centuries from studies of strategies for gambling games. From these beginnings probability evolved to have applications in many areas of life. Life insurance companies use probability to estimate how long a person is likely to live, doctors use probability to predict the success of a treatment, and meteorologists use probability to forecast weather conditions.

NCTM Standards

One trend in education in recent years has been to increase emphasis on probability and statistics in the elementary grades. NCTM's *Curriculum and Evaluation Standards for School Mathematics* supports this trend by including statistics and probability as a major strand in the standards for grades K to 4 (p. 54).

Collecting, organizing, describing, displaying, and interpreting data, as well as making decisions and predictions on the basis of that information, are skills that are increasingly important in a society based on technology and communication.

HISTORICAL HIGHLIGHT



Blaise Pascal, 1623–1662

The founders of the mathematical theory of probability were Blaise Pascal and Pierre de Fermat (1601–1665), who developed the principles of this subject in letters to each other during 1654. The initial problem that started their investigation was posed by Chevalier de Mere, a professional gambler. The problem was to determine how the stakes should be divided between two gamblers if they quit before the game was finished. The problem amounts to determining the probability each player has of winning the game at any given stage. The theory that originated in a gambler's dispute is now an essential tool in many disciplines.*

*E. T. Bell, *Men of Mathematics* (New York: Simon and Schuster, 1965), pp. 73–89.

Research Statement

The seventh national mathematics assessment found there was evidence of students' inability to use information from sample spaces, even from those that students correctly generated.

Zawojewski and Shaughnessy

PROBABILITIES OF OUTCOMES

Just as probability had its beginning in games of chance, it is often introduced in the early grades through simple games such as those involving spinners. Consider the experiment of spinning the spinner in Figure 8.1. There are four possible outcomes: blue, red, green, and yellow. We would expect the color blue to come up about $\frac{1}{4}$ of the time if we spin many times. That is, the probability of obtaining blue is $\frac{1}{4}$. This probability is indicated by writing, $P(\text{blue}) = \frac{1}{4}$.

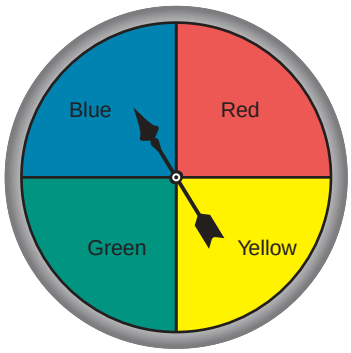


Figure 8.1

In general, an activity such as spinning a spinner, tossing a coin, or rolling a die is called an **experiment,** and the different results that can occur are called **outcomes.** The set of all outcomes of an experiment is called the **sample space.** For the preceding spinner, the sample space is the set of four outcomes: blue, red, green, and yellow.

EXAMPLE A

For each experiment, determine the sample space and the probability of the given outcome.

1. Rolling a regular six-sided die (faces are numbered from 1 to 6) once and obtaining a 2
2. Tossing a coin once and obtaining a head
3. Selecting a green marble on one draw from a box containing five green marbles and seven blue marbles

Solution **1.** The sample space contains the numbers 1 to 6, and $P(2) = \frac{1}{6}$. **2.** The sample space has two outcomes, heads (H) and tails (T), and $P(\text{H}) = \frac{1}{2}$. **3.** The sample space contains 12 outcomes, five that can be denoted by G_1, G_2, G_3, G_4, and G_5 for the five green marbles, and seven that can be denoted by B_1, B_2, B_3, B_4, B_5, B_6, and B_7 for the seven blue marbles. $P(\text{G}) = \frac{5}{12}$.

Research Statement

Recent research has concluded that a substantial number of students in grades 1 through 3 are not able to list the outcomes of a one-dimensional experiment (such as rolling a single die) even after instruction.

National Research Council

There are two methods of determining probabilities. One is to conduct experiments and observe the results. A probability derived in this fashion is called an **experimental probability.** For example, if a coin is tossed 500 times and 300 heads occur, the experimental probability of obtaining a head for this experiment is $\frac{300}{500}$, or $\frac{3}{5}$. The second method of determining probabilities is based on theoretical considerations. Since spinners, dice, coins, and other physical devices for determining random outcomes all have imperfections that lead to biased results, we assign **theoretical probabilities** to the outcomes of ideal experiments. Ideally, for example, the spinner shown in Figure 8.1 will be equally likely to stop on any of the four colors. So the theoretical probability of obtaining blue is $\frac{1}{4}$. From here on, the word *probability* will mean *theoretical probability*, unless otherwise stated.

The probability of obtaining one of a group of equally likely outcomes is defined as follows.

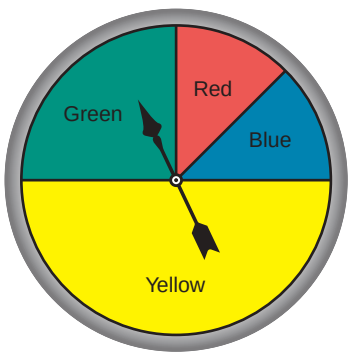
Probability of an Equally Likely Outcome If there are n equally likely outcomes, then the **probability** of any given outcome is $\frac{1}{n}$.

Outcomes are not always equally likely, as shown in Example B.

EXAMPLE B

Spinning this spinner will result in one of four outcomes: blue (B), red (R), green (G), or yellow (Y). Determine the following probabilities.

1. $P(\text{B})$ **2.** $P(\text{G})$ **3.** $P(\text{Y})$



Solution **1.** $P(\text{B}) = \frac{1}{8}$. **2.** $P(\text{G}) = \frac{1}{4}$. **3.** $P(\text{Y}) = \frac{1}{2}$.

PROBABILITIES OF EVENTS

Consider the experiment of rolling two ordinary dice. The 36 possible outcomes of the sample space are shown in Figure 8.2 on page 523, and since each outcome is equally likely, the probability of obtaining any given pair of numbers is $\frac{1}{36}$. Notice that there are two different outcomes for rolling a 1 and a 2: a red 1 and a yellow 2 is a different outcome than a yellow 1 and a red 2. So, the probability of rolling a sum of 3 is $\frac{2}{36}$. Similarly, there is a $\frac{2}{36}$ probability of rolling a sum of 11.

9-7 Theoretical and Experimental Probability

MAIN IDEA
Find and compare experimental and theoretical probabilities.

New Vocabulary
theoretical probability
experimental probability

Math Online
glencoe.com
- Extra Examples
- Personal Tutor
- Self-Check Quiz

MINI Lab



Follow the steps to determine how many times doubles are expected to turn up when two number cubes are rolled.

Step 1 Use the table to help you find the expected number of times doubles should turn up when rolling two number cubes 36 times. The top row represents one number cube, and the left column represents the other number cube.

	1	2	3	4	5	6
1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

Step 2 Roll two number cubes 36 times. Record the number of times doubles turn up.

1. Compare the number of times you *expected* to roll doubles with the number of times you *actually* rolled doubles.
2. Write the probability of rolling doubles out of 36 rolls using the number of times you *expected* to roll doubles from Step 1. Then write the probability of rolling doubles out of 36 rolls using the number of times you *actually* rolled doubles from Step 2.

In the Mini Lab above, you found both the theoretical probability and the experimental probability of rolling doubles using two number cubes. **Theoretical probability** is based on what *should* happen when conducting a probability experiment. This is the probability you have been using since Lesson 9-1. **Experimental probability** is based on what *actually* occurred during such an experiment.

Theoretical Probability

$\frac{6}{36}$ ← 6 rolls *should* occur

Experimental Probability

$\frac{n}{36}$ ← n rolls *actually* occurred

The theoretical probability and the experimental probability of an event may or may not be the same. As the number of times an experiment is conducted increases, the theoretical probability and the experimental probability should become closer in value.

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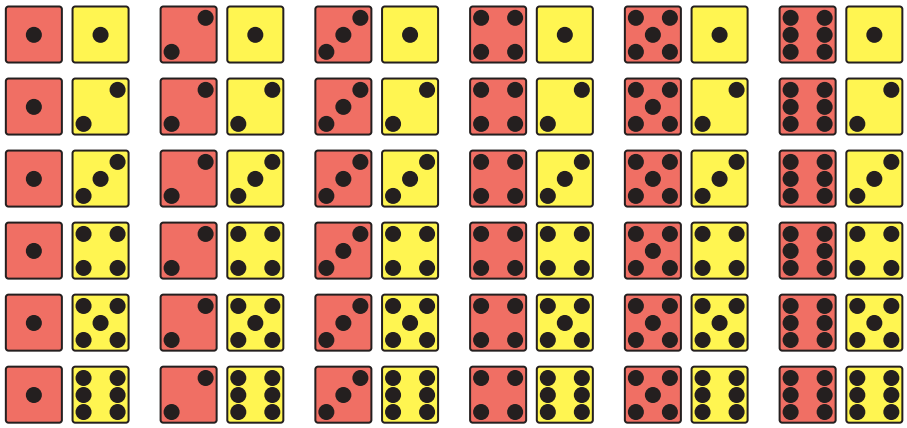


Figure 8.2
Thirty-six possible outcomes of rolling two dice

Once we know the complete sample space, as in Figure 8.2, it is possible to answer more difficult questions concerning tosses of two dice.

EXAMPLE C

Figure 8.2 illustrates the 36 equally likely outcomes for the toss of two dice. Determine the probabilities of the following outcomes.

1. Obtaining a sum less than 4
2. Obtaining a sum of 5
3. Obtaining a sum greater than or equal to 9

Solution **1.** There are 3 outcomes whose sum is less than 4, (1, 1), (2, 1), and (1, 2), so $P(\text{sum} < 4) = \frac{3}{36} = \frac{1}{12}$. **2.** There are 4 outcomes whose sum is 5, so $P(\text{sum} = 5) = \frac{4}{36} = \frac{1}{9}$. **3.** There are 10 outcomes whose sum is greater than or equal to 9, so $P(\text{sum} \geq 9) = \frac{10}{36} = \frac{5}{18}$.

Notice that in part 1 of Example C the probability of obtaining a sum less than 4 involves more than one outcome: The pairs (1, 1), (1, 2), and (2, 1) all have a sum less than 4. A subset of outcomes in a sample space is called an **event.** For example, the event with sums of 5 has 4 of the 36 outcomes, and the event with sums greater than or equal to 9 has 10 of the 36 outcomes.

Example C suggests the following rule for obtaining the probability of an event.

Probability of an Event If all the outcomes of a sample space $\boldsymbol{S}$ are equally likely, the **probability of an event** $\boldsymbol{E}$ is

$$\boldsymbol{P(E)} = \frac{\text{number of outcomes in } \boldsymbol{E}}{\text{number of outcomes in } \boldsymbol{S}}$$

Let's use this rule to determine the probability of the event obtaining a sum of 7 on a toss of two dice. Figure 8.2 shows that there are 6 ways of rolling a 7 (6 favorable outcomes), so

$$P(\text{sum} = 7) = \frac{6}{36} = \frac{1}{6}$$

Listing all the outcomes of a sample space (as in Figure 8.2) and counting the favorable outcomes is a common method of determining the probability of an event when there are relatively few outcomes. This approach is used in Examples D and E.

EXAMPLE D

List the sample space for the experiment of tossing three coins and observing heads or tails for each coin. Then determine the probabilities of the following events.

1. Obtaining exactly 2 heads
2. Obtaining at least 2 heads

Solution There are 8 equally likely outcomes in the sample space:

HHH HHT HTH THH TTT TTH THT HTT

1. There are 3 outcomes with exactly 2 heads (HHT, HTH, and THH), so

$$P(\text{exactly 2 heads}) = \frac{3}{8}$$

2. There are 4 outcomes with at least 2 heads (HHT, HTH, THH, and HHH), so

$$P(\text{at least 2 heads}) = \frac{4}{8} = \frac{1}{2}$$

EXAMPLE E

Five tickets numbered 1, 2, 3, 4, and 5 are placed in a box, and two are selected at random. List the sample space and determine the following probabilities:

1. Obtaining a 1 or a 2 or both
2. Obtaining two odd numbers
3. Obtaining the number 6 in the pair
4. Obtaining only numbers less than 6

Solution There are 10 equally likely outcomes in the sample space:

1, 2 1, 3 1, 4 1, 5 2, 3 2, 4 2, 5 3, 4 3, 5 4, 5

1. There are 7 outcomes containing either a 1 or a 2 or both, so

$$P(1 \text{ or } 2 \text{ or both}) = \frac{7}{10}$$

2. There are 3 outcomes in which both numbers are odd, so

$$P(\text{both odd}) = \frac{3}{10}$$

3. There are no outcomes that include the number 6, so

$$P(6) = 0$$

4. All 10 outcomes include only numbers less than 6, so

$$P(\text{number} < 6) = \frac{10}{10} = 1$$

NCTM Standards

Students should come to understand and use 0 to represent the probability of an impossible event and 1 to represent the probability of a certain event, and they should use common fractions to represent the probability of events that are neither certain nor impossible. p. 181

In part 3 of Example E, the event of obtaining the number 6 is the empty set. In this case the event is called an **impossible event,** and it has a probability of 0. At the opposite extreme, the event in part 4 of Example E contains all possible outcomes. Such an event is called a **certain event** and has a probability of 1. Since the number of favorable outcomes is always less than or equal to the total number of outcomes, the *probability of an event is always less than or equal to 1.* These observations are summarized in the following inequalities, which hold for any event E.

$$0 \le P(E) \le 1$$

We have been computing the probabilities of events by dividing the number of favorable outcomes by the total number of outcomes. Perhaps you have noticed that the probability

of an event can also be found by adding the probabilities of the various outcomes in the event. In part 2 of Example E, there are three outcomes in which both numbers are odd, so the probability of selecting a pair of odd numbers is $\frac{3}{10}$. The probability of this event can also be found by adding the probabilities of each outcome:

$$\frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{3}{10}$$

This is a special case of the following property.

Probability of Events The **probability of an event *E*,** which has outcomes $e_1, e_2, \ldots, e_n$, is the sum of the probabilities of the outcomes.

$$P(E) = P(e_1) + P(e_2) + \ldots + P(e_n)$$

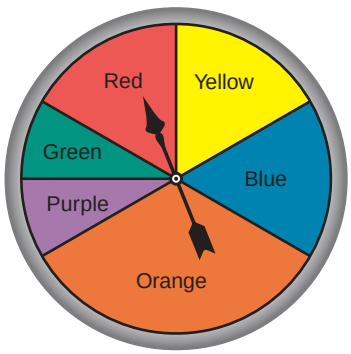
This property holds for events with equally likely outcomes as well as for those whose outcomes are not equally likely, as in Example F.

EXAMPLE F

The outcomes of spinning the spinner shown below have these probabilities:

$P(\text{purple}) = \frac{1}{12}$ $P(\text{blue}) = \frac{1}{6}$ $P(\text{red}) = \frac{1}{6}$

$P(\text{yellow}) = \frac{1}{6}$ $P(\text{green}) = \frac{1}{12}$ $P(\text{orange}) = \frac{1}{3}$



Determine the probabilities of the following events.

1. E: Obtaining a primary color (red, blue, or yellow)
2. T: Obtaining a color with six letters in its name

Solution 1. $P(E) = P(\text{red}) + P(\text{blue}) + P(\text{yellow}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$.

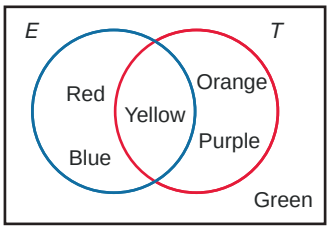
2. $P(T) = P(\text{purple}) + P(\text{orange}) + P(\text{yellow}) = \frac{1}{12} + \frac{1}{3} + \frac{1}{6} = \frac{7}{12}$.

NCTM Standards

Middle-grade students should learn and use appropriate terminology and should be able to compute probabilities for simple compound events, such as the number of expected occurrences of two heads when two coins are tossed 100 times. p. 51

PROBABILITIES OF COMPOUND EVENTS

An event that can be described in terms of the union, intersection, or complement of other sets is called a **compound event.** For the events E and T in Example F, $E \cup T$ (the event that E, or T, or both occur) and $E \cap T$ (the event that both E and T occur) are examples of compound events. Since E and T in Example F are not disjoint (the Venn diagram on the next page shows they have the outcome *yellow* in common), the probability of $E \cup T$



cannot be obtained by simply adding $P(E)$ and $P(T)$ because $P(\text{yellow})$ would be counted twice. The following equation confirms that $P(E \cup T)$ cannot be computed in this way because it gives $P(E) + P(T) > 1$ and the probability of $P(E \cup T)$ must be less than or equal to 1.

$$\begin{aligned} P(E) + P(T) &= P(\text{red}) + P(\text{blue}) + P(\text{yellow}) + P(\text{purple}) + P(\text{orange}) + P(\text{yellow}) \\ &= 1\frac{1}{12} \end{aligned}$$

However, $P(E \cup T)$ is equal to $P(E) + P(T) - P(\text{yellow})$. In general, we have the following property.

Addition Property If events A and B are not disjoint, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If events A and B are disjoint, then

$$P(A \cup B) = P(A) + P(B)$$

When events A and B are disjoint, they are called **mutually exclusive events.** In this case, $A \cap B = \varnothing$, and so $P(A \cap B) = 0$. This explains why the second equation of the *addition property* is a simple variation of the first equation.

EXAMPLE G

Several events containing the outcomes of spinning the spinner in Example F are defined as follows.

E: Obtaining a primary color (red, blue, or yellow)
T: Obtaining a color with six letters in its name
H: Obtaining a color with five letters in its name
K: Obtaining a color with four letters in its name
N: Obtaining a color with fewer than six letters in its name

Use these events to determine the following probabilities. Which refer to mutually exclusive events? You may find constructing Venn diagrams to be helpful.

1. $P(E \cup H)$ **2.** $P(T)$ **3.** $P(T \cup K)$ **4.** $P(N)$ **5.** $P(E \cup K)$ **6.** $P(E \cap K)$

Solution **1.** $P(E \cup H) = P(E) + P(H) = \frac{1}{2} + \frac{1}{12} = \frac{7}{12}$ since E and H are mutually exclusive. **2.** $P(T) = \frac{7}{12}$ **3.** $P(T \cup K) = P(T) + P(K) = \frac{7}{12} + \frac{1}{6} = \frac{9}{12} = \frac{3}{4}$ T and K are mutually exclusive. **4.** $P(N) = P(\text{red}) + P(\text{blue}) + P(\text{green}) = \frac{1}{6} + \frac{1}{6} + \frac{1}{12} = \frac{5}{12}$. **5.** $P(E \cup K) = \frac{1}{2} + \frac{1}{6} - \frac{1}{6} = \frac{1}{2}$. **6.** $P(E \cap K) = \frac{1}{6}$.

Did you notice that events T and N in Example G are complementary sets? That is, they have no outcomes in common, and their union contains all the outcomes of the sample space. Such events are called **complementary events.** Notice also that $P(T) + P(N) = \frac{7}{12} + \frac{5}{12} = 1$. In general, if A and B are complementary events, then we can determine the probability of one by knowing the probability of the other:

$$P(A) + P(B) = 1$$

For example, if a regular six-sided die is rolled, the event D of obtaining a number divisible by 3 and the event N of obtaining a number not divisible by 3 are complementary events. Since 3 and 6 are the only numbers divisible by 3, $P(D) = \frac{2}{6}$, and therefore

$$P(N) = 1 - P(D) = \frac{4}{6}$$

The *addition property* is sometimes used in analyzing data in tables.

Projected School Enrollments in 2018* (in millions of students)

	Public	Private	Total
K through Grade 8	38	5	43
Secondary	16	1	17
College	15	5	20
Total	69	11	80

Technology Connection

Coin Toss Simulation

How many tosses of a coin on the average would be needed to obtain three consecutive heads? Four consecutive heads? You can experiment with a coin or use the online 8.1 Mathematics Investigation to simulate tossing a coin until the desired outcome is obtained. Explore this and related questions in this investigation.



Mathematics Investigation
Chapter 8, Section 1
www.mhhe.com/bbn

Let S be the event "a student is in secondary school" and C be the event "a student is in college." If a student is chosen at random from the 80 million students, what is the probability that he or she is in secondary school or college? Using the addition property and the fact that the intersection of S and C is the empty set, we have

$$\begin{aligned} P(S \cup C) &= P(S) + P(C) - P(S \cap C) \\ &= \frac{17}{80} + \frac{20}{80} - 0 \\ &= \frac{37}{80} \approx .47 \end{aligned}$$

Let's consider another example, using the data in the preceding table. Let E be the event that a student is in a private educational institution. For a randomly chosen student, what is the probability that she or he is a college student or in a private educational institution?

$$\begin{aligned} P(C \cup E) &= P(C) + P(E) - P(C \cap E) \\ &= \frac{20}{80} + \frac{11}{80} - \frac{5}{80} \\ &= \frac{26}{80} \approx .33 \end{aligned}$$

If three of the four parts in the equation for the addition property are known, the equation can be solved for the missing part, as in the next example. Two symptoms of a common disease are a fever (F) and a rash (R). For people who have this disease, 20 percent will have the fever alone, 30 percent will have the rash alone, and 40 percent will have a fever or a rash or both. What is the probability that a randomly selected person with the disease will have both a fever and a rash?

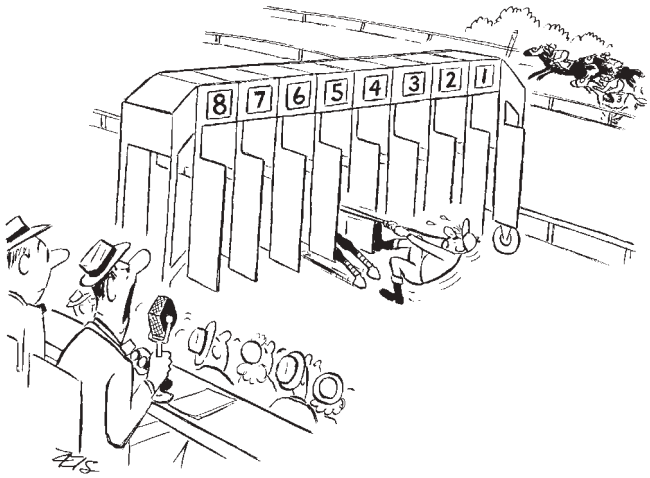
$$\begin{aligned} P(F \cup R) &= P(F) + P(R) - P(F \cap R) \\ .40 &= .20 + .30 - P(F \cap R) \\ P(F \cap R) &= .10 \end{aligned}$$

So, 10 percent will have both a fever and a rash.

ODDS

Racetracks state probabilities in terms of odds. Suppose that the odds against Blue Boy's winning are 4 to 1. This means that the racetrack management will match every $1 you bet

**Statistical Abstract of the United States,* 128th ed. (Washington, DC: Bureau of the Census, 2009), Table 214.



"Blue Boy seems to be holding back a bit."

on Blue Boy with $4. Each time you win, you receive the money you bet plus the money put up by the racetrack. That is, for $1 you receive $5, for $2 you receive $10, etc. The 4-to-1 odds indicate that the racetrack management expects Blue Boy to lose 4 out of every 5 races he runs. Thus, the probability of Blue Boy losing the race is $\frac{4}{5}$, and the probability of his winning is $\frac{1}{5}$.

This example shows the close relationship between odds and probability: They are different ways of presenting the same information. This relationship is illustrated in Figure 8.3. The bar has 5 equal parts, 4 to represent the unfavorable outcomes (Blue Boy's losing) and 1 to represent the favorable outcome (Blue Boy's winning). The odds of 4 to 1 are shown by the ratio of the 4 yellow parts to the 1 blue part. The probability of $\frac{4}{5}$ is the ratio of the 4 yellow parts to the whole (5 parts).

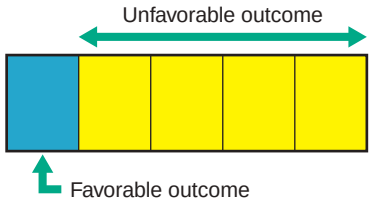


Figure 8.3

In general, **odds** are ratios. If the **odds in favor** of an event are ***n* to *m*,** then the probability of the event occurring is $\frac{n}{(n+m)}$ (Figure 8.4). In this case the **odds against** the event are ***m* to *n*,** and the probability of the event not occurring is $\frac{m}{(n+m)}$.

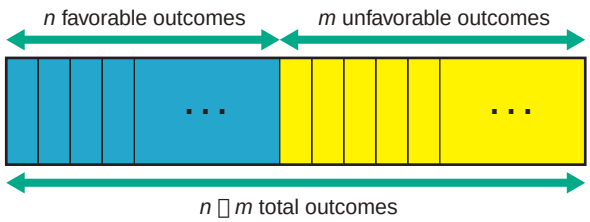


Figure 8.4

EXAMPLE H

One card is selected at random from an ordinary deck of 52 cards, which contains four aces. Determine the following odds and probabilities.

1. Odds of obtaining 1 ace
2. Probability of obtaining 1 ace
3. Odds of not obtaining 1 ace
4. Probability of not obtaining 1 ace

Solution **1.** Ratio of the number of favorable outcomes to the number of unfavorable outcomes = 4 to 48 = 1 to 12. **2.** $\frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}} = \frac{4}{52} = \frac{1}{13}$. **3.** Ratio of the number of unfavorable outcomes to the number of favorable outcomes = 48 to 4 = 12 to 1. **4.** $\frac{\text{Number of unfavorable outcomes}}{\text{Total number of outcomes}} = \frac{48}{52} = \frac{12}{13}$.

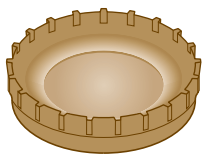
Example H helps us to see that if the odds in favor of an event happening are low, the probability is close to 0; and if the odds are high, the probability is close to 1. The odds of selecting an ace are low, 4 to 48, and the probability is $\frac{1}{13}$. Similarly, the odds of selecting a card that is not an ace are high, 48 to 4, and the probability is $\frac{12}{13}$.

EXPERIMENTAL PROBABILITY

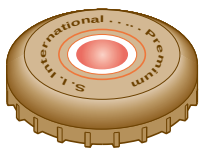
It is often more difficult to determine a theoretical probability than to determine an experimental probability. Moreover, experimental probabilities involve conducting repeated trials and observing and recording data, activities that are appropriate for students at all levels.

EXAMPLE I

Consider tossing a bottle cap to determine the experimental probability that it will land with its edge up (U) or its edge down (D). What is this experimental probability for the 50 tosses shown here?



Edge up



Edge down

D U U U D D U D U D
U U U U D U U D U U
U D U U U U U U D U
U U U D U U U D U U
D U U D U D U D U U

NCTM Standards

Although simulations can be useful, students also need to develop their probabilistic thinking by frequent experience with actual experiments. p. 254

Solution The bottle cap landed with its edge down 15 times out of 50, so the experimental probability for this experiment is $\frac{15}{50} = \frac{3}{10}$. If many repeated experiments yield approximately the same result, we can conclude that the experimental probability is approximately $\frac{3}{10}$.

In many fields the empirical approach is the only means of determining probability. Insurance companies measure the risks against which people are buying insurance in order to set premiums. A person's age and life expectancy are important factors. To compute the probability that a person 20 years old will live to be 65 years old, insurance companies gather birth and death records of large numbers of people and compile mortality tables. A table based on the births and deaths among policyholders of several large insurance companies appears in Figure 8.5 on the next page; it shows they estimate out of every 10 million people born in the United States, 6,800,531 will live to age 65.* Thus, their estimated probability that a newborn baby will live to age 65 is $\frac{6{,}800{,}531}{10{,}000{,}000}$, or about .68 (68 percent). According to the table, 9,664,994 people will live to age 20. Therefore, the experimental probability of a newborn baby living to age 20 is about .966, or 96.6 percent.

*Robert Mehr and Emerson Commack, *Principles of Insurance.*

Age	Number Living	Number Dying	Age	Number Living	Number Dying
0	10,000,000	70,800	35	9,373,807	23,528
1	9,929,200	17,475	36	9,350,279	24,685
2	9,911,725	15,066	37	9,325,594	26,112
3	9,896,659	14,449	38	9,299,482	27,991
4	9,882,210	13,835	39	9,271,491	30,132
5	9,868,375	13,322	40	9,241,359	32,622
6	9,855,053	12,812	41	9,208,737	35,362
7	9,842,241	12,401	42	9,173,375	38,253
8	9,829,840	12,091	43	9,135,122	41,382
9	9,817,749	11,879	44	9,093,740	44,741
10	9,805,870	11,865	45	9,048,999	48,412
11	9,794,005	12,047	46	9,000,587	52,473
12	9,781,958	12,325	47	8,948,114	56,910
13	9,769,633	12,896	48	8,891,204	61,794
14	9,756,737	13,562	49	8,829,410	67,104
15	9,743,175	14,225	50	8,762,306	72,902
16	9,728,950	14,983	51	8,689,404	79,160
17	9,713,967	15,737	52	8,610,244	85,758
18	9,698,230	16,390	53	8,524,486	92,832
19	9,681,840	16,846	54	8,431,654	100,337
20	9,664,994	17,300	55	8,331,317	108,307
21	9,647,694	17,655	56	8,223,010	116,849
22	9,630,039	17,912	57	8,106,161	125,970
23	9,612,127	18,167	58	7,980,191	135,663
24	9,593,960	18,324	59	7,844,528	145,830
25	9,575,636	18,481	60	7,698,698	156,592
26	9,557,155	18,732	61	7,542,106	167,736
27	9,538,423	18,981	62	7,374,370	179,271
28	9,519,442	19,324	63	7,195,099	191,174
29	9,500,118	19,760	64	7,003,925	203,394
30	9,480,358	20,193	65	6,800,531	215,917
31	9,460,165	20,718	66	6,584,614	228,749
32	9,439,447	21,239	67	6,355,865	241,777
33	9,418,208	21,850	68	6,114,088	254,835
34	9,396,358	22,551	69	5,859,253	267,241

Figure 8.5

Reproduced with permission of The McGraw-Hill Companies.

EXAMPLE J

Determine the experimental probabilities of the following events, using the table in Figure 8.5 on the previous page.

1. A newborn baby will live to age 60
2. A person aged 20 will live to be 60 years old
3. A person aged 50 will live to be 60 years old

Solution **1.** Approximately .77 **2.** Out of 10 million births, there are 9,664,994 people living at age 20 and 7,698,698 living at age 60, so the probability of a person surviving from age 20 to age 60 is $\frac{7{,}698{,}698}{9{,}664{,}994} \approx .797$, or approximately 80 percent. **3.** Out of 10 million births, there are 8,762,306 people living at age 50 and 7,698,698 living at age 60, so the probability of a person surviving from age 50 to 60 is $\frac{7{,}698{,}698}{8{,}762{,}306} \approx .879$, or approximately 88 percent.

NCTM Standards

Simulations afford students access to relatively large samples that can be generated quickly and modified easily. p. 254

SIMULATIONS

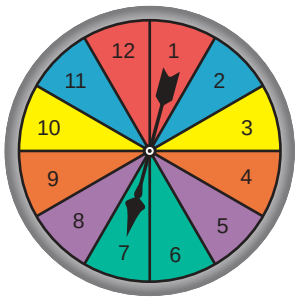
You may remember from Chapter 7 that a simulation is a procedure in which experiments that closely resemble a real situation are conducted repeatedly. In that chapter, simulations provided answers to questions in statistics. Simulations are also used to obtain approximations to theoretical probabilities.

EXAMPLE K

What is the probability that in a group of 5 people chosen at random at least 2 will have a birthday in the same month?

Solution One approach to this problem is to conduct experiments. Since polling a large number of people to determine their birth months is time-consuming, we will use a simulation. The following 20 groups of 5 numbers each were obtained by spinning a spinner labeled with whole numbers from 1 to 12. Since in 11 of the 20 groups the same number occurs 2 or more times, the experimental probability for this simulation is $\frac{11}{20} \approx \frac{1}{2}$.

9	3	9	9	11	7	2	5	4	4	6	9	9	11	12	1	1	10	8	5
1	2	6	12	10	3	9	10	9	1	9	5	9	11	11	7	11	12	2	6
2	7	5	9	2	5	11	4	9	7	11	12	5	8	6	12	3	2	12	8
2	6	3	11	2	7	10	11	7	9	7	2	6	3	12	2	12	5	1	7
7	6	10	9	7	3	8	2	3	12	9	12	4	8	7	6	9	1	4	12



The theoretical basis for approximating probabilities with simulations is called the **law of large numbers.** This law states that the more times a simulation is carried out, the closer the probability

$$\frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$

is to the theoretical probability.

NCTM Standards

The *Curriculum and Evaluation Standards for School Mathematics* (p. 111) suggest the use of a computer to carry out simulations:

> Once students have experimented with a problem, a computer can generate hundreds or thousands of simulated results. It is important that the computer simulation follow active student exploration. This follow-up broadens students' understanding and provides them with an opportunity to observe how a greater number of trials can refine the probability model.

Tables of random digits are also commonly used for simulations. A portion of such a table is shown in Figure 8.6.

57455	72455	93949	03017	33463	50612	65976	18630	26080	99135
01177	18110	31846	33144	99175	43471	29341	07096	69643	85566
25107	69058	16098	53085	88020	30108	81469	33487	55936	34594
73312	70522	45206	00165	06447	65724	29908	96532	14636	25790
72526	06721	23176	95705	10722	72474	01434	38573	08089	09806
68868	49240	16140	11046	38620	49148	80338	45266	39020	06304
45101	17710	54682	31812	76734	87045	96291	67557	18680	18886
12672	99918	24766	14132	63739	18576	80955	67381	60403	09892
12201	94684	41296	86044	83170	95446	14032	86602	34998	49065
46062	88535	71445	10422	72088	50200	55509	03741	73748	38899
12483	92564	43692	60562	93982	44567	62843	51987	11525	02695
33791	32729	88363	65524	45698	02573	97181	30352	10505	02352
78160	17311	24688	87381	00257	76315	69875	34128	01483	21765
43595	78341	07757	76471	37801	90306	20915	38132	91714	44436
92750	50923	26074	03327	57400	79251	04823	74914	11445	93818
96564	04624	46940	79735	27074	99264	32920	51271	57583	82685
55645	86878	27211	89358	30594	70161	26045	33370	19425	25961
32582	88628	11166	47654	62462	05080	51664	39828	01770	01607
07866	68988	70054	83887	31538	66864	58710	70349	65126	02265
97092	11334	78242	15410	99001	65756	23979	63446	84808	06072

Figure 8.6

NCTM Standards

Misconceptions about probability have been held not only by many students but also by many adults (Konold 1989). To correct misconceptions, it is useful for students to make predictions and then compare the predictions with actual outcomes. p. 254

EXAMPLE L

Try to answer the following question by using the table of random digits to carry out a simulation. What is the experimental probability that in a family of 5 children there are 3 boys and 2 girls?

Solution Let each odd digit from the table in Figure 8.6 represent a boy and each even digit represent a girl. Then examine groups of 5 digits, and circle those with 3 odd digits and 2 even digits. Starting at the beginning of the list, we find that the first group with 3 odd and 2 even digits is 72455. There are 30 groups out of the first 100 groups with 3 odd and 2 even digits. These are listed here. Thus, the experimental probability based on this simulation is $\frac{30}{100} = .3$. This ratio is an approximation to the theoretical probability of a family's having 3 boys and 2 girls among 5 children.

72455	03017	33463	65976	18110	33144	43471	29341	25107	53085
33487	34594	96532	25790	23176	45101	31812	76734	96291	14132
18576	80955	67381	83170	34998	88535	71445	03741	73748	38899

PROBLEM-SOLVING APPLICATION

Solutions to problems in probability quite often conflict with our intuition. For example, if 2 coins are tossed, the probability of obtaining exactly 1 head is 50 percent; however, if 4 coins are tossed, the probability of obtaining exactly 2 heads is not 50 percent.

Problem

To create interest in probability, a teacher asks for a volunteer to play the following game: Four coins will be tossed, and if exactly 2 heads are obtained, the student wins the coins; otherwise, the student loses. What is the probability that the student will win?

Understanding the Problem In order to win, the student must obtain exactly 2 heads.
Question 1: What happens if the student obtains 3 heads or 4 heads?

Devising a Plan One plan is to toss 4 coins, record the number of heads, and repeat this experiment many times. An experimental probability can then be determined by dividing the number of times exactly 2 heads appear by the total number of experiments. A second plan is to *use a simulation.* In a list of random digits, each even digit can be designated as a head (H) and each odd digit as a tail (T). **Question 2:** If the digits from 0 to 9 are used, are there equal numbers of heads and tails?

Carrying Out the Plan Use the list of random digits in Figure 8.6 on the previous page to carry out a simulation. One way to do this is to consider only the first 4 digits in each group of 5 digits. For example, in the first row of the table, 5 of the 10 groups have exactly 2 "heads" (see the following list). Continue the simulation, using the first 10 rows of the table. **Question 3:** For this simulation what is the experimental probability of obtaining exactly 2 heads in a toss of 4 coins?

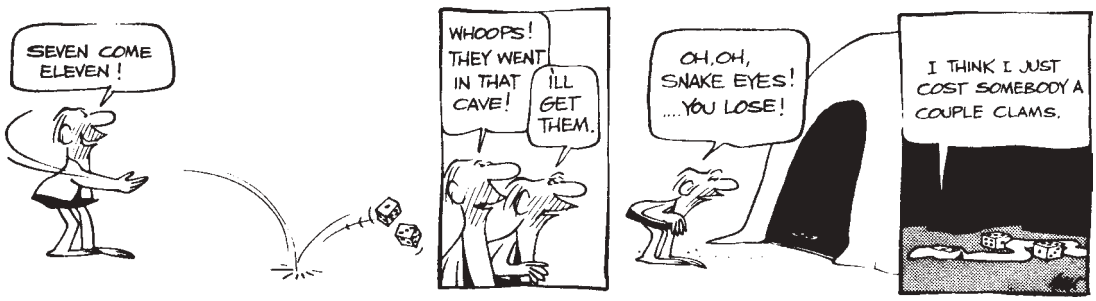
	2H			2H	2H	2H		2H		
57455	72455	93949	03017	33463	50612	65976	18630	26080	99135

Looking Back The theoretical probability of obtaining exactly 2 heads can be found by listing the 16 different outcomes of tossing 4 coins and counting those with exactly 2 heads (see following list). **Question 4:** What is this probability? How does it compare to the probability obtained from the simulation?

HTHT	HHHH	HHHT	HHTH	HTHH	THHH	HHTT	HTTH
TTHH	THTH	TTTT	TTTH	TTHT	THTT	HTTT	THHT

Answers to Questions 1–4 **1.** The student loses. **2.** Yes (remember, zero is an even number) **3.** Since 42 groups have exactly 2 even digits and 2 odd digits among the first 4 digits, the approximation to the theoretical probability of obtaining exactly 2 heads in tossing 4 coins is $\frac{42}{100} = .42$. **4.** Six of the 16 outcomes have exactly 2 heads, so the theoretical probability is $\frac{6}{16} = .375$. This is reasonably close to the probability of .42, which was obtained from the simulation.

Exercises and Problems 8.1



1. Out of 36 possible outcomes of tossing two dice, 6 produce a sum of 7. Complete the following table by computing the probability of rolling each of the other sums. (You may want to use the array of dice shown in Figure 8.2 at the beginning of this section.)

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability						$\frac{6}{36}$					

a. What is the probability of obtaining a sum greater than or equal to 8?

b. What is the probability of obtaining a sum greater than 4 and less than 8?

The dice game called *craps* has the following rules. If the player rolling the dice gets a sum of 7 or 11, he or she wins. If the player rolls a sum of 2, 3, or 12, she or he loses. If the first sum rolled is a 4, 5, 6, 8, 9, or 10, the player continues

to roll the dice. After the first roll, the player wins if she or he can obtain the first sum rolled before rolling a 7. Use this information in exercises 2 and 3.

2. a. What is the probability of rolling a 7 or an 11? (See the table in exercise 1.)
b. What is the probability of losing on the first roll?

3. a. What is the probability of rolling a 4, 5, 6, 8, 9, or 10 on the first roll?
b. Suppose a player rolls an 8 on the first roll. Which has the greater probability, rolling a 7 on the second roll or rolling another 8?

A box contains seven tickets numbered 1 through 7. One ticket will be selected at random from the box. Use this information in exercises 4 and 5.

4. a. List all the outcomes of the sample space for this experiment.
b. What is the probability of obtaining an even number?
c. What is the probability of obtaining a number greater than 3?

5. a. What is the probability of obtaining a prime number?
b. What is the probability of obtaining a number less than 5?
c. What is the probability of obtaining an odd number?

A chip is to be drawn from a box containing the following color chips: 8 orange, 5 green, 3 purple, and 2 red chips. Describe the sample space and determine the probabilities of selecting the types of chips in exercises 6 and 7.

6. a. A purple chip
b. A green or purple chip
c. A chip that is not orange

7. a. A red chip
b. A red or green chip
c. A chip that is not red

The following five dice are pictured here: tetrahedron (4 faces and yellow), cube (6 faces and pink), octahedron (8 faces and green), dodecahedron (12 faces and yellow), and icosahedron (20 faces and white). The faces of each of these dice are labeled with consecutive whole numbers beginning with 1. For example, the tetrahedron is labeled 1, 2, 3, 4, and the icosahedron is labeled 1 to 20. Since the tetrahedron is the only one of these dice that does not have a top face when rolled, it has three numbers on each of its faces and the number rolled is the number on the lower edges. In this photo, 4 was rolled on the tetrahedron.



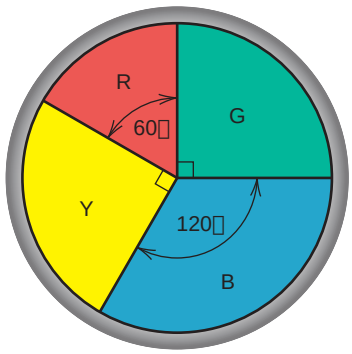
Determine the probabilities of rolling the numbers in exercises 8 and 9 with each type of die, and record the probabilities in a copy of the table given below.

8. a. A number less than 3 **b.** An even number

9. a. The number 2 **b.** A number greater than 3

	(8a)	(8b)	(9a)	(9b)
Tetrahedron				
Cube				
Octahedron				
Dodecahedron				
Icosahedron				

For an experiment consisting of spinning the spinner shown here, determine the probabilities in exercises 10 and 11.



10. a. $P(\text{B})$ **b.** $P(\text{Y})$
c. $P(\text{R})$ **d.** $P(\text{R or B})$

11. a. $P(\text{R or G})$ **b.** $P(\text{R or G or B})$
c. $P(\text{G})$ **d.** $P(\text{Y or G})$

12. A purse contains three identical-looking keys, but only two of the three keys will unlock the side door of a house. Answer the following questions if two of the three keys are randomly selected.
a. List all the outcomes of the sample space.
b. What is the probability of selecting one key that will open the door and one key that will not?
c. What is the probability of selecting both keys that will open the door?

13. Four identical chips lettered A, B, C, and D are placed in a box. An experiment consists of selecting two chips at random.
a. List all the outcomes of the sample space.
b. What is the probability that one of the two chips will be lettered B?
c. What is the probability that one chip will be lettered C and the other D?

14. A box contains 3 red marbles and 2 green marbles. An experiment consists of selecting 2 marbles at random from the box.
- **a.** List all the outcomes of the sample space.
- **b.** What is the probability of obtaining 1 red and 1 green marble?
- **c.** What is the probability that both marbles will be red?

15. An electric blender comes in blue, red, yellow, green, white, or pink. Two blenders are ordered, and the customer does not specify the color. Assume there is approximately the same number of blenders of each color and that the company randomly selects the blenders without regard to color for shipment.
- **a.** List all the outcomes of the sample space.
- **b.** What is the probability of receiving two blue blenders?
- **c.** What is the probability of receiving at least one green blender?
- **d.** What is the probability of not receiving a red blender?

An experiment consists of tossing a regular die with faces numbered 1 through 6. Use the following events to determine the probabilities in exercises 16 and 17.

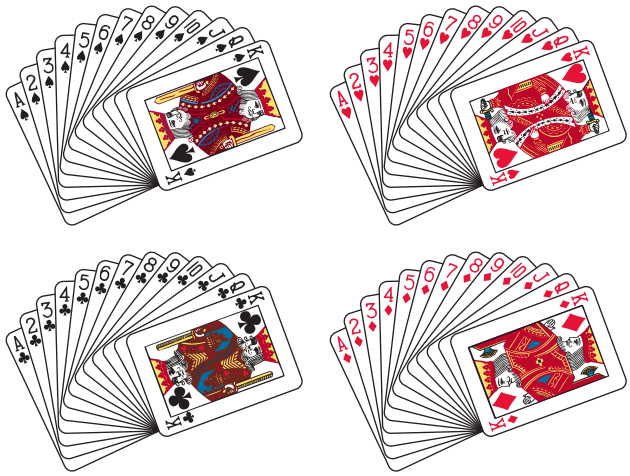
E: Obtaining an even number
F: Obtaining a prime number
G: Obtaining an odd number
H: Obtaining a number greater than 4

16.
- **a.** $P(E \cup G)$ and $P(E) + P(G)$
- **b.** $P(F \cup G)$ and $P(F) + P(G)$
- **c.** Which pairs, if any, of probabilities in parts a and b are equal? Explain why.

17.
- **a.** $P(F \cup H)$ and $P(F) + P(H)$
- **b.** $P(E \cup F)$ and $P(E) + P(F)$
- **c.** Which pairs, if any, of the probabilities in parts a and b are equal? What conditions must exist in order for the probability of the union of two events to equal the sum of the probabilities of the two events?

Consider a regular deck of 52 cards with 13 cards (including 3 face cards) in each of 4 suits (hearts, diamonds, spades, and clubs). Use the following events to determine the probabilities in exercises 18 and 19.

E: Selecting a face card (jack, queen, king)
F: Selecting an ace
G: Selecting a spade
H: Selecting a heart



18.
- **a.** $P(E)$
- **b.** $P(G)$
- **c.** $P(E \cup F)$
- **d.** $P(F \cup H)$
- **e.** $P(E \cap H)$

19.
- **a.** $P(F)$
- **b.** $P(H)$
- **c.** $P(G \cup H)$
- **d.** $P(E \cup H)$
- **e.** $P(G \cap E)$

The sum of the probability that an event will happen and the probability that the event will not happen is 1. Consider the experiment of selecting 1 card at random from a complete deck of 52 cards. Compute the probabilities of the events in exercises 20 and 21.

20.
- **a.** Not drawing a diamond
- **b.** Not drawing a red card
- **c.** Drawing a face card or an ace or a number less than 8
- **d.** Not drawing a 6

21.
- **a.** Not drawing an ace
- **b.** Not drawing a face card
- **c.** Drawing a club, heart, or diamond
- **d.** Not drawing a black face card

Compute the probability and odds of selecting each of the cards in exercises 22 and 23 at random from a complete deck of 52 cards.

22.
- **a.** An ace
- **b.** A face card
- **c.** A diamond
- **d.** A black face card

23.
- **a.** A club
- **b.** An 8 or a 9
- **c.** A red card
- **d.** A spade or a heart

Given the probabilities in exercises 24 and 25, determine the odds in favor of each event.

24.
- **a.** The probability of winning the lottery is $\frac{1}{1{,}000{,}000}$.
- **b.** The probability of selecting a defective part is 4 percent.

c. The probability that interest rates will increase is $\frac{1}{5}$.
d. There is a 90 percent chance that the operation will be successful.

25. a. The probability of living to age 65 is $\frac{7}{10}$.
b. The probability of selecting a person with type O blood is $\frac{3}{5}$.
c. The probability of winning a certain raffle is $\frac{1}{500}$.
d. The probability of rain on Monday is 80 percent.

Gambling syndicates predict the outcomes of sporting events in terms of odds. Convert each of the odds in exercises 26 and 27 to a probability.

26. a. The odds that the Packers will beat the Rams are 4 to 3.
b. The odds that the University of Michigan will defeat Ohio State are 7 to 5.

27. a. The odds that the Yankees will win the pennant are 10 to 3.
b. The odds of recovering the missing space capsule are 1 to 4.

Determine the experimental probabilities of the events in exercises 28 and 29 to the nearest .01.

28. a. Tossing a paper cup and having it land with its bottom down if it landed in this position 18 times in 150 tosses.
b. Spinning a spinner and obtaining the color green if this color was obtained in 46 out of 130 spins.

29. a. Tossing a tack and having it land with its point up if it landed in this position for 19 out of 90 tosses.
b. Selecting a nondefective part from an assembly line if 194 out of 200 of the previous selections have been nondefective.

The mortality table in Figure 8.5 on page 530 was computed on the basis of births and deaths among policyholders of several large insurance companies. Use the information in that table to answer exercises 30 and 31.

30. a. What is the experimental probability that a child will live to be 1 year old?
b. What is the experimental probability that a person will live to age 50?
c. What is the experimental probability that a person aged 34 will live to age 65?
d. If an insurance company has 2356 policyholders at age 60, how many death claims is the company likely to have to pay on behalf of those who do not reach age 61?

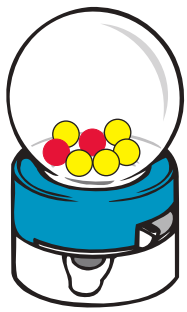
31. a. What is the experimental probability that a 60-year-old person will live to age 65?
b. What is the experimental probability that a child will live to be 12 years old?
c. What is the experimental probability that a person will live to be 70 years old?
d. If an insurance company has 7000 policyholders aged 28, how many death claims is the company likely to have to pay on behalf of those who do not reach age 29?

32. Use the table of random digits in Figure 8.6 on page 532 to carry out a simulation that approximates the probability of obtaining at least one 6 in five rolls of a die. Describe your simulation.

33. Use the table of random digits in Figure 8.6 on page 532 to carry out a simulation that approximates the probability of obtaining at least 5 heads in a toss of 10 fair coins. Describe your simulation.

Reasoning and Problem Solving

34. Featured Strategy: Using a Simulation. A machine that sells gumballs for a nickel apiece contains 5 yellow gumballs and 2 red gumballs. If 2 nickels are put into the machine, what is the probability of obtaining a red gumball if each gumball has an equal chance of being chosen?



a. Understanding the Problem. The problem is to determine the probability of obtaining at least 1 red gumball. If only 1 nickel is put into the machine, what is the probability of obtaining 1 red gumball?
b. Devising a Plan. One approach is to use a simulation to approximate the probability. Describe such a simulation.
c. Carrying Out the Plan. What probability do you obtain by using your simulation?
d. Looking Back. It is also possible to determine the theoretical probability by designating the gumballs as R_1, R_2, Y_1, Y_2, Y_3, Y_4, and Y_5, listing outcomes of the sample space, and counting those that contain at least 1 red. What is the theoretical probability obtained with this approach?

PS Use simulations to approximate the probabilities in exercises 35 to 38. Describe your method.

35. What is the probability that in a family of 3 boys and 2 girls, the 3 boys were born in succession?

36. On a certain quiz show, people are required to guess which of 3 identical envelopes contains a $1000 bill. What is the probability that exactly 4 out of 8 people will guess the correct envelope?

37. A baseball player has a batting average of .300, which means that on average the player gets 3 hits in 10 times at bat. What is the probability this player will get at least 3 hits in the next 5 times at bat?

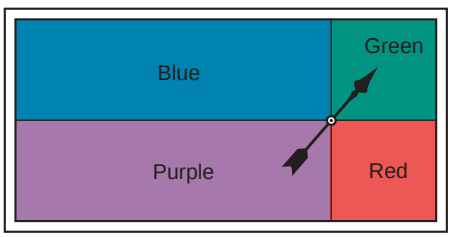
38. At a certain university it is required by state law that 3 out of 4 students be from the state. If 10 students are selected at random from this university, what is the probability that 4 or more will be from outside the state?

39. PS The problem-solving application on pages 532–533 shows that the probability of obtaining exactly 2 heads in a toss of 4 fair coins is .375.

a. Use a simulation to approximate the probability of obtaining exactly 3 heads in a toss of 6 fair coins. Describe your simulation.

b. Make a conjecture about the probability of obtaining exactly 10 heads in a toss of 20 fair coins. Will it be less than .5, equal to .5, or greater than .5?

40. PS Kyle made a spinner from the cover of a cardboard box by placing the pointer off the center, as shown in the following figure. One of his classmates claimed that this was not a fair spinner because the blue region and the purple region were both 3 times the size of the green and the red regions. Do you agree with his classmate? Explain your reasoning.



41. PS Maura is designing spinners for a middle school mathematics and science fair. Each spinner will have four regions, and each region will have one of the colors blue, green, purple, or red. Show with a detailed sketch a design for each of the following spinners.

a. A spinner such that the probability of obtaining the blue (or purple) region is 3 times the probability of obtaining the green (or red) region

b. A spinner such that the probability of obtaining the blue (or purple) region is 4 times the probability of obtaining the green (or red) region

c. A spinner such that the probability of obtaining the blue (or purple) region is 5 times the probability of obtaining the green (or red) region

42. PS Five students will be randomly selected from the seven top students in mathematics to represent their school in a regional competition. Angela and Brian are among the seven students, and they wish to determine the probability that they will both be chosen. Describe a sample space for determining this probability. What is the probability they will both be chosen?

43. PS Amanda and Dirk are renting in-line skates for a day in the park. They both take the same size shoe, and there are 6 pairs remaining in their size. Unbeknownst to them, 3 of the pairs of skates have defective bearings. If they randomly choose 2 pairs from the 6 pairs, what is the probability they will both get a pair of skates that are not defective? (*Hint:* List all of the outcomes of the sample space.)

44. PS A survey of the 400 students at Bishop Hill Middle School showed that 130 had their own computers, 174 owned graphing calculators, and 80 had both a computer and a graphing calculator. If a student is randomly selected from this school, what is the probability the student will own a computer or a graphing calculator?

45. PS To gather data involving their classmates, 24 fourth-graders each filled out a personal information survey sheet. In the category of pets, 5 students had a dog, 7 students had a cat, and 4 students had both a dog and a cat. If a student is randomly selected from this class, what is the probability the student will have a dog or a cat?

46. PS A hospital laboratory uses two tests to classify each sample of blood. The first test correctly identifies the blood type 65 percent of the time, and the second test correctly identifies the blood type 72 percent of the time. If the probability is .80 that at least one of the tests correctly identifies the blood type, what is the probability that both tests correctly identify the blood type?

47. PS The sales manager of a company believes there is a good chance she will be chosen as vice president of a new division, if the company opens a new division and if this year's sales are higher than last year's sales. She estimates the probability that the company will open a new division is .90, the probability that sales will be higher is .70, and the probability that at least one of these will happen is .75. Based on her estimates, what is the probability she will become vice president of the new division?

PS The table here shows the numbers to the nearest 100 boys and girls in school districts 1, 2, and 3. Determine the probability of each event in exercises 48 and 49, assuming that a student is selected at random from the three districts.

	District 1	District 2	District 3
Girls	1300	2300	1500
Boys	1400	2200	1300

48. **a.** Selected student is a girl.
b. Selected student is a boy from district 3.
c. Selected student is from district 1 or a girl.

49. **a.** Selected student is a boy.
b. Selected student is a girl from district 2.
c. Selected student is a boy or from district 3.

PS A survey of the teachers in a large school system yielded the information on gender and marital status shown in the table. Determine the requested probabilities in exercises 50 and 51 for a randomly chosen teacher from the school system.

	Married	Single	Divorced	Widowed
Male	12%	3%	8%	2%
Female	55%	12%	8%	0%

Events: M: teacher is a male; F: teacher is a female; S: teacher is a single person; D: teacher is a divorced person; and W: teacher is a widowed person.

50. **a.** $P(M \cap D)$ **b.** $P(F \cup D)$ **c.** $P(S \cup D)$

51. **a.** $P(F \cap S)$ **b.** $P(S \cup W)$ **c.** $P(M \cup S)$

Teaching Questions

1. Andrew has flipped a coin 50 times and tallied 15 heads and 35 tails. He asks you why he is not getting 25 heads and 25 tails because you said the probability of getting heads is $\frac{1}{2}$. How would you respond?

2. One group of students in your class determined that the probability of rolling a sum of 7 with a pair of dice is $\frac{1}{7}$. They demonstrated by showing you 21 possible outcomes, of which 3 outcomes (see below) had a sum of 7. What is incorrect about their reasoning and explain how you would address this problem.

3. Several of your students are having trouble getting started on the following simulation. Describe how you can help them.

   In a sports club with 12 students, 4 play soccer, 6 play tennis, 8 play basketball, 1 plays both soccer and tennis, 1 plays both soccer and basketball, and 4 play tennis and basketball. No member of the club plays all three sports. If three students are randomly selected from the club, what is the probability that each sport will be represented? Design a simulation for determining this probability.

4. One of your students told you that if a family has 3 children the probability that they are all 3 boys is $\frac{1}{4}$. She reasoned that there were four possibilities: all boys, all girls, 2 girls and a boy, or 2 boys and a girl. So the probability of all boys is $\frac{1}{4}$. Explain how you would help her understand why her solution is not correct.

5. Suppose that your fifth-grade class was working through the activity described in the Mini Lab from the **Elementary School Text** on page 522. One group of students who finished the activity with much enthusiasm wanted to find the expected number of times that triples (three of the same number) would turn up when rolling three number cubes.
   a. How big would their sample space be?
   b. Describe how you could help them by making a list of questions you would ask to help them solve this problem.

Classroom Connections

1. The **Spotlight on Teaching** at the beginning of this chapter focuses on probabilistic thinking and poses two example problems from the **Standards**—one about drawing numbered slips and the other about paper airplane throwing. Solve each of the problems and explain your procedures and reasoning.

2. Read the **Standards** statement on page 525 and explain how you would determine the number of expected occurrences. Describe what prerequisite experiences a sixth-grade class should have before investigating this problem.

3. What recommendation does the **PreK–2 Standards—Data Analysis and Probability** (see inside front cover) make about teaching probability in the early grades? Give examples of questions you would ask children that would satisfy this recommendation.

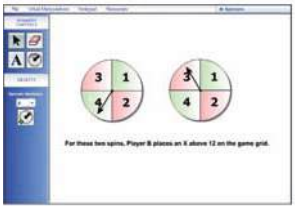
4. Give some examples of what is meant by the **Standards** statement on page 529.

Students' list of possible sums with a pair of dice

1 + 1	1 + 2	1 + 3 2 + 2	1 + 4 2 + 3	1 + 5 2 + 4 3 + 3	1 + 6 2 + 5 3 + 4	2 + 6 3 + 5 4 + 4	3 + 6 4 + 5	4 + 6 5 + 5	5 + 6	6 + 6

MATH ACTIVITY 8.2

Virtual Manipulatives

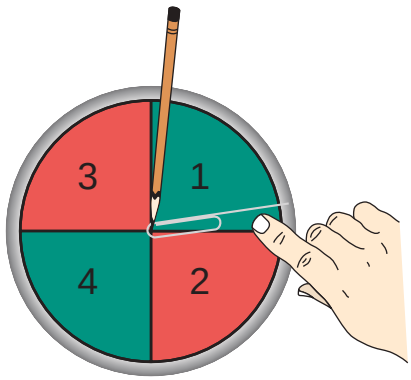


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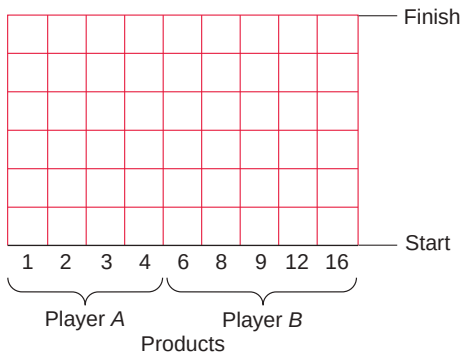
Determining the Fairness of Games

Purpose: Play and analyze games to determine whether or not they are fair.

Materials: 1-to-4 Spinner in the Manipulative Kit or Virtual Manipulatives and a grid from the website. For the cardstock spinner, bend a paper clip and hold a pencil at the center of the spinner, as shown below.



1. **Racing Game** Each player in turn spins the 1-to-4 spinner twice and computes the product of the 2 numbers. If the product is 1, 2, 3, or 4, player *A* puts an X on the square above that number in the grid shown below. In a similar manner player *B* records the products 6, 8, 9, 12, and 16. The game ends when one of the players has built a column of Xs up to the finish line and both players have completed the round. Each player receives 1 point for each of his or her Xs, and the winner is the player with the greater score. Play this game to form an opinion about whether player *A* or player *B* has an advantage.



   a. Based on playing this game, does it seem fair? That is, do both players seem to have an equal chance of winning?

   b. The multiplication table at the left shows the ways that products can occur. For example, the product 12 can occur in two ways: 3 on the first spin and 4 on the second spin; or 4 on the first spin and 3 on the second spin. Complete the table to determine the 16 possible ways products can occur.

Second spin

First spin \ X	1	2	3	4
1				
2				
3				12
4			12	

   c. Using the table and assuming that each product is equally likely to occur, can you make a case that any one of the numbers is more likely to reach the finish line first? Less likely to reach the finish line first?

   ***d.** Based on the table, is this a fair game? Write an explanation to support your conclusion.

2. **Racing Game** This game is similar to the preceding game, but the 1-to-4 spinner is used to obtain one number and the 1-to-6 spinner is used to obtain the second number for each product. Design a game board, and decide which products each player should use so that it is a fair game. Write a convincing argument as to why your game is fair.

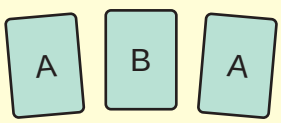
Section 8.2 MULTISTAGE EXPERIMENTS



Photo taken by the National Oceanic and Atmospheric Administration's (NOAA) weather satellite of the Americas showing Hurricane Andrew as it makes landfall on the Louisiana coast.

PROBLEM OPENER

Make three cards of equal size. Label both sides of one card with the letter A, both sides of the second with the letter B, and one side of the third with the letter A and the other side with the letter B. Select a card at random, and place it on a table. There will be either an A or a B facing up. What is the probability that the letter facing down on this card is different from the letter facing up?



Meteorologists use computers and probability to analyze weather patterns. In recent years meteorological satellites have improved the accuracy of weather forecasting. One of the

National Oceanic and Atmospheric Administration's weather satellites sent back the preceding photograph of North and South America showing storms approaching the East Coast of the United States and the Caribbean Islands. Weather forecasts are usually stated in terms of probability, and each probability may be determined from several others. For example, there may be one probability for a cold front and another probability for a change in wind direction. In this section, we will see how to use the probabilities of two or more events to determine the probability of a combination of events.

PROBABILITIES OF MULTISTAGE EXPERIMENTS

In Section 8.1 we studied **single-stage** experiments such as spinning a spinner, rolling a die, and tossing a coin. These experiments are over after one step. Now we will study combinations of experiments, called **multistage experiments.**

Suppose we spin spinner A and then spinner B in Figure 8.7. This is an example of a *two-stage experiment.*

NCTM Standards

Students should also explore probability through experiments that have only a few outcomes, such as using game spinners with certain portions shaded and considering how likely it is that the spinner will land on a particular color. p. 181

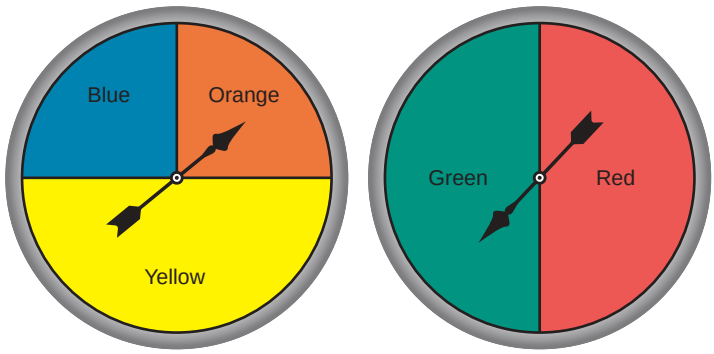


Figure 8.7

The different outcomes for multistage experiments can be determined by constructing *tree diagrams,* which were used in Chapter 3 as a model for the multiplication of whole numbers. Since there are 3 different outcomes from spinner A and 2 different outcomes from spinner B, the experiment of spinning first spinner A and then spinner B has $3 \times 2 = 6$ outcomes (Figure 8.8). This figure illustrates the following generalization.

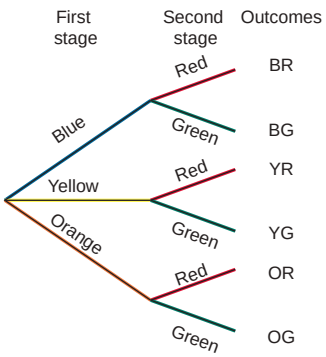


Figure 8.8

Multiplication Principle If event A can occur in m ways and then event B can occur in n ways, no matter what happens in event A, then event A followed by event B can occur in $m \times n$ ways.

The **Multiplication Principle** can be generalized to products with more than two factors. For example, if a third spinner C with 5 outcomes is added to Figure 8.7, then the total number of outcomes for spinning spinner A followed by spinner B followed by spinner C is $3 \times 2 \times 5 = 30$ outcomes.

Figure 8.9 on the next page shows the probabilities of obtaining each color and each outcome from Figure 8.8. Such a diagram is called a **probability tree.** The probability of each of the 6 outcomes can be determined from this probability tree. For example, consider the probability of obtaining BR (blue on spinner A followed by red on spinner B). Since blue occurs $\frac{1}{4}$ of the time on Spinner A and red occurs $\frac{1}{2}$ of the time on Spinner B, the probability of BR is $\frac{1}{4} \times \frac{1}{2}$, or $\frac{1}{8}$. This probability is the product of the two probabilities along the path that leads to BR. Similarly, the probability of YG (yellow followed by green) is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. Notice that the sum of the probabilities for all 6 outcomes is 1.

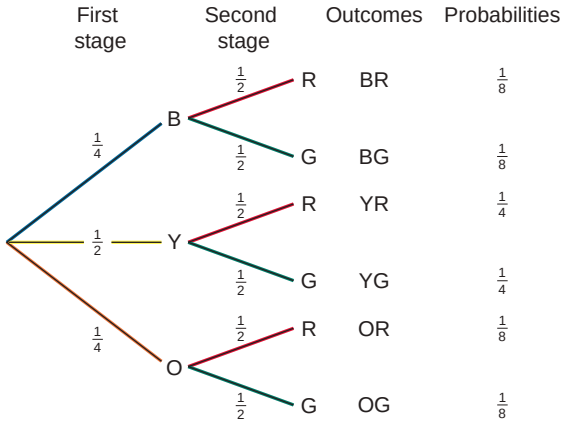


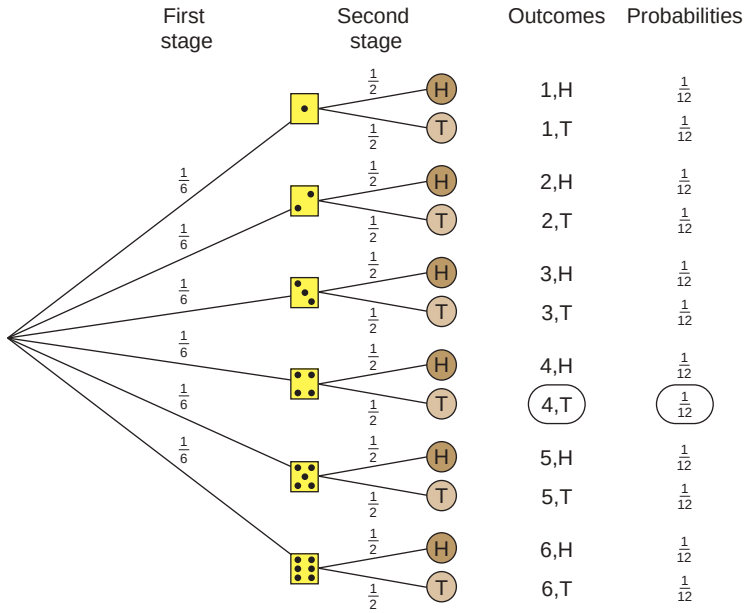
Figure 8.9

The principle of multiplying along the paths of a probability tree can be generalized as follows. *If the outcomes of an experiment can be represented as the paths of a tree diagram, then the probability of any outcome is the product of the probabilities on its path.*

EXAMPLE A

A die is rolled and a coin is tossed. Sketch the probability tree for this experiment, and determine the probability of rolling a 4 on the die and getting a tail on the coin toss.

Solution

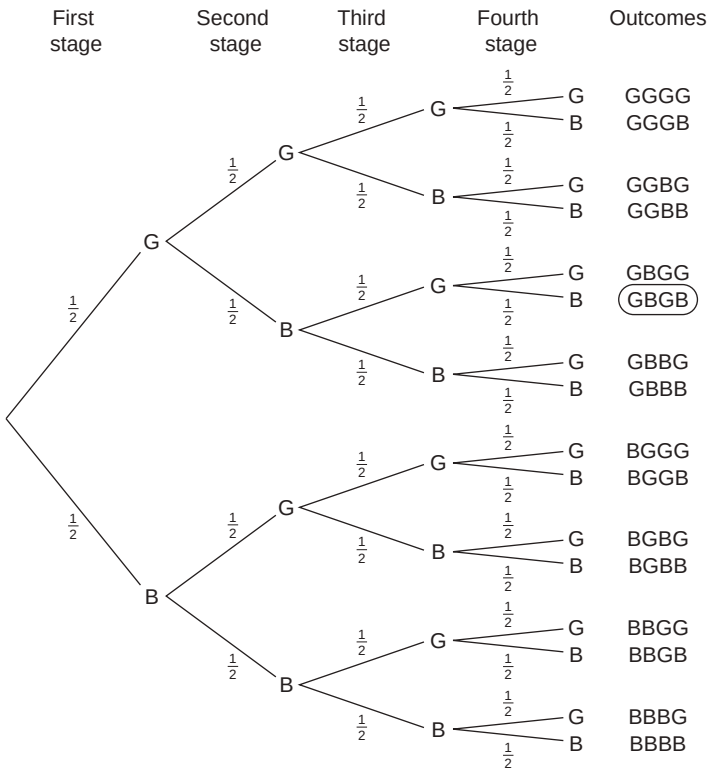


The probability of rolling a 4 and getting a tail is $\frac{1}{12}$. In this two-stage experiment there are $6 \times 2 = 12$ outcomes, and each outcome is equally likely.

An experiment may consist of several stages, as in Example B.

EXAMPLE B What is the probability that the children in a family of 4 children will be born in the order girl, boy, girl, boy?

Solution It is customary to assume that the probability of a baby being a girl is $\frac{1}{2}$ and the probability of a baby being a boy is also $\frac{1}{2}$.



The probability tree shows that all the outcomes are equally likely and that each has a probability of $\frac{1}{16}\left(\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\right)$. Since there is only one outcome with the order GBGB, the probability of a family having a girl, a boy, a girl, and a boy in this order is $\frac{1}{16}$.

PROBABILITIES OF EVENTS

Once probabilities have been assigned to the outcomes of a multistage experiment, the probabilities of specific events can be determined. Using the probability tree in Example B, we can determine the probabilities of several events. For example, let E be the event of a family having 3 girls and 1 boy (in any order). Since there are 4 such outcomes,

$$P(E) = \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{4}{16} = \frac{1}{4}$$

Or if F is the event of a family having 2 girls and 2 boys (in any order),

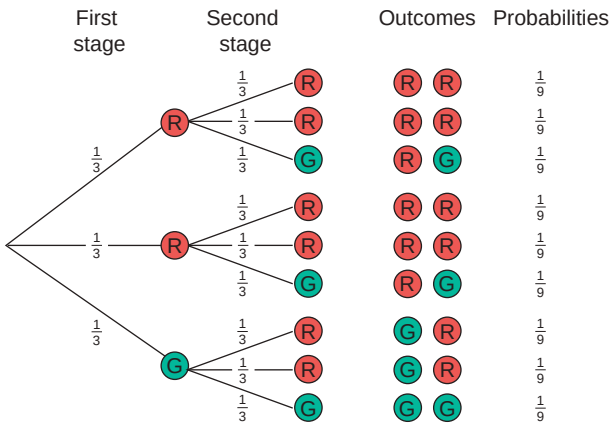
$$P(F) = \frac{6}{16} = \frac{3}{8}$$

since there are 6 outcomes with 2 girls and 2 boys.

EXAMPLE C

A box contains 2 red marbles and 1 green marble. A marble is randomly selected and returned to the box, and a second marble is randomly selected. What is the probability of selecting 2 red marbles?

Solution



The probability tree shows that there are 9 equally likely outcomes, each with a probability of $\frac{1}{9}$. Since there are 4 outcomes with 2 red marbles, the probability of this event is $\frac{4}{9}$.

The probability tree in Example C can be simplified by combining branches. Since we are interested in only whether the first marble is red or green, the first stage of the experiment can be represented by two branches, one with a probability of $\frac{2}{3}$ (selecting a red marble) and one with a probability of $\frac{1}{3}$ (selecting a green marble). Similarly, in the second stage we simply wish to distinguish between selecting red and green (Figure 8.10). Notice that the probability of obtaining 2 red marbles is the product of the probabilities along the top branch: $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$.

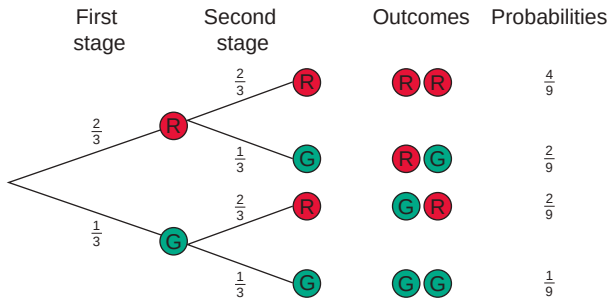


Figure 8.10

INDEPENDENT AND DEPENDENT EVENTS

In the experiment of rolling a die and tossing a coin (see Example A), the outcome from the die does not affect the outcome from the coin. In other words, rolling a die and tossing a coin are independent of each other. If neither of two events affects the probability of the occurrence of the other, we say the two events are **independent.** The stages of the multistage experiments in Examples A through C consist of independent events. The tree diagrams for these experiments show that the probability of an outcome is the product of the

probabilities on its path. These examples suggest the following property for finding the probability of events A and B both occurring:

Multiplication Property If A and B are independent events, then the probability of both events A and B occurring is:

$$P(A \cap B) = P(A) \times P(B)$$

EXAMPLE D

What is the probability of rolling a die and obtaining a 4 and then rolling it a second time and obtaining an even number?

Solution Since the outcome from the first roll does not affect the outcome on the second roll, these events are independent. The probability of obtaining a 4 on one roll of a die is $\frac{1}{6}$, and the probability of obtaining an even number is $\frac{1}{2}$. By the *multiplication property,* the probability of rolling a 4 and then rolling an even number is $\frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$.

The multiplication property enables us to compute the probability of independent events without sketching a probability tree. For example, the top path of the probability tree in Figure 8.10 on page 544 shows the probability of selecting 2 red marbles from a box that contains 2 red marbles and 1 green marble. In this experiment, event A, "obtaining a red marble on the first selection," and event B, "obtaining a red marble on the second selection," are independent events. Thus, the probability of A and B is

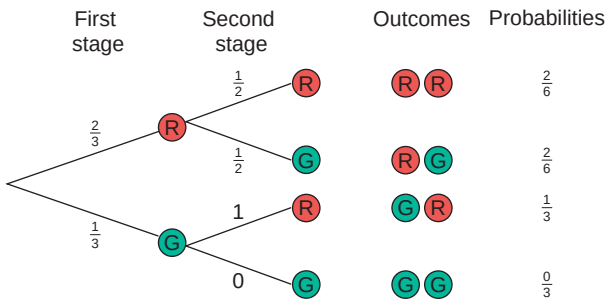
$$P(A) \times P(B) = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$$

Sometimes events are not independent. In Example E, the first marble selected from the box is *not replaced* for the second draw. In this case the event "obtaining a red marble on the first selection" and the event "obtaining a red marble on the second selection" are not independent. When one event affects the probability of the occurrence of the other, the two events are **dependent.**

EXAMPLE E

A box contains 2 red (R) marbles and 1 green (G) marble. A marble is selected at random but not returned to the box, and then a second marble is selected. What is the probability of selecting 2 red marbles?

Solution The probability of selecting a red marble on the first draw is $\frac{2}{3}$. So, the first stage of the probability tree is the same as that in Figure 8.10 on page 544. However, because the first marble is not replaced, the probabilities for the second stage are affected. If a red marble is selected on the first draw, then 1 red marble and 1 green marble are left, so the probability of choosing a red marble on the second draw is $\frac{1}{2}$. The top branches of the following probability tree show that the probability of selecting 2 red marbles in this case is $\frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$.

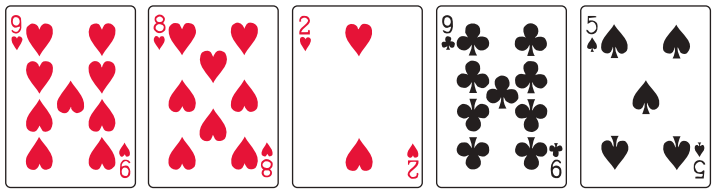


Notice in this example that there is one path leading to RG and one path leading to GR, so the probability of choosing 1 red marble and 1 green marble is $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ when the order in which they are selected is not important. Also, the bottom branch of the tree shows that the probability of selecting 2 green marbles is 0 (there is only 1 green marble in the box).

The probability of dependent events both occurring also can be computed by using the *multiplication property,* as suggested by the probability tree in Example E. However, care must be taken in determining the probability of an event that is affected by the outcome of the other. Examples F and G involve dependent events.

EXAMPLE F

What is the probability of randomly selecting 2 hearts from the 5 cards shown here?



Solution The probability of obtaining a heart on the first draw is $\frac{3}{5}$, and if the first card is a heart, the probability of obtaining a heart on the second draw is $\frac{2}{4}$. The probability of obtaining 2 hearts is $\frac{3}{5} \times \frac{2}{4} = \frac{6}{20} = \frac{3}{10}$.

EXAMPLE G

Consider the events of selecting 1 card from an ordinary deck of 52 cards and then selecting another card without replacing the first. Determine the following probabilities.

1. Selecting 2 clubs **2.** Selecting 2 face cards

3. Selecting 2 aces **4.** Selecting 2 red cards

Research Statement

Research has shown that when sixth and seventh graders were asked to determine conditional probabilities, the performance was dramatically lower when the task involved selection without replacement as compared to selection with replacement.

National Research Council

Solution **1.** There are 13 clubs: $\frac{13}{52} \times \frac{12}{51} = \frac{156}{2652} \approx .059$. **2.** There are 12 face cards: $\frac{12}{52} \times \frac{11}{51} = \frac{132}{2652} \approx .050$. **3.** There are 4 aces: $\frac{4}{52} \times \frac{3}{51} = \frac{12}{2652} \approx .005$. **4.** There are 26 red cards: $\frac{26}{52} \times \frac{25}{51} = \frac{650}{2652} \approx .245$.

The addition property introduced in Section 8.1 can now be refined using the multiplication property:

Addition Property For events A and B, the probability of A or B or both occurring is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If A and B are independent events, then

$$P(A \cup B) = P(A) + P(B) - P(A) \times P(B)$$

EXAMPLE H

A certain city's smog will be above acceptable levels 25 percent of the days, and its pollen will be above acceptable levels 20 percent of the days. Assume S is the event "smog above acceptable levels" and R is the event "pollen above acceptable levels."

1. If S and R are independent events, what is the probability the smog, pollen, or both will be above acceptable levels on any randomly selected day?
2. Suppose records actually show the probability of a day occurring with both smog and pollen above acceptable levels is .08. In this case: (a) Are S and R independent events? (b) What is the probability the smog, pollen, or both will be above acceptable levels on any randomly selected day?

Solution 1. $P(S \cup R) = P(S) + P(R) - P(S) \times P(R) = .25 + .20 - (.25 \times .20) = .40$. 2. $P(S \cap R) = .08 \neq P(S) \times P(R) = .05$; the events are not independent. $P(S \cup R) = P(S) + P(R) - P(S \cap R) = .25 + .20 - .08 = .37$.

The previous example suggests the following test to determine if two events are independent or not.

Test for Independence For events A and B, if

$$P(A \cap B) = P(A) \times P(B)$$

then A and B are independent events.

COMPLEMENTARY EVENTS

There are some problems in which the probability of an event can be most easily found by first computing the probability of its complement.

EXAMPLE I If a die is tossed 4 times, what is the probability of obtaining *at least* one 6?

Solution Let E be the event of obtaining at least one 6 (this includes the possibility of obtaining one, two, three, or four 6s), and let F be the event of not obtaining any 6s. Then E and F are complementary events, and $P(E) + P(F) = 1$. The probability of not obtaining a 6 on 1 roll of a die is $\frac{5}{6}$. So the probability of obtaining no 6s on 4 rolls is

$$P(F) = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{625}{1296} \approx .48$$

and the probability of obtaining at least one 6 is

$$P(E) \approx 1 - .48 = .52$$

That is, slightly more than half of the time you can expect to obtain at least one 6 in 4 tosses of a die.

Since in Example I obtaining at least one 6 includes several different possibilities (obtaining one 6, two 6s, three 6s, or four 6s), it is easier to consider the probability that this will not happen. The words *at least* are sometimes a clue to the fact that the probability of an event may be more easily found by first computing the probability of its complement.

EXAMPLE J 10 teachers have volunteered for a school committee; 7 are women and 3 are men. If 3 of these people are chosen randomly, what is the probability that *at least* 1 person is a man?

Solution The probability that a man will not be chosen (that is, that all 3 people will be women) is $\frac{7}{10} \times \frac{6}{9} \times \frac{5}{8} \approx .29$. Thus, the probability of at least 1 man being chosen is approximately $1 - .29 = .71$.

There is a well-known problem in probability whose solution often surprises people.

What is the smallest randomly chosen group of people for which there is better than a 50 percent chance that at least 2 of them will have a birthday on the same day of the year?

Surprisingly, the answer is only 23 people. In fact, there is a 70 percent probability that among 30 randomly chosen people, 2 will have birthdays on the same day. With 50 randomly chosen people the probability is about 97 percent. The graph in Figure 8.11 shows that in a group of more than 50 randomly chosen people we can be almost certain of finding 2 with birthdays on the same day.

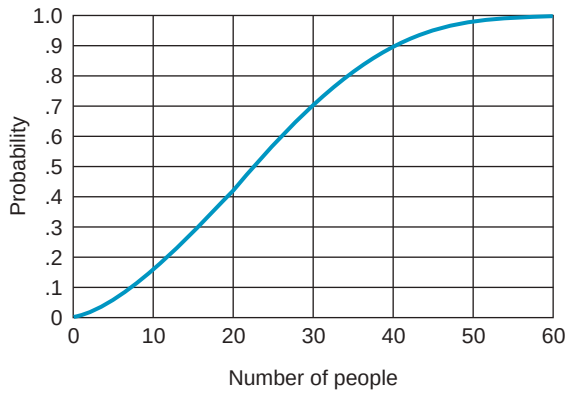


Figure 8.11

The probability that at least 2 out of 23 people have the same birthday can be found by computing the probability of a complementary event. That is, we can determine the probability that all 23 people have birthdays on different days and then subtract this probability from 1. To begin with, consider the problem for just 2 people. No matter when the first person was born, there is a probability of $\frac{364}{365}$ that the second person's birthday will not be on the same day. When a third person joins this group, the probability that his or her birth date will differ from those of the other 2 people is $\frac{363}{365}$. Therefore, the probability that each of 3 people has a different birth date is $\frac{364}{365} \times \frac{363}{365}$. Similarly, the probability that 23 people will have different birthdays is the following product of 22 numbers:

$$\frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \ldots \times \frac{344}{365} \times \frac{343}{365} \approx .49$$

Therefore, the probability that there will be 2 or more people with birthdays on the same day in a group of 23 people is approximately $1 - .49 = .51$. Make a prediction next time you're in a group of 23 or more people. The odds are in your favor that there will be at least 2 people with birthdays on the same day.

PROBLEM-SOLVING APPLICATION

For cases in which it is difficult to find the probability of a multistage experiment, the probability may be approximated by *carrying out a simulation.* At other times a simulation may be used to check on the reasonableness of a calculation that produces a theoretical probability.

Problem

A state lottery has a daily drawing in which 4 ping-pong balls are selected at random from among 10 balls numbered 0, 1, 2, . . . , 9. After a ball is selected, it is returned for the next selection. An elementary school student noticed that quite often two of the four digits drawn are equal. What is the probability that at least two out of four digits will be equal?

Understanding the Problem The condition *at least two* includes the possibilities that there may be two equal digits, three equal digits, or four equal digits.

Devising a Plan One way of determining the probability is to carry out a simulation using the table of random digits in Figure 8.6 (page 532). **Question 1:** How can this be done?

Carrying Out the Plan The first 12 groups of 4 digits (using consecutive digits) from the sixth row of the table in Figure 8.6 are shown here. Notice that six of these groups have two or more equal digits. Continue this simulation through five complete rows of the table. **Question 2:** What is the probability for this simulation?

6886 8492 4016 1401 1046 3862 0491 4880 3384 5266 3902 0063

Looking Back The theoretical probability for this problem can be found by first computing the probability for the complementary event, that is, the probability that all four digits are different. After the first digit is drawn, the probability that the second digit will not equal the first digit is $\frac{9}{10}$; the probability that the third digit will be different from the first two is $\frac{8}{10}$; and the probability that the fourth will be different from the first three is $\frac{7}{10}$. So the probability of selecting four different digits is

$$\frac{9}{10} \times \frac{8}{10} \times \frac{7}{10} = \frac{504}{1000} = .504$$

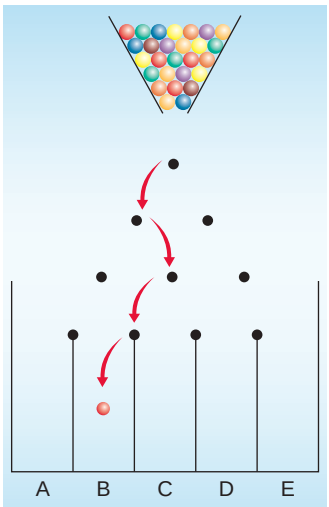
Question 3: What is the theoretical probability that at least two of the four digits drawn will be equal?

Answers to Questions 1–3 **1.** Start with any digit in the table and consider groups of four digits at a time. The number of groups with two or more equal digits divided by the total number of groups is the probability for this simulation. **2.** Since 31 out of 62 groups of four digits have two or more equal digits, the experimental probability for this simulation is $\frac{31}{62} = .5$. 3. The theoretical probability that at least two of the four digits drawn will be equal is $1 - .504 = .496$, or approximately .5.

Laboratory Connection

Probability Machines

A simplified probability machine is illustrated in this figure. As a ball drops and hits each peg, it has a 50 percent chance of going left or right. What is the probability it will fall into compartment B? Explore this and related questions in this investigation.



Mathematics Investigation
Chapter 8, Section 2
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EXPECTED VALUE

To evaluate the fairness of a game, we must consider the prize we can expect to gain as well as the probability of winning. The probability of winning may be fairly small, but if the prize is large enough, the game may be a good risk. Consider the game in Example K.

EXAMPLE K

A game involves rolling two dice. If the player obtains a sum of 7, she or he is paid \$5. Otherwise, the player pays \$1. Over time, can the player expect to gain money, lose money, or break even?

Solution The probability of rolling a sum of 7 is $\frac{1}{6}$. Thus, for 1 out of every 6 rolls, on average, the player can expect to receive \$5. However, for 5 out of 6 rolls, on average, the player can expect to pay \$1 per roll. Thus, over time the player can expect to break even.

The amounts to be won and lost in Example K can be expressed in an equation. Because there is a $\frac{1}{6}$ chance of winning \$5 and a $\frac{5}{6}$ chance of losing \$1, the net winnings from the game can be computed as follows:

$$\frac{1}{6}(5) + \frac{5}{6}(^{-}1) = \frac{5}{6} + \frac{^{-}5}{6} = 0$$

This equation expresses the *expected value* of the game; it is generalized in the following definition.

Expected Value If all possible outcomes of an experiment have values $v_1, v_2, \ldots, v_n$ and the outcomes have probabilities $p_1, p_2, \ldots, p_n$, respectively, then the **expected value** is

$$p_1v_1 + p_2v_2 + \ldots + p_nv_n$$

Sometimes the expected value involves several prizes, and each prize has its own probability of occurring.

EXAMPLE L

The sweepstakes ticket shown here has six hidden dollar amounts that might be any one of the following: \$50, \$20, \$5, or \$1. If you have three identical dollar amounts, you win that amount. There is also a \$10 bonus amount on the ticket if a Sun symbol is revealed. Suppose the probabilities of winning \$50, \$20, \$5, \$1, or the bonus of \$10 are $\frac{1}{500}$, $\frac{1}{200}$, $\frac{1}{50}$, $\frac{1}{10}$, and $\frac{1}{200}$, respectively.

1. What are the expected earnings of this lottery ticket?
2. If each ticket costs \$1, will the player who regularly buys these tickets gain or lose money over time?



Solution **1.** The expected earnings are computed by multiplying the amount of the prize by its probability of occurring and adding these products:

$$\frac{1}{500}(\$50) + \frac{1}{200}(\$20) + \frac{1}{50}(\$5) + \frac{1}{10}(\$1) + \frac{1}{200}(\$10) = \$.45$$

2. Since the expected earnings are \$.45 and each ticket costs \$1, the player will lose money over time. The expected value is a \$.55 loss for each ticket that is purchased for \$1. Notice that since $\frac{1}{500} + \frac{1}{200} + \frac{1}{50} + \frac{1}{10} + \frac{1}{200} = \frac{132}{1000}$ the probability of winning a prize is .132 and the probability of not winning a prize is .868.

A game is called a **fair game** if the expected value is zero. For example, the game in Example K is a fair game; however, the game in Example L is not a fair game. Most gambling games are not fair games.

EXAMPLE M

A roulette wheel has 38 compartments. Two are numbered 0 and 00 and are colored green. The remaining compartments are numbered 1 through 36; one-half of these are red, and one-half are black. With each spin of the wheel, a ball falls into one of the compartments. One way of playing this game is to bet on the red or black color.

1. What is the probability of obtaining a red number on one spin?
2. If a player bets \$1 on red and wins, the player is paid \$1 plus the \$1 the player bet. Otherwise the player loses their \$1. What is the expected value of this game?
3. Is this a fair game?



Solution **1.** The probability of obtaining a red number is $\frac{18}{38} = \frac{9}{19}$. **2.** The expected value of this bet is $\frac{18}{38}(\$1) + \frac{20}{38}(\$^{-}1) \approx {}^{-}\$.05$(or $^{-}5$ cents). **3.** This game is not fair. On average, a player will lose 5 cents on each spin.

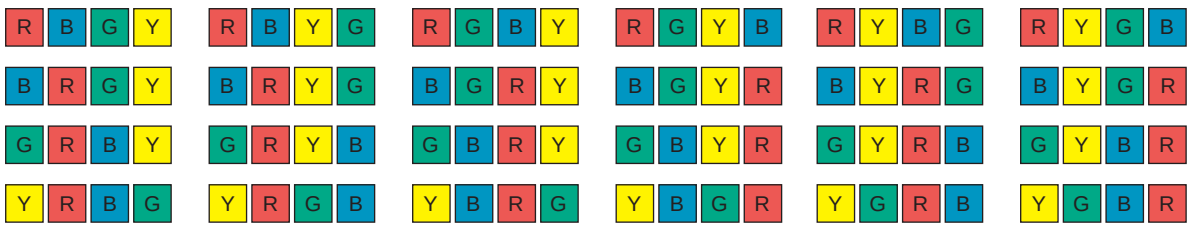
PERMUTATIONS AND COMBINATIONS

In Section 8.1 and the first part of Section 8.2, we were able to determine probabilities by listing the elements of sample spaces and by using tree diagrams. Some sample spaces have too many outcomes to conveniently list, so we will now consider methods of finding the numbers of elements for larger sample spaces.

EXAMPLE N

How many different ways are there to place four different colored tiles in a row? Assume the tiles are red, blue, green, and yellow.

Solution One method of solution is to place the four colored tiles in all possible different orders. There are 24 different arrangements, as shown here.



The *Multiplication Principle* (see page 541) can be used for a more convenient solution. Sketch four blank spaces and imagine placing one of the four tiles in each of the spaces:

1st tile	*2d tile*	*3d tile*	*4th tile*

Any one of the 4 tiles can be placed in the first space; any one of the 3 remaining tiles can be placed in the second space; any one of the 2 remaining tiles can be placed in the third space; and the remaining tile can be placed in the fourth space. By the *Multiplication Principle,* the number of arrangements is $4 \times 3 \times 2 \times 1 = 24$.

Example N illustrates a permutation. A **permutation** of objects is an arrangement of these objects into a *particular order*.

Notice that the solution for Example N involves the product of decreasing whole numbers. In general, for any whole number $n > 0$, the product of the whole numbers from 1 through n is written as $n!$ and called ***n* factorial.** It is usually more convenient to write $n!$ with the whole numbers in decreasing order. For example, in Example N, $4! = 4 \times 3 \times 2 \times 1 = 24$.

Technology Connection

Many calculators have a factorial [!] key. To use this key, enter n, enter [!] and enter =.

***n* factorial**

$$n! = n \times (n - 1) \times \ldots \times 2 \times 1$$

Special case: 0! is defined to be 1.

EXAMPLE O

How many different ways are there to place three different colored tiles chosen from a set of five different colored tiles in a row? Assume the five tiles are red, blue, green, yellow, and orange.

Solution Using the *Multiplication Principle* and three blank spaces:

1st tile	*2d tile*	*3d tile*

any one of 5 tiles can be placed in the first space, any one of the 4 remaining tiles in the second space, and any one of the 3 remaining tiles in the third space. So there are $5 \times 4 \times 3 = 60$ different arrangements or permutations of 5 colored tiles taken 3 at a time. The number of permutations of 3 objects from a set of 5 objects is abbreviated as ${}_5P_3$ and we have shown here that ${}_5P_3 = 60$.

Notice that the solution to Example O can also be expressed as follows by using factorials:

$${}_5P_3 = 60 = 5 \times 4 \times 3 = 5 \times 4 \times 3 \times \frac{2 \times 1}{2 \times 1} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \frac{5!}{2!} = \frac{5!}{(5 - 3)!}$$

This expression is a special case of the following formula for determining the number of permutations of ***n*** objects taken ***r*** at a time:

Technology Connection

Many calculators have a permutation [nPr] key. To use this key, enter n, enter [nPr], enter r and enter =.

Permutation Theorem The number of permutations of n objects taken r objects at a time, where $0 \leq r \leq n$, is

$${}_nP_r = \frac{n!}{(n - r)!}$$

The word *permutations* will usually not be given in the wording of a question, as illustrated in Examples N, O, and P.

EXAMPLE P

In a school soccer league with seven teams, in how many ways can the teams finish in the positions of winner, runner-up, and third place?

Solution In forming all the possible arrangements for the three finishing places, order must be considered. For example, having Team 4, Team 7, and Team 2 in the positions of winner, runner-up, and third place, respectively, is different from having Team 7, Team 2, and Team 4 in these three finishing spots. Using the *Multiplication Principle,* there are 7 possibilities for the winner, and then 6 possibilities are left for the runner-up, and then 5 possibilities are left for the third place. So, there are $7 \times 6 \times 5$ different ways the teams can finish in the positions of winner, runner-up, and third place.

7	×	6	×	5	=	210 possibilities
Winner		Runner-up		Third place		

Using the permutation formula for 7 teams taken 3 at a time,

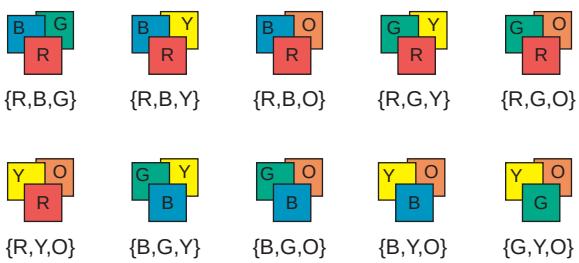
$$ {}_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} = 7 \times 6 \times 5 = 210 $$

In *permutations* the order of the elements is important. However, in forming collections, order is sometimes not important and can be ignored. A collection of objects for which order is not important is called a **combination.** Consider the following example using the five different colored tiles from Example O.

EXAMPLE Q

How many different collections of three tiles can be chosen from a set of five different colored tiles (red, blue, green, yellow, and orange)?

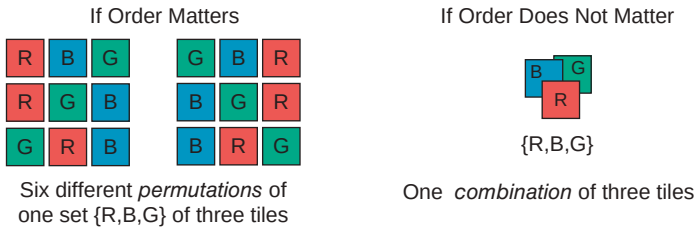
Solution Since order is not important here we can systematically list the different combinations to see there are 10 distinct combinations (each shown here as an unordered set of three tiles).



Note that each collection of three tiles is given as a set to indicate the order of the tiles does not matter.

The number of combinations of 3 things chosen from a collection of 5 objects can be abbreviated as ${}_5C_3$, and we have shown here that ${}_5C_3 = 10$.

It is interesting that the 3 tiles from each of the above combinations can be arranged in $3! = 3 \times 2 \times 1 = 6$ ways. For example, the 3 tile set from the first combination {R, B, G} can be arranged, if order matters, in 6 different permutations:



Six different *permutations* of one set {R,B,G} of three tiles

One *combination* of three tiles

Since each collection of three tiles can be arranged 3! ways this leads to the following observation:

$$3! \times {}_5C_3 = {}_5P_3 \qquad \text{or} \qquad {}_5C_3 = \frac{{}_5P_3}{3!} = \frac{5!}{(5-3)!3!}$$

This expression is a special case of the following formula for determining the number of *combinations* of n things taken r at a time.

Technology Connection

Many calculators have a combination $\boxed{{}_n\text{C}_r}$ key. To use this key, enter n, enter $\boxed{{}_n\text{C}_r}$, enter r, and enter $=$.

Combination Theorem The number of combinations of n objects taken r objects at a time, where $0 \le r \le n$, is

$$_nC_r = \frac{n!}{(n-r)!r!}$$

Examples O and Q show that the number of permutations of 5 objects taken 3 at a time is 6 times the number of combinations of 5 objects taken 3 at a time. In general, for n objects taken r at a time, there will be more permutations than combinations because considering the different orders of objects increases the number of outcomes.

The first step in solving problems involving permutations or combinations is determining whether or not it is necessary to consider the order of the elements. The two questions in Example R will help to distinguish between when to use permutations and when to use combinations.

EXAMPLE R

The school hiking club has 10 members.

1. In how many ways can 3 members of the club be chosen for the Rules Committee?
2. In how many ways can 3 members of the club be chosen for the offices of president, vice president, and secretary?

Solution

1. There is no requirement to consider the order of the people on the 3-person Rules Committee. So, the number of different committees can be found with the formula for combinations.

$${}_{10}C_3 = \frac{10!}{(10-3)!3!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) \times (3 \times 2 \times 1)} = \frac{10 \times 9 \times 8}{6} = 120$$

2. Order must be considered in choosing the three officers because it makes a difference as to who holds each office. So, the number of different possibilities for the three offices can be found with the formula for permutations.

$${}_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = 10 \times 9 \times 8 = 720$$

The following two examples use permutations and combinations to find probabilities. These two examples are very similar, but you may be surprised at their different probabilities.

EXAMPLE S

If 5 students are randomly chosen from a group of 12 students for the offices of president, vice president, secretary, treasurer, and activity director, what is the probability that group members Alice and Tom will be chosen for secretary and activity director, respectively?

Solution For any 5 students who are selected, it matters which students hold the various offices. That is, order is important, so the problem involves permutations.

1. The total number of different permutations of 12 students taken 5 at a time is:

$$_{12}P_5 = \frac{12!}{(12-5)!} = \frac{12\times 11\times 10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{7\times 6\times 5\times 4\times 3\times 2\times 1} = 12\times 11\times 10\times 9\times 8$$

2. If Alice is to be secretary and Tom is to be activity director, the 3 remaining students can be chosen in $_{10}P_3$ different ways.

$$_{10}P_3 = \frac{10!}{(10-3)!} = \frac{10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{7\times 6\times 5\times 4\times 3\times 2\times 1} = 10\times 9\times 8$$

Since there is only 1 way to choose Alice to be secretary and Tom to be activity director, there are

$$1\times {}_{10}P_3 = 1\times 10\times 9\times 8$$

ways to pick officers with Alice as secretary and Tom as activity director.

3. Using the results from 1 and 2 shows that the probability that Alice will be secretary and Tom will be activity director is:

$$\frac{10\times 9\times 8}{12\times 11\times 10\times 9\times 8} = \frac{1}{12\times 11} = \frac{1}{132} \approx .0075 \approx 1\% \text{ to the nearest percent}$$

EXAMPLE T

If 5 students are to be randomly chosen from a group of 12 students to form a committee for a class trip, what is the probability that group members Alice and Tom will be chosen for the committee?

Solution For any 5 students that are selected, the order of the students does not matter. That is, order is not important so the problem involves combinations.

1. The total number of different combinations of 12 students taken 5 at a time is:

$$_{12}C_5 = \frac{12!}{(12-5)!\,5!} = \frac{12\times 11\times 10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{(7\times 6\times 5\times 4\times 3\times 2\times 1)\times(5\times 4\times 3\times 2\times 1)} = \frac{12\times 11\times 10\times 9\times 8}{5\times 4\times 3\times 2\times 1} = 792$$

2. If Alice and Tom are to be on the committee, the 3 remaining students can be chosen in $_{10}C_3$ different ways.

$$_{10}C_3 = \frac{10!}{(10-3)!\,3!} = \frac{10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{(7\times 6\times 5\times 4\times 3\times 2\times 1)\times(3\times 2\times 1)} = \frac{10\times 9\times 8}{3\times 2\times 1} = 120$$

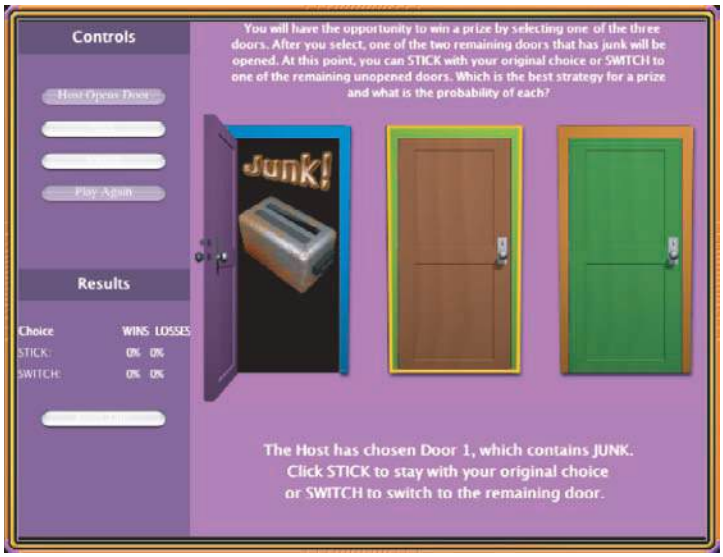
3. Using the results from 1 and 2 shows that the probability that Alice and Tom will be on the committee is:

$$\frac{120}{792} \approx .15 = 15\% \text{ to the nearest percent}$$

Notice that even though Examples S and T are similar, the probability for Example T is approximately 20 times greater than the probability for Example S.

Technology Connection

You may have heard of the TV game show where the contestant picks 1 of 3 doors in hopes there is a prize behind it. The host then opens one of the remaining doors with junk behind it and asks if the contestant wishes to stick with the original choice or switch. Would you STICK or SWITCH? This applet will help you discover the winning strategy. The results may be surprising.



Door Prizes Applet, Chapter 8, Section 2
www.mhhe.com/bbn

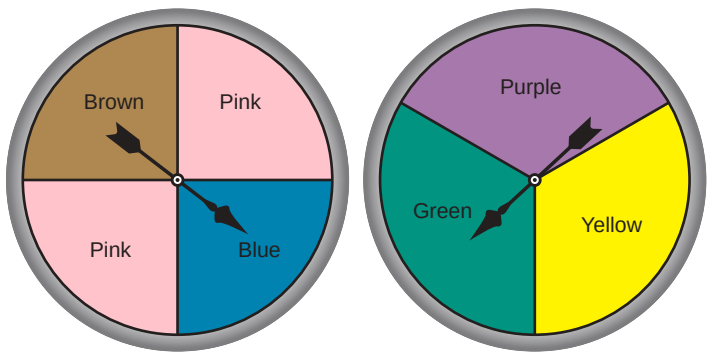
Exercises and Problems 8.2

For a test of 10 true–false questions, determine the probabilities of the events in 1 and 2 if every question is answered by guessing.

1. **a.** Getting the first two questions correct
   b. Getting the first five questions correct
   c. Getting all 10 questions correct

2. **a.** Getting the first three questions correct
   b. Answering the first two questions incorrectly
   c. Answering all the even-numbered questions correctly and all the odd-numbered questions incorrectly

Consider the two-stage experiment with the spinners shown here, spinning first the spinner on the left and then the spinner on the right for exercises 3 and 4.

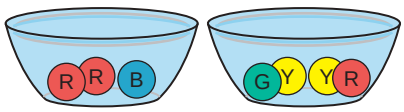


3. **a.** What is the probability of obtaining pink followed by yellow?
   b. What is the probability of obtaining pink or blue followed by yellow?

c. What is the probability of obtaining brown followed by purple or green?

4. a. What is the probability of obtaining blue followed by green?
b. What is the probability of obtaining pink followed by purple?
c. Sketch a probability tree showing all possible outcomes and their probabilities.

Consider the two-stage experiment of randomly selecting a marble from the bowl on the left and then a marble from the bowl on the right. Use this experiment in exercises 5 and 6.



5. a. What is the probability of selecting 2 red marbles?
b. What is the probability of selecting at least 1 red marble?
c. What is the probability of selecting 1 yellow marble?
d. Sketch a probability tree showing all possible outcomes and their probabilities.

6. a. What is the probability of selecting 1 green marble?
b. What is the probability of selecting 1 blue marble?
c. What is the probability of not selecting a red marble?
d. What is the probability of selecting 1 red marble and 1 green marble?

A family has 3 children. Assume that the chances of having a boy or a girl are equally likely in exercises 7 and 8.

7. a. What is the probability that the family has 3 girls?
b. What is the probability that the family has at least 1 boy?
c. What is the probability that the family has at least 2 girls?

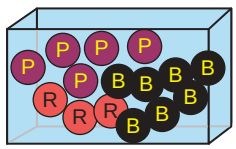
8. a. What is the probability that the family has 2 boys and 1 girl?
b. What is the probability that the family has at least 1 girl?
c. Draw a probability tree showing all possible combinations of boys and girls.

Exercises 9 and 10 use the fact that a fair coin is tossed 4 times.

9. a. What is the probability of obtaining 3 tails and 1 head?
b. What is the probability of obtaining at least 2 tails?
c. Draw a probability tree showing all possible outcomes of heads and tails.

10. a. What is the probability of obtaining 3 heads and 1 tail?
b. What is the probability of obtaining at least 2 heads?
c. What is the probability of obtaining 2 heads and 2 tails?

A box contains 7 black, 3 red, and 5 purple marbles. Consider the two-stage experiment of randomly selecting a marble from the box, replacing it, and then selecting a second marble. Determine the probabilities of the events in exercises 11 and 12.



11. a. Selecting 2 red marbles
b. Selecting 1 red then 1 black marble
c. Selecting 1 red then 1 purple marble

12. a. Selecting 2 black marbles
b. Selecting 1 black then 1 purple marble
c. Selecting 2 purple marbles

13. Suppose that in exercise 11, the first marble selected is not replaced before the second marble is chosen. Determine the probabilities of the events in 11a, b, and c.

14. Suppose that in exercise 12 the first marble is not replaced before the second marble is chosen. Determine the probabilities of the events in 12a, b, and c.

15. a. If you flipped a fair coin 9 times and got 9 heads, what would be the probability of getting a head on the next toss?
b. If you rolled a fair die 5 times and got the numbers 1, 2, 3, 4, and 5, what would be the probability of rolling a 6 on the next turn?

Classify the events in exercises 16 and 17 as dependent or independent and compute their probabilities.

16. a. Tossing a coin 3 times and getting 3 heads in a row
b. Drawing 2 aces from a complete deck of 52 playing cards if the first card selected is not replaced

17. a. Rolling 2 dice and getting a sum of 7 twice in succession
b. Selecting 2 green balls from a bag of 5 green and 3 red balls if the first ball selected is not replaced

Alice and Bill pay one bill each week, and it is determined by their "bill payment spinner." Assume each one of the 12 different outcomes on the spinner is equally likely for determining the probabilities in exercises 18 and 19.



18. **a.** What is the probability of making a fuel payment 2 weeks in a row?
b. What is the probability they will not make an electricity payment this week?
c. If they don't make an electricity payment within the next 3 weeks, their lights will be shut off. What is the probability they will lose their lights?

19. **a.** What is the probability of not making a fuel payment this week?
b. What is the probability of not making a fuel payment and not making an electricity payment this week?
c. If they don't make a fuel payment within the next 2 weeks, they will be charged interest on the outstanding balance of their bill. What is the probability that a payment will not be made within the next 2 weeks?

Determine the probabilities of the events in exercises 20 and 21. (*Hint:* Use complementary events.)

20. **a.** Getting a sum of 7 at least once on 4 rolls of a pair of dice
b. Getting at least one 6 on 4 rolls of a die

21. **a.** Getting at least one sum of 7 or 11 on 3 rolls of a pair of dice
b. Getting a sum of 9 or greater at least once in 5 rolls of a pair of dice

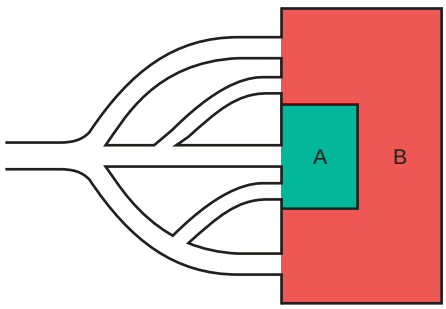
The typical slot machine has three wheels that operate independently of one another. Each wheel has six different symbols that occur various numbers of times, as shown in the chart below. If any one of the winning combinations appears, the player wins money according to the payoff assigned to each combination. Find the probabilities of the events in exercises 22 and 23.

	Wheel 1	Wheel 2	Wheel 3
Cherries	7	7	0
Oranges	3	6	7
Lemons	3	0	4
Plums	5	1	5
Bells	1	3	3
Bars	1	3	1
Totals	20	20	20

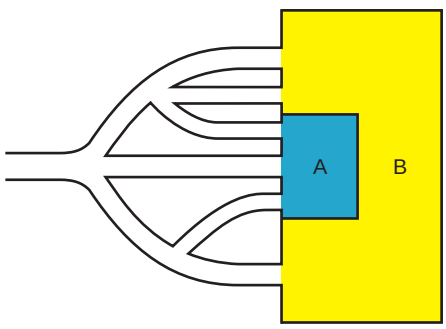
22. **a.** A bar on wheel 1
b. A bar on all three wheels

23. **a.** Bells on wheels 1 and 2 and a bar on wheel 3
b. Plums on wheels 1 and 2 and a bar on wheel 3

24. Assuming that at each branch point in the maze below, any branch is equally likely to be chosen, determine the probability of entering room A.



25. Assuming that at each branch point in the maze below, any branch is equally likely to be chosen, determine the probability of entering room B.



Reasoning and Problem Solving

26. A college student is considering 6 elective courses taught by 6 different professors. She must select 2 of the courses. The student is unaware that 2 of the 6 courses will be taught by professors who have received distinguished teaching awards. List the outcomes of the sample space, and then determine the probabilities of the following events, assuming the student chooses her 2 courses randomly.
- **a.** Not selecting any courses taught by the award-winning professors
- **b.** Selecting exactly 1 course taught by an award-winning professor
- **c.** Selecting at least 1 course taught by an award-winning professor

27. A consumer buys a package of 5 flashbulbs, not knowing that 1 of the bulbs is bad. List the outcomes in the sample space if 2 bulbs are selected randomly, and then find the probabilities of the following events.
- **a.** Both bulbs are good.
- **b.** One of the bulbs is bad.

28. A bureau drawer contains 10 black socks and 10 brown socks. Their wearer is a very early riser who selects the socks in the dark. Find the probabilities of the following events:
- **a.** Selecting 2 socks and having both black
- **b.** Selecting 2 socks and obtaining 1 black and 1 brown
- **c.** Selecting 3 socks and obtaining at least 2 of the same color

29. Mr. and Mrs. Petritz of Butte, Montana, have 5 children who were all born on April 15. Answer the following questions, assuming that it is equally likely that a child will be born on any of the 365 days of the year.
- **a.** After the first Petritz child was born, what was the probability the second child would be born on April 15 if the child was not a twin?
- **b.** If a couple has 1 child on April 15, what is the probability that their next 4 children will be born on April 15 if there are no multiple births?

Use a simulation and complementary events to solve problems 30 and 31. Describe your simulations.

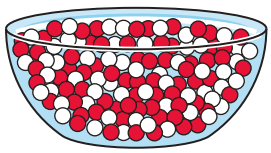
30. A system with three components fails if one or more components fail. The probability that any given component will fail is $\frac{1}{10}$. What is the probability that the system will fail?

31. A manufacturer of bubble gum puts a 5-cent coupon in 1 out of every 5 packages of gum. What is the probability of obtaining at least 1 of these coupons in 4 packages of gum?

32. Suppose there are 3 red, 4 blue, and 5 green chips in a bag and you win by selecting either a red or a green chip. If you get a red chip, you win \$3; a green chip pays \$2; and a blue chip pays \$0.
- **a.** What are the expected earnings of this game?
- **b.** If it costs \$1.50 to play this game, is it a fair game?

33. A game consists of rolling a die; the number of dollars you receive is the number that shows on the die. For example, if you roll a 3, you receive \$3.
- **a.** What are the expected earnings of this game?
- **b.** What should a person pay when playing in order for this to be a fair game?

34. Featured Strategy: Solving a Simpler Problem and Using a Simulation. Two players have invented a game. A bowl is filled with an equal number of white and red marbles. One player, called the *holder,* holds the bowl while the other player, called the *drawer,* is blindfolded and selects two marbles. The drawer wins if both marbles are the same color; otherwise, she loses. Which player has the better chance of winning?



- **a. Understanding the Problem.** Either the two marbles selected will both be white or both be red, or the colors will be different. The drawer feels that she has a chance of winning since there are 3 outcomes and 2 are favorable. Is this true?
- **b. Devising a Plan.** One approach is to simplify the problem and try to solve it for smaller numbers. What is the probability that the drawer will win if there are 3 red and 3 white marbles in the bowl? Another approach is to use a simulation by placing slips of paper representing marbles in a box and drawing them out 1 at a time.
- **c. Carrying Out the Plan.** Who has the better chance of winning this game? What happens to the probability if greater numbers of marbles are used?
- **d. Looking Back.** Suppose the game continues with the drawer selecting 2 marbles at a time until there are no marbles left. The drawer wins a point each time the marbles are the same color and otherwise loses a point. Does this game favor the drawer or the holder? (*Hint:* Try some experiments and determine an experimental probability.)

35. The following four prize amounts are hidden under the $ signs on the lottery ticket: $100, $50, $20, and $1. Each of these four prize amounts is numbered, and if the number of a prize amount matches one of the two winning numbers at the top of the ticket, you win that prize. If a Caboose symbol is revealed, you win $10. The probabilities of winning $100, $50, $20, $1, and $10 are $\frac{1}{1000}$, $\frac{1}{500}$, $\frac{1}{200}$, $\frac{1}{5}$, and $\frac{1}{100}$.

a. What are the expected earnings for one ticket?

b. If each ticket costs $1, is this a fair game?



One way you can bet in roulette is to place a $1 chip on a single number (see page 551). If the ball lands in the compartment with your number, the house pays you 35 chips plus the chip you bet. Use this information in problems 36 and 37.

36. **a.** What is the probability that the ball will land on 13?

b. If each chip is worth $1, what is the expected value in this game?

37. **a.** What is the probability that the ball will not land on 17?

b. Is the expected value for playing a color (as computed on page 551) greater than, less than, or equal to that of playing a particular number?

In an experiment designed to test estimates of probability, people were asked to select one of two outcomes that would be more likely to occur. Determine which outcome in a. or b. in problems 38 and 39 is more likely to occur. *Note:* Each box has only one winning ticket.

38. **a.** Obtaining a winning ticket by drawing once from a box of 10 tickets

b. Obtaining a winning ticket both times by drawing twice with replacement from a box of 5 tickets

39. **a.** Obtaining a winning ticket by drawing once from a box of 10 tickets

b. Obtaining a winning ticket at least once by drawing twice with replacement from a box of 20 tickets (*Hint:* Use complementary events.)

40. An environmental task force estimates that 6 percent of the streams suffer both chemical and thermal pollution, 40 percent suffer chemical pollution, and 30 percent suffer thermal pollution. Are chemical and thermal pollution of the streams independent events?

41. An experimental plane has two engines. The probability that the left one fails is .02, the probability the right one fails is .01, and the probability that neither fails is .98. Are the events "failure of the left engine" and "failure of the right engine" independent events?

42. A deep-sea diver has two independent oxygen systems. Suppose the probability that system *A* works is .9 and the probability that system *B* works is .8. What is the probability that system *A*, system *B*, or both systems will work?

43. A fire alarm in a school has two independent circuits. The alarm will function if one or both of the circuits are working. If the probability that the first circuit is working is .95 and the probability that the second circuit is working is .92, what is the probability that the fire alarm will function?

The following table shows the number of students at a college who were offered teaching positions before graduation. Determine the probability of each event in problems 44 and 45, assuming that one of the students represented in the table is randomly selected.

	Offer	No Offer
Female	104	54
Male	56	36

44. **a.** Student is a female who received a teaching position offer.

b. Student is a male, and he did not receive a teaching position offer.

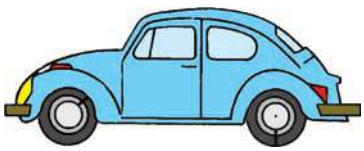
c. Student received a teaching position offer or is a male.

45. **a.** Student is a male who received a teaching position offer.

b. Student is a female, and she did not receive a teaching position offer.

c. Student received a teaching position offer or is a female.

PS In Sweden a motorist was accused of overparking in a restricted-time zone. A police officer testified that this particular parked car was seen with the tire valves pointing to 1 o'clock and to 6 o'clock. When the officer returned later (after the allowed parking time had expired), this same car was there with its valves pointing in the same directions—so a ticket was written. The motorist claimed that he had driven the car away from that spot during the elapsed time, and when he returned later, the tire values coincidentally came to rest in the same positions as before. The driver was acquitted, but the judge remarked that if the positions of the tire valves of all four wheels had been recorded and found to point in the same direction, the coincidence claim would be rejected as too improbable. Assume in problems 46 and 47 that because of variations in tire sizes, the tires will not turn the same amounts.



46. Using the 12-hour positions, determine the probability that two given tire valves of a car will return to their respective earlier positions when the car is reparked.

47. What is the probability that all four tire valves will return to the same positions as before?

Calculate each answer in exercises 48 through 51.

48. **a.** $\frac{12!}{9!}$ **b.** $\frac{10!}{(7! \times 3!)}$

49. **a.** $\frac{15!}{13!}$ **b.** $\frac{9!}{(4! \times 5!)}$

50. **a.** ${}_{12}C_4$ **b.** ${}_{12}P_4$

51. **a.** ${}_{12}C_8$ **b.** ${}_{8}P_3$

52. For the all-state cross-country meet, the coach will select 4 of the 11 top runners for the 4-person relay race. If the runners are assigned the positions of starter, second runner, third runner, and finisher, in how many ways can the relay team be selected?

53. Twelve students attend a meeting for the school play; in how many ways can 4 students be selected for the parts of chauffeur, teacher, coach, and parent?

54. An ice cream shop advertises 21 different flavors; how many different 3-scoop dishes of ice cream can you order?

55. A popular coffee shop has 12 flavors that can be added to a coffee latte. If you chose two flavors with each latte you ordered, how many different two-flavor latte drinks can you order?

In a standard deck of 52 playing cards there are 4 jacks, 4 queens, and 4 kings, called face cards. Assume that being dealt a hand in cards is like selecting those cards at random from the deck. Use this information in questions 56 through 58.

56. Four-card hands
- **a.** How many different 4-card hands are possible from a deck of 52 cards?
- **b.** How many different 4-card hands with 4 face cards are possible from a deck of 52 cards?
- **c.** What is the probability, to four decimal places, of being dealt a 4-card hand of all face cards from a deck of 52 cards?
- **d.** What is the probability, to four decimal places, of being dealt a 4-card hand of all kings from a deck of 52 cards?

57. Five-card hands
- **a.** How many different 5-card hands are possible from a deck of 52 cards?
- **b.** How many different 5-card hands with 5 face cards are possible from a deck of 52 cards?
- **c.** What is the probability, to four decimal places, of being dealt a 5-card hand with all face cards from a deck of 52 cards?
- **d.** What is the probability, to five decimal places, of being dealt a 5-card hand with no face cards from a deck of 52 cards?

58. Write an explanation for each of the following:
- **a.** Why the probability, to four decimal places, of being dealt a 5-card hand with exactly one ace and four kings is $\frac{4}{{}_{52}C_5}$.
- **b.** Why the probability, to four decimal places, of being dealt a 5-card hand with four aces is $\frac{48}{{}_{52}C_5}$.

The names of 10 fifth graders, including the top math and the top spelling student in the fifth grade class are placed in a hat and randomly selected to sit on the stage with the governor for his visit to the school. Use this information in questions 59 and 60.

59. Four students are randomly selected to join the governor.
- **a.** In how many ways can four students be selected for first chair, second chair, third chair, and fourth chair from the governor for this occasion?
- **b.** In how many ways can three more students be selected for the remaining chairs, if the top math student is to sit in the first chair?
- **c.** What is the probability that the top math student will sit in the first chair?

60. Five students are randomly selected to join the governor.

a. In how many ways can 5 students be selected for first chair, second chair, third chair, fourth chair, and fifth chair from the governor for this occasion?

b. In how many ways can three more students be selected for the third, fourth, and fifth chairs, if the top math student is to sit in the first chair and a different student who is the top spelling student is to sit in the second chair?

c. What is the probability, to two decimal places, that the top math student will sit in the first chair and the top spelling student will sit in the second chair?

Teaching Questions

1. In spite of experimental evidence to the contrary, one of your students still maintains that the probability of getting one head and one tail when two coins are dropped is $\frac{1}{3}$. Explain how you think the student has arrived at this conclusion and what you would do to help him reach the correct theoretical probability.

2. When drawing 2 chips from a container with 7 red chips and 3 blue chips, a student argues the following. "The chance of first drawing a red chip is over 50 percent. On a second draw the chance of getting another red chip is still over 50 percent even though I have not replaced the first chip I drew. So, the chance of getting 2 red chips must be more than 50 percent." Do you agree with the student? If not, how would you respond?

3. The students in a class were asked to consider an experiment that involved flipping a coin and rolling a die. They were asked to predict the probability of getting a head on the coin or an odd number on the die or both a head and an odd number. One student said the probability was certainty because the probability of getting a head was $\frac{1}{2}$ and the probability of rolling an odd number was $\frac{1}{2}$ and $\frac{1}{2} + \frac{1}{2} = 1$, which is certainty. How would you respond to the student's reasoning?

4. An elementary school teacher made two identical spinners; each with 10 equal parts and each part contained one of the digits from 0 through 9. The students were carrying out the activity of spinning first one spinner and then the other and adding the two digits. The students soon realized that the smallest sum of 0 and the largest sum of 18 did not occur very often. This led to the question of which of the possible sums would occur most often. The teacher asked each student to make a prediction. Describe an activity or game the students could carry out to answer this question. List some probability questions that could be answered from your activity and/or game.

Classroom Connections

1. The **Research** Statement on page 546 reports a result concerning student performance on conditional probabilities. Read the statement and explain, with examples, why you believe there is a difference in performance between "replacement" and "nonreplacement" conditional probabilities and what makes one more difficult for students than the other. Then suggest how you would teach these concepts to avoid that difficulty.

2. Design a fair *Racing Game,* as described in question 2 of the one-page **Math Activity** at the beginning of this section and explain why the game is fair.

3. In the **Grades 3–5 Standards—Data Analysis and Probability** (see inside front cover) select one of the three recommendations that apply to probability. Briefly describe how you would implement that standard in an elementary classroom by giving an example of an activity or a series of questions you could ask the students.

4. Read the **Standards** statement on page 541. Refer to the spinners on that page and assume that your students understand the probabilities for getting various colors on the individual spinners. Describe how you can structure an activity to help students understand the probability of a two-stage experiment—for example, the probability of getting a certain color on spinner A and another specific color on spinner B.

CHAPTER 8 REVIEW

1. **Probability and expected value**
   - **a.** Any activity such as spinning a spinner, tossing a coin, or rolling a die is called an **experiment.**
   - **b.** The different results that can occur from an experiment are called **outcomes.**
   - **c.** The set of all outcomes is called the **sample space.**
   - **d.** Probabilities determined from conducting experiments are called **experimental probabilities.**
   - **e.** Probabilities determined from ideal experiments are called **theoretical probabilities.**
   - **f.** If there are n equally likely outcomes, then the **probability of an outcome** is $\frac{1}{n}$.
   - **g.** If all the outcomes of a sample space S are equally likely, the **probability of an event E** is
   $$P(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S}$$
   - **h.** If all possible outcomes of an experiment have values $v_1, v_2, \ldots, v_n$ and the outcomes have probabilities $p_1, p_2, \ldots, p_n$, respectively, then the **expected value** of the experiment is
   $$p_1v_1 + p_2v_2 + \ldots + p_nv_n$$
   - **i.** A game is called a **fair game** if the expected value of the game is zero.

2. **Events and probabilities**
   - **a.** Any subset of an outcome is called an **event.**
   - **b.** If an event is the empty set, it is called an **impossible event** and its probability is 0.
   - **c.** If an event contains all possible outcomes, it is called a **certain event** and the probability of the event is 1. Therefore $0 \le P(E) \le 1$ for any event E.
   - **d.** The **probability of an event** is the sum of the probabilities of its outcomes.
   - **e.** An event that can be described in terms of the union, intersection, or complement of other sets is called a **compound event.**
   - **f. Addition property** If events A and B are not disjoint, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. If events A and B are disjoint, they are called **mutually exclusive events.** In this case $P(A \cup B) = P(A) + P(B)$.
   - **g.** If events A and B are complementary sets, they are called **complementary events.** In this case $P(A) + P(B) = 1$.

3. **Odds and simulations**
   - **a.** The **odds in favor** of an event are the ratio, n to m, of the number of favorable outcomes n to the number of unfavorable outcomes m. The probability of this event's occurring is $\frac{n}{(n+m)}$.
   - **b.** The **odds against** an event are the ratio, m to n, of the number of unfavorable outcomes m to the number of favorable outcomes n. The probability of this event's not occurring is $\frac{m}{(n+m)}$.
   - **c. Simulations** (used in Chapter 7 for statistical experiments) are also used to obtain approximations to theoretical probabilities.
   - **d.** The **law of large numbers:** The more times a simulation is carried out, the closer the experimental probability is to the theoretical probability.

4. **Single and multistage experiments**
   - **a.** An experiment that is over after one step such as spinning a spinner, rolling a die, or tossing a coin is a **single-stage** experiment. Combinations of experiments such as spinning a spinner and then rolling a die are called **multistage experiments.**
   - **b.** A tree diagram showing the outcomes of an experiment and their probabilities is called a **probability tree.**
   - **c.** If A and B are two events and the probability of B is not affected by the occurrence of event A, then these events are called **independent events:** otherwise, when one event affects the probability of the occurrence of the other, they are called **dependent events.**
   - **d. Multiplication property.** If A and B are independent events, then $P(A \cap B) = P(A) \times P(B)$.
   - **e.** If $P(A \cap B) = P(A) \times P(B)$, then A and B are independent events.

5. **Multiplication Principle, permutations and combinations**
   - **a. Multiplication Principle.** If event A can occur in m ways and then event B can occur in n ways, no matter what happens in event A, then event A followed by event B can occur in $m \times n$ ways.
   - **b.** The product of the whole numbers from 1 through n is written as $n!$ and called ***n* factorial.**
   - **c.** A **permutation of objects** is an arrangement of these objects into a particular order.
   - **d. Permutation theorem.** The number of permutations of n objects taken r objects at a time, where $0 \le r \le n$, is
   $${}_nP_r = \frac{n!}{(n-r)!}$$
   - **e.** A collection of objects for which order is not important is called a **combination.**
   - **f. Combination theorem.** The number of combinations of n objects taken r objects at a time, where $0 \le r \le n$, is
   $${}_nC_r = \frac{n!}{(n-r)!r!}$$

CHAPTER 8 TEST

1. A box contains six tickets lettered A, B, C, D, E, and F. Two tickets will be randomly selected from the box (without replacement).
- **a.** List all the outcomes of the sample space.
- **b.** What is the probability of selecting tickets A and B?
- **c.** What is the probability that one of the tickets will be ticket A?

2. A chip is selected at random from a box that contains 3 blue chips, 4 red chips, and 5 yellow chips. Determine the probabilities of selecting each of the following.
- **a.** A red chip
- **b.** A red chip or a yellow chip
- **c.** A chip that is not red

3. A box contains 3 green marbles and 2 orange marbles. An experiment consists of randomly selecting 2 marbles from the box (without replacement).
- **a.** List all the outcomes of the sample space.
- **b.** What is the probability of obtaining 2 green marbles?
- **c.** What is the probability of obtaining 2 orange marbles?
- **d.** What is the probability of obtaining 1 green and 1 orange marble?

4. The odds of a certain bill passing through a state senate are 7 to 5.
- **a.** What are the odds of the bill not passing?
- **b.** What is the probability that the bill will be passed?

5. The names *Eben, Evelyn, Eunice, Frieda,* and *Frank* are to be randomly chosen, and each has the same probability of being selected. Use events E, F, G, and H to determine the probabilities in parts a to d.

E: Selecting a name with first letter E
F: Selecting a name with first letter F
G: Selecting a name with fewer than six letters
H: Selecting a name with six letters

- **a.** $P(E \cup F)$
- **b.** $P(E \cap H)$
- **c.** $P(E \cup G)$
- **d.** $P(F \cap G)$

6. A box contains 4 red marbles and 2 yellow marbles. Consider the two-stage experiment of randomly selecting a marble from a box, replacing it, and then selecting a second marble. Determine the probabilities of the following events.
- **a.** Selecting 2 red marbles
- **b.** Selecting a red marble on the first draw and a yellow marble on the second
- **c.** Selecting at least 1 yellow marble

7. Suppose that in exercise 6 the first marble that is selected is not replaced. Determine the probabilities of the events in 6a, b, and c.

8. A family has 4 children.
- **a.** Draw a probability tree showing all possible combinations of boys and girls.
- **b.** What is the probability of the family having 2 boys and 2 girls?
- **c.** What is the probability of the family having at least 2 girls?

9. A contestant on a quiz show will choose 2 out of 7 envelopes (without replacement). If 2 of the 7 envelopes each contain $10,000, what is the probability the contestant will win at least $10,000?

10. The manufacturer of a certain brand of cereal puts a coupon for a free box of cereal in 20 percent of its boxes. If 3 boxes are purchased, what is the probability of obtaining at least 1 coupon?

11. Players in a die-toss game using a regular die with numbers 1 through 6 can win the following amounts: $2 for an even number; $1 for a 1; $3 for a 3; and $5 for a 5.
- **a.** What are the expected earnings of the game?
- **b.** In order for this to be a fair game, what should it cost to play?

12. A certain system fails to operate if any one of four relays overloads. The probability of a relay's overloading is .01. What is the probability that the system will fail?

13. An athlete enters three track and field events. She has a .9 probability of winning the 100-meter dash, a .9 probability of winning the low hurdles, and a .8 probability of winning the long jump.
- **a.** What is the probability that the athlete will win the hurdles and the long jump?
- **b.** What is the probability that she will win all three events?
- **c.** What is the probability that she will win at least 1 of the 3 events?

14. A school's computer cluster has access to two mainframe computers. The probability that the first mainframe can be accessed is .9 and the probability that the second mainframe can be accessed is .8. If the probability of accessing at least one of the mainframe computers is .98, do the mainframe computers operate independently of each other?

15. Use the information in this table to answer the questions below.

Educational Attainment of People over 24 Years Old (in millions)*

	Female	Male
Not a high school graduate	13	13.3
High school graduate without college	31.7	29.5
High school graduate with some college	27.8	23.2

a. If a person over 24 years old is chosen at random, what is the probability to the nearest .01 that the person is a high school graduate without college education or not a high school graduate?

b. What is the probability to the nearest .01 that a randomly chosen person over 24 years of age is a female or a high school graduate with some college education?

16. A family of a mother, father, older sister, brother, and younger sister will be randomly assigned seats A, B, C, D, and E, in a row, for their flight to Chicago. Seat A is next to the window.

a. In how many different ways can the family be assigned seats?

b. In how many different ways can the family be assigned seats, if the older sister is assigned Seat A next to the window?

c. What is the probability that the older sister will be assigned Seat A?

d. What is the probability that the older sister will not be assigned Seat A?

17. A jar contains 45 balls numbered 1 to 45.

a. How many different sets of 5 balls can be randomly taken from the jar?

b. How many different sets of 5 balls containing the ball numbered 42 can be taken from the jar?

c. What is the probability that a 5-ball set will contain the ball numbered 42?

d. What is the probability that a 5-ball set will not contain the ball numbered 42?

**Statistical Abstract of the United States:* 128th ed. (Washington: Bureau of the Census, 2009), Table 226.

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CHAPTER

9 Geometric Figures

Spotlight on Teaching

Excerpts from NCTM's Standard for School Mathematics Grades 3–5*

Students in grades 3–5 can explore shapes with more than one line of symmetry. For example:

> In how many ways can you place a mirror on a square so that what you see in the mirror looks exactly like the original square? Is this true for all squares?
>
> Can you make a quadrilateral with exactly two lines of symmetry? One line of symmetry? No lines of symmetry? If so, in each case, what kind of quadrilateral is it?

Although younger students often create figures with rotational symmetry with, for example, pattern blocks, they have difficulty describing the regularity they see. In grades 3–5, they should be using language about turns and angles to describe designs such as the one in Figure 9.A: "If you turn it 180 degrees about the center, it's exactly the same" or "It would take six equal small turns to get back to where you started, but you can't tell where you started unless you mark it because it looks the same after each small turn."

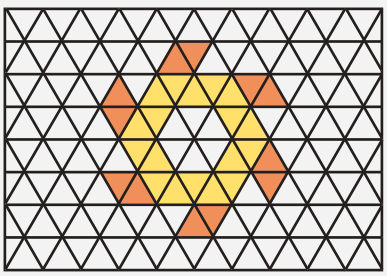


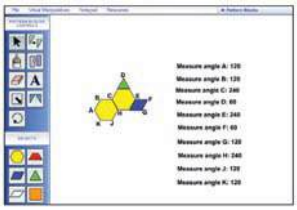
Figure 9.A
Pattern with rotational symmetry.

**Principles and Standards for School Mathematics,* p. 168.

MATH ACTIVITY 9.1

Angles in Pattern Block Figures

Virtual Manipulatives

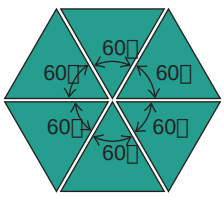


www.mhhe.com/bbn

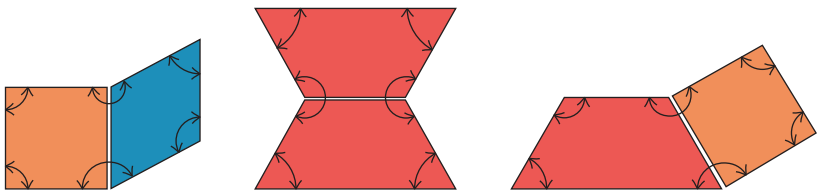
Purpose: Use pattern blocks to determine sums of interior angles in polygons.

Materials: Pattern Blocks in the Manipulative Kit or Virtual Manipulatives.

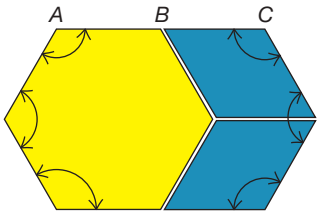
***1.** Each of the three angles of a pattern block triangle has a measure of 60°. This can be shown by placing the *vertices* (corners) of six triangles at a point and using the fact that there are 360° in a circle (see figure at left below). Since 6 angles meet at a point, there are $360 \div 6 = 60$ degrees in each angle. Determine the measures of the different interior angles of each of the pattern blocks, and draw sketches or trace pattern blocks to explain your reasoning.



2. Each of the following pattern block figures has six sides **(hexagon)** and six interior angles.

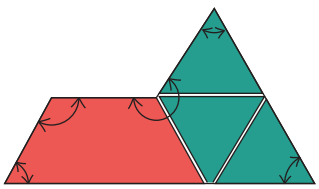


a. Find the measure of each interior angle, and compute the sum of the measures of all the interior angles of each polygon above. Form a conjecture about the sum of the measures of the interior angles of a hexagon.



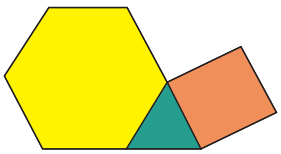
b. The figure at the left that is formed with a hexagon and two parallelograms also has six sides. These sides meet in six interior angles which have been marked. Notice that the edges of the pattern blocks from *A* to *C* lie on a straight line, so these edges are counted as only one side. Also, point *B* is not the vertex of an interior angle of the hexagon, because it is not the intersection of two sides. Find the measures of the interior angles of this figure. Does the sum of the measures of these angles support your conclusion in part a?

3. The figure at the left that is formed with a trapezoid and three triangles has five sides **(pentagon)** and five interior angles. Find the sum of the measures of the interior angles.



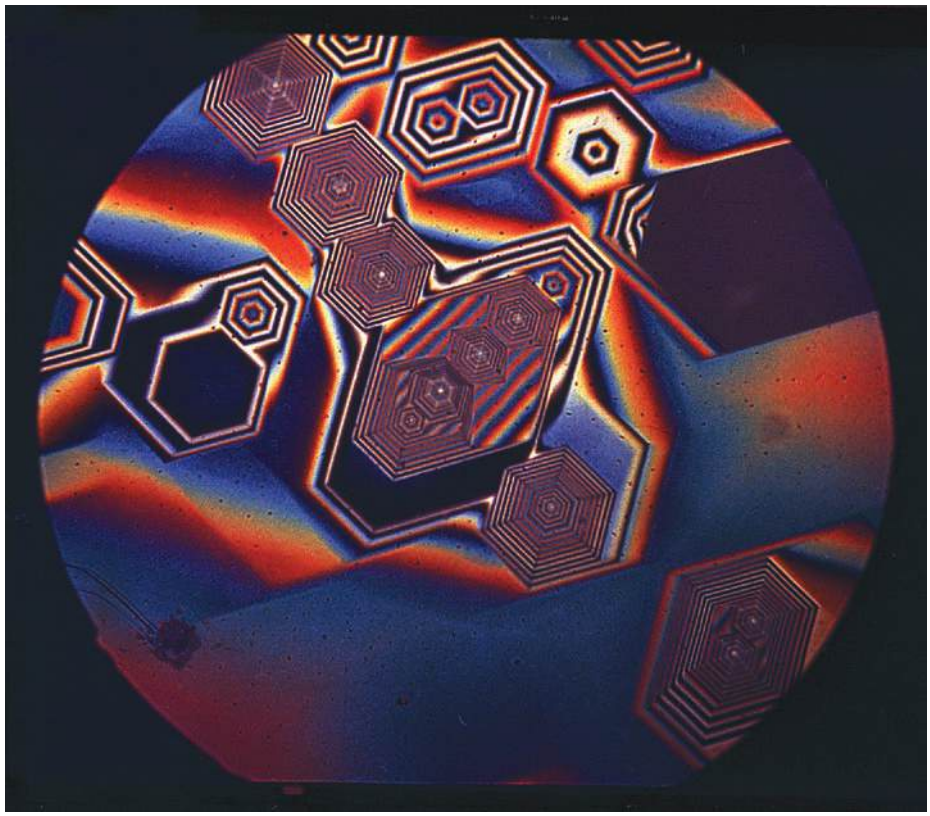
a. Use your pattern blocks to form other figures with five sides, and determine the sums of the measures of their interior angles. (*Suggestion:* It may help you to outline the five sides with bold lines as was done for the figure.)

b. Form a conjecture about the sum of the measures of the interior angles of a pentagon.



4. The figure at the left that is formed with a hexagon, triangle, and square has eight sides and eight interior angles. Use arcs to mark these eight angles. What is the sum of their measures? Use your pattern blocks to form other figures with various numbers of sides. Form a conjecture about the sum of the measures of the interior angles of a polygon, given the number of sides.

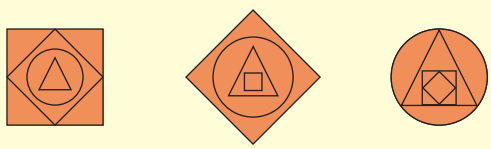
Section 9.1 PLANE FIGURES



Cross section of cadmium sulfide crystals—hexagons formed by nature

PROBLEM OPENER

Find a pattern in the three figures, and draw the next two figures according to the pattern.*



NCTM Standards

Geometry is more than definitions; it is about describing relationships and reasoning. p. 40

We have become so accustomed to hearing about the regularity of patterns in nature that we often take it for granted. Still, it is a source of wonder to see figures with straight edges and uniform angles, such as those in the preceding photograph, occurring in nature.

The study of relationships among lines, angles, surfaces, and solids is a major part of geometry, one of the earliest branches of mathematics. The word **geometry** is from the Latin *geometria,* which means *earth-measure.*

*"Problems of the Month," *Mathematics Teacher,* 80: 550.

MATHEMATICAL SYSTEMS

More than 5000 years ago, the Egyptians and Babylonians were using geometry in surveying and architecture. These ancient mathematicians discovered geometric facts and relationships through experimentation and inductive reasoning. Because of their approach, they could never be sure of their conclusions, and in some cases their formulas were incorrect. The ancient Greeks, on the other hand, viewed points, lines, and figures as abstract concepts about which they could reason deductively. They were willing to experiment in order to formulate ideas, but final acceptance of a mathematical statement depended on proof by deductive reasoning. The Greeks' approach was the beginning of mathematical systems.

A **mathematical system** consists of *undefined terms, definitions, axioms,* and *theorems.* There must always be some words that are undefined. *Line* is an example of an undefined term in geometry. We all have an intuitive idea of what a line is, but trying to define it involves more words, such as *straight, extends indefinitely,* and *has no thickness.* These words would also have to be defined. To avoid this problem of *circularity,* certain basic words such as *point* and *line* are **undefined terms.** These words are then used in **definitions** to define other words. Similarly, there must always be some statements, called **axioms,** that we assume to be true and do not try to prove. Finally, the axioms, definitions, and undefined terms are used together with deductive reasoning to prove statements called **theorems.**

Theorems
↑
Undefined terms
Definitions
Axioms

HISTORICAL HIGHLIGHT



Euclid, ca. 350 B.C.E.

The crowning achievement of Greek mathematical reasoning was *Euclid's Elements,* a series of 13 books written about 300 B.C.E. These books contain over 600 theorems, which were obtained by deductive reasoning from 10 basic assumptions called axioms. Although much of the material was drawn from earlier sources, the superbly logical arrangement of the theorems displays the genius of the author. *Euclid's Elements* stood as a model of deductive reasoning for over 2000 years, and few books have been more important to the thought and education of the western world.*

*D. M. Burton, *The History of Mathematics,* 7th ed. (New York: McGraw-Hill, 2010), pp. 143–170.

POINTS, LINES, AND PLANES

One fundamental notion in geometry is that of a *point.* All geometric figures are sets of points. **Points** are abstract ideas, which we illustrate by dots, corners of boxes, and tips of pointed objects. These concrete illustrations have width and thickness, but points have no dimensions. The following description of a point, from *Mr. Fortune's Maggot,* by Sylvia Townsend Warner, indicates some of the problems associated with teaching elementary school children the concept of a point.†

†Quoted in J. R. Newman, *The World of Mathematics,* 4th ed., p. 2254.

Calm, methodical, with a mind prepared for the onset, he guided Lueli down to the beach and with a stick prodded a small hole in it.

"What is this?"

"A hole."

"No, Lueli, it may seem like a hole, but it is a point."

Perhaps he had prodded a little too emphatically. Lueli's mistake was quite natural. Anyhow, there were bound to be a few misunderstandings at the start. He took out his pocket knife and whittled the end of the stick. Then he tried again.

"What is this?"

"A smaller hole."

"Point," said Mr. Fortune suggestively.

"Yes, I mean a smaller point."

"No, not quite. It is a point, but it is not smaller. Holes may be of different sizes, but no point is larger or smaller than another point."

A **line** is a set of points that we describe intuitively as being "straight" and extending indefinitely in both directions. The edges of boxes and taut pieces of string or wire are models of lines. The line in Figure 9.1 passes through points A and B and is denoted by $\overleftrightarrow{AB}$. The arrows indicate that the line continues indefinitely in both directions. If two or more points are on the same line, they are called **collinear.**



Figure 9.1

A plane is another set of points that is undefined. We describe a **plane** as being "flat" like the top of a table, but extending indefinitely. The surfaces of floors and walls are other common models for portions of planes. A plane can be illustrated by a drawing that uses arrows, as in Figure 9.2, to indicate that it extends and is not bounded.

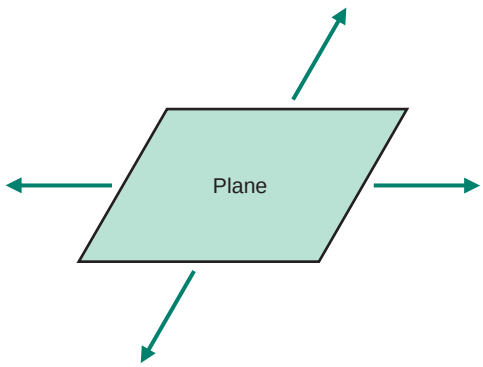


Figure 9.2

EXAMPLE A A standard sheet of paper is a model for part of a plane.

1. What part of a sheet of paper might be used as a model for a line?
2. What part of a sheet of paper might be used as a model for a point?
3. How can models of lines and points be obtained by folding a sheet of paper?

Solution **1.** Each edge of the paper is a model for part of a line. **2.** Each corner of the paper is a model for a point. **3.** The crease made by folding a sheet of paper is a model for part of a line. Two folds can produce parts of two lines that intersect in a point.

Points, lines, and *planes* are undefined terms in geometry that are used to define other terms and geometric figures. The following paragraphs contain some of the more common definitions and examples of figures that occur in planes.

HALF-PLANES, SEGMENTS, RAYS, AND ANGLES

Half-Planes A line in a plane partitions the plane into three disjoint sets: the points on the line and **two half-planes.** Line ℓ in Figure 9.3 partitions the plane into half-planes with point A in one half and point B in the other.

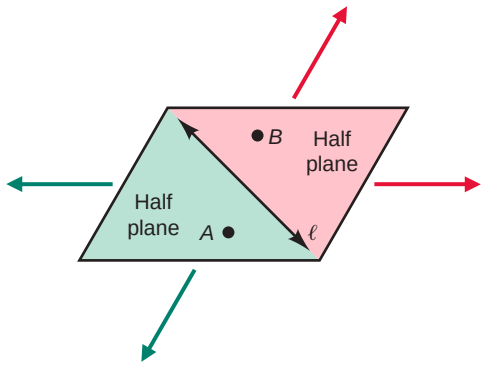


Figure 9.3

Line Segments A **line segment** consists of two points on a line and all the points between them (Figure 9.4). The line segment with **endpoints** A and B is denoted by $\overline{AB}$. It is customary to refer to the length of a line segment by removing the bar above the letters. So if segment $\overline{AB}$ is equal in length to segment $\overline{BC}$, we write $AB = BC$. To **bisect** a line segment means to divide it into two parts of equal length. The **midpoint** C bisects $\overline{AB}$.

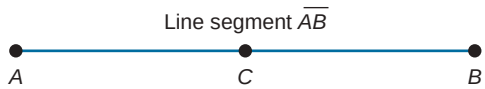


Figure 9.4

Half-Lines and Rays A point on a line partitions the line into three disjoint sets: the point and two **half-lines.** Figure 9.5a shows two half-lines that are determined by point P. A **ray** consists of a point on a line and all the points in one of the half-lines determined by the point. The ray in part b, which has D as an *endpoint* and contains point E, is denoted by $\overrightarrow{DE}$.

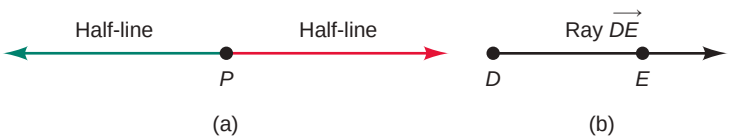


Figure 9.5

Angles An **angle** is formed by the union of two rays, with a common endpoint, as shown in Figure 9.6a, or by two line segments that have a common endpoint, as in part b. This endpoint is called the **vertex,** and the rays or line segments are called the **sides of the angle.** The angle with vertex G, whose sides contain points F and H, is denoted by $\measuredangle FGH$. Sometimes it is convenient to identify an angle by the letter of its vertex, such as $\measuredangle G$ in part a, or by a numeral, such as $\measuredangle 1$ in part b. If two angles have the same measure, we write, for example, $m(\measuredangle 1) = m(\measuredangle 2)$.

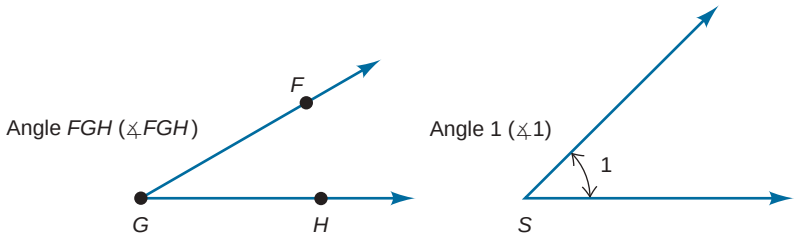


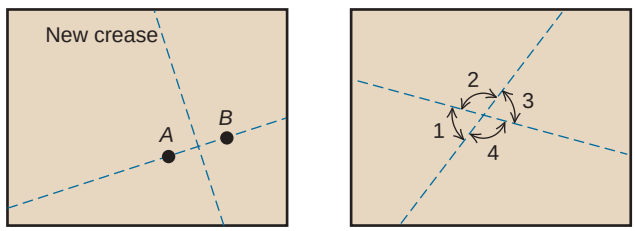
Figure 9.6

EXAMPLE B Fold a standard sheet of paper to create models of the following terms.

1. Parts of two opposite half-planes **2.** A bisected line segment

3. Part of a ray **4.** An angle

Solution **1.** Any crease creates two half-planes. **2.** Fold the paper to obtain a crease and draw a line in the crease, as shown in figure (a). Select two points A and B on the line, and fold the line onto itself so that point A coincides with point B. The point where the new crease intersects segment $\overline{AB}$ is the midpoint that bisects $\overline{AB}$ into two segments. **3.** Any crease creates a line, and selecting a point on the line determines two rays. **4.** Any two folds that form creases that intersect in a point create four angles having the point as a vertex. Figure (b) shows angles 1, 2, 3, and 4.



PROBLEM-SOLVING APPLICATION

The ability to determine the number of line segments whose endpoints are a given number of points has many practical applications. One of these became evident in the early days of the development of the telephone system. The fundamental problem was how to connect two people who wanted to talk. This was done by connecting cords and plugs for each pair of people. In 1884, Ezra T. Gilliland devised a mechanical system that would allow 15 subscribers to reach one another without the aid of an operator.

Problem

How many line segments are needed to connect 15 points in a plane so that each pair of points are the endpoints of a line segment?

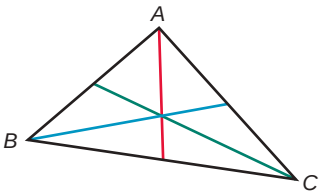
Understanding the Problem One line segment connects 2 points, and 3 line segments connect 3 points. **Question 1:** How many line segments are needed to connect 4 points?

Devising a Plan Let's examine a few more special cases. Perhaps the strategies of *solving a simpler problem* and *finding a pattern* will lead to a solution. In the figure on the next page, 6 line segments have the points A, B, C, and D as endpoints. **Question 2:** How many new

Technology Connection

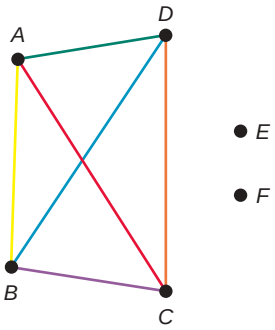
Properties of Triangles

If each vertex of a triangle is connected to the midpoint of the opposite side of the triangle, will the areas of the six smaller triangles ever be equal? This and similar questions are explored using Geometer's Sketchpad® student modules available at the companion website.



Mathematics Investigation
Chapter 9, Section 1
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line segments are needed to connect E to each of these 4 points, and what is the total number of line segments connecting the 5 points?



Carrying Out the Plan Placing a sixth point, F, in the diagram, we can see that there will be 5 new line segments from F to the other points and a total of 15 line segments for the 6 points. Find a pattern and complete the following table. **Question 3:** How many line segments are required to connect 15 points?

No. of points	2	3	4	5	6	7	8	9	10	15
No. of segments	1	3	6	10	15					

Looking Back You probably recognize the numbers 1, 3, 6, 10, 15, etc., in the table as triangular numbers (Chapter 1). Note that the first triangular number is associated with 2 points, the second with 3 points, etc. The formula for the nth triangular number is $\frac{n(n+1)}{2}$. Using this formula, you can determine the number of line segments needed to connect 20 points so that the points in each pair are the endpoints of a line segment. **Question 4:** What is the number?

Answers to Questions 1–4 **1.** 6 **2.** There will be 4 new line segments and a total of 10 line segments for the 5 points. **3.** 105 **4.** The number of line segments needed to connect 20 points is the 19th triangular number: $\frac{(19 \times 20)}{2} = 190$.

ANGLE MEASUREMENTS

NCTM Standards

Students in grades 3–5 can begin to establish some benchmarks by which to estimate or judge the size of objects. For example, they can learn that a "square corner" is called a *right angle* and establish this as a benchmark for estimating the size of other angles. p. 172

The ancient Babylonians devised a method for measuring angles by dividing a circle into 360 equal parts, called **degrees.** One degree (1°) is $\frac{1}{360}$ of a complete turn about a circle, as shown in Figure 9.7. Each degree can be divided into 60 equal parts, called **minutes,** and

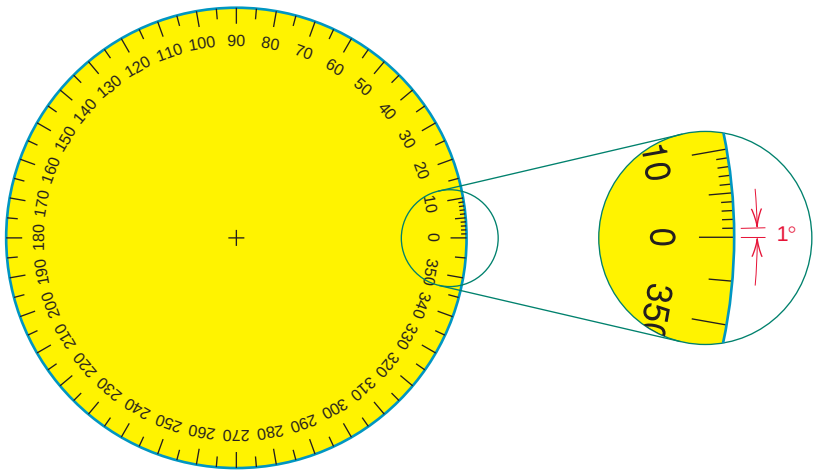


Figure 9.7

each minute can be divided into 60 equal parts, called **seconds.** This is the origin of the modern practice of dividing hours into minutes and seconds.

A **protractor** is a device for measuring angles (Figure 9.8). To **measure** an angle, place the center of the protractor on the vertex of the angle (B in this example), and line up one side of the angle ($\overline{BC}$) with the baseline of the protractor. The protractor in Figure 9.8 shows that $\measuredangle ABC$ has a measure of approximately 60°.

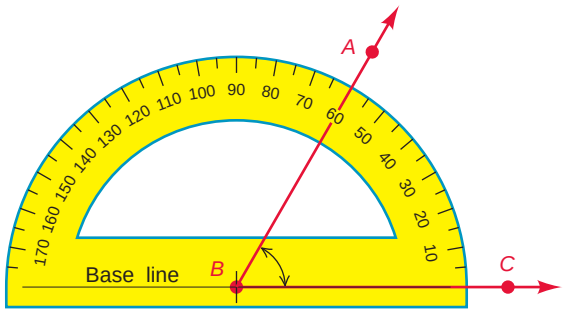


Figure 9.8

If an angle has a measure of 90°, as in Figure 9.9a, it is called a **right angle;** if it is less than 90°, and greater than 0°, as in part b, it is called an **acute angle;** if it is greater than 90° and less than 180°, as in part c, it is called an **obtuse angle;** and if it has a measure of 180° it is called a **straight angle.** It is customary to draw ⌝ at the vertex of a right angle. Occasionally we use angles with measures of more than 180° and less than 360°, as shown in Figure 9.9d. Such an angle is called a **reflex angle.** To indicate a reflex angle, we draw a circular arc to connect the two sides of the angle.

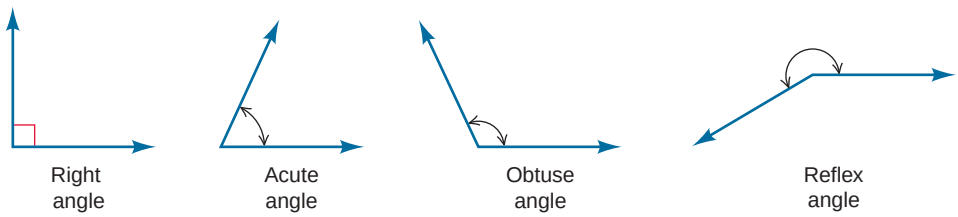


Figure 9.9

If the sum of two angles is 90°, the angles are called **complementary;** if their sum is 180°, they are called **supplementary.** Figure 9.10 shows special cases of complementary and supplementary angles in which the pairs of angles share a common side. If two angles have the same vertex, share a common side, and do not overlap, they are called **adjacent angles.** Angles 1 and 2 are adjacent complementary angles, and angles 3 and 4 are adjacent supplementary angles.

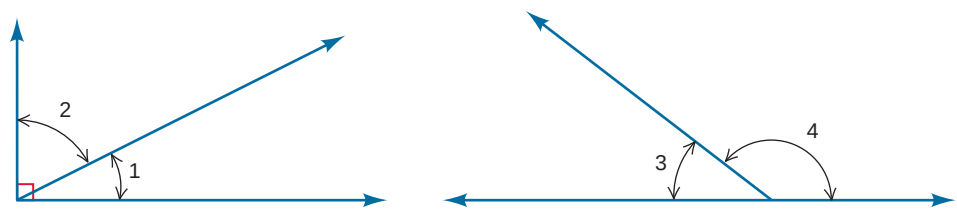


Figure 9.10

EXAMPLE C

Fold a standard sheet of paper to create models for the following terms.

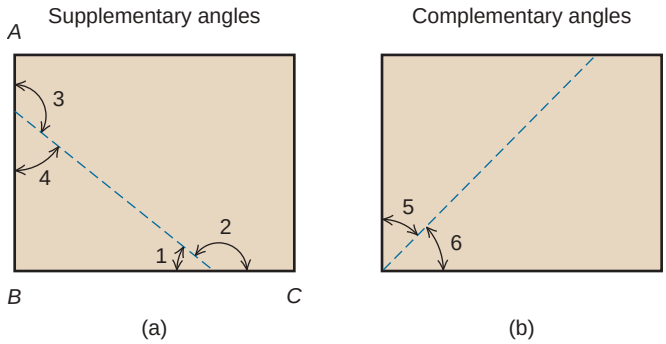
1. Acute angle
2. Obtuse angle
3. Supplementary angles
4. Complementary angles
5. Adjacent angles

Research Statement

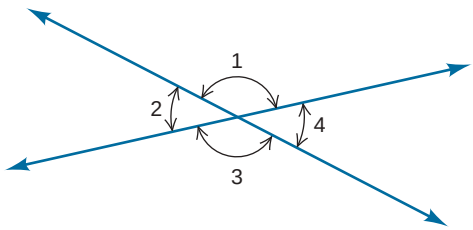
Findings from research studies suggest that students often have misconceived notions about angles and other geometric figures that are based solely on how these figures are oriented in textbooks.

Clements and Battista

Solution **1, 2, 3, 5.** Any crease that intersects an edge of the paper forms supplementary angles with the edges. For example, the crease in figure (a) intersects $\overline{BC}$, forming supplementary angles 1 and 2. These angles are also adjacent angles. The same crease intersects $\overline{AB}$ and forms adjacent supplementary angles 3 and 4. Angles 1 and 4 are acute, and angles 2 and 3 are obtuse. **4, 5.** Any crease through a corner of the paper forms adjacent complementary angles with the edges. Angles 5 and 6 in figure (b) are adjacent complementary angles.



Two intersecting lines form four pairs of adjacent supplementary angles. For example, $\measuredangle 1$ and $\measuredangle 4$ in Figure 9.11 are supplementary angles. Nonadjacent angles formed by two intersecting lines, such as $\measuredangle 2$ and $\measuredangle 4$ in Figure 9.11, are called **vertical angles** and vertical angles have equal measure.



Vertical angles
$\measuredangle 2$ and $\measuredangle 4$; $\measuredangle 1$ and $\measuredangle 3$

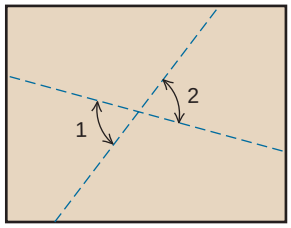
Figure 9.11

EXAMPLE D

1. Name four pairs of supplementary angles in Figure 9.11.
2. Which angles in Figure 9.11 have equal measure?
3. Fold a sheet of paper to create a model of two intersecting lines. Compare the measures of the vertical angles and use the folded paper to help explain why they have equal measure.

Solution **1.** The following pairs of angles are supplementary angles: $\measuredangle 1$ and $\measuredangle 4$; $\measuredangle 1$ and $\measuredangle 2$; $\measuredangle 2$ and $\measuredangle 3$; $\measuredangle 3$ and $\measuredangle 4$. **2.** The following pairs of angles have equal measure: $m(\measuredangle 2) = m(\measuredangle 4)$

and $m(\measuredangle 1) = m(\measuredangle 3)$. **3.** Two intersecting creases produce vertical angles. Angles 1 and 2 in the figure here are vertical angles. The fact that vertical angles have equal measure can be illustrated by folding the angles onto each other.



NCTM Standards

. . . students in grades 3–5 should be expanding their mathematical vocabulary. . . . As they describe shapes, they should hear, understand, and use mathematical terms such as parallel, perpendicular, face, edge, vertex, angle, trapezoid, prism, and so forth, to communicate geometric ideas with greater precision. p. 166

PERPENDICULAR AND PARALLEL LINES

If two lines intersect to form right angles, they are **perpendicular.** Lines m and n in Figure 9.12 are perpendicular; this is indicated by writing $m \perp n$. Two intersecting line segments, such as $\overline{AB}$ and $\overline{CD}$ in Figure 9.12 are **perpendicular line segments** if they lie on perpendicular lines. In this case, we write $\overline{AB} \perp \overline{CD}$.

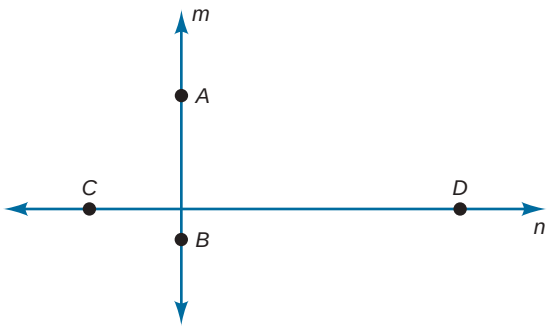


Figure 9.12

Perpendicular lines

If two lines are in a plane and they do not intersect, they are **parallel.** Lines m and n in Figure 9.13 are parallel; this is indicated by writing $m \parallel n$. Similarly, two line segments are **parallel line segments** if they lie on parallel lines. For example, segments $\overline{EF}$ and $\overline{GH}$ in Figure 9.13 are parallel, and we write $\overline{EF} \parallel \overline{GH}$.

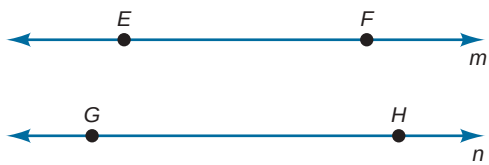


Figure 9.13

Parallel lines

If two lines ℓ and m are intersected by a third line t (see Figure 9.14 on the next page), we call line t a **transversal.** Two very special angles are created on the alternate sides of the transversal and are interior to lines ℓ and m (angles 1 and 2 in Figure 9.14). These angles are called **alternate interior angles.** If the two lines ℓ and m are parallel (as in Figure 9.14), *the alternate interior angles have the same measure.* The converse of this statement is also true: *If the alternate interior angles have the same measure,* lines ℓ and m are parallel. These statements are combined in the following property.

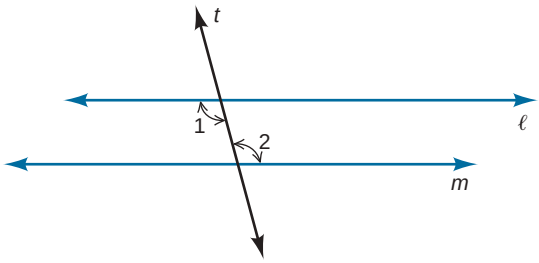


Figure 9.14

Alternate Interior Angles If two lines are intersected by a transversal, the lines are parallel if and only if the alternate interior angles created by the transversal have the same measure.

EXAMPLE E

Use a standard sheet of paper to model the following geometric terms: parallel lines, perpendicular lines, lines intersected by a transversal, and alternate interior angles having the same measure. Draw and label these on the paper.

Solution The opposite edges of the paper are parallel line segments, and any two edges that meet at a corner are perpendicular line segments. Any fold of the paper that intersects the opposite parallel edges of the paper will create alternate interior angles with the same measure.

There are other ways of obtaining parallel and perpendicular lines by folding paper. Two perpendicular lines can be obtained by folding the paper in half along one edge and then folding it in half along the other edge. Two parallel lines can be obtained by folding the paper in half along one edge and then folding it in half again along the same edge.

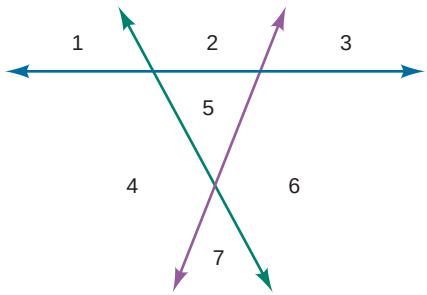
PROBLEM-SOLVING APPLICATION

Problem

What is the maximum number of regions into which a plane can be partitioned by 12 lines?

Understanding the Problem One line partitions a plane into 2 regions, and 2 intersecting lines partition a plane into 4 regions. **Question 1:** What is the maximum number of regions created by 3 lines in a plane?

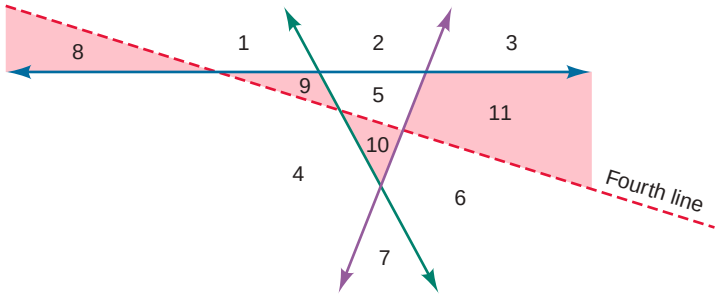
Devising a Plan It would be difficult to draw 12 lines and count the resulting regions. Let's *make a table* to record the numbers of regions for the first few lines. This approach may suggest a solution. Three lines divide the plane into 7 regions. **Question 2:** What is the maximum number of regions created by 4 lines?



Carrying Out the Plan The following table lists the maximum number of regions for 1, 2, 3, and 4 lines. Find a pattern and use inductive reasoning to predict the numbers of regions for the next few lines. **Question 3:** How many regions will there be for 12 lines?

No. of lines	1	2	3	4	5	6	7	8	9	10	11	12
No. of regions	2	4	7	11								

Looking Back When a fourth line that is not parallel to any of the first 3 lines is drawn on the plane, by definition it will intersect each of the 3 given lines. Also, it will cut across 4 regions, as shown in the figure below. This accounts for 4 new regions. **Question 4:** How many lines and how many regions will a fifth nonparallel line intersect?



Answers to Questions 1–4 **1.** 7 **2.** 11

3.

No. of lines	5	6	7	8	9	10	11	12
No. of regions	16	22	29	37	46	56	67	79

4. The fifth line will intersect 4 lines and 5 regions to create 5 new regions.

CURVES AND CONVEX SETS

We can draw a curve through a set of points by using a single continuous motion (Figure 9.15).

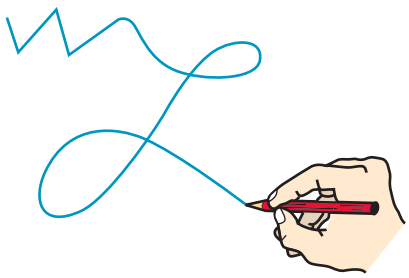
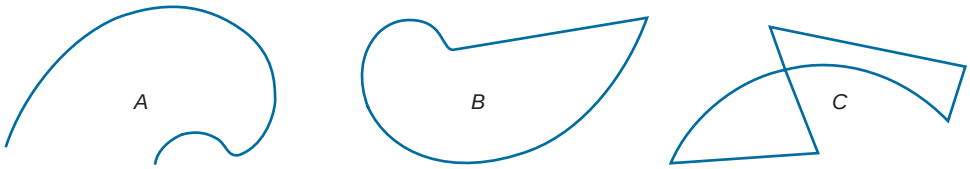


Figure 9.15

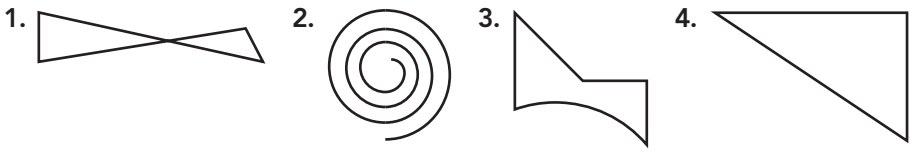
Several types of curves are shown in Figure 9.16 on the next page. Curve *A* is called a **simple curve** because it starts and stops without intersecting itself. Curve *B* is a **simple closed curve** because it is a simple curve that starts and stops at the same point. Curve *C* is a **closed curve,** but since it intersects itself, it is not a simple closed curve.



Simple curve | Simple closed curve | Closed curve

Figure 9.16

EXAMPLE F Classify each curve as simple, simple closed, or closed.



Solution **1.** Closed **2.** Simple **3.** Simple closed **4.** Simple closed

A well-known theorem in mathematics, called the **Jordan curve theorem,** states that every simple closed curve partitions a plane into three disjoint sets: the points on the curve, the points in the interior, and the points in the exterior. This means that if K is in the interior and M is in the exterior, then $\overline{KM}$ will intersect the curve (Figure 9.17).

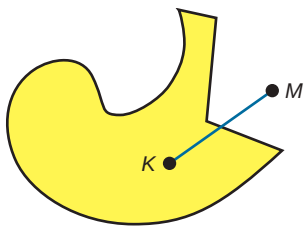
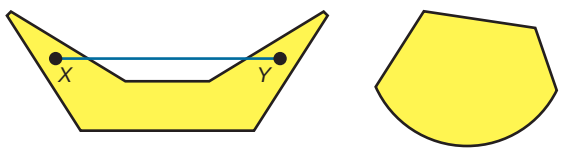


Figure 9.17

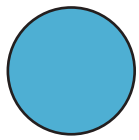
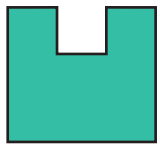
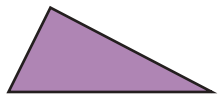
Convex Sets The union of a simple closed curve and its interior is called a **plane region.** Plane regions can be classified as *concave* or *convex*. You may have heard the word *concave*. It is from the Latin word *concavus,* meaning hollow. Intuitively, a concave set may be thought of as "caved in," as in Figure 9.18a. To be more mathematically precise, we say that a set is **concave** if it contains two points such that the line segment joining the points does not completely lie in the set. The set in Figure 9.18a is concave because $\overline{XY}$ is not completely in the set. If a set is not concave, it is called **convex.** An intuitive way of thinking about a convex set is to imagine enclosing the boundary of a figure with an elastic band. If the elastic touches all points on the boundary, as it will for the set in Figure 9.18b, the set is convex; and if not, as in Figure 9.18a, the set is concave (also sometimes called **nonconvex**).



Concave (a) | Convex (b)

Figure 9.18

EXAMPLE G Classify each region as concave or convex.

1. 
2. 
3. 

Solution 1. Convex 2. Concave 3. Convex

Circles A **circle** is a special case of a simple closed curve whose interior is a convex set (Figure 9.19). Each point on a circle is the same distance from a fixed point, called the **center.** A line segment from a point on the circle to its center is a **radius,** and a line segment whose endpoints are both on the circle is a **chord.** A chord that passes through the center is a **diameter.** The words *radius* and *diameter* are also used to refer to the lengths of these line segments. A line that intersects the circle in exactly one point is a **tangent.** The distance around the circle is the **circumference.** The union of a circle and its interior is called a **disk.**

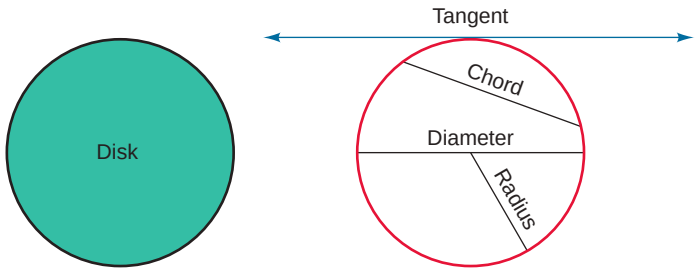


Figure 9.19

POLYGONS

A **polygon** is a simple closed curve that is the union of line segments. The union of a polygon and its interior is called a **polygonal region.** Polygons are classified according to their number of line segments. A few examples of polygonal regions are shown in Figure 9.20. The line segments of a polygon are called **sides,** and the endpoints of these segments are **vertices.** Two sides of a polygon are **adjacent sides** if they share a common vertex, and two vertices are **adjacent vertices** if they share a common side.

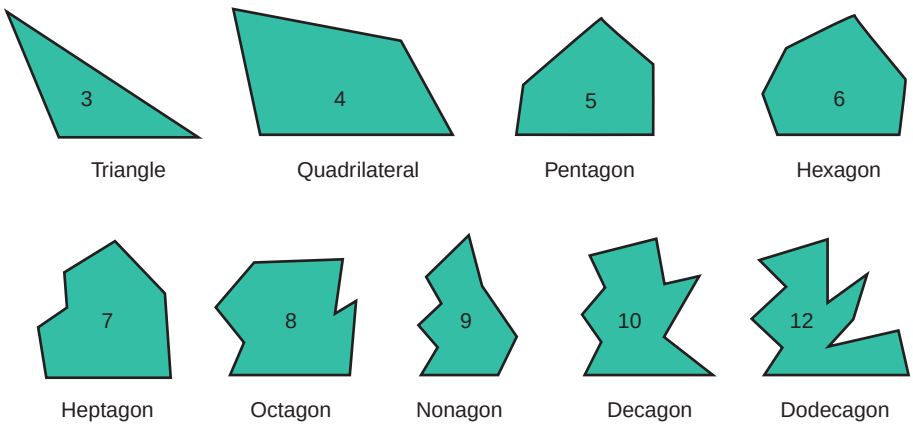
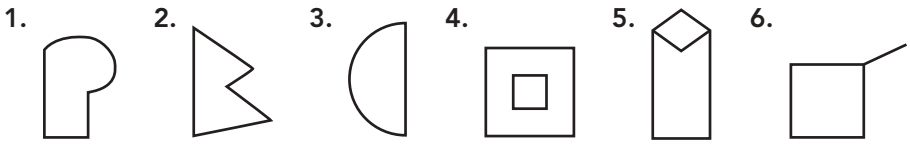


Figure 9.20

EXAMPLE H Which of the following figures are polygons?



Solution Figure (2) is the only polygon. Figures (1) and (3) are simple closed curves, but are not the union of line segments. Figures (4), (5), and (6) are not simple closed curves.

Any line segment connecting one vertex of a polygon to a nonadjacent vertex is a **diagonal.** Figure 9.21 shows a pentagon with its five diagonals.

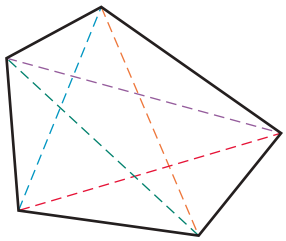


Figure 9.21

EXAMPLE I How many diagonals are there in each of the following polygons?

1. Quadrilateral **2.** Triangle **3.** Hexagon

Solution **1.** Two **2.** Zero **3.** Nine

Certain triangles and quadrilaterals occur often enough to be given special names. Several of these are shown in Figure 9.22.

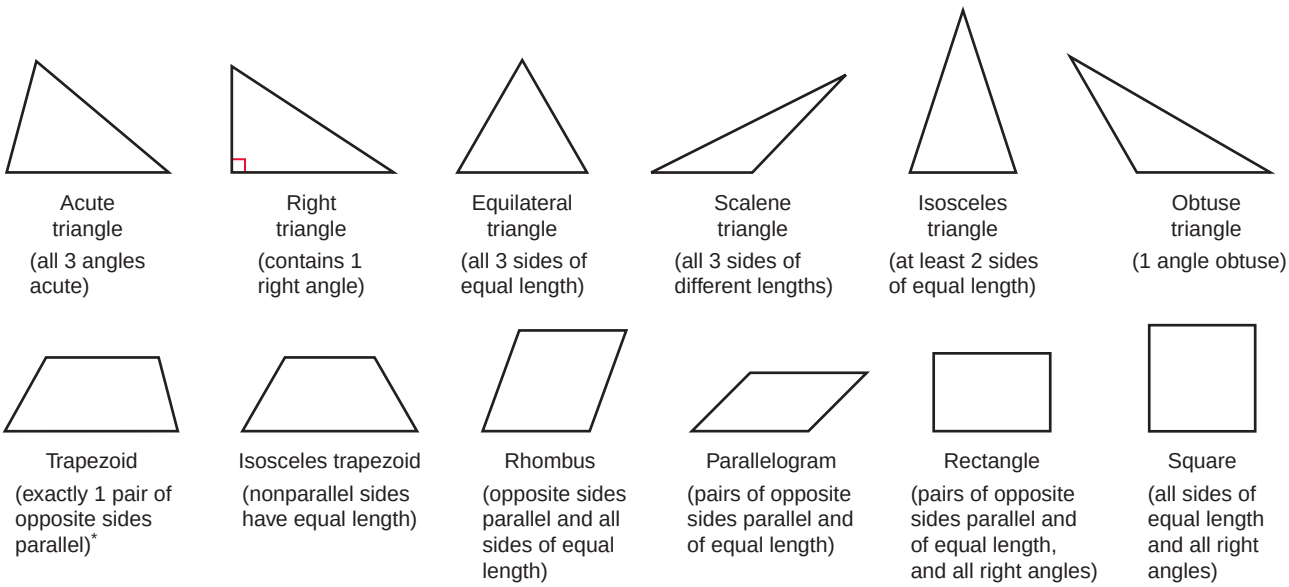


Figure 9.22

*Some books define a trapezoid as having *at least* one pair of opposite sides parallel. In this case, a parallelogram is also a trapezoid because it has at least one pair of parallel sides.

EXAMPLE J

Determine whether each statement is true or false, and state a reason.

1. Every square is a rectangle.
2. Every equilateral triangle is an isosceles triangle.
3. Some right triangles are isosceles triangles.
4. Some trapezoids are parallelograms.
5. Some isosceles triangles are scalene triangles.

Solution **1.** True. The opposite sides of a square are parallel and of equal length. **2.** True. An equilateral triangle has three sides of equal length, so it has *at least* two sides of equal length. **3.** True. A right triangle could have two legs of length 1 and a hypotenuse of length $\sqrt{2}$; the two equal sides would make it an isosceles triangle. **4.** False. Trapezoids have only one pair of opposite parallel sides; a parallelogram must have two pairs of opposite parallel sides. **5.** False. All three sides are of different lengths in a scalene triangle.

NCTM Standards

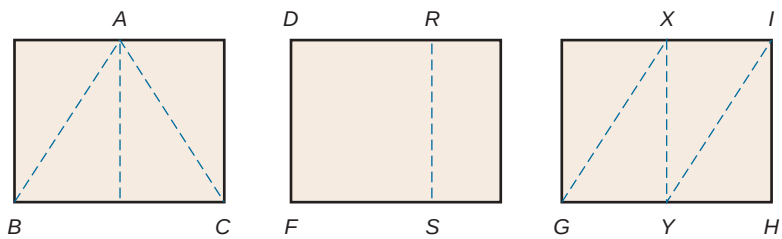
In the early grades, students will have classified and sorted geometric objects such as triangles or cylinders by noting general characteristics. In grades 3–5, they should develop more precise ways to describe shapes, focusing on identifying and describing the shape's properties and learning specialized vocabulary associated with these shapes and properties. p. 165

EXAMPLE K

Fold a standard sheet of paper to obtain a model of each geometric figure.

1. Isosceles triangle **2.** Square **3.** Parallelogram

Solution Here are some possibilities. There are other ways to obtain these figures. **1.** Fold the paper in half to obtain point A, as shown in figure (a). Then fold to obtain the crease $\overline{AB}$ and fold again to obtain the crease $\overline{AC}$. Line segment $\overline{AB}$ can be folded onto $\overline{AC}$ to show that triangle ABC is isosceles. **2.** Fold corner D [figure (b)] down to point S so that $\overline{DF}$ coincides with $\overline{FS}$. With the paper in this folded position, use edge $\overline{DR}$ to draw line $\overline{RS}$. Then figure $DRSF$ is a square. **3.** Fold the paper in half to obtain points X and Y, as shown in figure (c). Then fold to obtain the creases $\overline{GX}$ and $\overline{IY}$; $GYIX$ is a parallelogram.



(a) (b) (c)

NCTM Standards

The *Curriculum and Evaluation Standards for School Mathematics,* grades 5–8, *Geometry* (p. 113), observes that geometry has a vocabulary of its own and students need ample time to gain confidence with new terms:

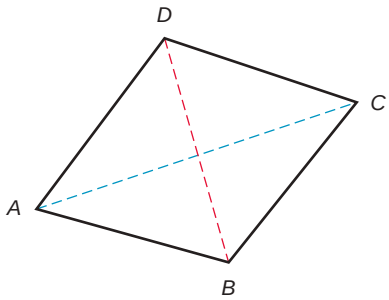
> Definitions should evolve from experiences in constructing, visualizing, drawing, and measuring two- and three-dimensional figures, relating properties to figures, and contrasting and classifying figures according to their properties.

PROBLEM-SOLVING APPLICATION

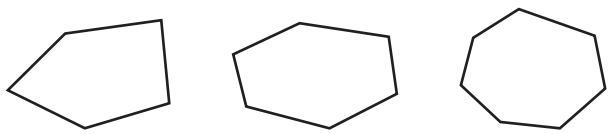
Problem

How many diagonals does a 15-sided polygon have?

Understanding the Problem A diagonal is a line segment connecting any two nonadjacent vertices of a polygon. Quadrilateral *ABCD*, shown here, has diagonals *AC* and *BD*. Sketch a concave quadrilateral. **Question 1:** Does such a quadrilateral have two diagonals?



Devising a Plan One approach is to *simplify the problem* by drawing a few polygons and counting the number of diagonals. By listing these in a table, we may be able to find a pattern. **Question 2:** How many diagonals are there in each of the following polygons?



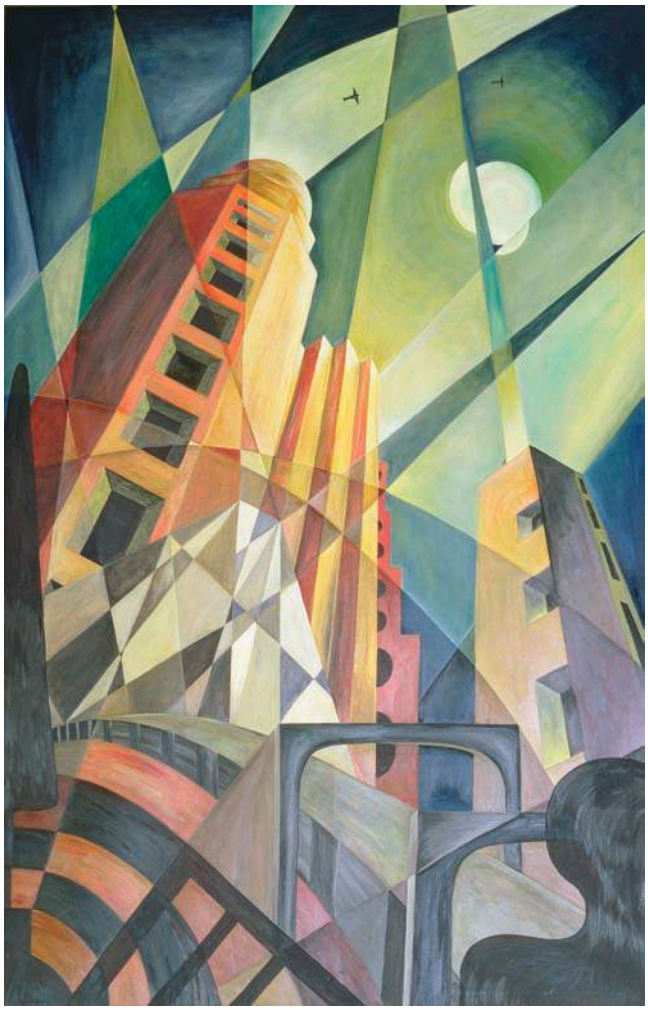
Carrying Out the Plan Fill in a few blanks of the table below and *look for a pattern.* Use your pattern and inductive reasoning to complete the table. **Question 3:** How many diagonals are there in a 15-sided polygon?

No. of sides	3	4	5	6	7	8	9	10	11	12
No. of diagonals	0	2	5	9						

Looking Back Another approach to this problem is to use the result from the problem-solving application on pages 573–574, in which we found the number of line segments connecting 15 points. Since there are 105 line segments connecting 15 points, the number of diagonals in a 15-sided polygon can be found by subtracting 15 (the number of sides in the polygon) from 105. Thus, there are 90 diagonals. **Question 4:** How many diagonals are there in a 25-sided polygon?

Answers to Questions 1–4 **1.** Yes **2.** Pentagon, 5 diagonals; hexagon, 9 diagonals; heptagon, 14 diagonals. **3.** 90 **4.** The number of line segments connecting 25 points is the 24th triangular number: $\frac{(24 \times 25)}{2} = 300$. Subtracting 25, the number of sides in the polygon, from 300 yields 275 diagonals.

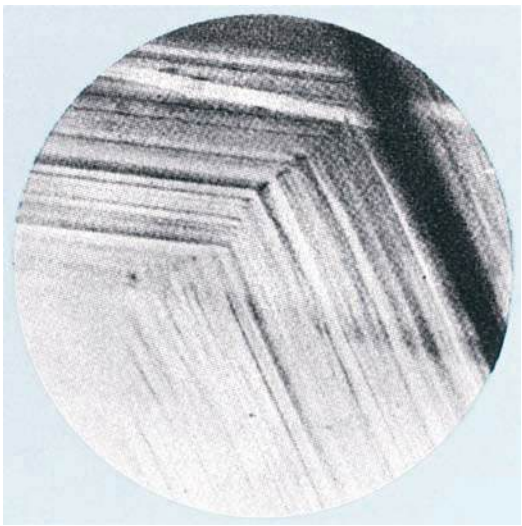
Exercises and Problems 9.1



"City in Shards of Light" by Carolyn Hubbard-Ford has many examples of geometric figures and angles. Find at least one example of each of the following.

1. **a.** Acute angle
   b. Trapezoid
   c. Right angle
   d. Convex pentagon

2. The photo of a growth structure in sapphire at the top of the next column shows angles that each have the same number of degrees.
   a. Are these angles acute or obtuse?
   b. Measure these angles. Approximately how many degrees are there in each of these angles?



3. List the four components of every mathematical system and briefly describe each.

4. **a.** List three undefined geometric terms.
   b. List three defined geometric terms whose definitions use one or more of the undefined terms.
   c. Explain why it is necessary to have undefined terms in geometry.

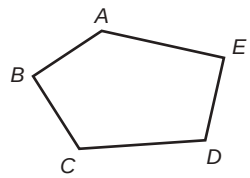
Give two examples of physical models that illustrate each term in exercises 5 and 6.

5. **a.** Line segment **b.** Triangle **c.** Plane

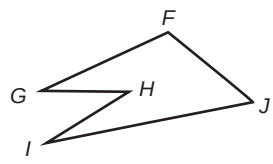
6. **a.** Angle **b.** Point **c.** Square

Place the corner of a sheet of paper or a file card on the angles of the polygons in exercises 7 and 8 to check for right angles.

7. **a.** Which angles, if any, are acute?
   b. Which angles, if any, are obtuse?
   c. Which angles, if any, are right angles?

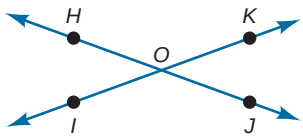


8. **a.** Which angles, if any, are acute?
   b. Which angles, if any, are obtuse?
   c. Which angles, if any, are reflex angles?

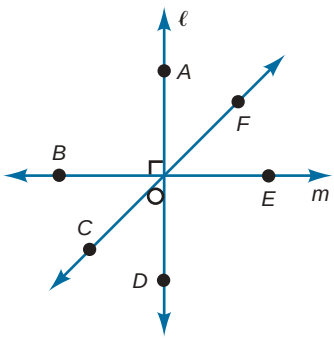


Use the angles in the figures in exercises 9 and 10 to identify the pairs of angles.

9. **a.** Three pairs of adjacent supplementary angles
b. Two pairs of vertical angles
c. Two pairs of angles with the same measure

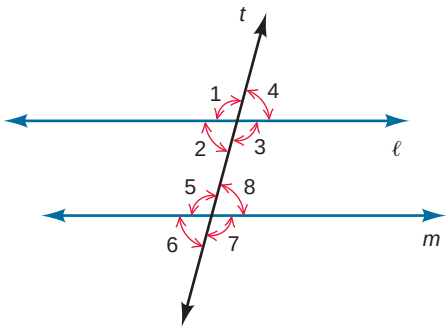


10. **a.** Three pairs of adjacent supplementary angles
b. Three pairs of vertical angles
c. Two pairs of adjacent complementary angles



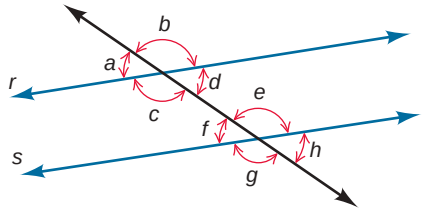
11. Draw a circle that illustrates each of the following geometric situations.
a. A diameter that is perpendicular to a chord
b. A line tangent to the circle at one end of a radius
c. Two chords that bisect each other

12. If ℓ and m are parallel lines, explain why the angles in each pair in parts a through d have the same measure.

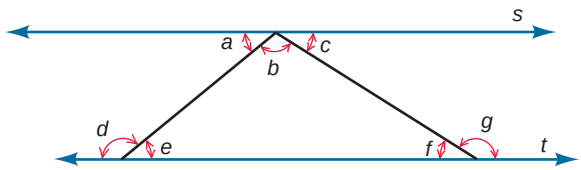


a. $\measuredangle 2$ and $\measuredangle 8$
b. $\measuredangle 2$ and $\measuredangle 4$
c. $\measuredangle 4$ and $\measuredangle 8$ (These angles are called **corresponding angles.**)
d. $\measuredangle 1$ and $\measuredangle 7$
e. Explain why $\measuredangle 3$ and $\measuredangle 8$ are supplementary angles.

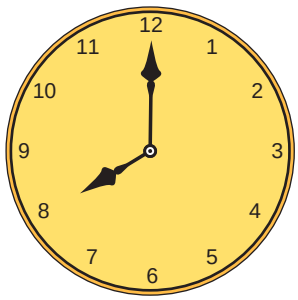
13. If r and s are parallel lines and the measure of $\measuredangle a$ is 34.5°, what is the measure of each of the following angles?
a. $\measuredangle e$ **b.** $\measuredangle h$
c. $\measuredangle c$ **d.** $\measuredangle f$



14. If s and t are parallel lines and the measure of $\measuredangle f$ is 32° and the measure of $\measuredangle a$ is 40°, determine the measure of each of the following angles.
a. $\measuredangle b$ **b.** $\measuredangle c$ **c.** $\measuredangle d$ **d.** $\measuredangle e$ **e.** $\measuredangle g$



Use the type of clock shown here to answer the questions in exercises 15 and 16.

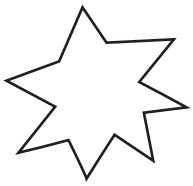
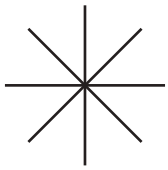


15. **a.** What is the measure of the obtuse angle formed by the hour and minute hands of a clock if the time is 8 o'clock?
b. How many degrees will the hour hand of the clock move through when the time changes from 8 o'clock to 10 o'clock?
c. How many minutes have passed when the minute hand has moved through 42°?

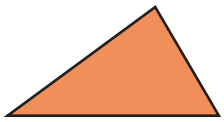
16. **a.** How many degrees will the minute hand of the clock move through when the time changes from 8 o'clock to 8:25?
b. How many hours will have passed when the hour hand has moved through 120°?
c. What is the measure of the obtuse angle formed by the hour hand and the minute hand if the time is 2:30?

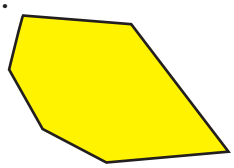
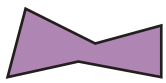
Classify each curve in exercises 17 and 18 as simple, simple closed, closed, or none of these.

17. **a.**  **b.**  **c.** 

18. **a.**  **b.**  **c.** 

Classify each region in exercises 19 and 20 as convex or concave.

19. **a.**  **b.**  **c.** 

20. **a.**  **b.**  **c.** 

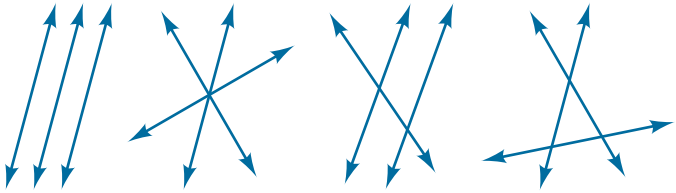
21. The following photograph shows three crystals of the mineral staurolite. These crystals are found in all parts of the world. They are especially common in the Shenandoah Valley. The crystal on the far right is known as the *Fairy Stone* of the Appalachian Mountains. This form and the one on the left are often imitated by jewelers.



a. The ridges on the top of the Fairy Stone form two lines that intersect to form four angles of equal measure. What is the measure of each of these angles?

b. The ridges on the top of the crystal on the left form three lines that intersect to form six angles of equal measure. What is the measure of each of these angles?

Three lines in a plane may intersect in 0, 1, 2, or 3 points. In exercises 22 and 23, draw lines to support your reasoning.



22. Determine all the different numbers of points of intersection that are possible with four lines in a plane.

23. Determine all the different numbers of points of intersection that are possible with five lines in a plane.

Draw some figures in exercises 24 and 25 to determine whether the following statements are true or false. For each false statement show a counterexample.

24. **a.** The two diagonals of a parallelogram have the same length.

b. Any two angles in a parallelogram that share a common side are supplementary.

c. The two diagonals of a rectangle have the same length.

25. **a.** If the two diagonals in a parallelogram have the same length, the parallelogram is a rectangle.

b. If the midpoints of the adjacent sides of a rectangle are connected, another rectangle is formed.

c. If the midpoints of the adjacent sides of a quadrilateral are connected, a parallelogram is formed.

Reasoning and Problem Solving

26. To prepare for their annual volleyball party, the Chase family has laid out a four-sided volleyball court in which two opposite sides have a length of 30 feet each and the other two opposite sides have a length of 60 feet each.

a. Explain why the court may not be rectangular. What shape might it have?

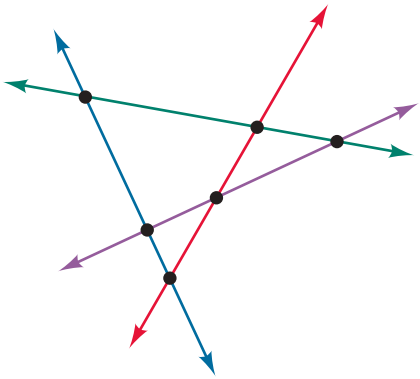
b. What additional directions need to be given to ensure the court is rectangular?

27. A 70-inch piece of pipe is to be cut at two points A and B such that A and B are not on the ends of the pipe and the length from A to B is 42 inches. How many possibilities are there for obtaining three pieces of pipe of different lengths if the lengths are whole numbers?

28. **Featured Strategies: Solving a Simpler Problem and Making a Table.** What is the maximum number of points of intersection for 12 lines?

a. **Understanding the Problem.** The problem asks for the greatest possible number of points of intersection. What is the minimum number of points of intersection for 12 lines?

b. Devising a Plan. The following figure shows that the maximum number of points of intersection for four lines is 6. Considering a few other cases for small numbers of lines may reveal a pattern. What is the maximum number of points of intersection for three lines?



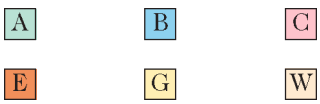
c. Carrying Out the Plan. Look for a pattern and complete the table here. What is the maximum number of points of intersection for 12 lines?

No. of lines	2	3	4	5	6	7	8	9	10	11	12
No. of intersections			6								

d. Looking Back. Use the pattern in part c to determine the maximum number of points of intersection for 50 lines.

29. In 1891, Almon B. Strowger patented a phone-dialing machine that could connect up to 99 subscribers. How many different two-party calls would such a machine permit?

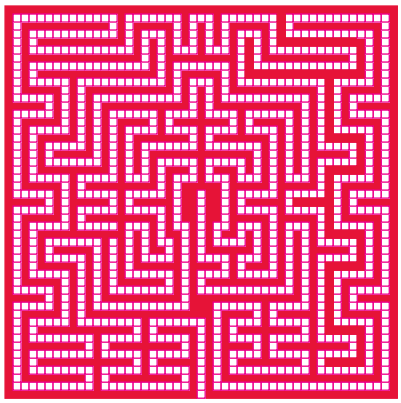
30. Suppose squares *A*, *B*, and *C* are houses and *E*, *G*, and *W* represent sources of electricity, gas, and water, respectively. Try to connect the houses with each utility by drawing lines or curves so that they do not cross one another. It is possible to make only eight of the nine connections. Draw these eight connections.



a. Some of your connections will form a simple closed curve with the remaining unconnected house and utility on opposite sides of this curve. Find this curve and mark it with dark lines.

b. How does the Jordan curve theorem show that nine connections cannot be completed with the given conditions?

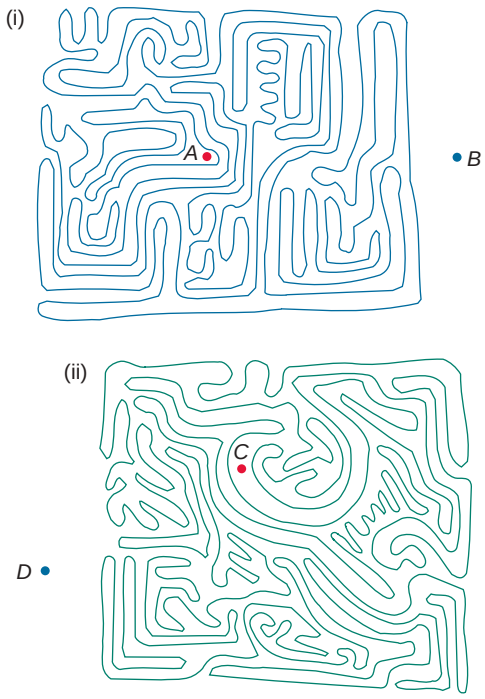
31. The white path in this ornament from the Middle Ages is a curve.



a. Is it a simple curve?

b. Is it a closed curve?

32. The curves shown below are simple closed curves. The Jordan curve theorem states that if a point inside a simple closed curve is connected to a point outside the curve, the connecting arc will intersect the curve. Use the fact that points *B* and *D* are outside the two curves in (i) and (ii) to answer the questions.

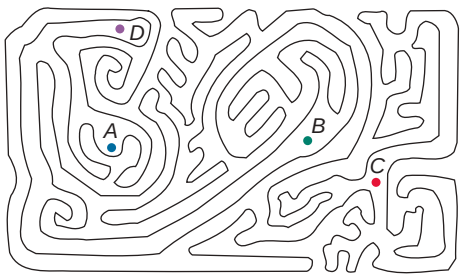


a. Can an arc be drawn from *A* to *B* that does not intersect the curve in (i)? Is *A* inside or outside the curve?

b. Can an arc be drawn from *C* to *D* that does not intersect the curve in (ii)? Is *C* inside or outside the curve in (ii)?

c. Draw line segment $\overline{AB}$. Count the number of times that $\overline{AB}$ intersects the curve in (i) on the previous page. How can this number be used to tell when a point is inside or outside a simple closed curve? (*Hint:* Draw a few simple closed curves.) Check your answer by drawing $\overline{CD}$ for the curve in (ii).

33. The following simple closed curve is from *Puzzles and Graphs* by John Fujii.* Determine whether the points in each pair below are on the same side of the curve.



a. A, B **b.** B, C **c.** D, C **d.** B, D

34. An equilateral triangle has three sides of equal length. Fold a sheet of paper to form an equilateral triangle, using the creases or edges of the paper. (*Hint:* Obtain a centerline by folding the paper in half.)

Teaching Questions

1. A student who built a pentagonal pattern block figure, as in question 3 in the one-page **Math Activity** in this section, says that the figure has seven sides. How do you think she got seven and how would you help her so she doesn't make that mistake again?

2. Students are sorting polygonal shapes into two groups, parallelograms and nonparallelograms. You see several students putting squares and rectangles into the nonparallelogram group. Describe ways you can help these students understand that squares and rectangles are parallelograms.

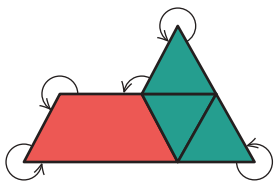
3. During a paper-folding activity in your class, a student concluded that any two line segments that do not intersect must be parallel. Is this student correct? How would you respond?

4. Pierre van Hiele and Dieke van Hiele-Geldof were teachers in the Netherlands who developed a theory about how students learn geometry. The "van Hiele theory"—that students learn geometry by progressing through five stages or levels of reasoning—prompted much research on the teaching and learning of geometry. Find information about the van Hiele levels of reasoning in geometry and describe each level and how the levels differ.

Classroom Connections

1. Read the first two recommendations for the **Grades 3–5 Standards—Geometry** (see inside front cover). Describe ways that you can help students in grades 3–5 begin to understand abstract relationships among quadrilaterals. As examples: All squares are rectangles but not all rectangles are squares; squares are all rhombuses, but not all rhombuses are squares; all squares, rhombuses, and rectangles are parallelograms, etc.

2. In the one-page **Math Activity** at the beginning of this section you discovered a way to predict the sum of the measures of the interior angles of a polygon. Use the pattern block figures on that page to predict the sum of the measures of the "outside angles" of any polygon. An example of the five outside angles of a pentagon is shown in this diagram.



3. Assume that you have a classroom set of circular protractors and have taught students to measure angles. The **Standards** statement on page 574 suggests that students learn to estimate angle measure. Design a two-person game that will help students learn to estimate angle measures. Write rules for the game in which students will check their estimates using the protractor.

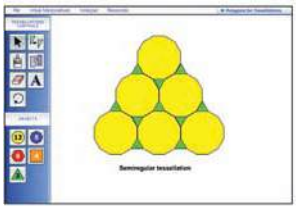
4. Read the **Research** statement on page 576. Explain what you believe the statement means for angles and other geometric figures. Include sketches of figures with your explanation.

*John Fujii, *Puzzles and Graphs*.

MATH ACTIVITY 9.2

Tessellations with Polygons

Virtual Manipulatives

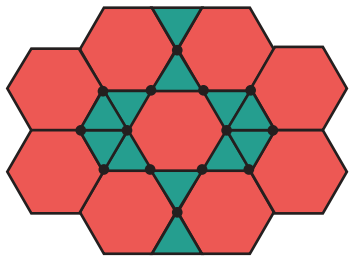


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Purpose: Arrange regular polygons to form regular and semiregular tessellations.

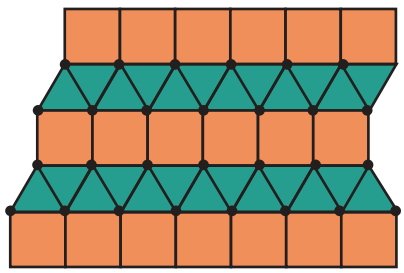
Materials: Polygons for Tessellations in the Manipulative Kit or Virtual Manipulatives.

1. An arrangement of nonoverlapping figures that are placed together to entirely cover a region is called a **tessellation.** A portion of a tessellation that uses triangles and hexagons is shown here. Form and sketch a tessellation that uses at least two different types of polygons from your set.



2. Each polygon in your set is called **regular** because all of its sides are congruent and all of its angles are congruent. If a tessellation can be formed with just one type of regular polygon, it is called a **regular tessellation.** Experiment with your polygons to find those that can be used to form a regular tessellation.

*3. The tessellation shown next is **semiregular** because it uses more than one type of regular polygon and each *vertex point* of the tessellation (bold point) is surrounded by the same arrangement of polygons. The pictured tessellation is denoted by the **code** 3, 3, 3, 4, 4 because a triangle has three sides and a square has four sides and each vertex is surrounded by three triangles and two squares in clockwise order. Explain why the tessellation in activity 1 is not semiregular.



4. There are other semiregular tessellations that can be formed by using two different types of regular polygons. Experiment with your polygons to find some of these. Sketch a portion of each tessellation, and write the numbers for its code.

5. There are two semiregular tessellations which use three different types of polygons. Find one of these and sketch a portion of its tessellation.

6. Is there a semiregular tessellation which uses four different types of polygons from your set? Explain your reasoning.

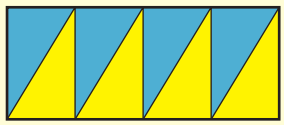
Section 9.2 POLYGONS AND TESSELLATIONS



Cross section of the gem tourmaline—triangles formed by nature

PROBLEM OPENER

This rectangular region is cut into eight congruent pieces. In how many ways can a rectangular region be cut into eight congruent pieces?



8 congruent pieces

The triangles in the preceding photograph are another amazing example of geometric figures that occur in nature. In each triangle, the sides have the same length, and the angles have the same measure. Such special types of polygons are discussed in this section.

ANGLES IN POLYGONS

The vertex angles of a polygon with four or more sides can be any size between 0° and 360°. In the hexagon in Figure 9.23, $\measuredangle B$ is less than 20° and $\measuredangle D$ and $\measuredangle A$ are both greater than 180°. In spite of this range of possible sizes, there is a relationship between the sum of all the angles in a polygon and its number of sides.

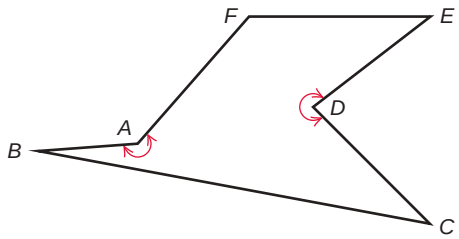


Figure 9.23

In any triangle, the sum of the three angle measures is 180°. This fact was proved by Greek mathematicians in the fourth century B.C.E. One way of demonstrating this theorem is to draw an arbitrary triangle and cut off its angles, as shown in Figure 9.24. When these angles are placed side by side with their vertices at a point, they form one-half of a revolution (180°) about the point.

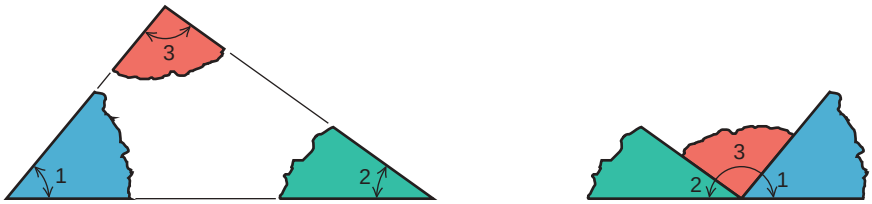


Figure 9.24

Research Statement

A recent national mathematics assessment found that only half (54%) of eighth grade students could find the measure of the missing angle of a triangle, given measures of the other two angles.

2007 NAEP in *Mathematics*

The sum of the angles in a polygon of four or more sides can be found by subdividing the polygon into triangles so that the vertices of the triangles are the vertices of the polygon. The quadrilateral in Figure 9.25 is partitioned into two triangles whose angles are numbered from 1 through 6. The sum of all six angles is $2 \times 180°$, or 360°. Therefore, the sum of the four angles of the quadrilateral is 360°.

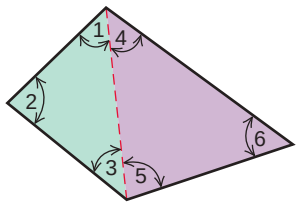


Figure 9.25

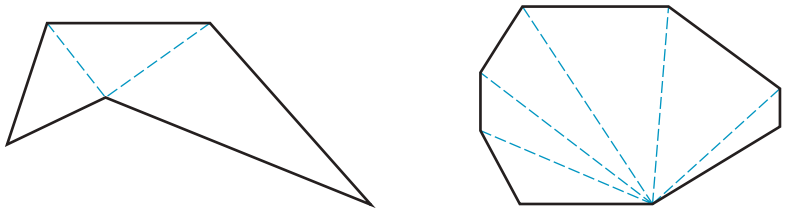
An infinite variety of quadrilaterals can be formed, some convex and others concave. However, since each quadrilateral can be partitioned into two triangles such that the vertices of the triangles are also the vertices of the quadrilateral, the sum of the angles of a quadrilateral will always be 360°. A similar approach can be used to find the sum of the angles in any polygon.

EXAMPLE A

Find the sum of all the angles in each polygon.

1. Pentagon **2.** Octagon

Solution **1.** A pentagon can be subdivided into three triangles, as shown in figure (a). So, the total number of degrees in its angles is $3 \times 180° = 540°$. **2.** An octagon can be subdivided into six triangles, as shown in figure (b). So, the total number of degrees in its angles is $6 \times 180° = 1080°$.



(a) (b)

CONGRUENCE

The idea of congruence is quite simple to understand intuitively: Two plane figures, such as those in Figure 9.26, are **congruent plane figures** if one can be placed on the other so that they coincide. Another way to describe congruent plane figures is to say that they have the same size and shape. (Congruence is presented in greater detail in Sections 11.1 and 11.2.)

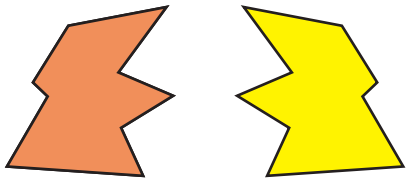


Figure 9.26

Congruent plane figures

We can be more precise at this point about congruence of line segments and angles. Two **line segments are congruent** if they have the same length, and two **angles are congruent** if they have the same measure (Figure 9.27).

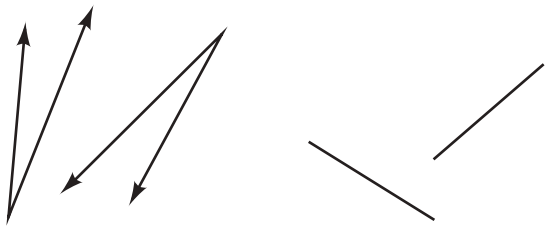


Figure 9.27

Congruent angles Congruent line segments

EXAMPLE B Fold a standard sheet of paper so that it is partitioned into the following figures.

1. Four congruent rectangles
2. Two congruent right triangles and a rectangle
3. Four congruent right triangles
4. Sixteen congruent rectangles

Solution Here are some methods. There are others. **1.** Fold the paper twice: once in half perpendicular to one edge and again in half perpendicular to an adjacent edge. **2.** Fold a corner of the paper down to obtain a rectangle and the largest possible square. The fold forms the diagonal of the square and bisects it into two right triangles. **3.** Fold the paper in half to obtain a rectangle, and then fold along the diagonal of the rectangle. **4.** Fold the paper in half perpendicular to the edges a total of four times.

REGULAR POLYGONS

Sections 9.1 and 9.2 opened with photographs of hexagons and triangles that grow naturally with congruent line segments and congruent angles. The figures in those photographs are examples of regular polygons. A polygon is called a **regular polygon** if it satisfies both of the following conditions:

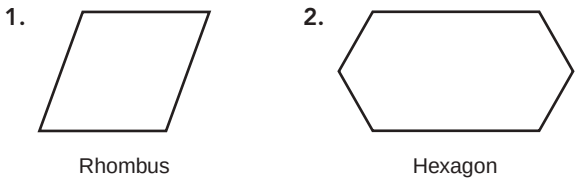
1. All angles are congruent.
2. All sides are congruent.

A few regular polygons are shown in Figure 9.28.



Figure 9.28

EXAMPLE C The following figures satisfy only one of the two conditions for regular polygons. For each polygon determine which condition is satisfied and which condition is not satisfied.



Solution **1.** The four sides are congruent, but the four angles are not congruent. **2.** The six angles are congruent, but the six sides are not congruent.

DRAWING REGULAR POLYGONS

There are three special angles in regular polygons (see Figure 9.29). A **vertex angle** is formed by two adjacent sides of the polygon; a **central angle** is formed by connecting the center of the polygon to two adjacent vertices of the polygon; and an **exterior angle** is formed by one side of the polygon and the extension of an adjacent side.

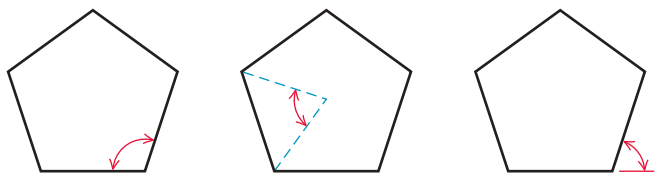
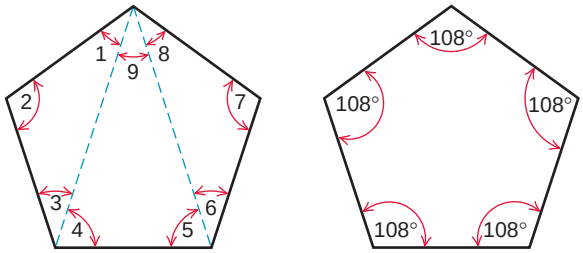


Figure 9.29

The sum of the measures of the angles in a polygon can be used to compute the number of degrees in each vertex angle of a regular polygon: Simply divide the sum of all the measures of the angles by the number of angles. For example, Figure 9.30a on the next page shows a regular pentagon that is subdivided into three triangles. Since each vertex of each triangle is a vertex of the pentagon, the sum of the nine angles in the triangles equals the sum of the five angles in the pentagon. So the sum of the angles in the pentagon is $3 \times 180°$, or $540°$. Therefore, each angle in a regular pentagon is $540° \div 5$, or $108°$, as shown in Figure 9.30b.



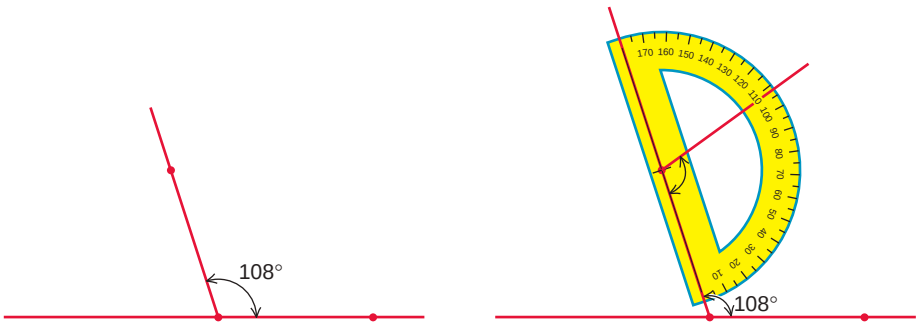
(a) (b)

Figure 9.30

Figure 9.31 shows the first four steps for drawing a regular pentagon. The process begins in step 1, where a line segment is drawn and a point for the vertex of the angle is marked. Then in step 2 the baseline of a protractor is placed on the line segment so that the center of the protractor's baseline is at the vertex point, and a 108° angle is drawn. In step 3 two sides of the pentagon are marked off, and in step 4 the protractor is used to draw another 108° angle. This process can be continued to obtain a regular pentagon.



Step (1) Draw a line segment and mark a vertex. Step (2) Measure off a 108° angle.



Step (3) Mark off two sides of equal length. Step (4) Measure off a second angle of 108°.

Figure 9.31

Another approach to drawing regular polygons begins with a circle and uses central angles. The number of degrees in the central angle of a regular polygon is 360 divided by the number of sides in the polygon. A decagon has 10 sides, so each central angle is 360° ÷ 10, or 36° (Figure 9.32).

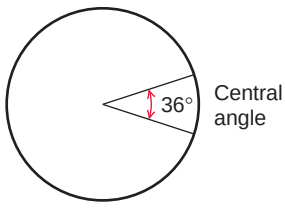
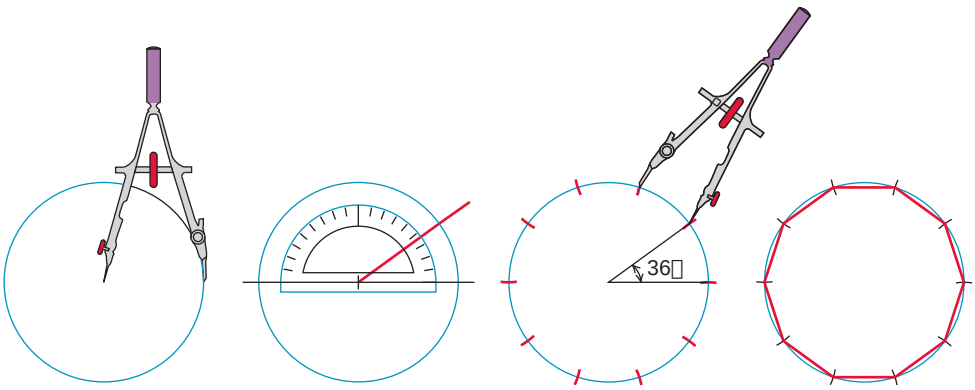


Figure 9.32

A four-step sequence for drawing a regular decagon is illustrated in Figure 9.33. The first and third steps use a **compass,** a device for drawing circles and arcs and marking off equal lengths. The decagon that is obtained is said to be *inscribed in the circle.* Any polygon whose vertices are points of a circle is called an **inscribed polygon.**



Step (1) Draw a circle with a compass.

Step (2) Measure a 36° angle with a protractor.

Step (3) Mark 10 equal lengths with a compass.

Step (4) Connect the points to form a decagon.

Figure 9.33

TESSELLATIONS WITH POLYGONS

The hexagonal cells of a honeycomb provide another example of regular polygons in nature (Figure 9.34). The cells in this photograph show that regular hexagons can be placed side by side with no uncovered gaps between them. Any arrangement in which nonoverlapping figures are placed together to entirely cover a region is called a **tessellation.** Floors and ceilings are often *tessellated,* or *tiled,* with square-shaped material, because squares can be joined together without gaps or overlaps. Equilateral triangles are also commonly used for tessellations. These three types of polygons—regular hexagons, squares, and equilateral triangles—are the only regular polygons that will tessellate.



Figure 9.34
Honeycomb with bees

HISTORICAL HIGHLIGHT



Mary Fairfax Somerville, 1780–1872

Mary Fairfax Somerville has been called "one of the greatest women scientists England ever produced." In spite of her parent's attempts to prevent her from studying mathematics, she acquired *Euclid's Elements of Geometry* and memorized much of the first six books. When she solved a prize problem on Diophantine equations in a mathematical journal, the editor advised her of the classics that would give her a sound background in mathematics. Thus, at age 32 she finally acquired a small library of mathematics books to pursue her studies. Somerville published many papers including two on solar rays that appeared in the *Philosophical Transactions of the Royal Society of London* and the *Edinburgh Philosophical Journal.* Her books *Mechanisms of the Heavens* and *The Connection of the Physical Sciences* brought her the greatest fame of all her works. Critics called the latter of these two books the best general survey of physical sciences published in England. Somerville belonged to a group of scientists who pioneered the efforts to arouse England's interest in mathematics and scientific progress. At the time of her death at age 92, she was engaged in several mathematical writing projects.*

*L. M. Osen, *Women in Mathematics* (Cambridge, MA: The MIT Press, 1974), pp. 95–116.

From ancient times tessellations have been used as patterns for rugs, fabrics, pottery, and architecture. The Moors, who settled in Spain in the eighth century, were masters of tessellating walls and floors with colored geometric tiles. Some of their work is shown in Figure 9.35, a photograph of a room and bath in the Alhambra, a fortress palace built in the middle of the fourteenth century for Moorish kings.

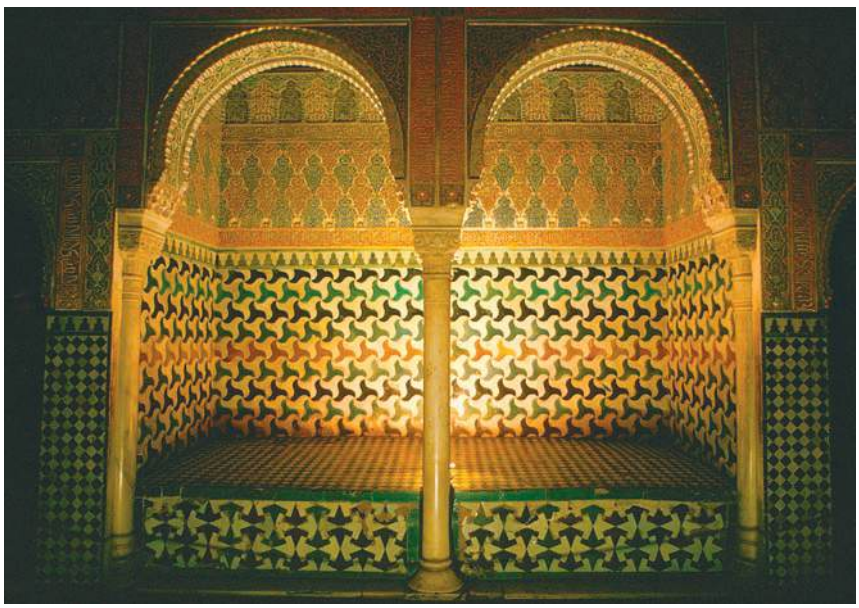


Figure 9.35
The Sala de las Camas (Room of the Beds), a beautifully tiled room in the Alhambra, the palace of fourteenth-century Moorish kings, in Granada, Spain.

The two tessellations in the center of the above photograph are made up of nonpolygonal (curved) figures. In the following paragraphs, however, we will concern ourselves only with polygons that tessellate. The triangle is an easy case to consider first. You can see

NCTM Standards

When teachers point out geometric shapes in nature or in architecture, students' awareness of geometry in the environment is increased. p. 101

that any triangle will tessellate by simply putting two copies of the triangle together to form a parallelogram (see the shaded region of Figure 9.36). Copies of the parallelogram can then be moved horizontally and vertically. The points at which the vertices of the triangle meet are called the **vertex points** of the tessellation. Since the sum of the angles in a triangle is 180°, the 360° about each vertex point of the tessellation will be covered by using each angle of the triangle twice. In the tessellation shown in Figure 9.36, angles 1, 2, and 3 occur twice about each vertex point.

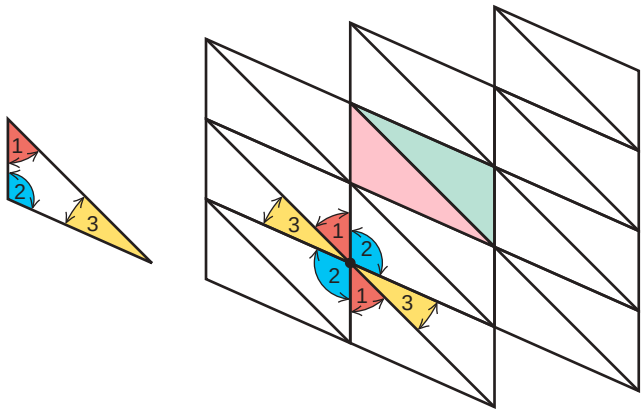


Figure 9.36

The sizes of the angles in a polygon and the sums of these angles will determine whether the polygon will tessellate. The fact that the sum of the angles in a quadrilateral is 360° suggests that a quadrilateral has the right combination of angles to fit around each vertex point of a tessellation. In the tessellation in Figure 9.37, each angle of the quadrilateral (angles 1, 2, 3, and 4) occurs once about each vertex point of the tessellation.

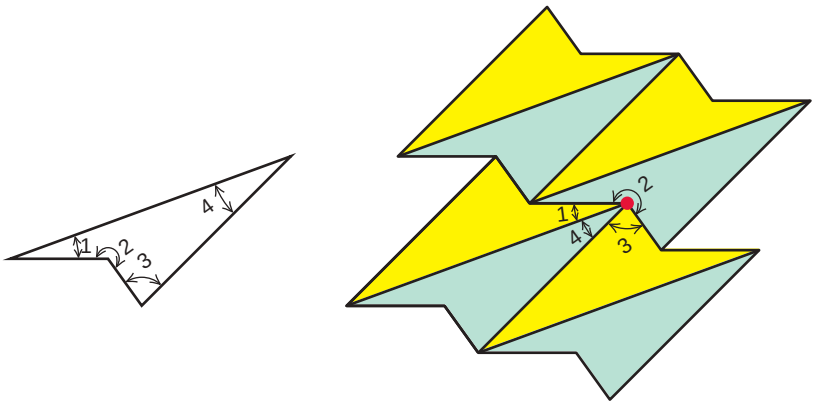


Figure 9.37

The quadrilateral in the tessellation in Figure 9.37 is concave. It is quite surprising that every quadrilateral, convex or concave, will tessellate. This is not true for polygons with more than four sides. Although there are some pentagons that will tessellate, there are others that will not tessellate. Similarly, some hexagons will tessellate (for example, a regular hexagon), but not all hexagons will.

If we consider only convex polygons, it can be proved that *no polygon with more than six sides* will tessellate. However, there are countless possibilities for tessellations of concave polygons of more than six sides. The tessellation in Figure 9.38 on the next page was made using a 12-sided concave polygon.

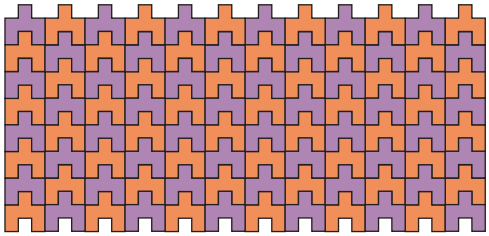


Figure 9.38

The only regular polygons that will tessellate by themselves are the equilateral triangle, the square, and the regular hexagon. However, if we allow two or more regular polygons in a tessellation, there are other possibilities. Two such tessellations are shown in Figure 9.39. The tessellation in part a uses three different regular polygons. Notice that each vertex is surrounded by the same arrangement of polygons: hexagon, square, triangle, and square. This can be denoted by a **code:** the number of sides of each polygon around a **vertex point.** For the tesselation in Figure 9.39a, the code is 6, 4, 3, 4 around every vertex.

A tessellation of two or more noncongruent regular polygons, in which each vertex is surrounded by the same arrangement of polygons, is called a **semiregular tessellation.** Figure 9.39b is a tessellation of regular polygons, but it is *not* semiregular, because some vertices are surrounded by two dodecagons and a triangle (12, 12, 3; see vertex *A*) and others by a dodecagon, two triangles, and a square (12, 3, 4, 3; see vertex *B*).

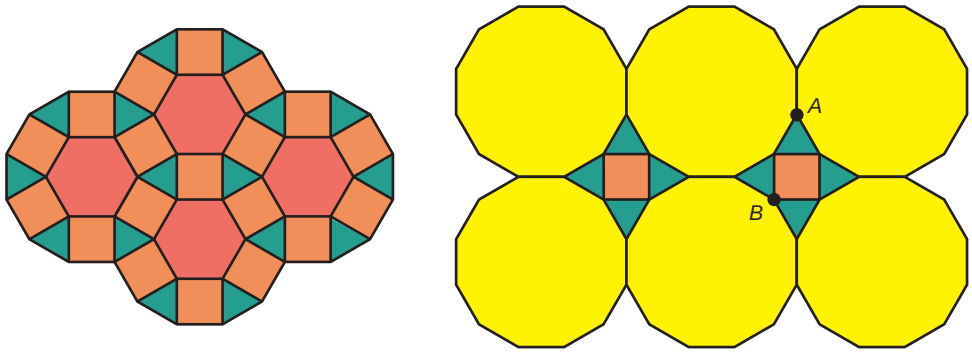
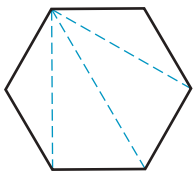


Figure 9.39 (a) (b)

PROBLEM-SOLVING APPLICATION

Problem

What is the measure of each vertex angle in a regular 50-sided polygon?



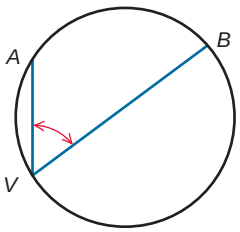
Understanding the Problem Consider a regular polygon with fewer sides. A regular hexagon has six congruent vertex angles, and since it can be partitioned into four triangles (see figure at left), the sum of all its angles is $4 \times 180° = 720°$. **Question 1:** What is the number of degrees in one of its vertex angles?

Devising a Plan The number of degrees in each vertex angle of a regular polygon can be determined once we know the sum of the degrees of all its angles. The total number of degrees in the angles of any polygon can be found by first partitioning the polygon into

Technology Connection

Inscribed Angles in Circles

What happens to the measure of angle *AVB* as point *V* moves around the circle while points *A* and *B* stay fixed? Use Geometer's Sketchpad® student modules available at the companion website to explore this surprising result and related questions in this investigation.



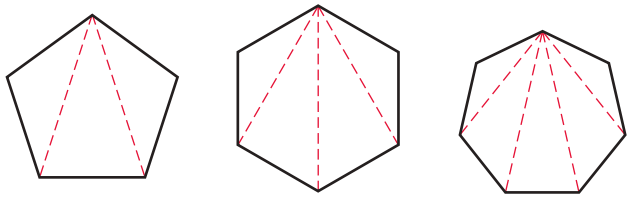
Inscribed angle

Mathematics Investigation
Chapter 9, Section 2
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triangles so that the vertices of the triangles are the same as the vertices of the polygon. Let's make a *table* to determine the number of such triangles for the first few polygons. **Question 2:** What is the number of triangles for a heptagon?

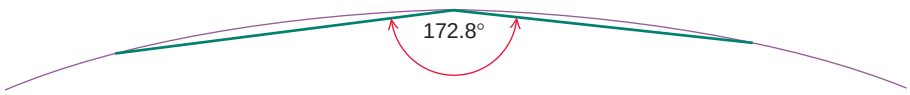
No. of sides	3	4	5	6	7	8	9
No. of triangles	1	2	3				
Total no. of degrees	180	2(180)	3(180)				

Carrying Out the Plan Pentagons, hexagons, and heptagons can be subdivided into three, four, and five triangles, respectively, as shown here. Notice that connecting one vertex of a polygon to each of the other nonadjacent vertices produces exactly one triangle for each of the nonadjacent vertices. This suggests that the number of triangles is 2 less than the number of vertices. Using this observation, we can calculate that the sum of the measures of the angles in a 50-sided polygon is $48 \times 180° = 8640°$. **Question 3:** What is the size of each vertex angle in a regular 50-sided polygon?



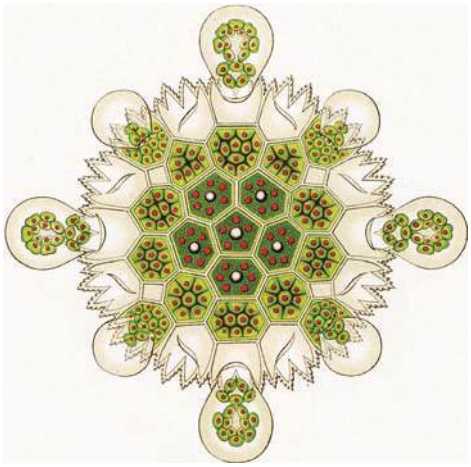
Pentagon Hexagon Heptagon

Looking Back As the number of sides in a regular polygon increases, the shape of the polygon gets closer to a circle and the size of each vertex angle gets closer to 180°. One of the vertex angles for a 50-sided regular polygon is shown in the figure below. **Question 4:** What is the number of degrees in each vertex angle of a regular 100-sided polygon?



Answers to Questions 1–4 **1.** 120° **2.** 5 **3.** 172.8° **4.** 176.4°

Exercises and Problems 9.2



Drawing of algae (sea life) by the German biologist Ernst Haeckel (1834–1919)

1. The polygons in Haeckel's drawing of algae form the beginning of a tessellation.
- **a.** There is a regular pentagon at the center. What polygons are adjacent to this pentagon? Are they regular?
- **b.** There is a second ring of polygons surrounding the inner six. What kind of polygons are these and what is interesting about how their numbers of sides vary?

Find the sum of all the vertex angles for each polygon in exercises 2 and 3. Use sketches to support your solutions.

2. **a.** A concave hexagon
b. Octagon

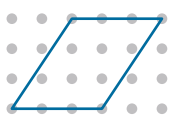
3. **a.** Decagon
b. A concave 15-sided polygon

The figures in exercises 4 and 5 have been drawn on a square lattice of dot paper. Determine whether each figure is a regular polygon. If it is not, write the condition or conditions that it does not satisfy.

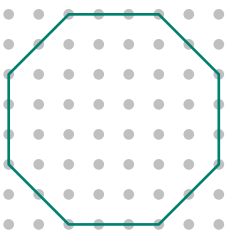
4. **a.**



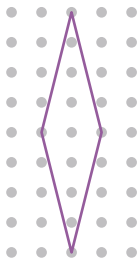
b.



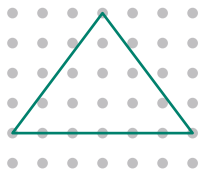
c.



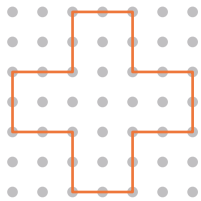
5. **a.**



b.

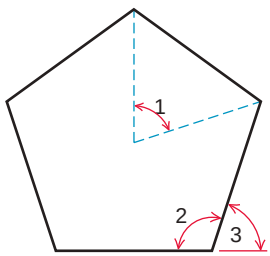


c.



6. The following figure is a regular pentagon, and $\measuredangle 1$ is a central angle. Determine the number of degrees in each of the following angles.

a. $\measuredangle 1$ **b.** $\measuredangle 2$ **c.** $\measuredangle 3$

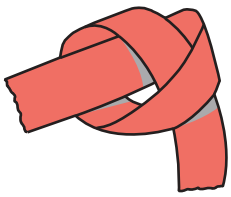


7. Write the number of degrees in the central angles of the regular polygons in the following table.

	Triangle	Quadrilateral	Pentagon	Hexagon	Heptagon	Octagon	Nonagon	Decagon		
No. of sides	3	4	5	6	7	8	9	10	20	100
Central angle	120°	90°								

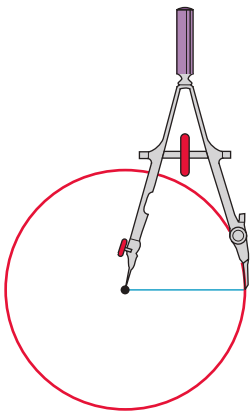
8. What regular polygons will be formed by the following methods?

a. Tie a long rectangular strip of paper into a knot, pull it tight, and smooth it down. (See drawing.)



b. Cut out an equilateral triangle and fold each vertex into the center.

9. a. Draw a circle with a compass. Open the compass an amount equal to the radius of the circle, use this distance to mark off points on the circumference, and connect adjacent points to form a polygon. What polygon results from your construction?



b. Connect pairs of nonadjacent points in part a to form a polygon. What polygon results from your construction?

The numbers of degrees in exercises 10 and 11 are the measures of the central angles of regular polygons. Determine the number of sides for each polygon.

10. a. 18° **b.** 10° **c.** 5°

11. a. 24° **b.** 20° **c.** 72°

Draw some figures to determine whether the statements in exercises 12 and 13 are true or false. For each false statement, show a counterexample.

12. a. In any quadrilateral the sum of the opposite angles is 180°.

b. A line segment from a vertex of a triangle to the midpoint of the opposite side is called a **median.** The three medians of a triangle meet in a point.

c. The diagonals of a regular hexagon are congruent.

13. a. If the midpoints of the adjacent sides of any regular hexagon are connected to form a simple closed curve, this curve is a regular hexagon.

b. A line segment from a vertex of a triangle that is perpendicular to the opposite side is called an **altitude.** The lines containing the three altitudes of a triangle meet in a point.

c. If the midpoints of the sides of any triangle are connected, an equilateral triangle is formed.

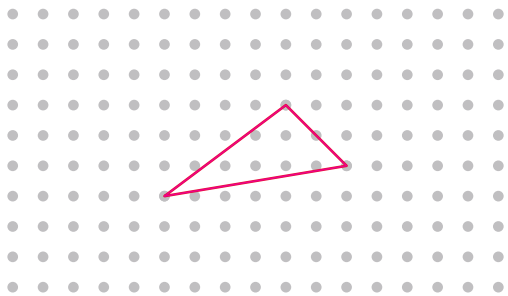
Which of the regular polygons in exercises 14 and 15 will tessellate by themselves? Illustrate and explain your answers.

14. a. Equilateral triangle **b.** Square **c.** Regular pentagon

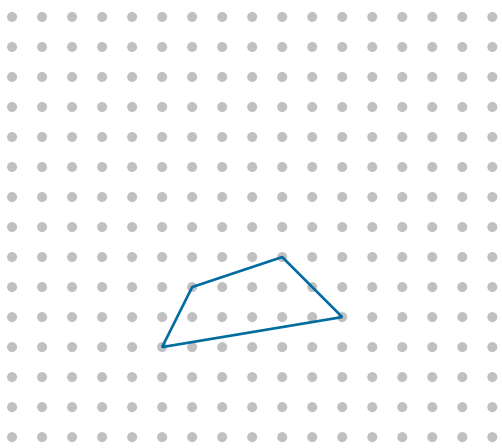
15. a. Regular hexagon **b.** Regular heptagon **c.** Regular octagon

16. What condition must be satisfied by the vertex angles of a regular polygon in order for the polygon to tessellate?

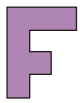
17. Form as large a tessellation as possible on the following grid, using the triangle. Explain why every triangle will tessellate. (The dot grid can be copied from the website.)



18. Form as large a tessellation as possible on the following grid, using the quadrilateral. Explain why every quadrilateral will tessellate. (The dot grid can be copied from the website.)



19. Draw a portion of a tessellation that can be made by using each of the following letters.

a.  b.  c. 

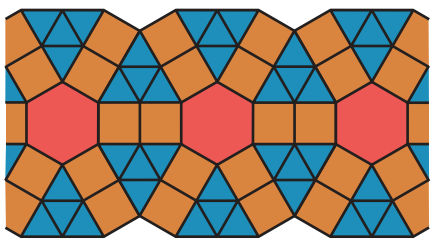
20. Find an uppercase block letter from the first half of the alphabet that will tessellate and that is different from those in exercise 19. Sketch a portion of your tessellation.

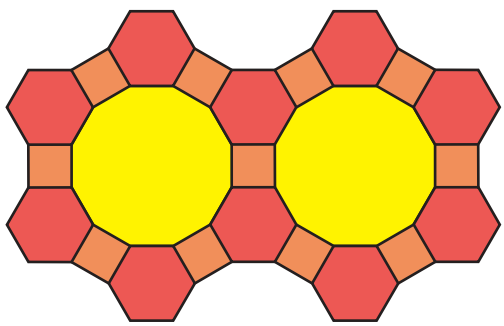
21. Find an uppercase block letter from the second half of the alphabet that will tessellate and that is different from those in exercise 19. Sketch a portion of your tessellation.

22. A polygon with more than six sides will not tessellate if it is convex. The following polygons have more than six sides, but they are concave. Sketch a portion of a tessellation for each of these polygons. (*Hint:* Trace and cut out one copy of the figure.)

a.  b.  c. 

23. Which of the following tessellations is semiregular? Explain why and give the vertex point codes.

a. 

b. 

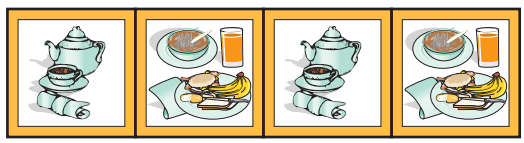
Reasoning and Problem Solving

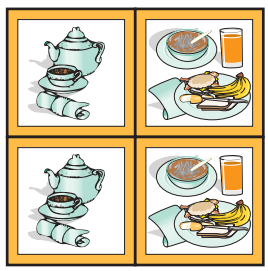
24. Featured Strategies: Solving a Simpler Problem and Making a Drawing. What is the smallest number of tacks needed to hold up 36 square pictures of the same size, so that each picture can be seen and each corner is tacked?

a. Understanding the Problem. If two pictures are tacked up separately, as shown in figure i, eight tacks will be required. How many tacks will be needed if the two pictures are placed side by side and slightly overlapping, as in figure ii?

i.  ii. 

b. Devising a Plan. Simplifying the problem and making a few drawings may provide some ideas. Consider only four pictures. How many tacks are needed for each of the following arrangements?

iii. 

iv. 

c. Carrying Out the Plan. The square arrangement of four pictures in figure iv suggests that we want as many "clusters" like this as possible so that one tack can be used for the corners of four pictures. Thus, we might conjecture that placing the 36 pictures in a square array as shown in figure v on the next page will minimize the number of tacks. How many tacks does this arrangement require?

v.

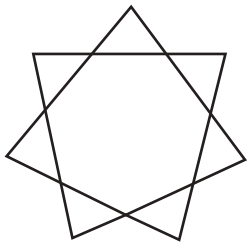


d. Looking Back. The grid in figure v suggests a method for finding the number of tacks for any square number of pictures. What is the smallest number of tacks needed for an 8 × 8 array of pictures?

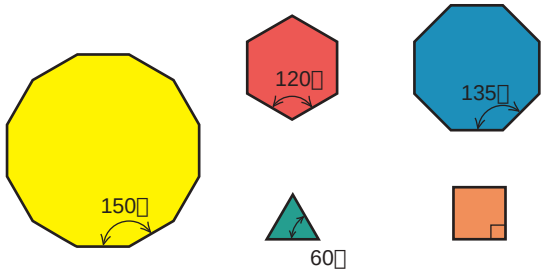
25. The Canadian nickel shown below is a regular dodecagon (12 sides). Assume that you have been asked to design a large posterboard model of this coin such that each side of the dodecagon has a length of 1 inch. Describe a method for constructing such a polygon.



26. The seven-pointed star shown below has a regular heptagon at its center. What is the measure of each angle at the tips of the star to the nearest .1°?



27. Semiregular tessellations can be made by using two or more of the following regular polygons. Sketch a portion of a semiregular tessellation that is different from the one shown in Figure 9.39a on page 599 and give the vertex point code for each. (*Hint:* Use the given measures of the vertex angles and Polygons for Tessellations in the Manipulative Kit or Virtual Manipulatives.)



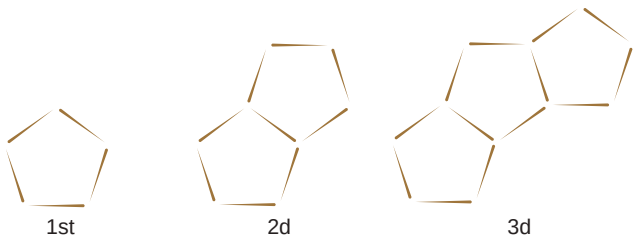
28. The following square figures are made of toothpicks.

a. How many toothpicks are needed to build the fourth figure?

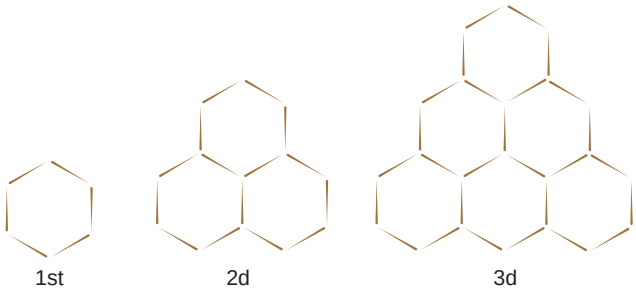
b. How many toothpicks are needed to build the 20th figure?



29. The first three figures in the following pattern contain pentagons formed by toothpicks. How many toothpicks will be needed to build the 30th figure in this pattern?



30. The first three figures in the following pattern contain hexagons formed by toothpicks. How many toothpicks will be needed to build the 10th figure?



31. Gestalt psychology, developed in Germany in the 1930s, is concerned primarily with the laws of perception. What is represented by the following polygons and their background?



Fold a piece of paper to obtain angles with the degree measurements in exercises 32 and 33. Label each angle, and explain how it was obtained.

32. **a.** 60° **b.** 45° **c.** 30°

33. **a.** 15° **b.** 150° **c.** 120°

Teaching Questions

1. A teacher showed her students that when three angles of a triangle are cut off and placed side by side with the vertices at a point, they form a straight angle of 180°, that is, one-half of a revolution about the point. A student asks if the same method works for any polygon. Try this method with a quadrilateral and pentagon. Describe your results and how you would respond to that student.

2. The following steps for tessellating with an arbitrary quadrilateral were suggested by a student.

1st Draw the quadrilateral on a piece of paper.
2nd Cut a copy of that quadrilateral from a separate piece of paper.
3rd Place the cutout copy on the drawing.
4th Rotate the copy 180° about the midpoint of an edge and trace around the copy.
5th Repeat 3rd and 4th steps for all sides of the original drawing.
6th Repeat 3rd and 4th steps for all sides of the newly created figures to extend the tessellation. Try this procedure with a convex and a concave quadrilateral. Summarize your results using illustrations and explain why you believe this method works or fails.

3. One of your students says that you can find the sum of the measures of the interior angles of any polygon by putting a point any place inside the polygon, connecting that point to each vertex, counting the number of triangles formed and multiplying that number by 180, and then subtracting 360. Try various convex and concave polygons to test this conjecture. Summarize your results using illustrations as you explain why you believe this method works or fails.

4. Nathan asked his teacher if it is possible to have an angle greater than 360 degrees. How would you respond to this question?

Classroom Connections

1. The **Standards** statement on page 598 stresses the importance of connecting geometric shapes to the world around us. Make a small photo collection of geometric shapes from nature and architecture that can be found in your community (from magazines, newspapers, brochures, etc.), and that could be shown to elementary students as examples of geometry in architecture and nature. Label each photo with a description of the geometric shape it represents.

2. As you travel around your local community make sketches of the various road signs that you see. Record the shape and statement on each sign. Try to make a connection between the general shape of the signs and the message each conveys. Do similar shapes convey similar types of messages? Explain.

3. Reflect back to your own middle-school math experiences. Write a brief statement about what you think accounts for the eighth-grade students' general lack of ability in computing angle measure in triangles, as indicated in the **Research** Statement on page 592.

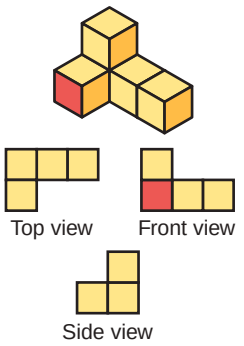
4. Studying and creating tessellations is a very popular topic in mathematics classes. Look through the **Grades 3–5 Standards—Geometry** (see inside front cover) and list the expectations that you believe are satisfied by studying tessellations. Explain how studying tessellations satisfies these expectations.

MATH ACTIVITY 9.3

Views of Cube Figures

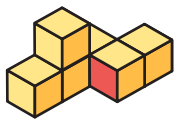
Purpose: Explore space figures by building and drawing views of cube figures.

Materials: Cubes for constructing figures; grid paper (copy from the website) for drawing views and blueprints.

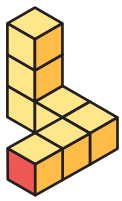


1. The cube figure at the left was constructed from five cubes. Looking at the figure from the top, the front (one face colored red), and the right side, gives three two-dimensional views of the cube figure. Build each figure in parts a, b, and c and sketch the top, front, and right side view of each figure. (The only hidden cubes are those supporting other cubes.)

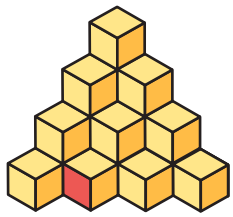
***a.**

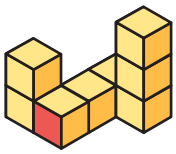


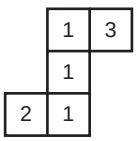
b.



c.



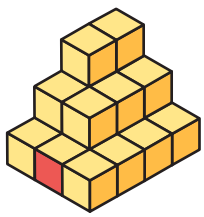




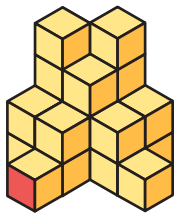
Blueprint

2. A *blueprint* for a cube figure can be made by showing a top view of the figure and by writing a number in each square to indicate the number of cubes in that column of the cube figure. For example, a cube figure and its blueprint are shown at the left. Sketch a blueprint for each of the cube figures in parts a, b, and c.

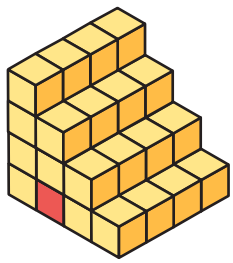
***a.**



b.



c.

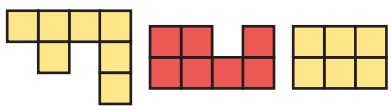


NCTM Standards

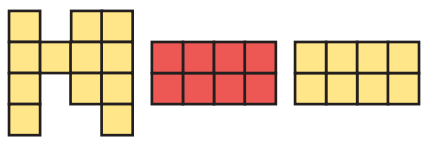
By representing three-dimensional shapes in two dimensions and constructing three-dimensional shapes from two-dimensional representations, students learn about the characteristics of shapes. p. 168

3. The top view, front view, and side views of two different cube figures are given below. Use the *fewest* number of cubes possible to construct each corresponding figure. Sketch a blueprint and give the total number of cubes used for each cube figure in parts a and b.

a.



b.

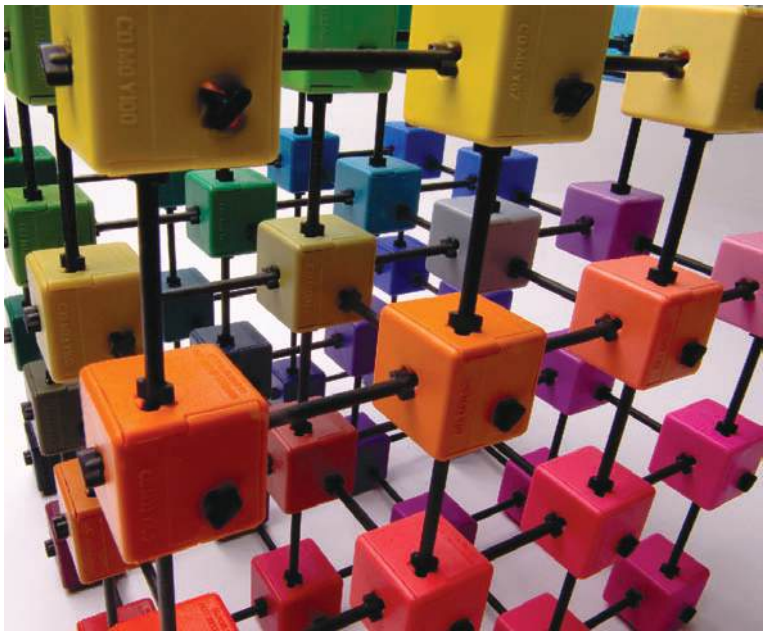


4. a. In activity 2, the term blueprint is defined for the TOP VIEW of a cube figure. Why does this give more information than defining blueprint for the FRONT VIEW or defining blueprint for the SIDE VIEW? Illustrate your explanation with an example.

b. If two people were each given a blueprint for a cube figure, would it be possible for each person to correctly build a corresponding cube figure and have the two cube figures be different? Illustrate your reasoning with examples.

Section **9.3**

SPACE FIGURES

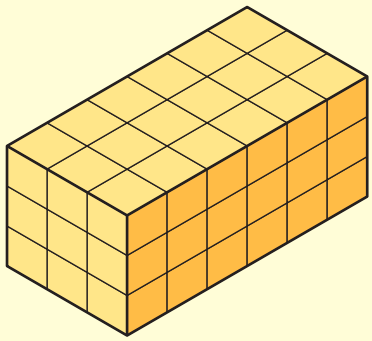


COLORCUBE 3D Color Puzzle

© 2000 Spittin' Image Software, Inc. New Westminster, British Columbia www.colorcube.com. Photograph by UNH Photo Graphic Services/ McGraw-Hill.

PROBLEM OPENER

This is a sketch of a three-dimensional figure that contains 54 small cubes. If the outside of the figure is painted and then the figure is disassembled into 54 individual cubes, how many cubes will have paint on one face, two faces, three faces, and no faces?



The COLORCUBE photo above shows a $4 \times 4 \times 4$ figure of 64 cubes and connecting rods. Imagine similar larger and larger cube structures that extend outwards in all directions to occupy, or fill, larger and larger regions of space.

The notion of **space** in geometry is an undefined term, just as the ideas of point, line, and plane are undefined. We intuitively think of space as three-dimensional and of a plane as only two-dimensional. In his theory of relativity, Einstein tied together the three dimensions of space and the fourth dimension of time. He showed that space and time affect each other and give us a four-dimensional universe.

NCTM Standards

The *Curriculum and Evaluation Standards for School Mathematics,* grades K–4, *Geometry and Spatial Sense* (p. 48), stresses the importance of spatial understanding:

> Insights and intuitions about two- and three-dimensional shapes and their characteristics, the interrelationships of shapes, and the effects of changes to shapes are important aspects of spatial sense. Children who develop a strong sense of spatial relationships and who master the concepts and language of geometry are better prepared to learn number and measurement ideas, as well as other advanced mathematical topics.

HISTORICAL HIGHLIGHT



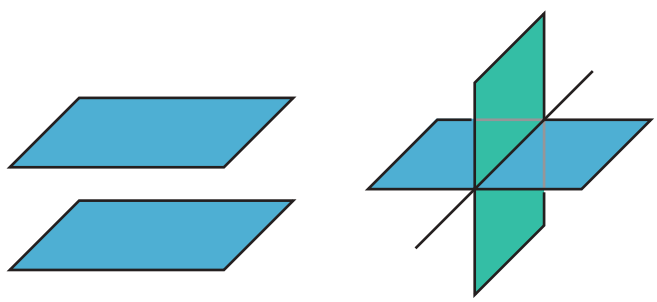
Sonya Kovalevsky, 1850–1891

The Russian mathematician Sonya Kovalevsky is regarded as the greatest woman mathematician to have lived before 1900. Since women were barred by law from institutions of higher learning in Russia, Kovalevsky attended Heidelberg University in Germany. Later she was refused admission to the University of Berlin, which also barred women. Even the famous mathematician Karl Weierstrass, who claimed she had "the gift of intuitive genius," was unable to obtain permission for Kovalevsky to attend his lectures. She obtained her doctorate from the University of Göttingen but was without a teaching position for nine years, until the newly formed University of Stockholm broke tradition and appointed her to an academic position. Kovalevsky's prominence as a mathematician reached its peak in 1888, when she received the famous Prix Bordin from the French Académie des Sciences for her research paper "On the Rotation of a Solid Body about a Fixed Point." The selection committee "recognized in this work not only the power of an expansive and profound mind, but also a great spirit of invention."*

*D. M. Burton, *The History of Mathematics,* 7th ed. (New York: McGraw-Hill, 2010) pp. 615–617.

PLANES

In two dimensions, the figures (lines, angles, polygons, etc.) all occur in a plane. In three dimensions, there are an infinite number of planes. Each plane partitions space into three disjoint sets: the points on the plane and two **half-spaces.** Representations of a few planes are shown in Figure 9.40. Any two planes either are *parallel,* as in part a, or *intersect* in a line, as in part b.



(a) (b)

Figure 9.40

When two planes intersect, we call the angle between the planes a **dihedral angle.** Figure 9.41 shows three dihedral angles and their measures. A dihedral angle is measured by measuring the angle whose sides lie in the planes and are perpendicular to the line of intersection of the two planes. Parts a, b, and c of Figure 9.41 show examples of obtuse, right, and acute dihedral angles, respectively.

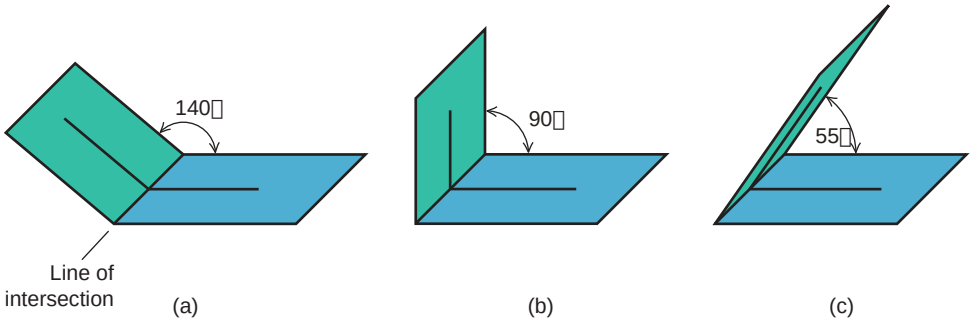


Figure 9.41

When a line m in three-dimensional space does not intersect a plane P, it is **parallel to the plane,** as in Figure 9.42a. A line n is **perpendicular to a plane** Q at a point K if the line is perpendicular to every line in the plane that contains point K, as in Figure 9.42b.

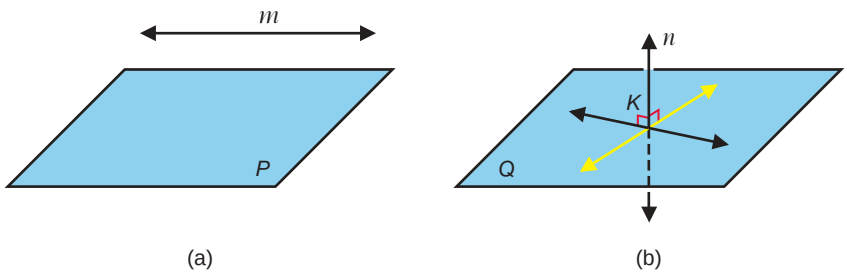


Figure 9.42

POLYHEDRA

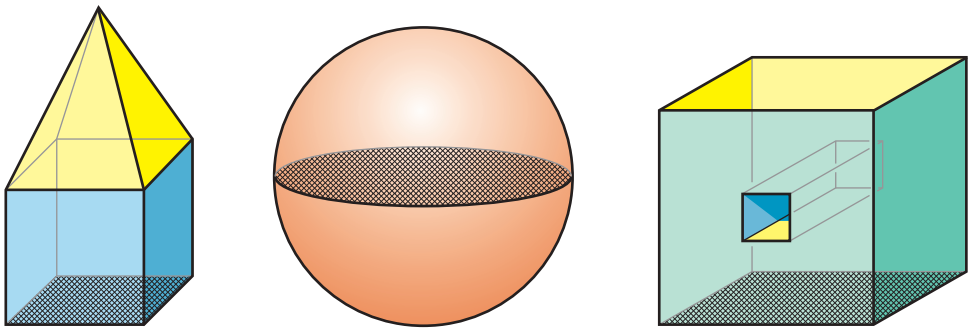
The three-dimensional object with flat sides in Figure 9.43 is a crystal of pyrite that is embedded in rock. Its 12 flat pentagonal sides with their straight edges were not cut by people but were shaped by nature.



Figure 9.43
Crystal of pyrite

The surface of a figure in space whose sides are polygonal regions, such as the one in Figure 9.43, is called a **polyhedron** (*polyhedra* is the plural). The polygonal regions are called **faces,** and they intersect in the **edges** and **vertices** of the polyhedron. The union of a

polyhedron and its interior is called a **solid.** Figure 9.44 shows examples of a polyhedron and two figures that are not polyhedra. The figure in part a is a polyhedron because its faces are polygonal regions. The figures in parts b and c are not polyhedra because one has a curved surface and the other has two faces that are not polygons.



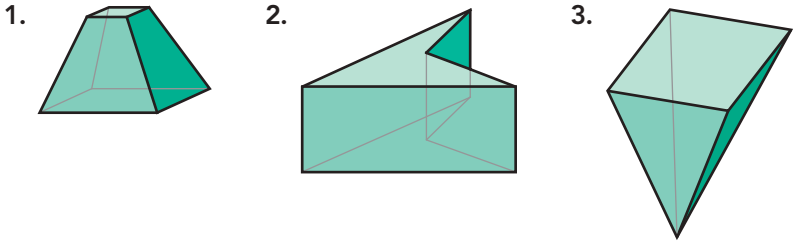
(a) (b) (c)

Figure 9.44

A polyhedron is **convex** if the line segment connecting any two of its points is contained inside the polyhedron or on its surface.

EXAMPLE A

Classify the following polyhedra as convex or concave.



Solution Polyhedra 1 and 3 are convex; 2 is concave.

NCTM Focal Points

Students relate two-dimensional shapes to three-dimensional shapes and analyze properties of polyhedral solids, describing them by the number of edges, faces, or vertices as well as types of faces. p. 33

REGULAR POLYHEDRA

The best known of all the polyhedra are the *regular polyhedra,* or *Platonic solids.* A **regular polyhedron** is a convex polyhedron whose faces are *congruent regular polygons,* the same number of which meet at each vertex. The ancient Greeks proved that there are only five regular polyhedra. Models of these polyhedra are shown in Figure 9.45. The **tetrahedron** has 4 triangles for faces; the **cube** has 6 square faces; the **octahedron** has 8 triangular faces; the **dodecahedron** has 12 pentagons for faces; and the **icosahedron** has 20 triangular faces.

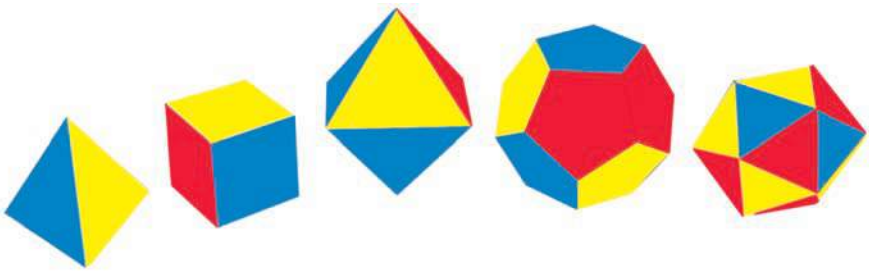


Figure 9.45
From left to right: tetrahedron, cube (hexahedron), octahedron; dodecahedron, icosahedron

The first four of the regular polyhedra shown on the preceding page in Figure 9.45 can be found in nature as crystals of pyrite. The cube and octahedron are shown in Figure 9.46 below and the dodecahedron is shown in Figure 9.43 on page 609. The cube, which is embedded in rock, was found in Vermont, and the octahedron is from Peru. The other regular polyhedron, the icosahedron, does not occur as crystals but has been found in the skeletons of microscopic sea animals called *radiolarians.*



Figure 9.46
Crystals of pyrite

Semiregular Polyhedra Some polyhedra have two or more different types of regular polygons for faces. The faces of the boracite crystal in Figure 9.47 are squares and equilateral triangles. This crystal, too, developed its flat, regularly shaped faces naturally, without the help of machines or people. Polyhedra whose faces are two or more regular polygons with the same arrangement of polygons around each vertex are called **semiregular polyhedra.** The boracite crystal is one of these. Each of its vertices is surrounded by three squares and one equilateral triangle.

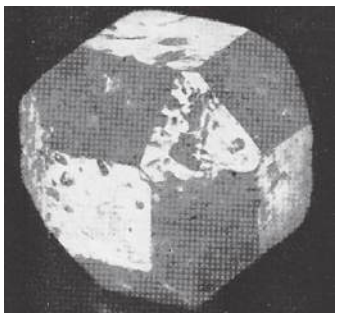


Figure 9.47
Crystal of boracite

Several other semiregular polyhedra are shown in Figure 9.48. You may recognize the combination of hexagons and pentagons in part a as the pattern used on the surface of soccer balls.

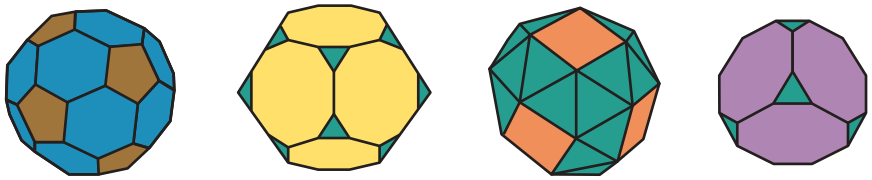


Figure 9.48

EXAMPLE B

For each semiregular polyhedron in Figure 9.48 on the previous page, list the polygons in the order in which they occur about any vertex.

Solution Part a: hexagon, hexagon, pentagon; part b: decagon, decagon, triangle; part c: triangle, triangle, triangle, triangle, square; part d: octagon, octagon, triangle.

PYRAMIDS AND PRISMS

Chances are that when you hear the word *pyramid,* you think of the monuments built by the ancient Egyptians. Each of the Egyptian pyramids has a square base and triangular sides rising up to the vertex. This is just one type of pyramid. In general, the **base of a pyramid** can be any polygon, but its sides are always triangular. Pyramids are named according to the shape of their bases. Church spires are familiar examples of pyramids. They are usually square, hexagonal, or octagonal pyramids. The spire in the photograph in Figure 9.49 is a hexagonal pyramid that sits on the octagonal roof that is supported by eight columns of the housing for the bell.



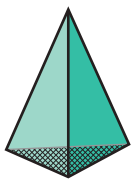
Figure 9.49
Community Church of Durham, New Hampshire

Several pyramids with different bases are shown in the following example. Pyramids whose sides are isosceles triangles, as in Figures (1), (3), and (4) of Example C, are called **right pyramids.** Otherwise, as in Figure (2) of Example C, the pyramid is called an **oblique pyramid.** The vertex that is not contained in the pyramid's base is called the **apex.**

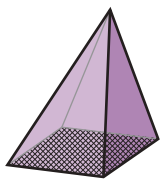
EXAMPLE C

Determine the name of each pyramid.

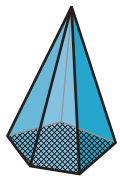
1.



2.



3.



4.



Solution **1.** Triangular pyramid. **2.** Oblique square pyramid. **3.** Pentagonal pyramid. **4.** Hexagonal pyramid.

Prisms Prisms are another common type of polyhedron. You probably remember from your science classes that a prism is used to produce the spectrum of colors ranging from violet to red. Because of the angle between the vertical faces of a prism, light directed into one face will be bent when it passes out through the other face (Figure 9.50).

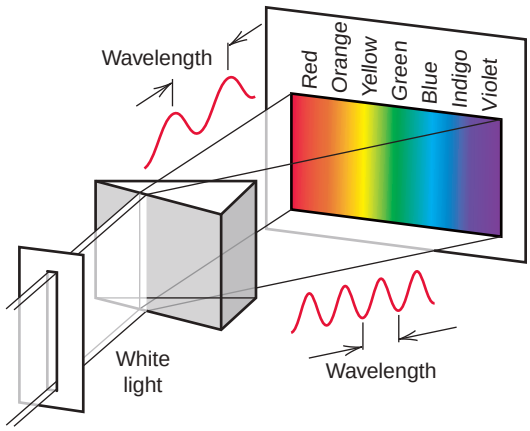


Figure 9.50

A **prism** has two parallel **bases,** upper and lower, which are congruent polygons. Like pyramids, prisms get their names from the shape of their bases. If the lateral sides of a prism are perpendicular to the bases, as in the case of the triangular, quadrilateral, hexagonal, and rectangular prisms in Figure 9.51, they are rectangles. Such a prism is called a **right prism.** A rectangular prism, which is modeled by a box, is the most common type of prism. If some of the lateral faces of a prism are parallelograms that are not rectangles, as in the pentagonal prism, the prism is called an **oblique prism.** The union of a prism and its interior is called a **solid prism.** A rectangular prism that is a solid is sometimes called a **rectangular solid.**

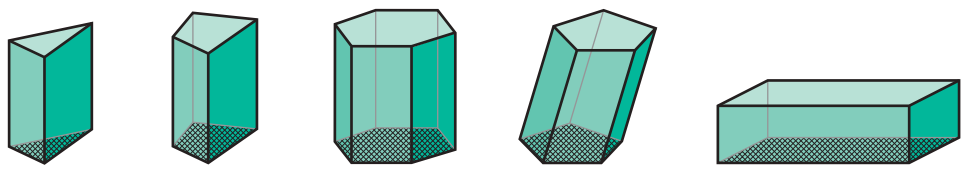
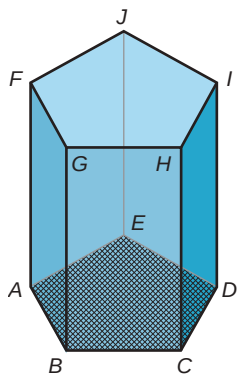


Figure 9.51

EXAMPLE D The following figure is a right prism with bases that are regular pentagons.



Research Statement

In order to develop a conceptual understanding of geometry, students need to be placed in situations that allow them to apply deductive, inductive, and spatial reasoning.

Geddes and Fortunato

1. What is the measure of the dihedral angle between face *ABGF* and face *BCHG*?
2. What is the measure of the dihedral angle between face *GHIJF* and face *CDIH*?
3. Name two faces that are in parallel planes.

Solution **1.** It is the same as the measure of $\measuredangle FGH$, which is 108°. **2.** 90°. Since this is a right prism, the top base is perpendicular to each of the vertical sides. **3.** *ABCDE* and *FGHIJ*.

The two oblique hexagonal prisms in Figure 9.52 are crystals that grew with these flat, smooth faces and straight edges. Their lateral faces are parallelograms.

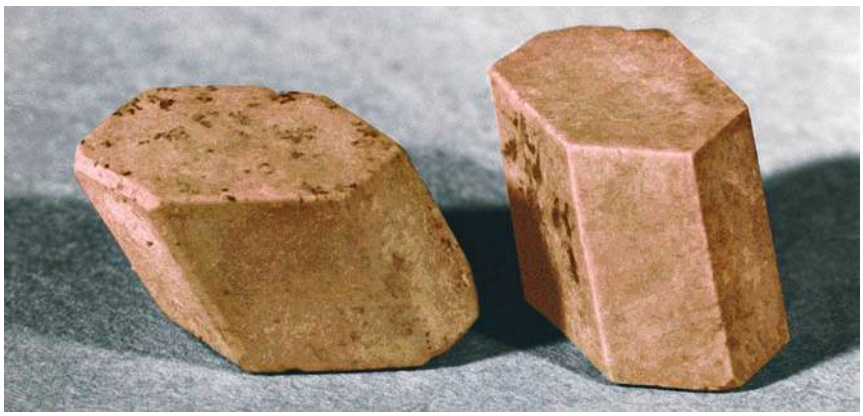


Figure 9.52
Prisms of the crystal orthoclase feldspar

CONES AND CYLINDERS

Cones and cylinders are the circular counterparts of pyramids and prisms. Ice cream cones, paper cups, and party hats are common examples of cones. A cone has a circular region (disk) for a **base** and a lateral surface that slopes to the **vertex (apex).** If the vertex lies directly above the center of the base, the cone is called a **right cone** or usually just a cone; otherwise, it is an **oblique cone** (Figure 9.53).

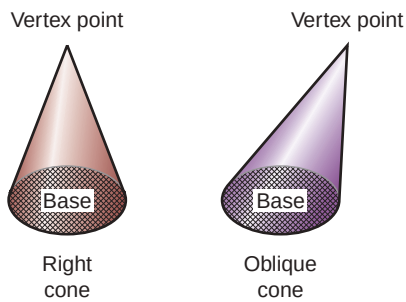


Figure 9.53

Ordinary cans are models of cylinders. A **cylinder** has two parallel circular **bases** (disks) of the same size and a lateral surface that rises from one base to the other. If the centers of the upper base and lower base lie on a line that is perpendicular to each base, the cylinder is called a **right cylinder** or simply a cylinder; otherwise, it is an **oblique cylinder** (Figure 9.54). Almost without exception, the cones and cylinders we use are right cones and right cylinders.

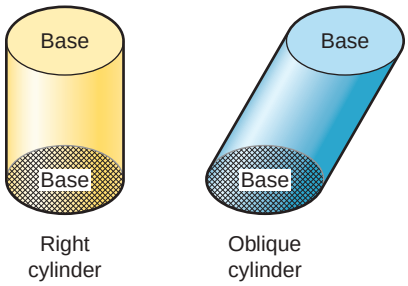


Figure 9.54

SPHERES AND MAPS

The photograph in Figure 9.55 is a view of Earth showing its almost perfect spherical shape. It was photographed from the *Apollo 17* spacecraft during its 1972 lunar mission. The dark regions are water. The Red Sea and the Gulf of Aden are near the top center, and the Arabian Sea and Indian Ocean are on the right.



Figure 9.55
View of Earth as seen by the *Apollo 17* crew traveling toward the Moon. This view extends from the Mediterranean Sea to the Antarctica south polar ice cap. Almost the entire coastline of Africa is visible and the Arabian Peninsula can be seen at the northeastern edge of Africa. The large island off the coast of Africa is the Malagasy Republic and the Asian mainland is on the horizon toward the northeast.

Sphere A **sphere** is the set of points in space that are the same distance from a fixed point, called the **center.** The union of a sphere and its interior is called a **solid sphere.**

A line segment joining the center of a sphere to a point on the sphere is called a **radius.** The length of such a line segment is also called the **radius of the sphere.** A line segment containing the center of the sphere and whose endpoints are on the sphere is called a **diameter,** and the length of such a line segment is called the **diameter of the sphere.**

The geometry of the sphere is especially important for navigating on the surface of the Earth. You may have noticed that airline maps show curved paths between distant cities. This is because the shortest distance between two points on a sphere is along an arc of a *great circle.* In the drawing of the sphere in Figure 9.56 on page 617, the red arc between points *X* and

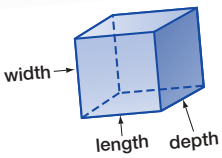
Explore 11-8

Geometry Lab
Three-Dimensional Figures

MAIN IDEA

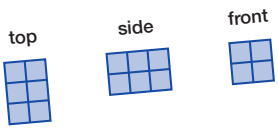
Build three-dimensional figures given the top, side, and front views.

Cubes are examples of three-dimensional figures because they have length, width, and depth. In this lab, you will use centimeter cubes to build other three-dimensional figures.



ACTIVITY

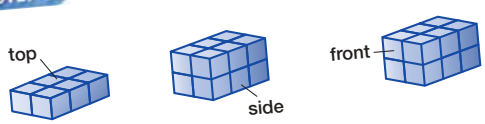
The top view, side view, and front view of a three-dimensional figure are shown below. Use centimeter cubes to build the figure. Then make a sketch of the figure.



STEP 1 Use the top view to build the figure's base.

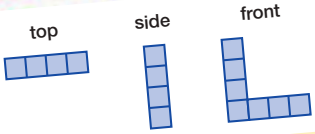
STEP 2 Use the side view to complete the figure.

STEP 3 Use the front view to check the figure.

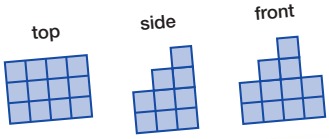


CHECK Your Progress

a.



b.

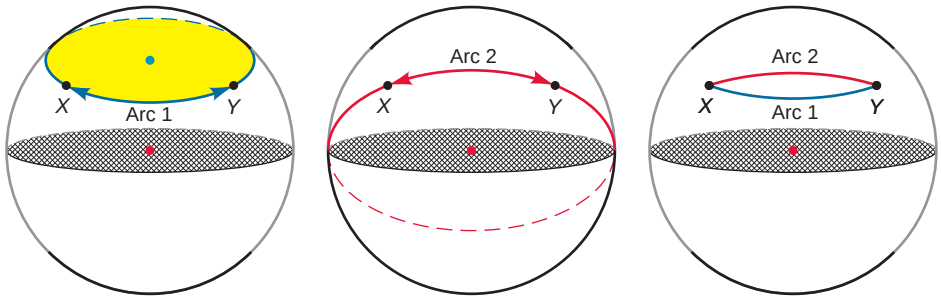


Analyze the Results

1. Explain how you began building the figures in Exercises a and b.
2. Determine whether there is more than one way to build each model. Explain your reasoning.
3. Build two different models that would look the same from two views, but not the third view. Draw a top view, side view, and front view of each model.
4. Describe a real-world situation where it might be necessary to draw a top, side, and front view of a three-dimensional figure.

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Y, Figure 9.56b, is the arc of a **great circle,** because the center of the red circle is also the center of the sphere. However, the arc of the blue circle between points *X* and *Y,* Figure 9.56a, is not the arc of a great circle. So the distance between points *X* and *Y* along the red arc on the sphere is less than the distance between these points along the blue arc. Although Arc 1 and Arc 2 have different lengths on a 3-D sphere, the lengths of these arcs do not appear to be significantly different in Figure 9.56c because we are representing the arcs on a 2-D surface.



Arc 1 is the portion marked by arrows on the blue circle

(a)

Arc 2 is the portion marked by arrows on the red (great) circle

(b)

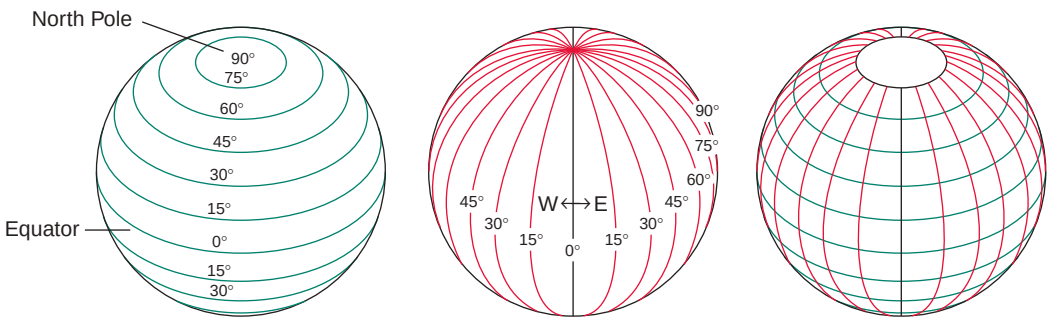
On a 3-D sphere, the length of arc 2 is less than the length of arc 1

(c)

Figure 9.56

Locations on the Earth's surface are often given by naming cities, streets, and buildings. A more general method of describing location uses two systems of circles (Figure 9.57). The circles that are parallel to the equator are called **parallels of latitude** and are shown in part a. Except for the equator, these circles are not great circles. Each parallel of latitude is specified by an angle from 0° to 90°, both north and south of the equator. For example, New York City is at a northern latitude of 41°, and Sydney, Australia, is at a southern latitude of 34°.

The second system of circles is shown in part b. These circles pass through the north and south poles and are called **meridians of longitude.** These are great circles, and each is perpendicular to the equator. Since there is no natural point at which to begin numbering the meridians of longitude, the meridian that passes through Greenwich, England, was chosen as the zero meridian. Each meridian of longitude is given by an angle from 0° to 180°, both east and west of the zero meridian. The longitude of New York City is 74° west, and that of Sydney, Australia, is 151° east. These parallels of latitude and meridians of longitude, shown together in part c, form a grid or coordinate system for locating any point on the Earth.



Parallels of latitude

(a)

Meridians of longitude

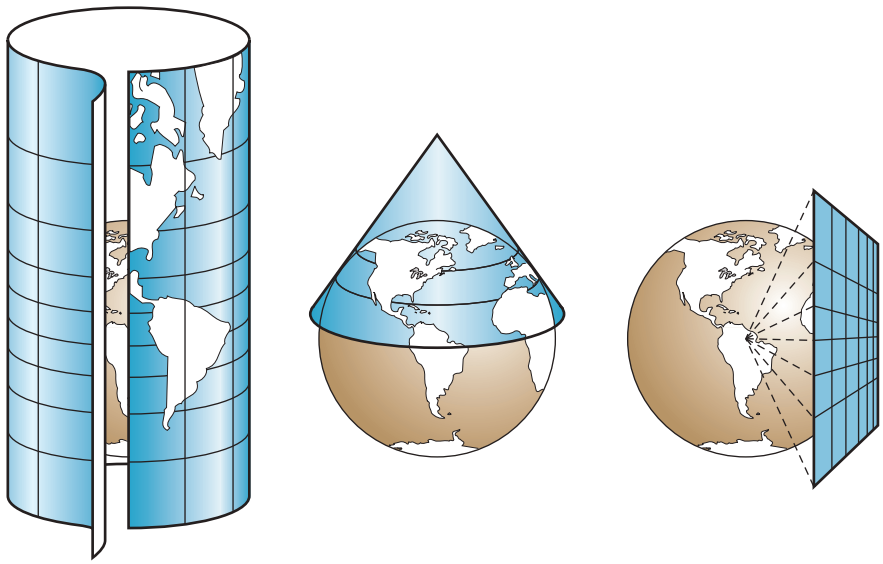
(b)

Grid formed by both types of circles

(c)

Figure 9.57

Map Projections The globe is a spherical map of the Earth. While such a map accurately represents the Earth's shape and relative distances, we cannot see the whole globe at one time, nor can distances be measured easily. Maps on a flat surface are much more convenient. However, since a sphere cannot be placed flat on a plane without separating or overlapping some of its surface, making flat maps of the Earth is a problem. There are three basic solutions: copying the Earth's surface onto a cylinder, a cone, or a plane (Figure 9.58). These methods of copying are called **map projections.** In each case, some distortions of shapes and distances occur.



Cylindrical projection (a) Conic projection (b) Plane projection (c)

Figure 9.58

A **cylindrical projection** (part a), also called a **mercator projection,** is obtained by placing a cylinder around a sphere and copying the surface of the sphere onto the cylinder. The cylinder is then cut to produce a flat map. Regions close to the equator are reproduced most accurately. The closer we get to the poles, the more the map is distorted.

A **conic projection** (part b) is produced by placing a cone with its apex over one of the poles and copying a portion of the surface of a sphere onto the cone. The cone is then cut and laid flat. This type of map construction is commonly used for countries that lie in an east-west direction and are middle latitude countries, as opposed to those near the poles or equator. The maps of the United States that are issued by the American Automobile Association are conical projections.

A **plane projection** (part c), also called an **azimuthal projection,** is made by placing a plane next to any point on a sphere and projecting the surface onto the plane. To visualize this process, imagine a light at the center of the sphere, and think of the boundary of a country as being pierced with small holes. The light shining through these holes, as shown by the dashed lines in part c, forms an image of the country on the plane. Less than one-half of the sphere's surface can be copied onto a plane projection, with the greatest distortion taking place at the outer edges of the plane. A plane projection, unlike cylindrical and conical projections, has the advantage that the distortion is uniform from the center of the map to its edges. Plane projections are used for hemispheres and maps of the Arctic and Antarctic. To map the polar regions, a plane is placed perpendicular to the Earth's axis in contact with the north or south pole.

PROBLEM-SOLVING APPLICATION

There is a remarkable formula that relates the numbers of vertices, edges, and faces of a polyhedron. This formula was first stated by René Descartes about 1635. In 1752 it was discovered again by Leonhard Euler and is now referred to as **Euler's formula.** See if you can discover this formula, either before or as you read the parts of the solution presented below.

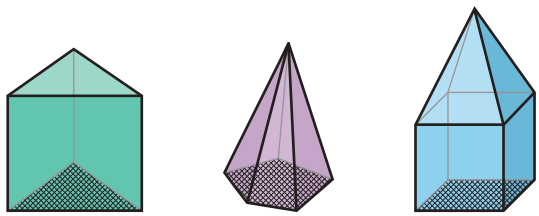
Problem

What is the relationship among the numbers of faces, vertices, and edges of a polyhedron?



Understanding the Problem Euler's formula holds for all polyhedra. Let's look at a specific example. A die is a cube that has six faces. **Question 1:** How many vertices and edges does it have?

Devising a Plan Let's *make a table;* list the numbers of faces, vertices, and edges for several polyhedra; and look for a relationship. **Question 2:** What are the numbers of faces, vertices, and edges for the polyhedra in figures (a), (b), and (c)?



(a) (b) (c)

Carrying Out the Plan The following table contains the numbers of faces, vertices, and edges for the cube in the margin above and the preceding polyhedra in figures (a) through (c). Using F for the number of faces, V for the number of vertices, and E for the number of edges, we can construct Euler's formula from these data. **Question 3:** What is Euler's formula?

	F	V	E
Cube	6	8	12
Figure (a)	5	6	9
Figure (b)	6	6	10
Figure (c)	9	9	16

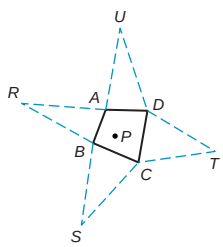
Looking Back You may remember that an icosahedron has 20 triangular faces, but may not remember the number of edges or vertices. Altogether, 20 triangles have a total of 60 edges. Since every two edges of a triangle form one edge of an icosahedron, this polyhedron has $60 \div 2 = 30$ edges. Given the numbers of faces and edges for the icosahedron and Euler's formula $F + V - 2 = E$, we can determine the number of vertices. **Question 4:** How many vertices are there?

Answers to Questions 1–4 **1.** 8 vertices and 12 edges. **2.** Figure (a): 5 faces, 6 vertices, 9 edges; figure (b): 6 faces, 6 vertices, 10 edges; figure (c): 9 faces, 9 vertices, 16 edges. **3.** $F + V - 2 = E$. **4.** 12; $20 + V - 2 = 30$.

Laboratory Connections

Pyramid Patterns

Determine how to construct a pattern that will fold to make a pyramid with a given polygonal base and an apex above a given point in the base. Explore this and related questions in this investigation.



Mathematics Investigation
Chapter 9, Section 3
www.mhhe.com/bbn

HISTORICAL HIGHLIGHT



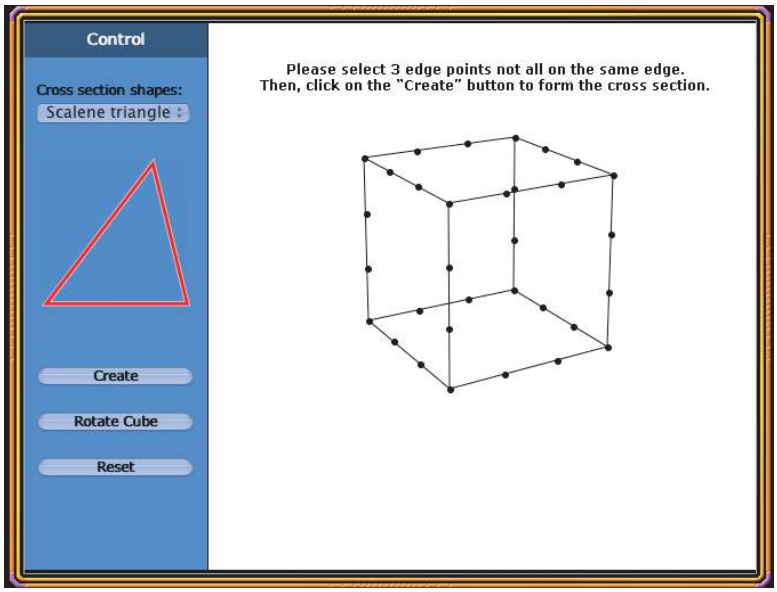
Leonhard Euler,
1707–1783

Switzerland's Leonhard Euler is considered to be the most prolific writer in the history of mathematics. He published over 850 books and papers, and most branches of mathematics contain his theorems. After he became totally blind at the age of 60, he continued his amazing productivity for 17 years by dictating to a secretary and writing formulas in chalk on a large slate. On the 200th anniversary of his birthday in 1907, a Swiss publisher began reissuing Euler's entire collected works; the collection is expected to run to 75 volumes of about 60 pages each.*

*H. W. Eves, *In Mathematical Circles* (Boston: Prindle, Weber and Schmidt, 1969), pp. 46–49.

Technology Connection

How would you cut this cube into two parts with one straight slice so that the cross section is a triangle? A trapezoid? This applet lets you select points on the edges of the cube for your slices and then rotate the cube for a better perspective of the resulting cross section.



Cross-Sections of a Cube Applet, Chapter 9, Section 3
www.mhhe.com/bbn

Exercises and Problems 9.3

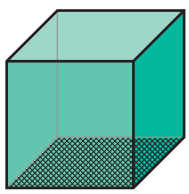


Crystals of calcite—hexagonal prisms formed by nature

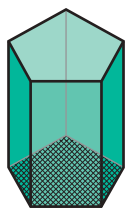
1. The crystals crowded together in the photograph are growing with flat polygonal faces.
- **a.** What type of polygon is the top face of these crystals?
- **b.** What type of polyhedron is formed by these crystals?

Which of the figures in exercises 2 and 3 are polyhedra?

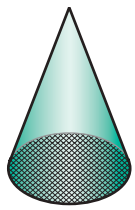
2. a.



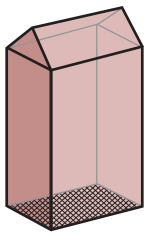
b.



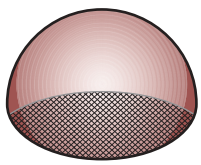
c.



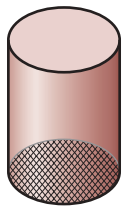
3. a.



b.

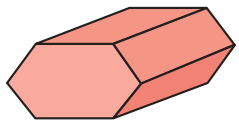


c.

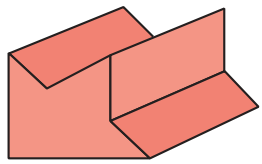


Classify the polyhedra in exercises 4 and 5 as convex or concave.

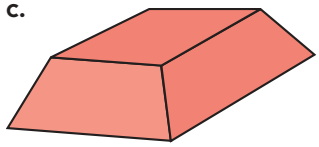
4. a.



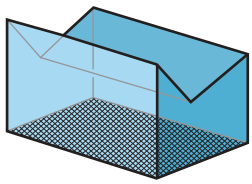
b.



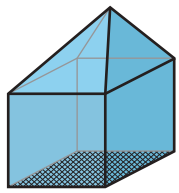
c.



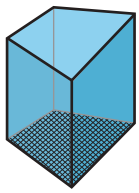
5. a.



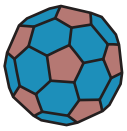
b.



c.



The semiregular polyhedra are classified according to the arrangement of regular polygons around each vertex. Proceeding counterclockwise, list the polygons about a vertex of each polyhedron in exercises 6 and 7.

6. a. 

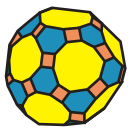
20 hexagons
12 pentagons

b. 

32 triangles
6 squares

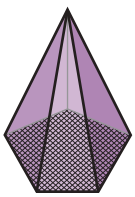
7. a. 

8 triangles
6 squares

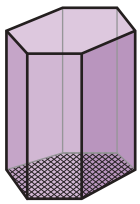
b. 

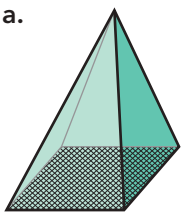
20 hexagons
30 squares
12 decagons

Name each of the figures in exercises 8 and 9.

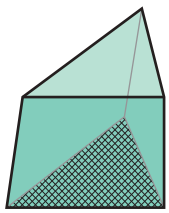
8. a. 

b. 

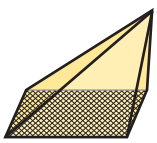
c. 

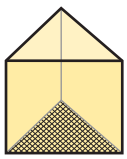
9. a. 

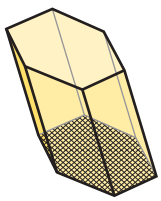
b. 

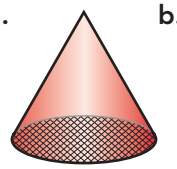
c. 

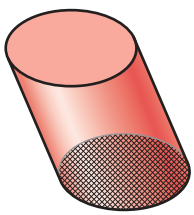
Name the figures in exercises 10 and 11, and also state whether they are right or oblique.

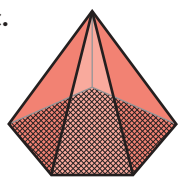
10. a. 

b. 

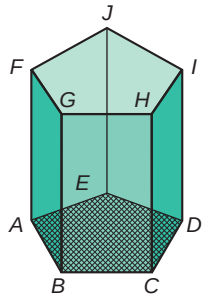
c. 

11. a. 

b. 

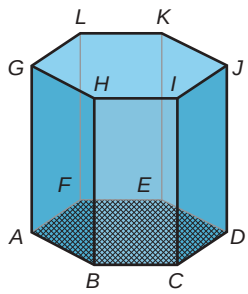
c. 

12. The polyhedron below is a right pentagonal prism whose bases are regular polygons.



a. What face is parallel to face *ABCDE*?
b. What is the measure of the dihedral angle between face *ABGF* and face *BCHG*?
c. What is the measure of the dihedral angle between face *FGHIJ* and face *EDIJ*?

13. The polyhedron shown here is a right prism, and its bases are regular hexagons.



a. What face is parallel to face *GHIJKL*?
b. What face is parallel to face *IJDC*?
c. What is the measure of the dihedral angle between face *ABHG* and face *ABCDEF*?
d. What is the measure of the dihedral angle between face *ABHG* and face *BCIH*?

Which of the three types of projections is best suited for making flat maps of the regions in exercises 14 and 15?

14. a. Australia
b. North, Central, and South America
c. The entire equatorial region between 30° north latitude and 30° south latitude

15. a. Arctic region
b. Western hemisphere between 20° north and 20° south
c. United States

Each of the geometric shapes listed in exercises 16 and 17 can be seen in the photograph. Locate these objects.



Thompson Hall, University of New Hampshire

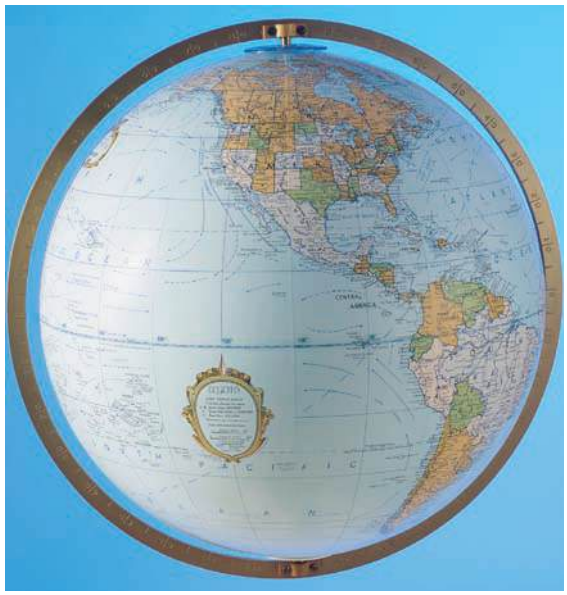
16. **a.** Cone **b.** Pyramid **c.** Cylinder
d. Sphere **e.** Circle

17. **a.** Obtuse angle **b.** Rectangle **c.** Semicircle
d. Square **e.** Isosceles triangle

18. Use the photo with exercise 19 and your knowledge of the spherical coordinate system to match each of the following cities with its longitude and latitude.

Tokyo	38°N and 120°W
San Francisco	56°N and 4°W
Melbourne	35°N and 140°E
Glasgow	35°S and 20°E
Capetown	38°S and 145°E

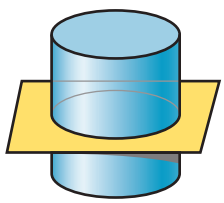
19. Two points on the Earth's surface that are on opposite ends of a line segment through the center of the Earth are called **antipodal points.** The coordinates of such points are nicely related. The latitude of one point is as far above the equator as that of the other is below, and the longitudes are supplementary angles (in opposite hemispheres). For example, (30°N, 15°W) is off the west coast of Africa near the Canary Islands, and its antipodal point (30°S, 165°E) is off the eastern coast of Australia.



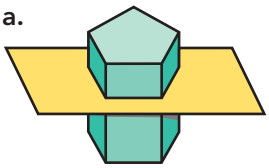
a. This globe shows that (20°N, 120°W) is a point in the Pacific Ocean just west of Mexico. Its antipodal point is just east of Madagascar. What are the coordinates of this antipodal point?
b. The point (30°S, 80°E) is in the Indian Ocean. What are the coordinates of its antipodal point? In what country is it located?

20. China is bounded by latitudes of 20°N and 55°N and by longitudes of 75°E and 135°E. It is playfully assumed that if you could dig a hole straight through the center of the Earth, you would come out in China. For which of the following starting points is this true?
a. Guayaquil, Ecuador (2°S, 79°W)
b. Buenos Aires (35°S, 58°W)
c. New York (41°N, 74°W)
d. Rio de Janeiro, Brazil (22°S, 43°W)

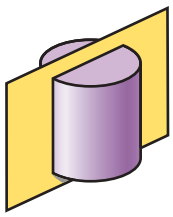
The intersection of a plane and a three-dimensional figure is called a **cross section.** The cross section produced by the intersection of a plane and a right cylinder, where the plane is parallel to the base of the cylinder (see figure), is a circle. Determine the cross sections of the figures in exercises 21 and 22.



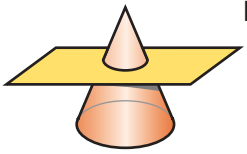
21. a.



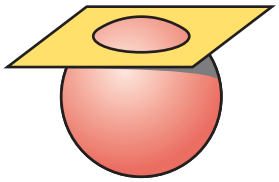
b.



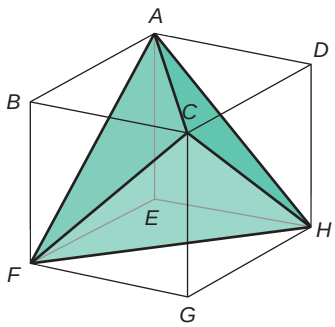
22. a.



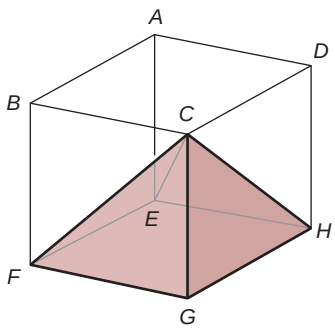
b.



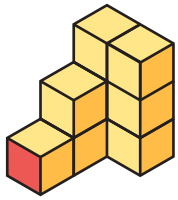
23. A cube can be divided into triangular pyramids in several ways. Pyramid *FHCA* divides this cube into five triangular pyramids. Name the four vertices of each of the other four pyramids.

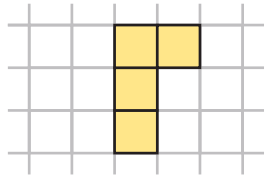


24. *E*, *F*, *G*, *H*, and *C* are the vertices of a square pyramid inside this cube. Name the five vertices of two more square pyramids that, together with the given pyramid, divide the cube into three pyramids.

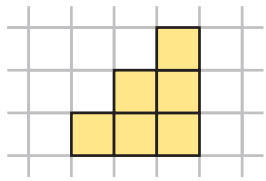


One method of describing a three-dimensional figure is to make a drawing of its different views. There are nine cubes in the first figure in the next column (two are hidden), and the top, right, and front views are shown.

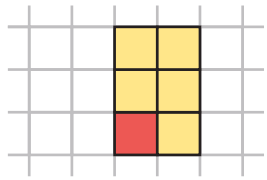




Top view



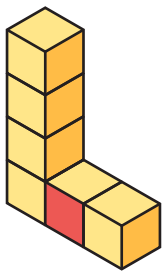
Side view (right)



Front view

Sketch the top, front, and side views of each of the figures in exercises 25 and 26. (*Note:* The only hidden cubes are those supporting other cubes. The red faces of the cubes are part of the front views of the figures.)

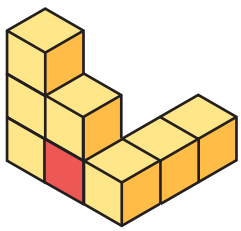
25. a.



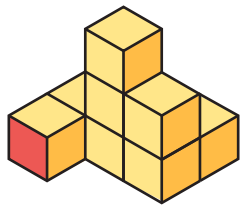
b.



26. a.



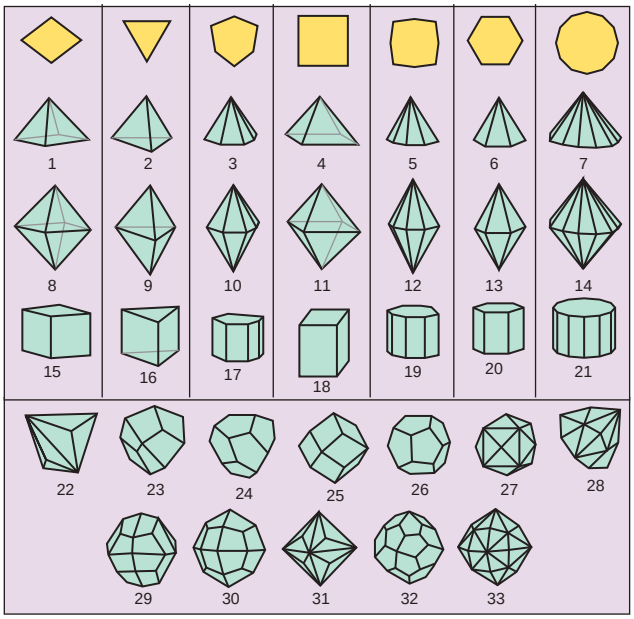
b.



The table of polyhedra on the next page illustrates some of the forms that crystals may take in nature. The polygons at the tops of the columns are the horizontal cross sections of the polyhedra in the columns. Use this table in exercises 27 and 28.

27. a. List the numbers of the polyhedra that are pyramids.
b. Which of the polyhedra is most like a dodecahedron?

28. **a.** List the numbers of the polyhedra that are prisms.
b. Which of the polyhedra is most like an octahedron?

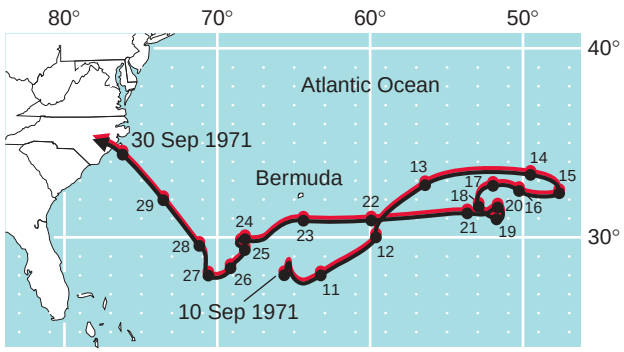


Use Euler's formula in exercises 29 and 30 to determine the missing numbers for each polyhedron. For each set of conditions, find a polyhedron from those numbered from 1 to 21 in the table above that has the given number of faces, vertices, and edges.

29. **a.** 7 faces, 7 vertices, _____ edges
b. 16 faces, _____ vertices, 24 edges
c. _____ faces, 5 vertices, 8 edges

30. **a.** 6 faces, _____ vertices, 9 edges
b. _____ faces, 8 vertices, 12 edges
c. 14 faces, 24 vertices, _____ edges

Hurricane Ginger was christened on September 10, 1971, and became the longest-lived Atlantic hurricane on record at that time. This tropical storm formed approximately 275 miles south of Bermuda and reached the U.S. mainland 20 days later. Use this map in exercises 31 and 32.



Erratic path of Hurricane Ginger

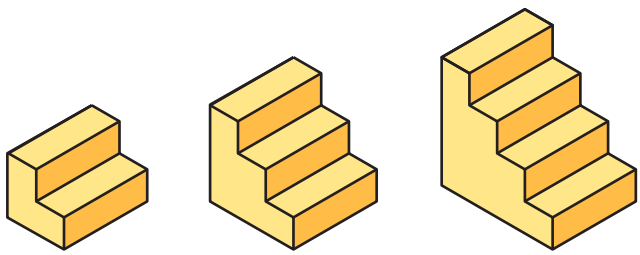
31. The storm's coordinates on September 10 were (28°N, 66°W). What were its coordinates on September 15, September 23, and September 30?

32. At this latitude on the Earth's surface, each degree of longitude spans a distance of approximately 60 miles. About how many miles did this hurricane travel between September 10 and September 30? (*Hint:* Use a piece of string.)

Reasoning and Problem Solving

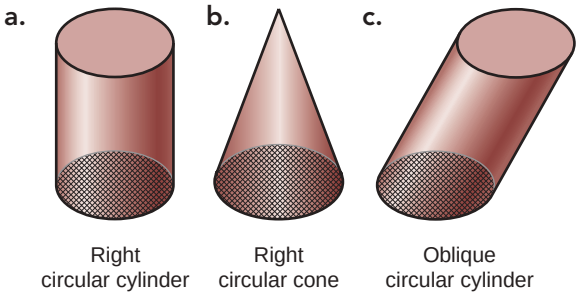
33. Erica is designing a science experiment that requires two different three-dimensional figures such that one fits inside the other and both figures have at least one cross section that is the same for both figures (see exercises 21 and 22). Find such a pair of figures.

34. Here are the first three figures in a staircase pattern. These staircases are polyhedra.



a. The number of faces for the polyhedron in the first figure is 8. How many faces are there for the polyhedron in the 35th figure?
b. The number of edges for the polyhedron in the first figure is 18. How many edges are there for the polyhedron in the 35th figure?
c. The number of vertices for the polyhedron in the first figure is 12. How many vertices are there for the polyhedron in the 35th figure?

35. Sketch and describe how to form a piece of paper into the following figures (without bases).

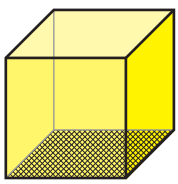


36. Featured Strategies: Making a Drawing and Using a Model. The five regular polyhedra and the numbers and shapes of their faces are shown in the following table. Determine the missing numbers of vertices and edges.

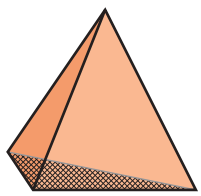
Polyhedron	Vertices	Faces	Edges
Tetrahedron		4 triangles	
Cube	8	6 squares	12
Octahedron		8 triangles	
Dodecahedron		12 pentagons	
Icosahedron		20 triangles	

a. Understanding the Problem. The cube is the most familiar of the regular polyhedra. Its 6 faces meet in 12 edges, and its edges meet in 8 vertices (see figure i). How many vertices and edges does a tetrahedron have?

i.



ii.



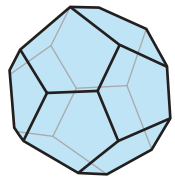
b. Devising a Plan. One approach is to use a model or a sketch of the polyhedra and to count the numbers of vertices and edges. Or, once we determine either the number of vertices or the number of edges, the missing number can be obtained by using Euler's formula $F + V - 2 = E$.

Another approach that avoids counting is to use the fact that each pair of faces meets in exactly one edge. For example, since a dodecahedron has 12 pentagons for faces and each pair of pentagons shares an edge, the number of edges is $\frac{(12 \times 5)}{2} = 30$. Using Euler's formula, determine the number of vertices in a dodecahedron.

c. Carrying Out the Plan. Continue to find the numbers of edges by multiplying the number of faces by the number of sides on the face and dividing by 2. For example, what is the number of edges in an icosahedron? Fill in the rest of the table above.

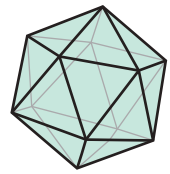
d. Looking Back. The number of vertices for each regular polyhedron can also be found directly from the number of edges that meet at each vertex. For example, three edges meet at each vertex of the dodecahedron, as shown in the following figure iii. Since there are 12 faces and each face has 5 vertex points, the dodecahedron has $\frac{(12 \times 5)}{3} = 20$ vertex points. Use this approach to determine the number of vertices for the icosahedron in figure iv.

iii.



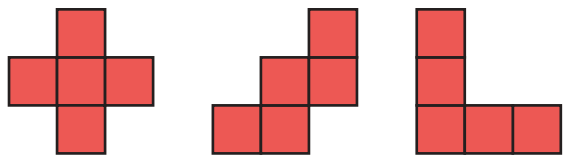
Dodecahedron

iv.



Icosahedron

37. Each of the following shapes contains five squares. There are only 12 such shapes that can be formed in the plane by joining five squares along their edges, and they are called **pentominoes.**



a. Which two of these pentominoes will fold into an open-top box, so that each face of the box is one of the squares?

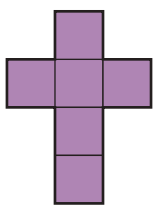
b. Eight of the 12 pentominoes will fold into an open-top box. Find another one of these.

38. The following shapes were formed by joining six squares along their edges. There are 35 such shapes, and they are called **hexominoes.**

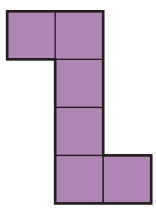
a. Which two of these hexominoes will fold into a cube so that each face of the cube is one of the squares?

b. Eleven of the 35 hexominoes will fold into a cube. Find another such hexomino.

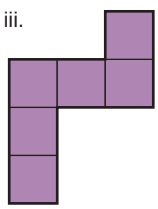
i.



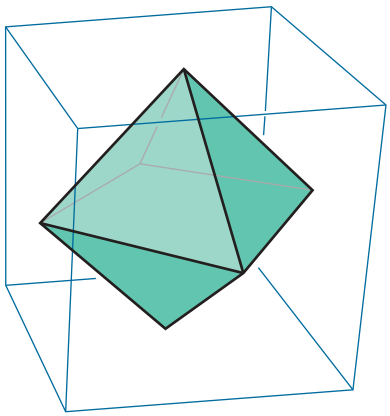
ii.



iii.



39. The centers of the faces of a cube can be connected to form a regular octahedron. Also, the centers of the faces of an octahedron can be connected to form a cube. Such pairs of polyhedra are called **duals.**



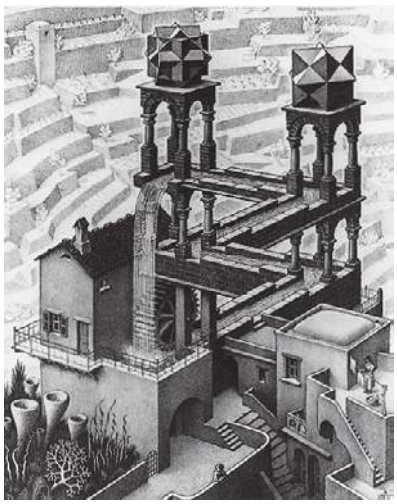
a. How is this dual relationship suggested by the table in exercise 36?

b. Find two other regular polyhedra that are duals of each other.

c. Which regular polyhedron is its own dual?

40. There are six categories of illusions.* One category, called *impossible objects,* is produced by drawing three-dimensional figures on two-dimensional surfaces. For example, trace the complete circuit of water flow in Escher's "Waterfall". Find the impossible feature in each of these figures.

a.

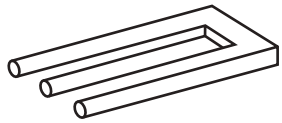


M. C. Escher's "Waterfall"

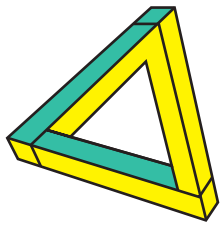
© 2008 The M. C. Escher Company – Baarn – Holland. All rights reserved. www.mcescher.com

*P. A. Rainey, *Illusions*, pp. 18–43.

b.

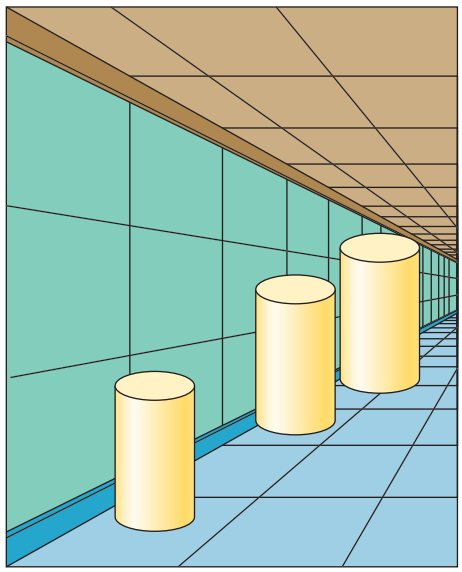


c.

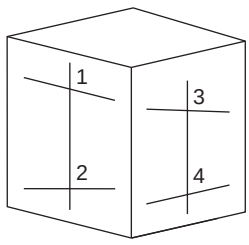


41. A second type of illusion involves depth perception. We have accustomed our eyes to see depth when three-dimensional objects are drawn on two-dimensional surfaces. Answer questions a and b by disregarding the depth illusions.

a. Is one of these cylinders larger than the others?



b. Which of the four numbered angles below is the largest? Which are right angles? (*Hint:* Use a corner of a piece of paper.)



Teaching Questions

1. Natalie asked her teacher why, when you blow bubbles, they are round like a sphere and not cubes or other shapes. Research the question and then write a response you can give to this student.
2. One of your students wants to make cone-shaped birthday hats and asks you to show him how to do it. Explain, with diagrams, how you would go about this.
3. After using your classroom Earth globe to illustrate a sphere, a student asks how they get the map of the Earth on a flat piece of paper. Research the question and then write a response that would make sense to this student.
4. Researchers have concluded that students need more experiences with concrete models. Suppose you are a new teacher going into a school that does not have three-dimensional manipulatives. Compile a list of objects you could acquire, or make, to bring into your classroom to successfully teach spatial concepts and illustrate the three-dimensional objects referred to in this section.

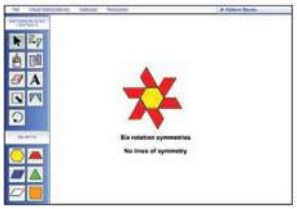
Classroom Connections

1. The **Standards** statement on page 606 suggests constructing three-dimensional shapes from two-dimensional representations. Hexominoes are polygons formed by joining six squares along their edges. (Three examples are shown in the exercise set for this section.) There are 35 hexominoes and 11 of them will fold into a cube. Find a method of identifying those 11 hexominoes and describe your procedure with diagrams of all 11 hexominoes that fold into a cube.
2. Repeat the question posed in the **Problem Opener** for this section for a $3 \times 3 \times 3$ cube made up of 27 of the smaller cubes. Repeat for a $4 \times 4 \times 4$ cube and then for an $n \times n \times n$ cube.
3. Read through the activity on the **Elementary School Text** example on page 616. (a) Use the fewest number of cubes possible to build each of the three-dimensional figures described in parts a and b under *Check Your Progress.* Record your results using a *blueprint* as described in Math Activity 9.3 on page 606. (b) The directions in this student activity ask the students to sketch the figures they constructed. Draw sketches of your two cube figures from part a and describe what type of experiences you think students would need to sketch figures like this.
4. One of the recommendations in the **Grades 3–5 Standards—Geometry** (see inside front cover) states: "Identify and build a three-dimensional object from a two-dimensional representation of that object." Explain what this means and give examples of how you would accomplish this.

MATH ACTIVITY 9.4

Symmetries of Pattern Block Figures

Virtual Manipulatives

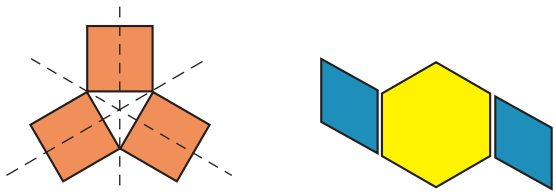


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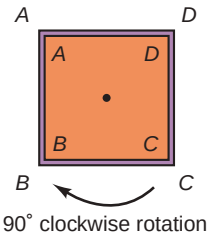
Purpose: Explore line and rotational symmetry using pattern block figures.

Materials: Pattern Blocks in the Manipulative Kit or Virtual Manipulatives.

1. The first pattern block figure shown below has three *lines of symmetry* (dotted lines), because when the figure is folded about any of these lines, it will coincide with itself. The second figure has no lines of symmetry, as can be shown by tracing pattern blocks and paper folding the resulting figure.



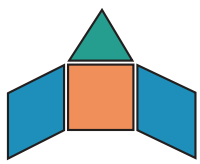
Construct pattern block figures that have exactly one, two, three, and four lines of symmetry. Record your figures and lines of symmetry.



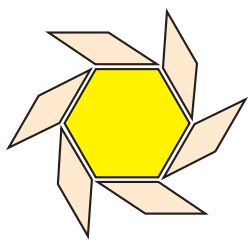
***2.** A purple frame has been traced about the square pattern block at the left, and each corner of the frame and the corresponding corner of the square have the same letter. If the square is rotated 90° clockwise, about the center of the square, it will fit back into the frame with *A* moving to corner *D* of the frame, and *D*, *C*, and *B* moving to corners *C*, *B*, and *A*, respectively, of the frame. The square is said to have 90° *rotation symmetry.* It also has 180°, 270°, and 360° rotation symmetries.

Determine all the rotation symmetries less than or equal to 360°, if there are two or more, for each of the following pattern block figures and the number of degrees for each rotation. (*Suggestion:* Trace each figure to form its frame.)

a.



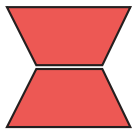
b.



c.



d.



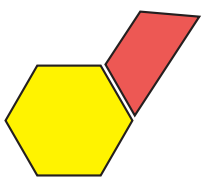
NCTM Standards

Teachers should guide students to recognize, describe, and informally prove the symmetric characteristics of designs through the materials they supply and the questions they ask. Students can use pattern blocks to create designs with line and rotational symmetry or use paper cutouts, paper folding, and mirrors to investigate lines of symmetry. p. 100

3. Build figures with two or more pattern blocks to satisfy each of the following conditions.

- **a.** Two lines of symmetry, two rotation symmetries
- **b.** Three rotation symmetries, no lines of symmetry
- **c.** Six lines of symmetry, six rotation symmetries

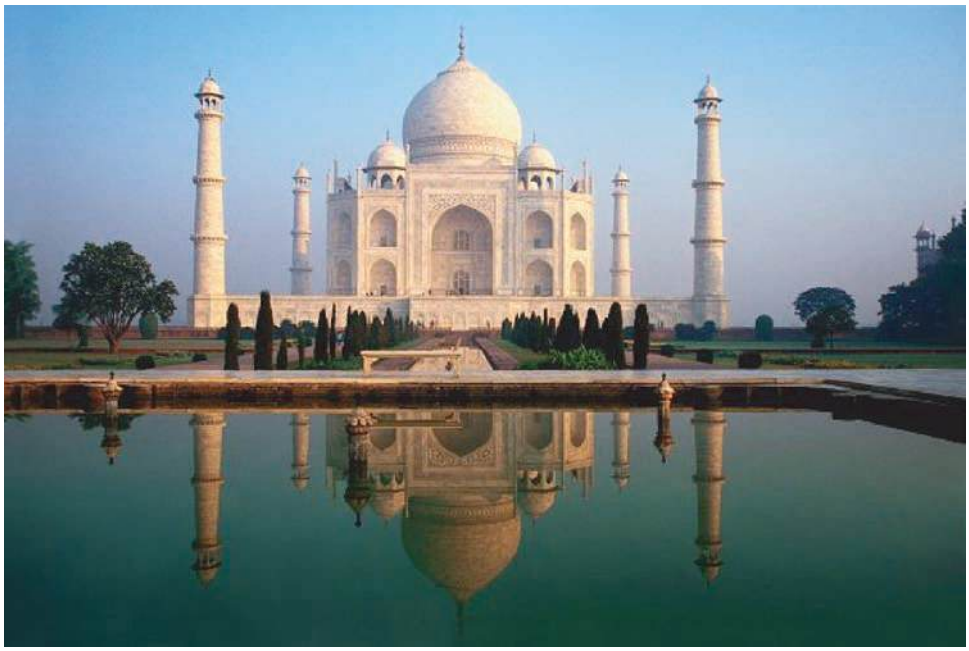
4. A trapezoid has been attached to a hexagon in the figure at the left.



- **a.** In how many different ways can a second trapezoid be attached to this figure to form a figure that has a line of symmetry? Show sketches and lines of symmetry.
- **b.** In how many different ways can one or more trapezoids be attached to this figure to form a figure that has more than one rotation symmetry?

Section **9.4**

SYMMETRIC FIGURES

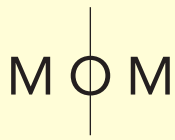


The Taj Mahal, built between 1630 and 1652 on the banks of the Yamuna River in Agra, India

PROBLEM OPENER

A vertical line can be drawn through the word MOM so that the left and right sides are mirror images of each other.

Find a word that can be cut by a horizontal line so that the bottom and top halves are mirror images of each other.



The Taj Mahal is considered by many to be the most beautiful building in the world. It is made entirely of white marble and is surrounded by a landscaped walled garden on the banks of the Yamuna River in Agra, India. It is an octagonal building, and four of its eight faces contain massive arches rising to a height of 33 meters (108 feet). The form and balance of the Taj Mahal can be described by saying it is *symmetric*. The human race has always found order and harmony in symmetry. Perhaps the most influential factor in our desire for symmetry is the shape of the human body. Even children in their earliest drawings show an awareness of body symmetry.

REFLECTION SYMMETRY FOR PLANE FIGURES

Many years before it became popular to teach geometric ideas in elementary school, cutting out symmetric figures was a common classroom activity. The procedure is to fold a piece of paper and draw a figure that encloses part of, or all of the crease, as shown in Figure 9.59a on the next page. When the figure is cut out and unfolded, it is symmetric (see part b). The crease is called a **line of symmetry,** and the figure is said to have **reflection symmetry.**

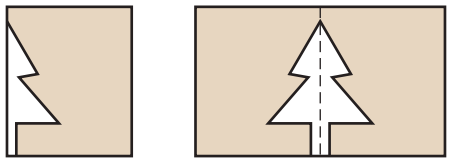


Figure 9.59

NCTM Standards

Young children come to school with intuitions about how shapes can be moved. Students can explore motions such as slides, flips, and turns by using mirrors, paper folding, and tracing. p. 43

Intuitively we understand the idea of reflection symmetry to mean that the two halves of the figure will coincide if one is folded onto the other. The word *reflection* is a natural one to use because of the mirror test for symmetry. If the edge of a mirror is placed along a line of symmetry, the half-figure and its image from the mirror will look like the whole figure. You can verify the line of symmetry for the photograph of the church in Figure 9.60 by placing the edge of a mirror along the vertical centerline of the photograph. With the mirror in this position, one-half of the church and its reflection will look like the whole church. Since this is the only way the mirror can be placed so that this will happen, the photograph of the church has only one line of symmetry.



Figure 9.60
Holy Trinity Lutheran Church, Newington, New Hampshire

The Mira is a convenient device for locating lines of symmetry for plane figures. It is made of Plexiglas so that the user can see through it and at the same time see reflections. If a figure has a line of symmetry, as does the hexagon in Figure 9.61 on the next page, and the Mira is placed so that the reflection of the figure coincides with the part of the figure behind the Mira, then the edge of the Mira lies on a line of symmetry.

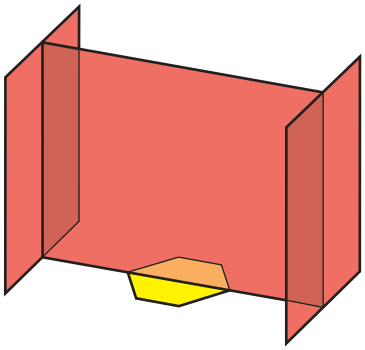


Figure 9.61

Some figures have more than one line of symmetry. To produce a figure with two such lines, fold a sheet of paper in half and then in half again. Then draw a figure whose endpoints touch the creases as in Figure 9.62a. If the figure is cut out and the paper is opened, the two perpendicular creases will be lines of symmetry for the figure, as shown in Figure 9.62b.

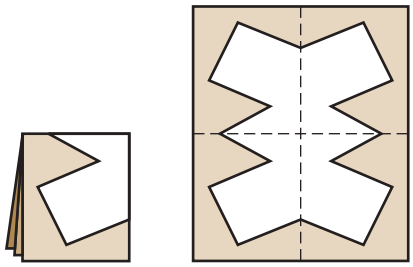
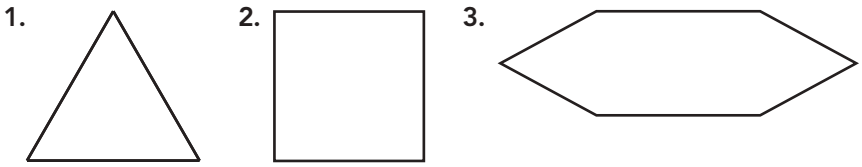


Figure 9.62

EXAMPLE A Each of the following polygons has two or more lines of symmetry. Determine these lines for each figure.



Solution **1.** An equilateral triangle has three lines of symmetry: one line through each vertex perpendicular to the opposite side. **2.** A square has four lines of symmetry: one horizontal line and one vertical line through the midpoints of opposite sides and two lines containing the diagonals. **3.** This figure has two lines of symmetry: one horizontal line through opposite vertices and one vertical line through the midpoints of opposite sides.

The idea of symmetry can be made more precise by adopting the term *image,* which is suggested by mirrors. If a line can be drawn through a figure so that each point on one side of the line has a matching point on the other side at the same perpendicular distance from the line, it is a **line of symmetry.** If two points on opposite sides of this line match up, one is called the **image** of the other. A few points and their images have been labeled in Figure 9.63 on the next page, where A corresponds to A', B to B', C to C', and D to D'. Each line segment connecting a point and its image is perpendicular to the line of symmetry.

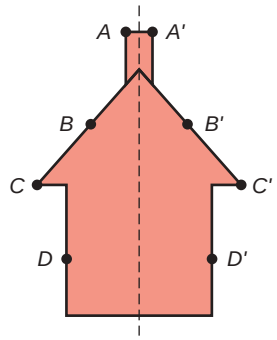
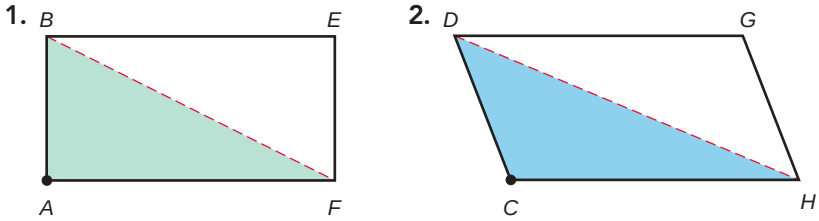


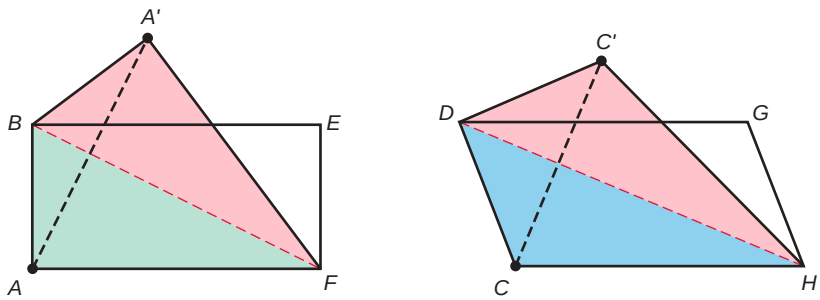
Figure 9.63

EXAMPLE B

For each of the following figures, show that the diagonal (red line) is not a line of symmetry by finding the image of points A and C for reflections about the diagonals.



Solution **1, 2.** The reflected image of the lower half of the rectangle and parallelogram can be formed by folds about the diagonals, as indicated by the pink shaded regions below. Since these images (pink regions) do not coincide with the upper halves of the original rectangle and parallelogram, the diagonals are not lines of symmetry. For example, the image A' of A lies outside the upper half of the rectangle, and the image C' of C lies outside the upper half of the parallelogram. This shows that even though the diagonals divide the rectangle and parallelogram into two parts that are congruent, the diagonals are not lines of symmetry.



Laboratory Connections

Mirror Cards

How can a mirror be placed on the figure at the left to obtain the figure on the right? Explore creating interesting figures and answering related questions in this investigation.



Mathematics Investigation
Chapter 9, Section 4
www.mhhe.com/bbn

ROTATION SYMMETRY FOR PLANE FIGURES

Figure 9.64 on the next page may look like a drawing of a plant, but it is a drawing of a type of jellyfish called *Aurelia.* It seems to have the form and balance of a symmetric figure, but it has no lines of reflection. It does, however, have **rotation symmetry,** because it can be turned about its center so that it coincides with itself. For example, if it is rotated 90° clockwise, the top "arm" will move to the 3 o'clock position, the bottom "arm" will move to the 9 o'clock position, etc.

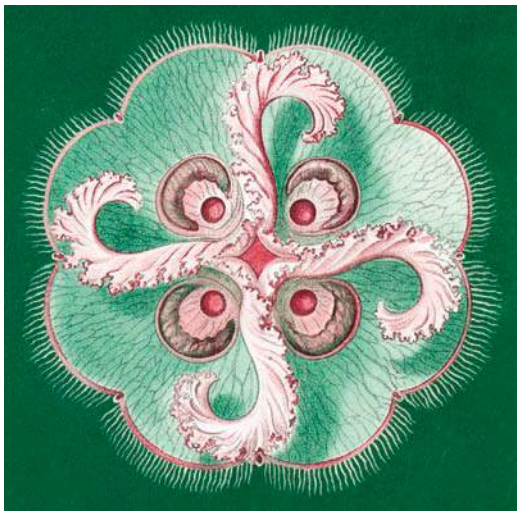


Figure 9.64
Aurelia, the common coastal jellyfish

Let's consider another example of rotation symmetry. Trace Figure 9.65 and mark the center X and the arms A, B, and C. Cut it out and place it on the page so that both figures coincide. If it is held down by a pencil at point X, the top figure can be rotated clockwise so that A goes to B, B to C, and C to A. This is an example of rotation symmetry, and X is called the **center of rotation.** Since the figure is rotated 120° (one-third of a full turn), it has a 120° *rotation symmetry*. From its original position, this figure can also be made to coincide with itself after a 240° clockwise rotation, with A going to C, B to A, and C to B. This is a 240° *rotation symmetry*. Since the figure can be rotated back onto itself after a 360° rotation, the figure also has a 360° *rotation symmetry. Note:* Any figure can be rotated 360° by using any point as the center of rotation. Thus, we will be interested in a 360° rotation symmetry only when a figure has other rotation symmetries as well.

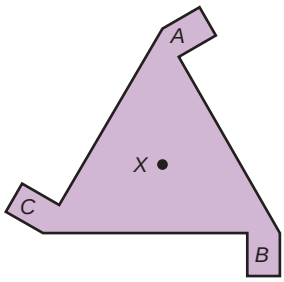
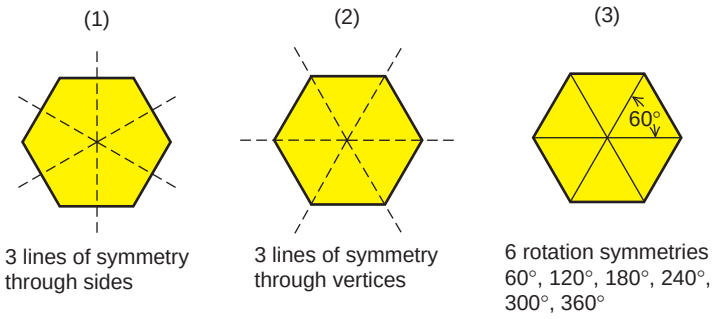


Figure 9.65

Some figures have both reflection symmetry and rotation symmetry. The regular polygons have both types. The central angles of these polygons determine the angles for the rotation symmetries.

EXAMPLE C Find all the reflection and rotation symmetries for a regular hexagon.

Solution Every regular hexagon has six reflection symmetries. Figure (1) on the next page shows three lines of symmetry passing through opposite pairs of parallel sides, and figure (2) shows three lines of symmetry passing through opposite pairs of vertices. Since the central angle in figure (3) has a measure of $360° \div 6 = 60°$, the figure has rotation symmetries of 60°, 120°, 180°, 240°, 300°, and 360°.



Snow crystals have the reflection and rotation symmetries of the hexagon, as can be seen from the three in Figure 9.66. Notice the six congruent central angles in the first of these two snow crystals. Despite the similarity that results from having six reflection and six rotation symmetries, there is a myriad of different details in snow crystals.

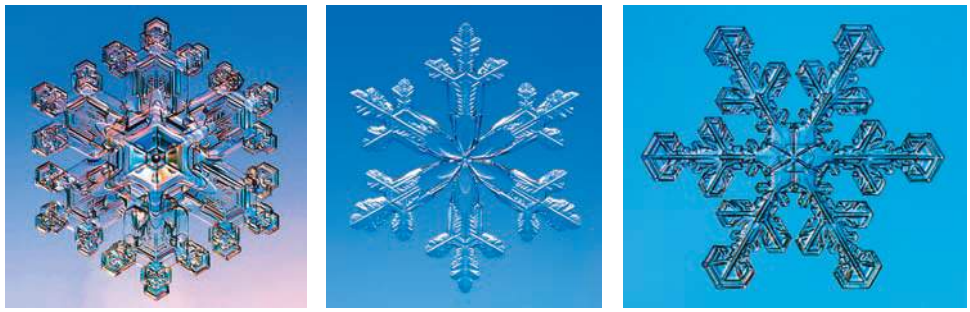
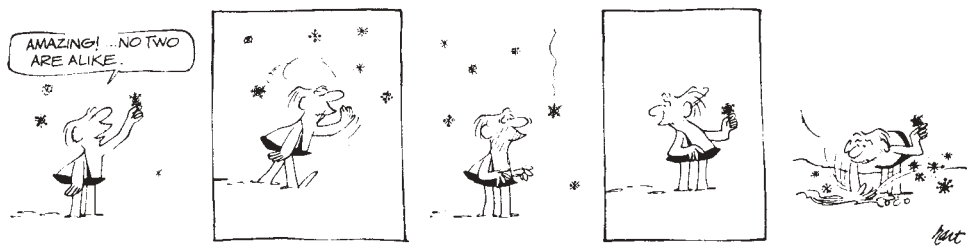


Figure 9.66



REFLECTION SYMMETRY FOR SPACE FIGURES

NCTM Standards

Some of the ways in which symmetry occurs around us are listed in the *Curriculum and Evaluation Standards for School Mathematics,* grades 5–8, *Geometry* (p. 115):

> Symmetry in two and three dimensions provides rich opportunities for students to see geometry in the world of art, nature, construction, and so on. Butterflies, faces, flowers, arrangements of windows, reflections in water, and some pottery designs involve symmetry. Turning symmetry is illustrated by bicycle gears. Pattern symmetry can be observed in the multiplication table, in numbers arrayed in charts, and in Pascal's triangle.

The idea of reflection symmetry for three-dimensional objects is similar to that for plane figures. With plane figures we found lines such that one-half of the figure was the reflection of the other. With figures in space there are *planes of symmetry* such that the points on one side of a plane are the reflection of the points on the other side. Consider, for example, the antique chair in Figure 9.67. The plane running down the center of the back and across the seat to the front of the chair divides it into left and right halves, which are mirror images of each other. Such a plane is called a **plane of symmetry.** The chair is said to have reflection symmetry.



Figure 9.67
Ornate antique armchair

Reflection symmetry for figures in space can be mathematically defined by requiring that for each point on the left side of the chair, there is a corresponding point on the right side such that both points are the same perpendicular distance from the plane of symmetry. For the antique chair, point A corresponds to A' and B corresponds to B'. These points are called **images** of each other, and the segments $\overline{AA'}$ and $\overline{BB'}$ are perpendicular to the plane of symmetry.

Two-sided symmetry, such as that of the antique chair in Figure 9.67 and the Longhorn Beetle and Scarlet Mormon butterfly in Figure 9.68, is sometimes called **vertical symmetry** because the plane of symmetry is perpendicular to the ground. Look around and you may be surprised at the number of things that have vertical symmetry.



Figure 9.68
Longhorn beetle and Scarlet Mormon butterfly

EXAMPLE D Determine the planes of symmetry for each of the following objects.







Solution The square-top table has four vertical planes of symmetry: one from front to back, one from side to side, and one through each diagonal of the top surface. The lamp has six vertical planes of symmetry because its shade has six congruent sections: three planes bisect opposite pairs of sections of the lampshade, and three planes pass through opposite pairs of seams of the shade. The top of the table has eight vertical planes of symmetry, since it is a regular octagon. However, since it has only four legs and these are centered below every other vertex of the table, the table together with its legs has only the four planes of symmetry passing through the vertices. Two of these planes will bisect the centers of opposite pairs of legs and the other two planes will pass through vertices not containing legs.

ROTATION SYMMETRY FOR SPACE FIGURES

Some three-dimensional objects, such as the table shown in Figure 9.69, have rotation symmetry. If the table is rotated 120°, the legs will change places and the table will be back in the same location or position. That is, leg A will go to the position of leg B, B to C, and C to A. In this example, the table can be rotated about line ℓ, which passes through the center of the table's top and its base. Line ℓ is called the **axis of symmetry,** and the table is said to have **rotation symmetry.** Since the dihedral angles formed by adjacent legs of this table have measures of 120°, the table has 120°, 240°, and 360° rotation symmetries.

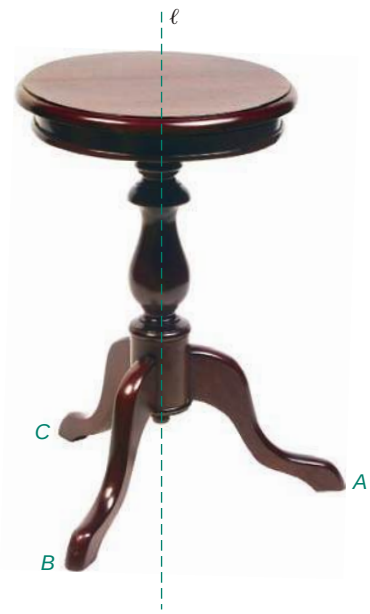


Figure 9.69

The three-legged table in Figure 9.69 on the previous page also has three vertical planes of symmetry, one passing through each leg. It is not difficult to find objects with both planes of symmetry and axes of symmetry. The two tables and the lamp in Example D all have both types of symmetry. Occasionally, however, you will see space figures that have rotation symmetry but no plane of symmetry.

EXAMPLE E Disregarding the stick, list all the plane and rotational symmetries of the paper windmill.



Solution The paper windmill has rotation symmetries of 90°, 180°, 270°, and 360° about its axis, which is the line through the center of the windmill and perpendicular to its surface. It has no planes of symmetry.

PROBLEM-SOLVING APPLICATION

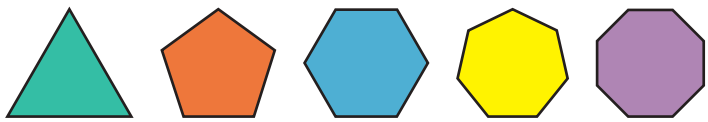
Problem

For every plane figure with two or more reflection symmetries, there is a relationship between the number of these symmetries and the number of rotation symmetries. What is this relationship?

Understanding the Problem There are plane figures with both rotation and reflection symmetries. For example, a rectangle has two lines of symmetry. **Question 1:** How many rotation symmetries does a rectangle have?

Devising a Plan *Making a table* and comparing the numbers of reflection and rotation symmetries may reveal a pattern. A square has four reflection symmetries. **Question 2:** How many rotation symmetries does a square have?

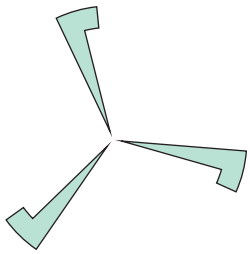
Carrying Out the Plan The numbers of lines of symmetry for several figures are shown below. Determine the numbers of rotation symmetries for these figures and record them in the table. **Question 3:** What does this result suggest?



3 lines of symmetry | 5 lines of symmetry | 6 lines of symmetry | 7 lines of symmetry | 8 lines of symmetry

No. of reflection symmetries	2	3	4	5	6	7	8	9	10
No. of rotation symmetries	2		4						

Looking Back As the results in the table suggest, if a figure has two or more reflection symmetries, it will have the same number of rotation symmetries. The converse, however, is not true. **Question 4:** What symmetries does the following figure have?



Answers to Questions 1–4 **1.** 2 **2.** 4 **3.** If a figure has two or more reflection symmetries, it will have the same number of rotation symmetries. **4.** The figure has rotation symmetries of 120°, 240°, and 360°. It has no reflection symmetries.

Exercises and Problems 9.4



The Alhambra, built in the fourteenth century for Moorish kings, Granada, Spain

1. The pool, building, and fortress in the section of the Alhambra shown in the photograph have a vertical plane of symmetry, about which their left sides are the reflections of their right sides.
   a. List five objects in this photograph that have images about this plane of symmetry.
   b. Physical objects can never be perfectly symmetric. In this scene, for example, there are several objects that deviate from perfect symmetry. List two objects that do not have an image for the vertical plane of symmetry.
   c. Several individual items in this photograph have vertical lines of symmetry. Name an object in this photograph that has a horizontal line of symmetry.

2. The base of a Navaho hogan is a regular octagon. The logs forming the walls are joined at the corners by notching. The octagonal roof is formed by laying poles at each of the eight corners to create a structure with many strong isosceles triangles.

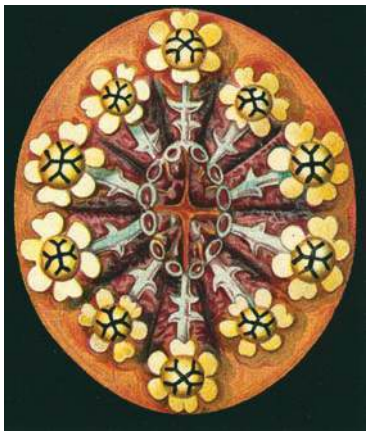




   a. How many rotation symmetries does a regular octagon have?
   b. How many degrees are there in the smallest rotation symmetry of a regular octagon?
   c. How many lines of symmetry does a regular octagon have?

The following sketches of sea life have reflection and rotation symmetries. Determine the number of lines of symmetry and the number of rotation symmetries for each figure in exercises 3 and 4. (Because these are sketches of natural sea life, please ignore minor imperfections in their shapes or colors.)

3. a.



b.



4. a.



b.



Draw all possible lines of symmetry and find the number of rotation symmetries for each polygon in exercises 5 and 6.

The subject of beauty has been discussed for thousands of years. Aristotle felt that the main elements of beauty are order and symmetry. The U.S. mathematician George Birkhoff (1884–1944) developed a formula for rating the beauty of objects.* Part of his formula involves counting symmetries. If only symmetry is used to rate the beauty of polygons, which polygon in exercise 5 (or exercise 6) has the highest rating (counting all lines of reflection and rotation symmetries) and which has the lowest rating?

5. a.



b.



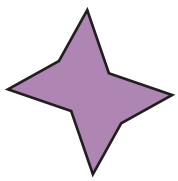
c.



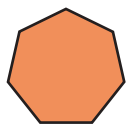
d.



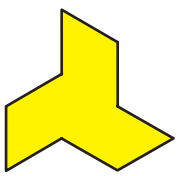
6. a.



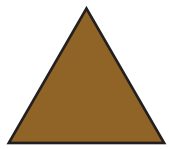
b.



c.



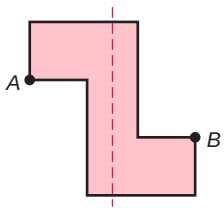
d.



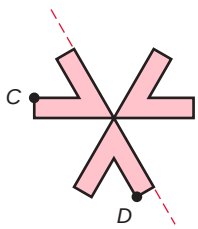
*G. D. Birkhoff, *Aesthetic Measure* (Cambridge, MA: Harvard University Press, 1933), pp. 33–46.

Show that the dashed lines in the figures in exercises 7 and 8 are not lines of symmetry by finding the images of the lettered points.

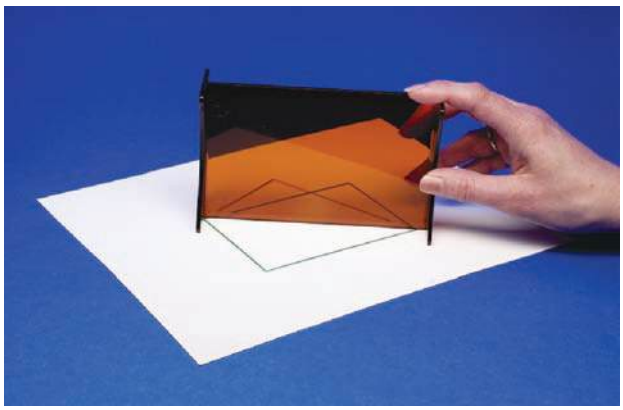
7.



8.



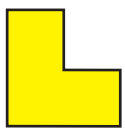
The mirror test for lines of symmetry is very effective when the reflecting is done with a Mira.* To find a line of symmetry, move the Mira until the image reflected from the front of the Plexiglas coincides with the portion of the figure behind it. In this photo the Mira has been placed along the diagonal of a rectangle, and since the reflected image from the front does not coincide with the portion of the rectangle behind the Mira, the diagonal of the rectangle is not a line of symmetry. Which of the figures in exercises 9 and 10 have a line of symmetry?



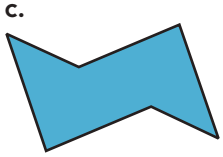
9. a.



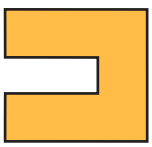
b.



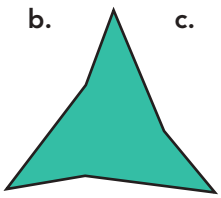
c.



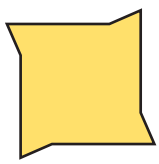
10. a.



b.



c.



*E. Woodward, "Geometry with a Mira," *The Arithmetic Teacher*, pp. 117–118. The Mira is distributed by several companies that produce educational materials.

11. a. Which uppercase letters have two lines of symmetry?
b. Which letters have two rotation symmetries but no lines of symmetry?

ABCDEFGHIJKLM
NOPQRSTUVWXYZ

12. If you write the letter P on a piece of paper and hold it in front of a mirror, it will look reversed.
a. Which uppercase letters will not appear reversed when they are held in front of a mirror?
b. What type of symmetry do these letters have?
c. Use some of the letters from part a to write a word whose reflection in a mirror is also a word.

The figures in exercises 13 and 14 were formed on circular geoboards. Which of these figures have no lines of symmetry? Determine the number of lines of symmetry for the remaining figures. Find the number of rotation symmetries for each figure that has two or more such symmetries, and give the number of degrees for each.

13. a.



b.



c.



d.



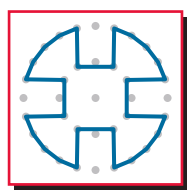
14. a.



b.



c.



d.



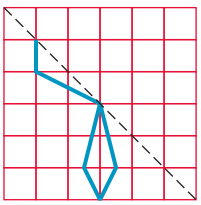
Sketch figures in exercises 15 and 16 with the given symmetries, as they would appear on a circular geoboard. Download circular geoboard paper from the website or use the Virtual Circular Geoboards.

15. a. Two rotation symmetries and two reflection symmetries
b. Three rotation symmetries and no reflection symmetries
c. 12 rotation symmetries and 12 reflection symmetries

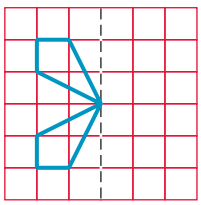
16. a. One rotation symmetry and one reflection symmetry
b. Eight rotation symmetries and no reflection symmetries
c. Six rotation symmetries and six reflection symmetries

17. Trace the sketches here and complete the figures so that they are symmetric about the dashed line. You might want to first find the image with a mirror or Mira.

a.

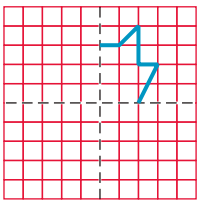


b.

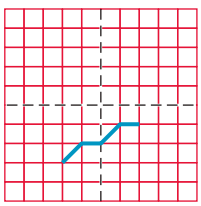


18. Trace the sketches below and complete the figures so that they are symmetric about the two perpendicular dashed lines.

a.



b.



Determine the number of rotation symmetries, if there are two or more, and the number of planes of symmetry for the objects in 19 and 20.

19. a. Consider only the front wagon wheel in the photo.



b. Determine the symmetries (i) with the window colored as shown and (ii) disregarding the color on the window.



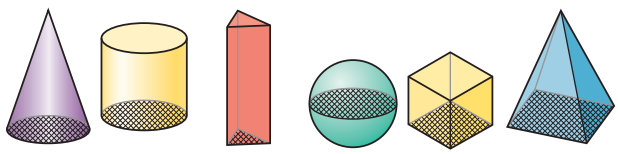
20. a.



b.



The 3-dimensional figures below are highly symmetric. Use these figures in exercises 21 and 22.



21. a. Which figures have at least one horizontal axis of symmetry?
b. For each figure that has a horizontal axis of symmetry, give the number of rotation symmetries for an axis of symmetry.

c. Which of these figures has the following number of planes of vertical symmetry: Exactly two? Exactly three? Exactly four?

22. a. Which figures on the previous page have a horizontal plane of symmetry?
b. Does each of these solids have at least one vertical plane of symmetry?
c. Give the number of rotation symmetries for each vertical axis of symmetry.

23. List all the rotation symmetries and planes of symmetry for the figure below.

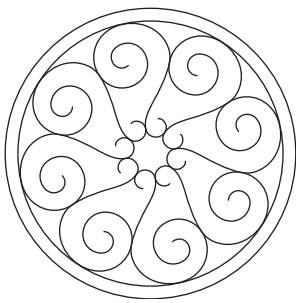


Sphere on pentagon

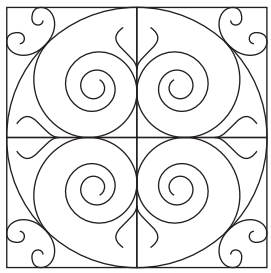
The metalwork designs in exercises 24 and 25 have many pleasing symmetries. How many rotation symmetries and lines of symmetry are there for each figure?

24.

a.

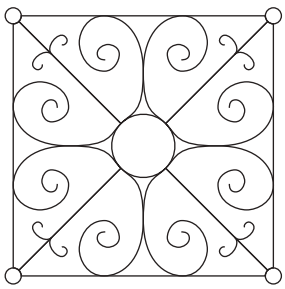


b.

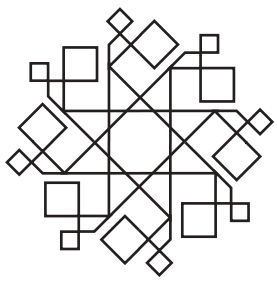


25.

a.



b.



How many rotation symmetries are there for each of the Japanese crests in exercises 26 and 27?

26. a.



b.



27. a.



b.



Find the number of rotation symmetries, if there are two or more, and the number of lines of symmetry for the signs and logos in exercises 28 and 29. How many of these logos can you identify?

28. a.



b.



c.



d.



e.



29. a.  b. 

c.  d. 

e. 

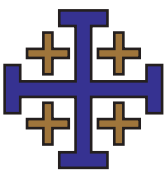
Determine the number of rotation symmetries, if there are two or more, and the number of lines of symmetry for the symbols in exercises 30 and 31. (Disregard the square background for 30a and 31b and color for 30c and 31b.)

30. a.  b. 

c.  d. 

e. 

31. a.  b. 

c.  d. 

e. 

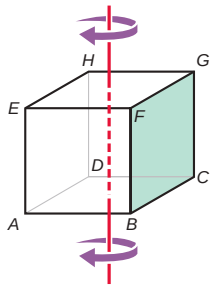
Reasoning and Problem Solving

32. Featured Strategy: Using a Model. Crystals are classified into different types according to the number of axes of rotation they have. This photograph shows several cubes of a galena crystal. How many axes of symmetry does a cube have? *Note:* It will be helpful to have a cube (wood block, die, etc.) while working through this question.

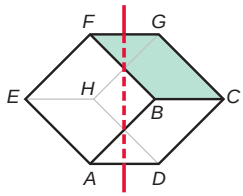


Intersecting cubes of galena crystals

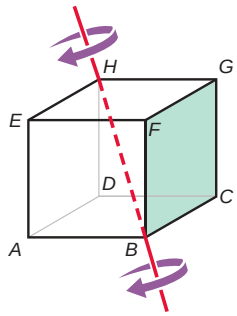
a. Understanding the Problem. One axis of symmetry in the cube below runs through the centers of faces *EFGH* and *ABCD*. What is the total number of axes of symmetry through the faces of the cube? Describe each by listing the pairs of faces.



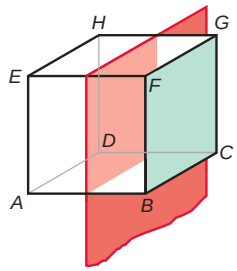
b. Devising a Plan. A posterboard or paper model of a cube that can be pierced by a wire is a helpful device for determining rotations that take the cube back onto itself. The following figure suggests some other possibilities for axes of symmetry. One axis passes through the edges $\overline{FG}$ and $\overline{AD}$. How many axes of symmetry pass through the edges of a cube? Describe each by listing pairs of edges.



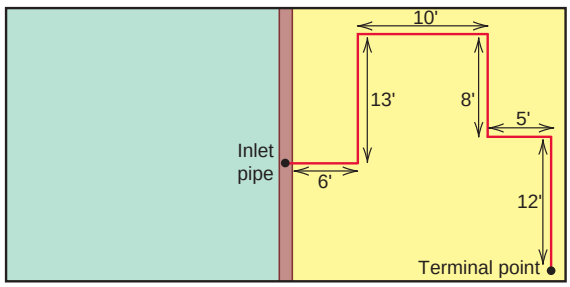
c. Carrying Out the Plan. A model will help to show that a cube has three types of axes of symmetry: through the faces, through the edges, and through the vertices. The following figure shows the axis through the pair of vertices H and B. How many axes of symmetry are there through pairs of vertices, and what is the total number of axes of symmetry for the cube?



d. Looking Back. A cube also has many planes of symmetry. The figure below shows a plane that bisects four edges of the cube: $\overline{AB}$, $\overline{DC}$, $\overline{HG}$, and $\overline{EF}$. How many planes of symmetry bisect edges of the cube? Describe each by listing the four edges.

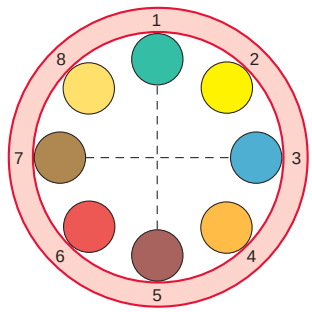


33. Two adjacent apartments that are the same size and on the same level are separated by a dividing wall as shown below. A water inlet pipe to the apartments is centered at the base of the dividing wall. The plans for one apartment show the path of the pipe and its length. If this path is symmetric about the dividing wall to the path of the piping in the adjacent apartment, how far apart are the two terminal points of the pipes in these apartments?

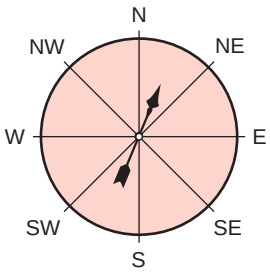


34. Nikita lives in a city where the avenues run north and south and the streets run east and west. She lives on Washington Avenue and has a jogging route that takes her two blocks east, three blocks north, four blocks east, five blocks south, and four blocks west. At this point she stops jogging and walks back to her apartment. One day she decides to jog a new route in such a way that the old and new routes form a path that has a 180° rotation symmetry about the place where she lives. How many blocks (edges of blocks) must she walk to get from the end of her new jogging route to the end of her old jogging route, if she does not cut diagonally?

35. Ms. Harris designed a beanbag game so her fourth grade students could become familiar with symmetry. Here are the rules: On each player's turn, five beanbags are tossed at eight cups that are placed about a circle (see figure below). Only one beanbag will fit in a cup. If one or more beanbags land in cups, the player receives the sum of the numbers on the cups plus the following points: 5 points if two beanbags land in cups that are symmetric about the vertical line through the centers of cups 1 and 5; 5 points if two beanbags land in cups that are symmetric about the horizontal line through the centers of cups 7 and 3. What is the greatest score a player can receive on one turn? (*Note:* There are some pairs of cups that are symmetric about both lines of symmetry.)



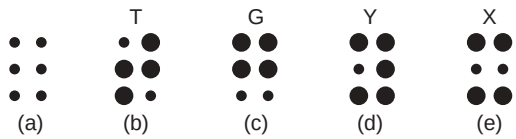
36. Sharon and Justin belong to an outdoor orienteering club. Sharon uses her compass (shown at the top of the next page) to walk over the trail given by the following directions from her orienteering assignment sheet: Start at the Wilderness Club House; go north 1 mile; go east 2 miles; go southeast 1 mile; go south 1 mile; go northeast 1.5 miles; and go southwest 2 miles. If Justin's route is symmetric to Sharon's route about the east-west line through the clubhouse, will Justin's route end north or south of the clubhouse?



Orienteering Compass

37. Louis Braille, a Frenchman living in the nineteenth century, invented an alphabet for use by blind people. Each letter of this alphabet consisted of a 2×3 grid having from one to six raised dots that were placed in the six positions shown in figure (a). Figures (b) through (e) show the letters for T, G, Y, and X. Note that the symbol for T has a 180° rotation symmetry; the symbol for G has a vertical line of symmetry; the symbol for Y has a horizontal line of symmetry; and the symbol for X has both horizontal and vertical lines of symmetry. There is a total of 64 different possible Braille symbols using from zero to six dots of two different sizes in the positions shown in figure (a). Find at least two more Braille-type symbols having each of the following types of symmetry.

a. A vertical line of symmetry
b. A horizontal line of symmetry
c. A 180° rotation symmetry, but no lines of symmetry



Teaching Questions

1. A student concludes that if a figure has two lines of symmetry, it also has rotation symmetry. How would you respond to this observation?

2. Use a set of pattern blocks and a small rectangular mirror to devise and write a set of instructions for elementary school students to discover which pattern block pieces have line symmetry. Extend your activity to include some shapes formed by joining two or more pattern block pieces to form new polygonal shapes. Include shapes in your activity that have no lines of symmetry and shapes that have more than one line of symmetry.

3. A student says that every two-dimensional and every three-dimensional figure has rotational symmetry because you can turn it 360 degrees. Do you agree? Explain why or why not.

4. Suppose you were asked to teach a lesson on rotational symmetry to a class unfamiliar with the concept. Outline a plan for introducing the concept of rotational symmetry. In your plan, include the materials you would need, activities you would have students do, and questions you would ask.

Classroom Connections

1. Obtain a small mirror that you can place on a piece of paper. Use the mirror to answer the symmetry of quadrilateral questions posed in the **Spotlight on Teaching** at the beginning of this chapter. Include sketches and explanations with your answers.

2. Examine both the **PreK–2** and **Grades 3–5 Standards—Geometry** (see inside front cover) to see where the topic of symmetry is mentioned. List the recommendations pertaining to symmetry for both levels.

3. Read the **Standards** quote on page 635. Gather a collection of photographs appropriate for elementary school-children that illustrate some of the subjects mentioned in that statement.

4. Read the following quote about line symmetry from the **Standards.** Illustrate the ideas in this statement with diagrams accompanied by a narrative that explains the observations that can be made by looking at symmetry in an isosceles trapezoid.

> Looking at line symmetry in certain classes of shapes can also lead to interesting observations. For example, isosceles trapezoids have a line of symmetry containing the midpoints of the parallel opposite sides (often called the bases). Students can observe that the pair of sides not intersected by the line of symmetry (often called the legs) is congruent, as are the two opposite pairs of angles. Students can conclude that the diagonals are the same lengths, since they can be reflected onto each other, and that several pairs of angles related to those diagonals are also congruent. (Standards, p. 237)

CHAPTER 9 REVIEW

1. **Mathematical systems**
   - **a.** A **mathematical system** consists of undefined terms, definitions, axioms, and theorems.
   - **b.** In every mathematical system there must be **undefined words.**
   - **c.** **Definitions** are stated in terms of undefined words or previously defined words.
   - **d.** **Axioms** are statements that are assumed to be true.
   - **e.** **Theorems** are statements that are proved by using definitions and axioms together with deductive reasoning.

2. **Plane figures**
   - **a.** The terms **point, line,** and **plane** are undefined.
   - **b.** **Half-planes, line segments, endpoints, half-lines, rays, angles,** angle **side** and **vertex, parallel lines** and **line segments, perpendicular lines** and **line segments,** and **collinear points** are defined.
   - **c.** Each **angle** is measured in degrees. A **degree** is $\frac{1}{360}$ of a complete turn about a circle.
   - **d.** A **protractor** is a device for measuring angles.
   - **e.** Angles are classified as **right, obtuse, acute, straight,** or **reflex.**
   - **f.** Two angles are **complementary** if the sum of their measures is 90° and **supplementary** if the sum of their measures is 180°. If two angles have the same vertex, share a common side, and do not overlap, they are called **adjacent angles.**
   - **g.** Two intersecting lines form pairs of congruent **vertical angles.**
   - **h.** If two lines are intersected by a third line called a **transversal,** the two lines are parallel if and only if the **alternate interior angles** are congruent. Angles on the same side of a transversal and both above or both below the two lines cut by the transversal are called **corresponding angles.** If the lines are parallel, the corresponding angles are congruent.
   - **i.** Curves are classified as **simple, simple closed,** or **closed.**
   - **j.** The union of a simple closed curve and its interior is called a **plane region.**
   - **k.** Plane regions are classified as **convex** or **concave.**
   - **l.** A **circle** is a special type of simple closed curve. **Radius, diameter, circumference, chord, tangent,** and **disk** are defined terms associated with circles.

3. **Polygons**
   - **a.** A **polygon** is a simple closed curve that is the union of line segments. The union of a polygon and its interior is called a **polygonal region,** the line segments of a polygon are called **sides** and the endpoints of these segments are called **vertices.**
   - **b.** Two **sides of a polygon are adjacent** if they share a common vertex, and two **vertices of a polygon are adjacent** if they share a common side. Any line segment connecting one vertex of a polygon to a nonadjacent vertex is a **diagonal.** A **vertex angle** is formed by two adjacent sides of the polygon, a **central angle** is formed by connecting the center of the polygon to two adjacent vertices of the polygon, and an **exterior angle** is formed by one side of the polygon and the extension of an adjacent side.
   - **c.** Polygons are named according to the number of sides: **triangle, quadrilateral, pentagon,** etc.
   - **d.** Triangles are classified according to their attributes: **acute, obtuse, equilateral, right, isosceles,** and **scalene.**
   - **e.** Quadrilaterals are classified according to their attributes: **square, rectangle, rhombus, parallelogram, trapezoid,** and **isosceles trapezoid.**
   - **f.** A line segment from a vertex of a triangle to the midpoint of the opposite side is called a **median.** A line segment from a vertex of a triangle that is perpendicular to the opposite side is called an **altitude.**
   - **g.** Two plane figures are **congruent plane figures** if one can be placed on the other so that they coincide.
   - **h.** Two **line segments are congruent** if they have the same length, and two **angles are congruent** if they have the same measure.
   - **i.** A polygon is called a **regular polygon** if all its angles are congruent and all its sides are congruent.
   - **j.** The first three regular polygons are the **equilateral triangle,** the **square,** and the **regular pentagon.**
   - **k.** The measure of the **central angle** of a regular n-sided polygon is $\frac{360}{n}$ and the measure of the vertex angle of a regular n-sided polygon is $\frac{(n-2)180}{n}$.
   - **l.** Any polygon whose vertices are points of a circle is called an **inscribed polygon.**
   - **m.** An arrangement of nonoverlapping figures that can be placed together to entirely cover a region is called a **tessellation.** The points at which the vertices of the figures meet are called the **vertex points.**
   - **n.** The **equilateral triangle, square,** and **regular hexagon** are the only regular polygons that will tessellate.

o. A tessellation with two or more noncongruent regular polygons in which each vertex is surrounded by the same arrangement of polygons (written as a **code,** the number of sides in the polygons listed in order) is called a **semiregular tessellation.**

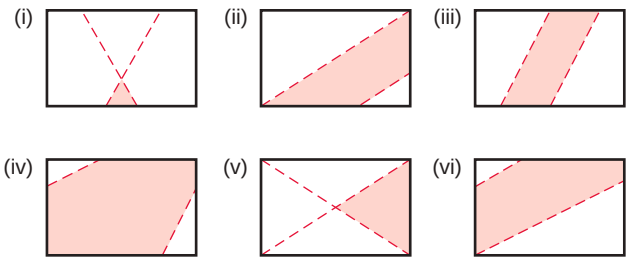
4. **Space figures**
   a. The term **space** is undefined. **Half-space, parallel,** and **perpendicular planes** are defined.
   b. The angle between two intersecting planes is called a **dihedral angle.** Lines **parallel** or **perpendicular to a plane** are defined.
   c. The surface of a three-dimensional figure whose sides are polygonal regions is called a **polyhedron.** The polygonal regions are called **faces,** and they intersect in the **edges** and **vertices** of the polyhedron. The union of a polyhedron and its interior is called a **solid.**
   d. Polyhedra are classified as **convex** or **concave.**
   e. A convex polyhedron whose faces are congruent regular polygons and that has the same arrangement of polygons at each vertex is called a **regular polyhedron** or a **Platonic solid.** The Platonic solids are the **tetrahedron, cube, octahedron, dodecahedron,** and **icosahedron.**
   f. A polyhedron whose faces are two or more noncongruent regular polygons and that has the same arrangement of polygons at each vertex is called a **semiregular polyhedron.**
   g. **Pyramids** are named by their base. The **base of a pyramid** can be any polygon, but its sides are always triangular. Pyramids whose sides are isosceles triangles are called **right pyramids;** otherwise, they are called **oblique pyramids.**
   h. **Prisms** are named by two parallel **bases,** upper and lower, which can be any congruent polygons. The sides of a **right prism** are rectangular, and the sides of an **oblique prism** are nonrectangular parallelograms.
   i. A **cone** has a circular region (disk) for a **base** and a lateral surface that slopes to the **vertex (apex).** If the vertex lies directly above the center of the base, the cone is called a **right cone** or usually just a cone; otherwise, it is an **oblique cone.**
   j. A **cylinder** has two parallel circular **bases** (disks) of the same size and a lateral surface that rises from one base to the other. If the centers of the upper base and lower base lie on a line that is perpendicular to each base, the cylinder is called a **right cylinder** or simply a cylinder; otherwise, it is an **oblique cylinder.**

5. **Spheres and maps**
   a. A **sphere** is the set of points in space that are the same distance from a fixed point, called the **center.** The union of a sphere and its interior is called a **solid sphere.** A line segment joining the center of a sphere to a point on the sphere is called a **radius.** A line segment containing the center of the sphere and whose endpoints are on the sphere is called a **diameter.**
   b. A **great circle** is a circle on a sphere whose center is also the center of the sphere.
   c. Points on the Earth's surface are located by two systems of circles: The circles that are parallel to the equator are **parallels of latitude,** and the circles that pass through the north and south poles are called **meridians of longitude.**
   d. **Cylindrical, conic,** and **plane projections** are three types of maps of the Earth's surface.
   e. Two points on the Earth's surface that are on opposite ends of a line segment through the center of the Earth are called **antipodal points.**

6. **Symmetry**
   a. A plane figure has **reflection symmetry** if there is a **line of symmetry** so that two halves of the figure will coincide if one is folded onto the other. If two points on opposite sides of the line of symmetry match up, one is called the **image** of the other.
   b. A plane figure has **rotation symmetry** if it can be turned about its center so that it coincides with itself. The point a figure is rotated about is called the **center of rotation.**
   c. A regular polygon with n sides has n reflection symmetries and n rotation symmetries.
   d. Every plane figure with reflection symmetries also has rotation symmetries other than 360°.
   e. A plane figure may have rotation symmetries but no reflection symmetries.
   f. A space figure with a plane that divides the figure into two halves, which are mirror images of each other, is said to have **reflection symmetry** and the plane is called a **plane of symmetry.**
   g. A space figure that can be rotated about a line which passes through the center of the figure, called the **axis of symmetry,** is said to have **rotation symmetry.**

CHAPTER 9 TEST

1. These figures were obtained by folding rectangular sheets of paper.



Write the numbers of the sheet(s) whose shaded region illustrates the polygon.

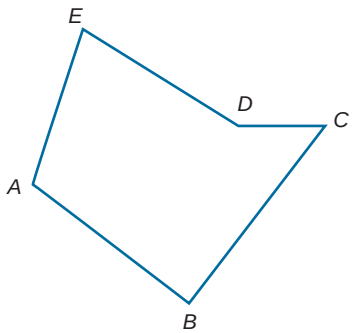
a. Hexagon **b.** Parallelogram
c. Trapezoid **d.** Equilateral triangle
e. Pentagon **f.** Isosceles triangle

2. Sketch an example of each of the following figures.

a. Concave pentagon
b. Simple closed curve
c. Convex decagon
d. A closed curve that is not simple

3. Identify the following types of angles in the pentagon.

a. Acute **b.** Reflex
c. Right **d.** Obtuse



4. Determine whether the following statements are true or false. Explain.

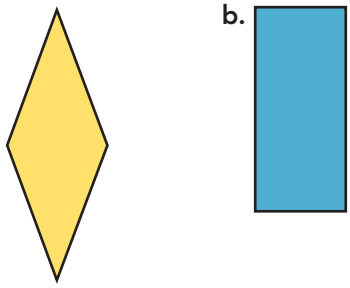
a. Every square is a rectangle.
b. Some scalene triangles are right triangles.
c. Every parallelogram is a rectangle.
d. Every rectangle is a parallelogram.
e. Some right triangles are equilateral triangles.

5. Determine the number of degrees in each angle.

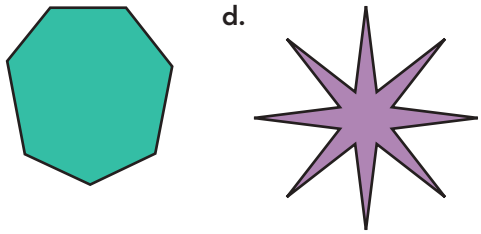
a. The central angle of a regular octagon
b. A vertex angle of a regular hexagon
c. An exterior angle of a regular pentagon

6. Determine whether each figure is a regular polygon. If it is not, state the condition it does not satisfy.

a. **b.**



c. **d.**



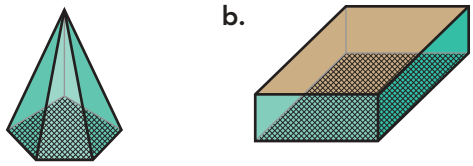
7. State whether each of the following polygons will tessellate.

a. Regular octagon **b.** Isosceles triangle
c. Regular hexagon **d.** Concave quadrilateral
e. Regular pentagon

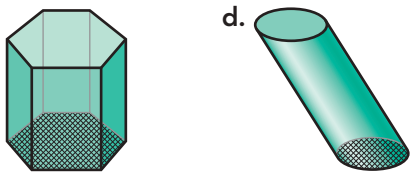
8. Can an equilateral triangle, a square, and a regular octagon, all of whose sides have the same length, be used together for a semiregular tessellation? Explain your answer.

9. Name each of the following figures, and classify each as right or oblique.

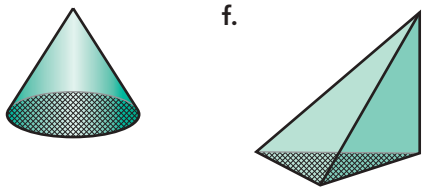
a. **b.**



c. **d.**



e. **f.**



10. Classify each figure as a polyhedron or a nonpolyhedron.

a. Sphere **b.** Prism
c. Pyramid **d.** Cone
e. Cube **f.** Dodecahedron

11. Determine the number of vertices in each of the following polyhedra.

a. An icosahedron (it has 30 edges)
b. A semiregular polyhedron with 14 faces and 36 edges

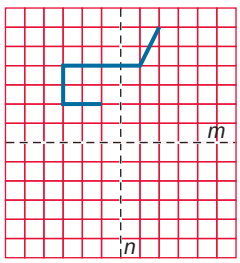
12. Sketch or describe each of the following plane figures.

a. A figure with three lines of symmetry
b. A figure with two rotation symmetries but no lines of symmetry
c. A figure with five rotation symmetries and five reflection symmetries

13. Determine the number of planes of symmetry for each figure.

a. A right prism whose base is a regular octagon
b. A right cone
c. A pyramid whose base is a regular pentagon

14. Finish sketching this figure so that it is symmetric about lines m and n.



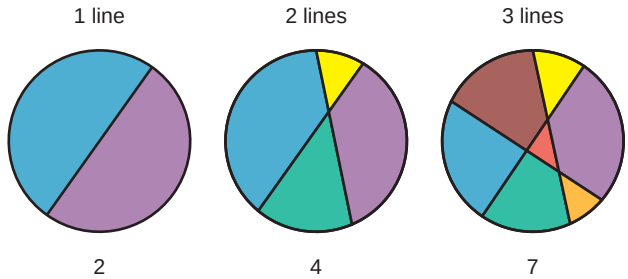
15. Determine the number of lines of symmetry and the number of rotation symmetries for each figure.

a. A rectangle
b. A regular heptagon
c. An equilateral triangle
d. A parallelogram

16. Seven points in a plane can be endpoints for a total of how many line segments?

17. What is the number of degrees in one vertex angle of a regular polygon with 40 sides?

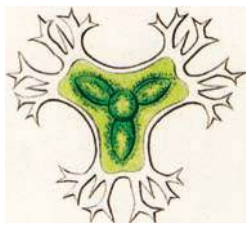
18. Suppose the interior of a circle is to be partitioned into the maximum number of regions by line segments. One line will divide it into two regions; two lines will divide it into four regions; and three lines will divide it into seven regions.



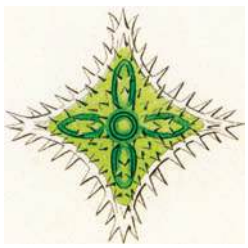
a. What is the maximum number of regions that can be created by four lines?
b. Find a pattern and use inductive reasoning to predict the maximum number of regions that can be created by 10 lines.

19. Determine the number of lines of symmetry and the number of rotation symmetries for each of the following sea organisms.

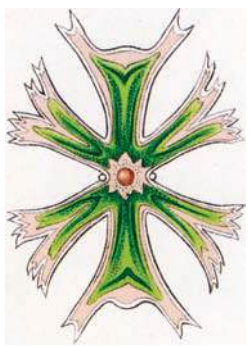
a.



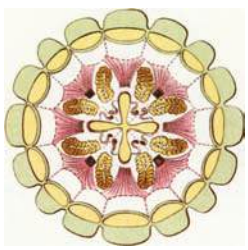
b.



c.



d.



CHAPTER

10 Measurement

Spotlight on Teaching

Excerpts from NCTM's Standards for School Mathematics Grades 3–5*

As they study ways to measure geometric objects, students will have opportunities to make generalizations based on patterns. For example, consider the problem in Figure 5.5. Fourth graders might make a table (see Figure 5.6) and note the iterative nature of the pattern. That is, there is a consistent relationship between the surface area of one tower and the next-bigger tower: "You add four to the previous number." Fifth graders could be challenged to justify a general rule with reference to the geometric model, for example, "The surface area is always four times the number of cubes plus two more because there are always four square units around each cube and one extra on each end of the tower." Once a relationship is established, students should be able to use it to answer questions like, "What is the surface area of a tower with fifty cubes?" or "How many cubes would there be in a tower with a surface area of 242 square units?"

Figure 5.5.
Finding surface area of tower of cubes

What is the surface area of each tower of cubes (include the bottom)? As the towers get taller, how does the surface area change?

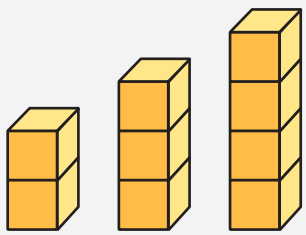


Figure 5.6.
A table used in the "tower of cubes" problem

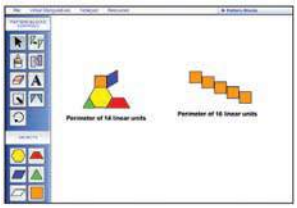
Number of cubes (N)	Surface area in square units. (S)
1	6
2	10
3	14
4	18

**Principles and Standards for School Mathematics*, p. 160.

MATH ACTIVITY 10.1

Perimeters of Pattern Block Figures

Virtual Manipulatives



www.mhhe.com/bbn

Purpose: Explore perimeters of pattern block figures using a nonstandard unit.

Materials: Pattern Block in the Manipulative Kit or Virtual Manipulatives.



1 linear unit

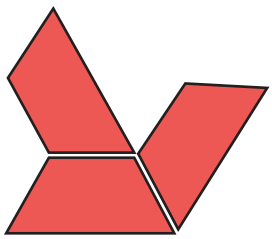
1. If the edge of the pattern block square is 1 linear unit, then its perimeter is 4 linear units. Determine the perimeter of each of the other pattern blocks.

2. The three trapezoid figure at the left has a perimeter of 11 units. Build pattern block figures that satisfy the following conditions. Sketch each figure and label its perimeter.

   a. Four triangles, perimeter of 6 units.

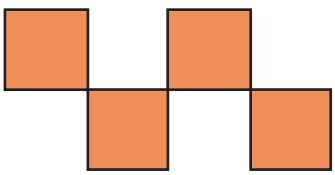
   b. Two hexagons and four triangles, perimeter of 10 units.

   c. Four tan parallelograms and four blue parallelograms, perimeter of 10 units.



Perimeter of 11 units

*3. The greatest perimeter of a figure with four pattern block squares is 16 units (see figure at left), and the least perimeter is 8 units. By using four pattern block squares, it is possible to build figures with perimeters for all the whole numbers from 8 to 16 units. Use your pattern block squares to build these figures. Sketch each figure and label its perimeter.

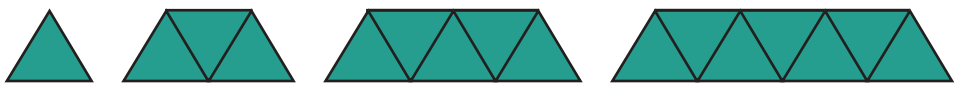


Perimeter of 16 units

4. Use your pattern blocks to build a figure with 14 triangles. What are the greatest and least possible perimeters for such figures? Repeat this activity for eight tan parallelograms to determine the greatest and least possible perimeters and for six hexagons to determine the greatest and least possible perimeters.

5. What generalization do activities 3 and 4 suggest about the shape of figures for obtaining the greatest perimeter as compared to the least perimeter?

6. The following pattern block figures are the first four in a sequence. The perimeter of the second figure is 5 units.



1st 2d 3d 4th

   a. Find a pattern and predict the perimeter for the fifth figure in this sequence. Then build the fifth figure and check your prediction.

   b. What is the perimeter of the 30th figure in this sequence?

   c. Write an algebraic expression for the number of triangles in the *n*th figure of this sequence.

   d. Write an algebraic expression for the perimeter of the *n*th figure in this sequence.

Section 10.1 SYSTEMS OF MEASUREMENT



Stonehenge, Salisbury Plain, England, believed to have been constructed between 1900 and 1700 B.C.E.

PROBLEM OPENER

Train A and train B are on the same track, headed toward each other. Both trains are traveling at 75 miles per hour. When the trains are 300 miles apart, a fly flies from the front of train A to the front of train B, then back to the front of train A, etc., returning back and forth until it is finally crushed by the colliding trains. How long is the fly in flight between the two trains?

NCTM Standards

Measurement is one of the most widely used applications of mathematics. It bridges two main areas of school mathematics—geometry and number. p. 103

The daily rotations of the Earth, the monthly changes of the Moon, and our planet's yearly orbits about the Sun provided some of the first units of measure. The day was divided into parts by sunrise, midday, and sunset; the year was divided into seasons. Some believe that the construction of the prehistoric monument known as Stonehenge (Figure 10.1), in southern England, was an early attempt to measure the length of a year and its seasons. By studying the shadows of the stones, druid priests may have been able to predict the arrival of the summer solstice and the occurrence of eclipses. Eventually, the sundial was invented to measure smaller periods of time, and it remained the principal method of measuring time until the fifteenth century. Now modern atomic clocks measure time to within 1 ten-millionth of a second.

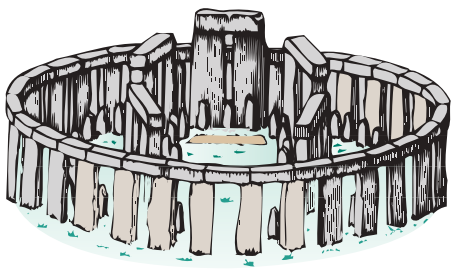


Figure 10.1
Stonehenge as it might have looked 4000 years ago

NCTM Standards

In today's schools, children first learn about concepts of measure with nonstandard units. The *Curriculum and Evaluation Standards for School Mathematics* (p. 52) supports this practice:

> If students' initial explorations use nonstandard units, they will develop some understandings about units and come to recognize the necessity of standard units in order to communicate.

Later, students should be taught both the English and metric units because both types of units are used in the United States. In this section we will look at examples of nonstandard units of measure as well as units of measure in the *English system* and the *metric system.*

NONSTANDARD UNITS OF LENGTH

The process of **measuring** consists of three steps:

1. Select an object and an attribute to be measured (length, weight, temperature, etc.).
2. Choose a **unit of measure** (any reproducible unit that can be used to measure a physical property).
3. Compare the unit to the object to determine the number of units, called the **measurement.**

Many of the first units of measure were parts of the body. The early Babylonian and Egyptian records indicate that the **span,** the **foot,** the **hand,** and the **cubit** were all units of measure (see Figure 10.2). The hand was used as a basic unit of measure by nearly all ancient civilizations and is the basis of the unit used today to measure the heights of horses. The height of a horse is measured by the number of hands from the ground to the horse's shoulders, and the hand has been standardized as 4 inches. Use the ruler on page 656 to compare the width of your hand to 4 inches.

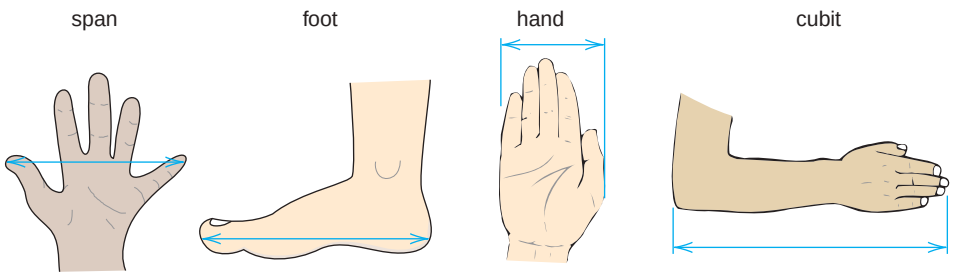


Figure 10.2

Elementary school experiences with measuring should provide the chance to relive our early measurement history through measuring activities with body parts.

EXAMPLE A

Choose two units of measure from Figure 10.2 with which to measure one of the following items: the length of a table (or desk), the height of a table, the length or width of a room, or the length of this book. List a few observations from this activity.

Solution The measurements will vary depending on the units chosen. The smaller the unit, the larger the measure; and the larger the unit, the smaller the measure. The measurement may not be a whole number. Two people may both choose the same unit, such as their hands, and obtain different measurements.

NCTM Standards

In preschool through grade 2, students should begin their study of measurement by using nonstandard units. They should be encouraged to use a wide variety of objects, such as paper clips to measure length, square tiles to measure area, and paper cups to measure volume. p. 45

Evidence of other early units of measure still exists today. Seeds and stones were common units for measuring weight. The word **carat,** which is the name of a unit of weight for precious stones, was derived from the word for the carob seeds of Mediterranean evergreen trees. *Carat* also expresses the fineness of a gold alloy. "Fourteen carat" means 14 parts of gold to 10 parts of alloy, or that 14 out of 24 parts are pure gold. The **grain,** a unit based on the average weight of grains of wheat and later standardized as .002285 ounce, is another unit of weight used by jewelers. Until recently the **stone** (14 pounds) was a common unit of weight in England and Canada. A newborn baby would weigh about one-half stone.

Such historical examples of units are helpful in understanding the concept of measure and suggest that nonstandard units of measure can be readily invented.

EXAMPLE B

1. Select an object to be used as a nonstandard unit of measure (paper clip, pencil, pen, handspan, etc.) to measure the length or width of a table, desk, chair, or other object near you. First guess, then check your answer.
2. Use the length of the following safety pin to measure the length of the pencil.



Solution **2.** The pencil is approximately $5\frac{1}{2}$ safety pin units long.

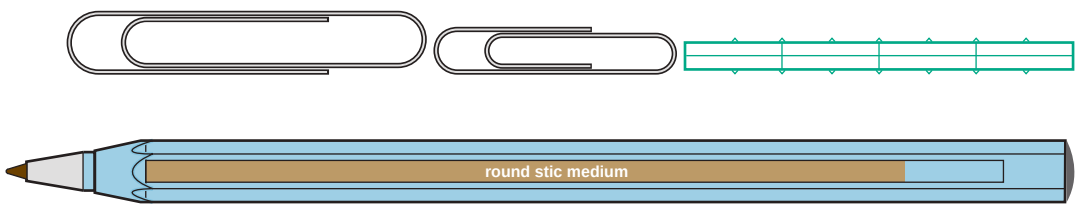
It is possible to find the length of an object by counting the number of times a chosen unit can be marked off on the object. Once a unit has been chosen, we assign it a length of 1 length (or linear) unit.

Since it is unlikely that the chosen unit will be marked off a whole number of times, three choices are possible for dealing with the part that is left over: (1) Estimate what fraction of the unit is left over, (2) create a smaller unit, or (3) subdivide the unit into an equal number of smaller parts to measure the part that is left over.

EXAMPLE C

Measure the length of the pen shown here, using each of the following units.

1. The length of the large paper clip.
2. The length of the small paper clip.
3. The length of the plastic twist-tie. The twist-tie has been bent into four equal parts.



Solution **1.** The pen is approximately 3 large paper-clip units long. **2.** The pen is approximately $4\frac{1}{2}$ small paper-clip units long. **3.** The pen is approximately $2\frac{3}{4}$ twist-tie units long.

ENGLISH UNITS

Length As societies evolved, measures became more complex. Since most units of measure had developed independently of one another, it was difficult to change from one unit to another. The English system, for example, arose from a hodgepodge of nonstandard units: The **inch** was the length of 3 barleycorns placed end to end, the **foot** was the length of a human foot, and the **yard** was the distance from the nose to the end of an outstretched arm. In the twelfth century, the yard was established by royal decree of King Henry I of England as the distance from his nose to his thumb (Figure 10.3). Gradually, the English system of measurements was standardized. The common units for length are shown in Figure 10.4.

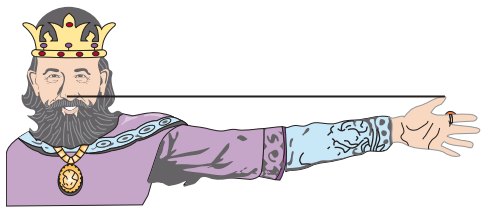


Figure 10.3

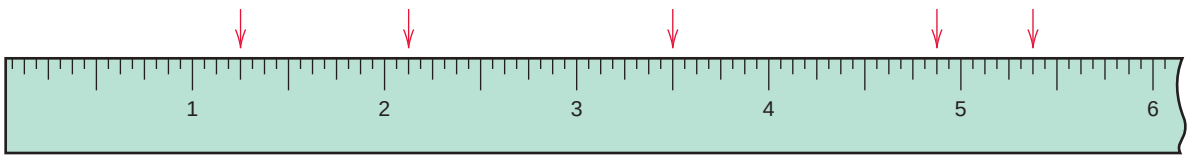
English Units for Length

inch	in	$\frac{1}{12}$ foot
foot	ft	12 inches
yard	yd	3 feet
mile	mi	5280 feet

Figure 10.4

EXAMPLE D Use the ruler pictured here to answer these questions.

1. What is the length from the tip of your index finger (or thumb) to the first joint?
2. What is the measure of the pencil in Example B?
3. What is the measure of the pen in Example C?
4. Write the measurement indicated by each arrow above the ruler.



Solution **1.** On the average adult hand, this length is approximately 1 inch. **2.** Approximately $5\frac{7}{8}$ inches. **3.** Approximately $5\frac{9}{16}$ inches. **4.** $1\frac{1}{4}$ inches; $2\frac{1}{8}$ inches; $3\frac{1}{2}$ inches; $4\frac{7}{8}$ inches; $5\frac{3}{8}$ inches.

Volume There are two methods of measuring volume in the English system. One uses cubes whose edges have lengths of 1 inch, 1 foot, or 1 yard. For example, a cubic inch is the volume of a cube whose edges are each 1 inch long (see Figure 10.5 on the next page).

Research Statement

The 7th national mathematics assessment found that fourth-grade students have difficulty with all items that asked them to use a ruler to measure an object or to draw a shape with particular dimensions.

Martin and Strutchens

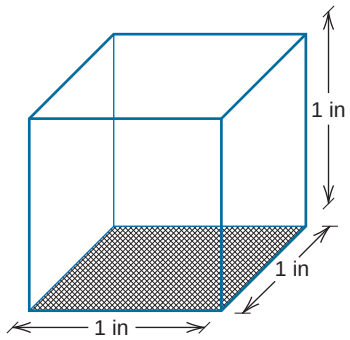


Figure 10.5

The other method of measuring volume uses measures that evolved from an ancient doubling system. Five of these measures are listed in Figure 10.6.

English Units for Volume

ounce	oz	$\frac{1}{8}$ cup
cup	c	8 ounces
pint	pt	2 cups
quart	qt	2 pints
gallon	gal	4 quarts

Figure 10.6

One inconvenience of the English system is that it is difficult to convert from one unit to another.

EXAMPLE E

One gallon is equal to 231 cubic inches. Determine the number of cubic inches in each of the following volume measures.

1. 1 quart **2.** 1 cup

Solution **1.** Since 4 quarts = 1 gallon and 231 ÷ 4 = 57.75, there are 57.75 cubic inches in 1 quart. **2.** Since 4 cups = 1 quart and 57.75 ÷ 4 = 14.4375, there are 14.4375 cubic inches in 1 cup.

HISTORICAL HIGHLIGHT

Units of Volume

2 mouthfuls = 1 jigger
2 jiggers = 1 jack (jackpot)
2 jacks = 1 jill
2 jills = 1 cup
2 cups = 1 pint
2 pints = 1 quart
2 quarts = 1 pottle
2 pottles = 1 gallon
2 gallons = 1 pail

The *mouthful* is a unit of measure for volume used by the ancient Egyptians. It was also part of an English doubling system: 2 mouthfuls equal 1 jigger; 2 jiggers equal 1 jack; 2 jacks equal 1 jill; etc. The familiar nursery rhyme that begins "Jack and Jill went up the hill" mentions three units of volume: the *jack,* the *jill,* and the *pail*. The rhyme was composed as a protest against King Charles I of England for his taxation of the jacks, or jackpots, of liquor sold in taverns. Charles's success at accumulating revenue from the taxes on liquor is the origin of the expression to *hit* the *jackpot*. The phrase *broke his crown* in the nursery rhyme refers to Charles I. Not only did he lose his crown, but also he lost his head in Britain's civil war not many years after he began taxing jackpots.*

*A. Kline, *The World of Measurements* (New York: Simon and Schuster, 1975), pp. 32–39.

Weight The English system has two systems for measuring weight: one for precious metals, in which there are 12 ounces in a pound **(troy unit);** and one for everyday use, in which there are 16 ounces in a pound **(avoirdupois unit).** We will use the avoirdupois unit. The common English units for weight are shown in Figure 10.7.

Figure 10.7

English Units for Weight

ounce	oz	$\frac{1}{16}$ pound
pound	lb	16 ounces
ton	tn	2000 pounds

EXAMPLE F

1. 14.3 pounds equals how many ounces?
2. 3200 pounds equals how many tons?

Solution **1.** 228.8 ounces ($14.3 \times 16 = 228.8$). **2.** 1.6 tons ($3200 \div 2000 = 1.6$).

Temperature In 1714, Gabriel Fahrenheit, a German instrument maker, invented the first mercury thermometer. The lowest temperature he was able to attain with a mixture of ice and salt he called zero degrees (0°). He used the normal temperature of the human body, which he selected to be 96 degrees (96°), for the upper point of his scale. (With today's more accurate thermometers, we know that human body temperature is about 98.6° on the Fahrenheit scale.) On this scale of temperatures, water freezes at 32° and boils at 212°. This scale is called the **Fahrenheit scale** (Figure 10.8).

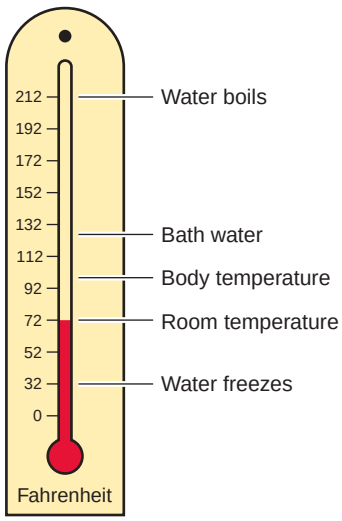


Figure 10.8

METRIC UNITS

In 1790, in the midst of the French Revolution, the metric system was developed by the French Academy of Sciences. To create a system of "natural standards," the scientists subdivided the length of a meridian from the equator to the north pole into 10 million parts to obtain the basic unit of length, the **meter** (see Figure 10.9 on the next page).

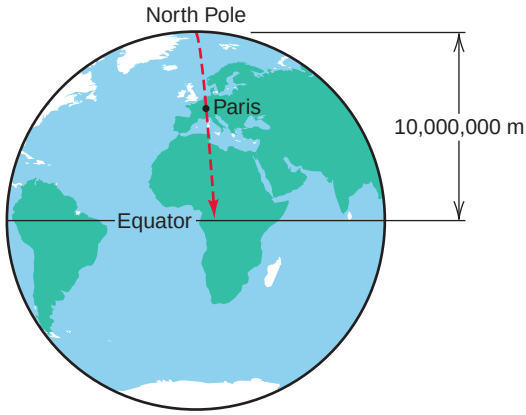


Figure 10.9

Once a basic unit is established in the metric system, smaller units are obtained by dividing the basic unit into 10, 100, and 1000 parts. Larger units are 10, 100, and 1000 times the basic unit. These units are named by attaching prefixes (Figure 10.10) to the name of the basic unit. The prefixes marked with asterisks are commonly used in everyday non-scientific measurement.

Metric Prefixes

Greek prefixes	kilo*	1000
	hecto	100
	deka	10
Latin prefixes	deci*	$\frac{1}{10}$
	centi*	$\frac{1}{100}$
	milli*	$\frac{1}{1000}$

Figure 10.10

There are additional prefixes for naming both larger and smaller units that are becoming common, especially in scientific articles. For example, **mega, giga,** and **tera** are prefixes meaning *million, billion,* and *trillion,* respectively, and **micro, nano,** and **pico** are prefixes meaning *one-millionth, one-billionth,* and *one-trillionth,* respectively. The fact that metric prefixes designate powers of ten is a major advantage of the metric system because we use a base-ten numeration system.

Length Figure 10.11 shows how the metric system prefixes are used with the meter to obtain other metric units. Notice that as we move from the millimeter to the kilometer, each unit is 10 times greater than the preceding unit. (The common lengths are marked with asterisks.) Several units of length that are less than a millimeter are shown on page 665.

Metric Units for Length

kilometer*	km	1000 m
hectometer	hm	100 m
dekameter	dam	10 m
meter*	m	1 m
decimeter	dm	$\frac{1}{10}$ m
centimeter*	cm	$\frac{1}{100}$ m
millimeter*	mm	$\frac{1}{1000}$ m

Figure 10.11

NCTM Standards

Measurement lends itself especially well to the use of concrete materials. In fact, it is unlikely that children can gain a deep understanding of measurement without handling materials, making comparisons physically, and measuring with tools. p. 44

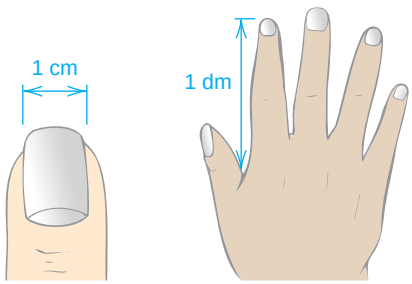
To acquire a feeling for the metric units of length, it is helpful to visualize objects having a given length. A *meter* is roughly the distance from the floor to the waist of an adult (b) or the distance from one shoulder to the fingertips of the opposite outstretched arm (c), as shown in Figure 10.12. The distance from the floor to a doorknob (a) is usually a little less than 1 meter. A meter might be used to measure the length of a house, a car, or an athletic field.



(a) (b) (c)

Figure 10.12

A **centimeter** is $\frac{1}{100}$ meter. This is the common unit for such body measurements as height, waist, and hat size. The width of a middle-school student's thumbnail (a) might be approximately 1 centimeter, as shown in Figure 10.13a. If the thumb is extended, as shown in part b, the length from the tip of the index finger to the bottom of the V shape (b) is approximately 1 decimeter (10 centimeters).



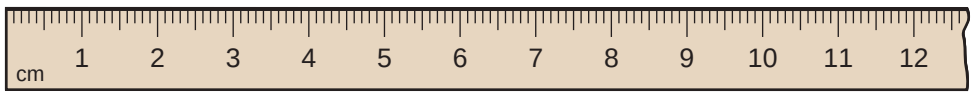
(a) (b)

Figure 10.13

Occasionally we need a measure smaller than a centimeter. One-tenth of a centimeter is a **millimeter.** The thickness of a pencil lead is approximately 2 millimeters. The ruler shown in Example G has a length between 12 and 13 centimeters, and each centimeter is divided into 10 millimeters.

EXAMPLE G

1. What is your handspan to the nearest centimeter?
2. Which of your finger widths is approximately 1 centimeter?
3. What is the diameter of a penny to the nearest millimeter?



NCTM Standards

Since the customary English system of measurement is still prevalent in the United States, students should learn both customary and metric systems and should know some rough equivalences between the metric and customary systems. p. 45

Solution **1.** The handspans of adults usually range from 18 to 23 centimeters. **2.** The little-finger width for an adult is close to 1 centimeter. **3.** The diameter of a penny is 1 centimeter and 9 millimeters, or 1.9 centimeters.

A **kilometer** is 1000 meters. Distances between cities and countries (and even planets) are measured in kilometers. A kilometer is shorter than a mile, approximately three-fifths of a mile, as indicated in Figure 10.14.†

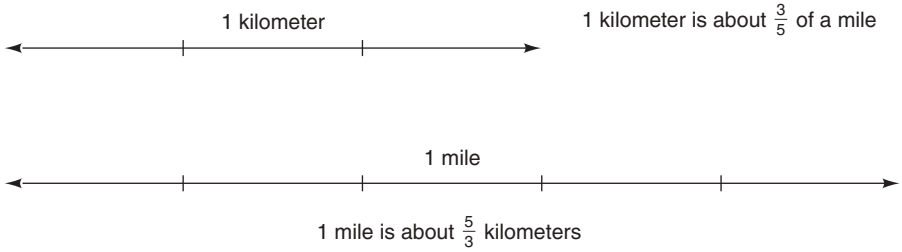


Figure 10.14

EXAMPLE H

1. If a person walks 3 miles per hour (1 mile every 20 minutes), approximately how many kilometers per hour does the person walk?

2. If a car is traveling 90 kilometers per hour, what is its speed in miles per hour?

Solution **1.** Since 1 mile $\approx \frac{5}{3}$ kilometer, 3 miles is 3 times $\frac{5}{3}$ kilometer:

$$3 \times \frac{5}{3} = \frac{15}{3} = 5$$

So, the person walks approximately 5 kilometers per hour.

2. Since 1 kilometer $\approx \frac{3}{5}$ mile, 90 kilometers is 90 times $\frac{3}{5}$ mile:

$$90 \times \frac{3}{5} = \frac{270}{5} = 54$$

So, the speed of the car is approximately 54 miles per hour.

Volume The basic unit of volume in the metric system is the **liter.** A liter is slightly larger than a quart. The capacities of fuel tanks, aquariums, and milk containers are measured in liters. For volumes that are less than a liter, such as those of small bottles or jars, the **milliliter** ($\frac{1}{1000}$ liter) is the common measure. Larger volumes, such as a community's reserve water supply, are measured in **kiloliters** (1000 liters).

Figure 10.15 lists the metric system units for volume, which are shown in relationship to the liter. Accepted abbreviations for liter are either uppercase (L) or lowercase (l). (The common volumes are marked with asterisks.)

Metric Units for Volume

kiloliter*	kL	1000 L
hectoliter	hL	100 L
dekaliter	daL	10 L
liter*	L	1 L
deciliter	dL	$\frac{1}{10}$ L
centiliter	cL	$\frac{1}{100}$ L
milliliter*	mL	$\frac{1}{1000}$ L

Figure 10.15

†Note: A more exact measurement is 1 kilometer = .621371192 mile.

EXAMPLE I

1. 1.3 liters equals how many milliliters?

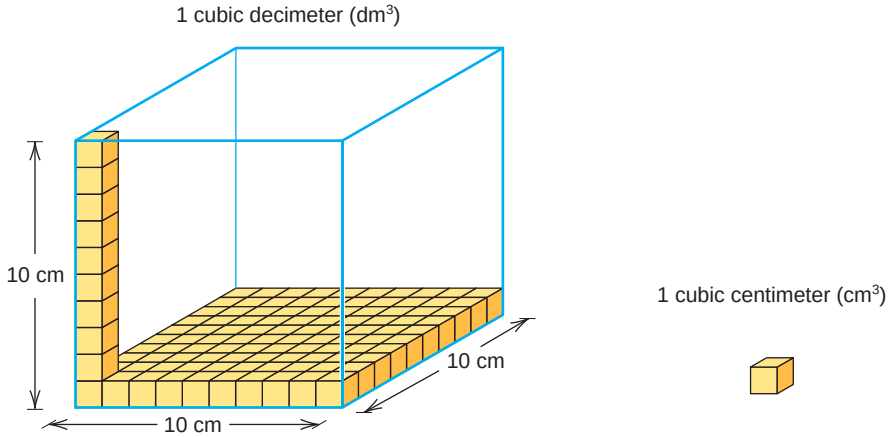
2. 245 milliliters equals how many liters?

3. 3487 liters equals how many kiloliters?

Solution **1.** 1300 milliliters. **2.** .245 liter. **3.** 3.487 kiloliters.

Notice in Example I how convenient it is to change from one unit of volume to another. Since 1 liter equals 1000 milliliters, the number of milliliters in 1.3 liters is 1.3×1000. Similarly, to change from 245 milliliters to liters, we divide by 1000. With metric units, conversions can be done mentally by multiplying and dividing by powers of 10.

A liter is the volume of a cube whose sides each have a length of 10 centimeters (part a in Figure 10.16). Such a cube is called a **cubic decimeter.** The dimensions of the small cube in part b are each 1 centimeter, and this cube is called a **cubic centimeter.**



(a) 1 liter equals 1000 cubic centimeters (b) 1 cubic centimeter equals 1 milliliter

Figure 10.16

Imagine filling the large cube in Figure 10.16 with the smaller cubes. The floor of the large cube is 10 centimeters $\times$ 10 centimeters and can be covered by 100 cubic centimeters. Since 10 layers of 100 cubes will fill the large cube, 1 liter has a volume of 1000 cubic centimeters.

Recall that 1 milliliter is $\frac{1}{1000}$ liter. Since a cubic centimeter is also $\frac{1}{1000}$ liter, 1 cubic centimeter equals 1 milliliter.

EXAMPLE J

1. 45 cubic centimeters equals how many milliliters?

2. 1.35 liters equals how many cubic centimeters?

3. 800 cubic centimeters equals how many liters?

Solution **1.** 45 milliliters. **2.** 1350 cubic centimeters. **3.** .8 liter.

Mass In the metric system the word *mass* and *weight* are different, the difference being due to the effect of gravity. We can think of **mass** as the amount of matter that makes up the object. **Weight,** however, is the force that gravity exerts on the object, and it varies with different locations from the center of the Earth. An object will weigh more at sea level than on top of a mountain, because the Earth's gravity exerts a greater force on it at lower altitudes. The same object in a spaceship would weigh practically nothing. Yet, in each of these three locations, the amount of material in the object hasn't changed! Because of this situation, the mass of an object is a measurement that does not change as the object is moved farther from the center of the Earth. At sea level the mass and weight of an object are essentially equal, and the variation in an object's weight between sea level and our highest mountains is very small (.1 percent difference).

The basic unit of mass in the metric system is the **kilogram.** A kilogram is 1000 grams and weighs approximately 2.2 pounds in the English system. A gram is a relatively small measure, approximately the mass of a medium-size paper clip or a dollar bill. Many items in grocery stores are measured in grams. Heavier objects are measured in kilograms. To acquire a feeling for a kilogram, it helps to know the approximate metric measurements of a few objects.

EXAMPLE K Use the fact that 1 kilogram is equivalent to approximately 2.2 pounds to determine each mass in kilograms.

1. Your mass.
2. The mass of a 10-pound bag of potatoes.
3. The mass of 1 pound of hamburger.

Solution **1.** A person who weighs 125 pounds has a mass of approximately 57 kilograms. **2.** Approximately 4.5 kilograms. **3.** Approximately .5 kilogram.

Figure 10.17 shows the metric units for mass and their relationships to the gram. (The common measures are marked with asterisks.) The metric ton, not shown in this table, is 1000 kilograms.

Metric Units for Mass

kilogram*	kg	1000 g
hectogram	hg	100 g
dekagram	dag	10 g
gram*	g	1 g
decigram	dg	$\frac{1}{10}$ g
centigram	cg	$\frac{1}{100}$ g
milligram*	mg	$\frac{1}{1000}$ g

Figure 10.17

A gram is the mass of 1 cubic centimeter of water. (Since water contracts and expands as its temperature changes, the technical definition calls for water to be at its densest state.) Since 1 cubic centimeter equals 1 milliliter ($\frac{1}{1000}$ liter), 1 milliliter of water has a mass of 1 gram. This simple relationship between mass, length, and volume (see Figure 10.18 on the next page) is another advantage of the metric system over the English system.

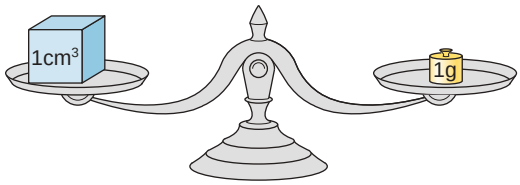


Figure 10.18

EXAMPLE L

1. 1 liter of water has a mass of how many kilograms?
2. 48.2 milliliters of water has a mass of how many grams?
3. 1500 cubic centimeters of water has a mass of how many kilograms?

Solution **1.** 1 kilogram. **2.** 48.2 grams. **3.** 1.5 kilograms.

Temperature In 1742, about 50 years before the development of the metric system, the Swedish astronomer Anders Celsius devised a temperature scale by selecting 0 as the freezing point of water and 100 as the boiling point. He called this system the **Centigrade** (100 grades) scale thermometer, but it came to be called the **Celsius scale** in his honor. Some examples of temperatures on the Celsius scale are shown in Figure 10.19.

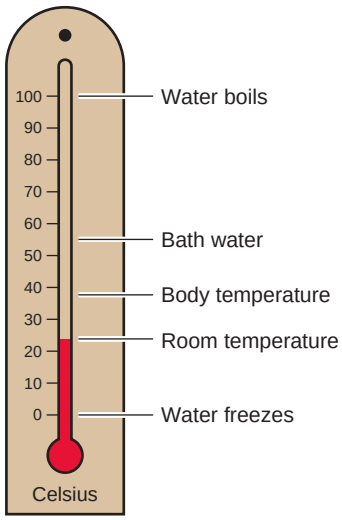


Figure 10.19

Heat is related to the motion of molecules: the faster their motion, the greater the heat. All movement of molecules stops at $^{-}273.15^{\circ}$ Celsius. The British mathematician and physicist William Thomson, known as Lord Kelvin, called this temperature *absolute zero* and devised the **Kelvin scale,** which increases 1 unit for each increase of 1° Celsius. Thus, 273.15 on the Kelvin scale is 0° on the Celsius scale. Both the Celsius and Kelvin scales are part of the metric system. The Celsius scale is used for weather reports, cooking temperatures, and other day-to-day needs; the Kelvin scale is used for scientific purposes.

EXAMPLE M

1. 270.15 on the Kelvin scale equals how many degrees on the Celsius scale?
2. 100° on the Celsius scale equals how many units on the Kelvin scale?

Solution **1.** $^{-}3^{\circ}$ **2.** 373.15

HISTORICAL HIGHLIGHT

The metric system was a radical change for the French people and met with widespread resistance. Finally in 1837, the French government passed a law forbidding the use of any measures other than those of the new system. Steadily, other nations adopted the metric system. In 1866, the Congress of the United States enacted a law stating that it was lawful to employ the weights and measures of the metric system and that no contract or dealing could be found invalid because of the use of metric units. Today, more than 200 years after its creation, the metric system has been adopted by almost every country. The U.S. Metric Conversion Act of 1975 set a policy of voluntary conversion with no overall timetable.

PRECISION AND SMALL MEASUREMENTS

Figure 10.20 shows a strand of DNA. Each DNA molecule is so small that 100,000 of them lined up side by side will fit into the thickness of this page.



Figure 10.20
DNA strand

NCTM Standards

In the middle grades, students should also develop an understanding of precision and measurement error. By examining and discussing how objects are measured and how the results are expressed, teachers can help their students to understand that a measurement is precise only to one-half of the smallest unit used in the measurement. p. 243

Measurements with this type of precision are possible with the transmission electron microscope. More recently, with the development of the field ion microscope, scientists have been able to view atoms that are one-tenth the size of DNA molecules. The scale in Figure 10.21 shows nine measures in decreasing order from a centimeter down to an angstrom, each being one-tenth of the size of the preceding one.

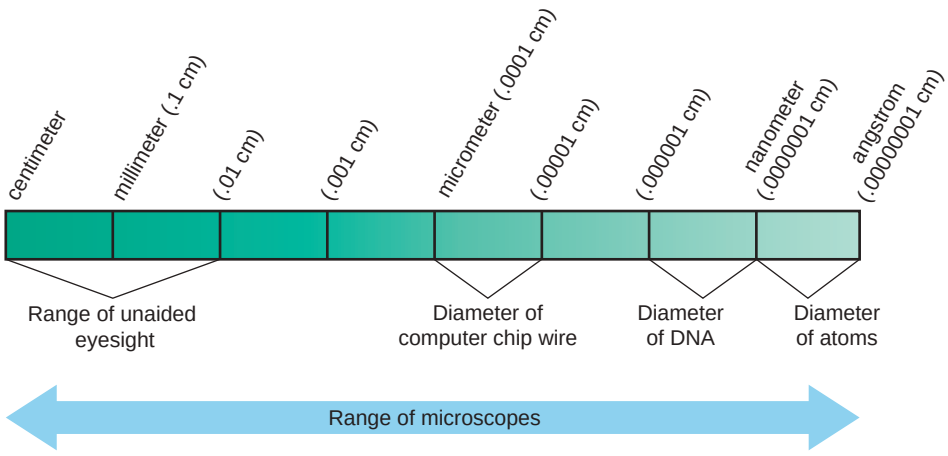
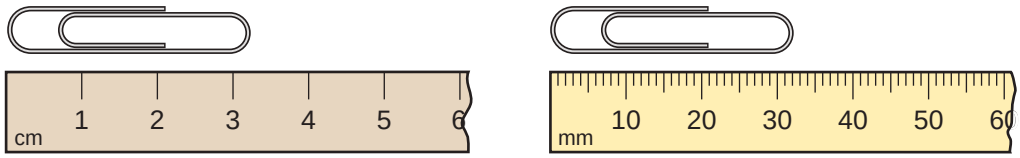


Figure 10.21

The amount of precision that is possible in taking measurements depends on the smallest unit of the measuring instrument. Using a centimeter ruler, we can determine that the paper clip in part a of Figure 10.22 has a length of just over 3 centimeters. If a ruler is marked off in millimeters, as in part b, the length of the paper clip can be measured as about 32 millimeters, or 3.2 centimeters. With instruments that are calibrated in smaller units, we might measure the length of this paper clip to be 3.24, 3.241, or 3.2412 centimeters. It would never be possible, however, to measure its length or the length of any other object exactly.



(a) (b)

Figure 10.22

If the smallest unit on the measuring instrument is a millimeter, then the measurement can be approximated to the nearest millimeter. This means that the measurement could be off by $\frac{1}{2}$ millimeter, either too much or too little. In general, the **precision** of any measurement is to within one-half of the smallest unit of measure being used.

Conversely, if a measurement is given as 14.5 centimeters, we can assume that it was measured to the nearest .1 centimeter and that it is closer to 14.5 centimeters than to 14.4 centimeters or 14.6 centimeters. In other words, it is 14.5 $\pm$ .05 centimeters, as shown in Figure 10.23 (.05 is one-half of one-tenth).

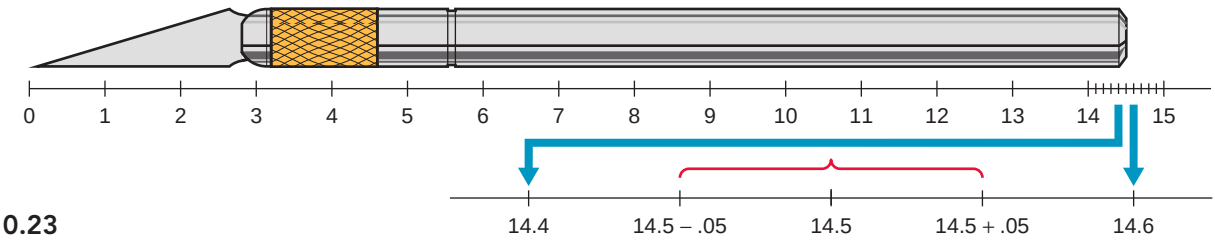


Figure 10.23

Writing a measurement as 7.62 centimeters indicates that the measurement has been obtained to the nearest .01 centimeter and may be off by as much as .005 centimeter (.005 is one-half of one-hundredth). Sometimes you will see a measurement such as 15.0 centimeters. A zero following the decimal point means that the measurement is accurate to the nearest .1 centimeter; 15.0 centimeters implies more precision than does 15 centimeters.

EXAMPLE N

Find the minimum and maximum measurements associated with each of the following measurements.

1. An oven temperature of 246.3° Celsius.
2. A baseball bat with length of 82 centimeters.
3. A bag of flour that is labeled 2.27 kilograms.

Solution **1.** One-half of one-tenth is .05, so the temperature is between 246.25 and 246.35° Celsius. **2.** One-half of 1 is .5, so the length of the bat is between 81.5 and 82.5 centimeters. **3.** One-half of .01 is .005, so the weight of the flour is between 2.265 and 2.275 kilograms.

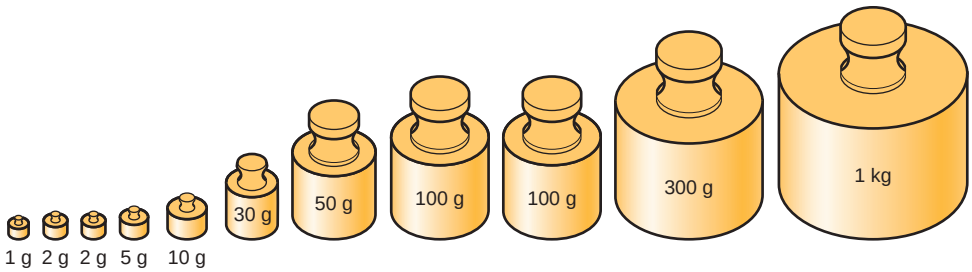
INTERNATIONAL SYSTEM OF UNITS

The **International System of Units** is a modern version of the metric system that was established by international agreement. Officially abbreviated as SI (for Système, International d'Unités), this system is built on the metric units discussed previously, but it also includes units for time **(second),** electric current **(ampere),** light intensity **(candela),** and the molecular weight of a substance **(mole).** This system provides a logical and interconnected framework for all measurements. To enable the type of precision needed in science today, the meter is now defined in SI units as the distance light travels in $\frac{1}{299{,}792{,}458}$ second, and 1 second of time is defined as the time required for a wave of the cesium-133 atom to complete 9,192,631,770 cycles.

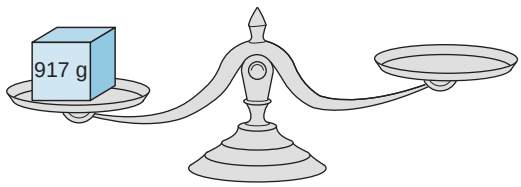
PROBLEM-SOLVING APPLICATION

Problem

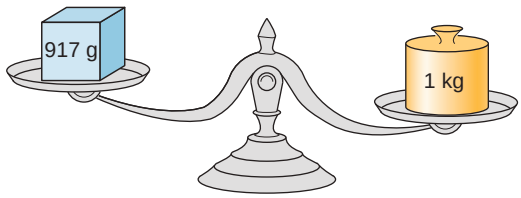
This standard set of 11 brass metric measures can be used with a balance scale to weigh any object whose mass is a whole number of grams from 1 to 1600. How can these measures be used to determine that an object has a mass of 917 grams?



Understanding the Problem The object to be weighed must be placed on one side of the scale, as shown below. The problem is that the 1-kilogram measure is greater than 917 grams and the sum of the remaining 10 measures is less than 917 grams. **Question 1:** What is the sum of the remaining 10 brass measures?



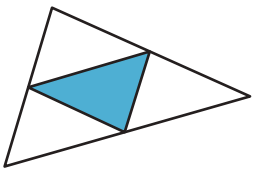
Devising a Plan One approach is to *guess and check* by experimenting with the brass measures. If the kilogram measure is placed on the right side of the scale, the scale will tip down on the right. **Question 2:** What additional brass measures will then be needed on the left side of the scale for a balance?



Technology Connection

Measuring Angles and Areas

If an inner triangle is formed by joining the midpoints of the sides of the larger triangle, what relationships are there between the inner triangle and the larger triangle? Use Geometer's Sketchpad® student modules available at the companion website to explore this and similar questions in this investigation.



Mathematics Investigation
Chapter 10, Section 1
www.mhhe.com/bbn

Carrying Out the Plan The left side of the scale will need measures totaling 83 grams, together with the object weighing 917 grams, to balance 1 kilogram. **Question 3:** How can measures totaling 83 grams be obtained from the set of brass measures?

Looking Back The key to solving this problem is to place brass measures on both sides of the scale. This is not always necessary. For example, an object with a mass of 18 grams can be balanced by placing measures of 10, 5, 2, and 1 gram on one side of the scale. **Question 4:** What is the lightest object whose mass is a whole number of grams for which brass measures must be placed on both sides of the scale to determine the mass of the object?

Answers to Questions 1–4 **1.** 600 grams. **2.** 83 grams. **3.** 50 grams, 30 grams, 2 grams, 1 gram. **4.** *Forming an organized list,* beginning with the smallest brass measures, shows that any object whose mass is between 1 and 20 grams can be measured by placing brass measures on just one side of the scale. The first object that requires the brass measures on both sides of the scale is an object whose mass is 21 grams.

Mass of Objects (grams)	Sets of Measures (grams)
1	1
2	2
3	2, 1
4	2, 2
.	.
.	.
.	.
19	10, 5, 2, 2
20	10, 5, 2, 2, 1

Exercises and Problems 10.1

Although almost all educators agree that we should not teach the metric system by converting back and forth from English units to metric units, it is sometimes helpful to make rough comparisons between the two systems. Make the comparisons in exercises 1 and 2.

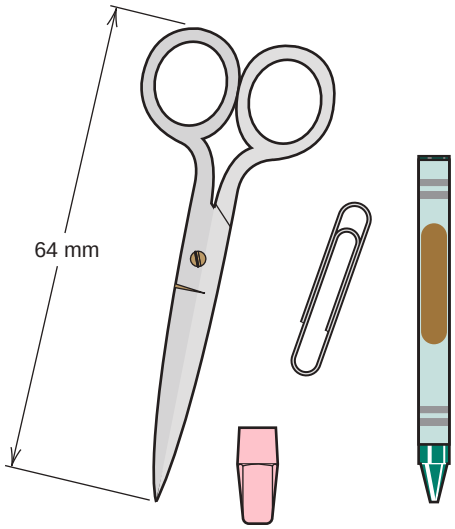
1. a. What is a 55-mile-per-hour speed limit approximately equal to in kilometers per hour?

b. The distance from Jersey City to New York is 24 miles. Approximately what is this distance in kilometers?

2. a. It has been suggested that the speed limit be set at 100 kilometers per hour. Approximately what would this speed limit be in miles per hour?

b. A 10-kilometer race is approximately how many miles?

Measure the length of the scissors shown in the figure, using each of the units in exercises 3 and 4.



3. **a.** The length of the paper clip
b. The length of the eraser

4. **a.** The length of the crayon
b. If you had to use one of these three units to measure other lengths, which do you think would consistently give you the most "accurate" measurements? Explain.

5. Express the amount the weight lifter in the figure is raising in each of the following units.
a. grams
b. milligrams
c. pounds (approximately)



6. Estimate each of the following to the nearest indicated unit. Then use the metric ruler on page 660 to check your estimate.
a. The thickness of pencil lead (millimeters)
b. The width of a pencil (millimeters)
c. The diameter of a dime (millimeters)
d. The length of a dollar bill (centimeters)
e. The width of a standard sheet of typing paper (centimeters)

7. Estimate (without direct calculation) each item in the preceding exercise, using inches and fractions or decimals for parts of an inch. Then check your estimate by using the inch ruler on page 656.

Recipes that use the English system of measurement include teaspoons (tsp) and tablespoons (tbsp or T) as units of measure. Complete exercises 8 and 9, using 16 tablespoons = 1 cup and 3 teaspoons = 1 tablespoon.

8. **a.** $\frac{1}{2}$ cup = _____ tablespoons
b. $\frac{1}{3}$ cup = _____ teaspoons
c. 2 quarts = _____ cups

9. **a.** 1 gallon = _____ pints
b. 1 gallon = _____ cups
c. $\frac{1}{2}$ pint = _____ cups

Complete the statements in exercises 10 and 11.

10. **a.** 1 mile = _____ yards
b. 4800 pounds = _____ tons
c. 7.5 gallons = _____ quarts

11. **a.** 12.6 feet = _____ yards
b. 56 ounces (oz) = _____ pounds
c. 40 cups = _____ gallons

12. List the following units in order of increasing length: meter, inch, centimeter, kilometer, yard, foot, mile, hectometer.

Choose the most realistic measure for each item in exercises 13 and 14.

13. **a.** Length of a ski: 200 millimeters, 200 centimeters, 200 meters
b. Mass of a person: 75 milligrams, 75 grams, 75 kilograms
c. Volume of an automobile gas tank: 48 milliliters, 48 liters, 48 kiloliters

14. **a.** Mass of a toothpick: 450 milligrams, 450 grams, 450 kilograms
b. Height of the Eiffel Tower: 300 centimeters, 300 meters, 300 kilometers
c. Amount of blood in the human body: 4 milliliters, 4 liters, 4 kiloliters



15. The first unit of measure of which there are historical records is the cubit, which is the distance from elbow to fingertips. This unit was used more than 4000 years ago by the Egyptians and Babylonians. The ancient Egyptian cubit (ca. 1550–1069 B.C.E.) shown above measures 52.5 centimeters and is preserved in the Louvre in Paris.

a. How does the Egyptian cubit compare in length with your cubit? What is the difference to the nearest centimeter?

b. The dimensions of Noah's Ark, as described in the Bible in the sixth chapter of Genesis, are listed in the table. Convert these measures to the nearest meter, using the length of the Egyptian cubit. Then convert the measures to the nearest foot (2.54 centimeters ≈ 1 inch).

	cubits	meters	feet
Length	300		
Breadth	50		
Height	30		

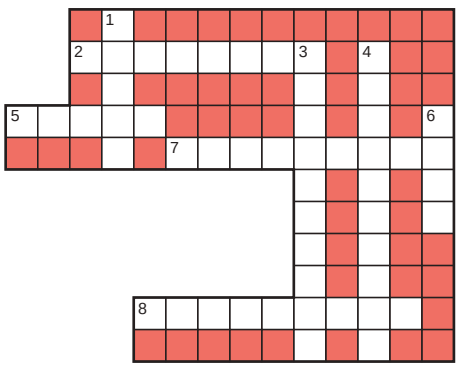
16. Complete the crossword puzzle by determining the most appropriate metric unit to use for the measurement in each clue.

Across

2. Mass of a truck
5. Length of a building
7. Volume of a city water supply
8. Length of a river

Down

1. Volume of a gasoline tank
3. Volume of a perfume bottle
4. Width of a television screen
6. Mass of a 50-cent coin



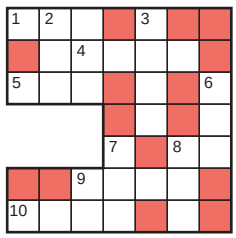
17. Complete the statement in each clue, and write your answers in the cross-number puzzle, placing one digit in each square.

Across

1. 16.5 cm = _____ mm
4. 3.15 L of water has mass of approximately _____ g
5. .12 L = _____ mL
8. _____ kg = 92,000 g
9. _____ mL of water has mass of approximately 7.920 kg
10. _____ m = 5.55 km

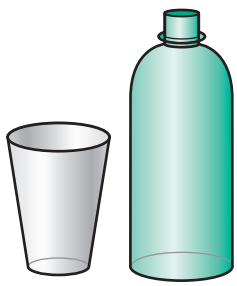
Down

2. 632,000 L = _____ kL
3. 4.5 km = _____ m
6. _____ g is the approximate mass of 432 mL of water
7. _____ kg is the approximate mass of 190 L of water
8. _____ mg = .9 g
9. .75 m = _____ cm

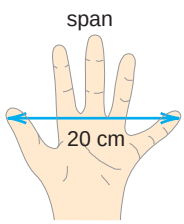


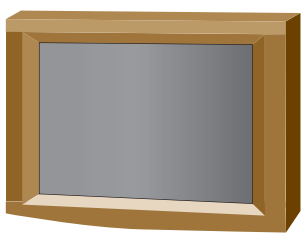
Measurements can be estimated by comparing the unknown quantity with familiar or known measurements. Obtain the estimations in exercises 18 and 19, using the given information.

18. a. Estimate the volume of the glass.



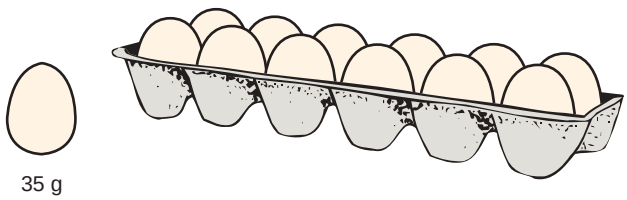
b. Estimate the length and width of the TV screen.





c. Estimate the distance between two towns if a car travels from one town to the other at an average rate of 55 miles per hour and it takes approximately $1\frac{1}{2}$ hours to make the trip.

19. a. Estimate the mass of a dozen eggs, including the mass of the carton.



b. Estimate the length of a garden hose if it is stretched out in a straight line from the house to the flower beds and an adult takes 30 walking steps from the faucet at the house to the end of the hose.

c. Estimate the height of the room shown in the figure here, assuming that the man is 6 feet tall.



20. The Celsius (C) and Fahrenheit (F) temperature scales are related by the following formulas:

$$C = \frac{5(F - 32)}{9} \quad \text{and} \quad F = \frac{9C}{5} + 32$$

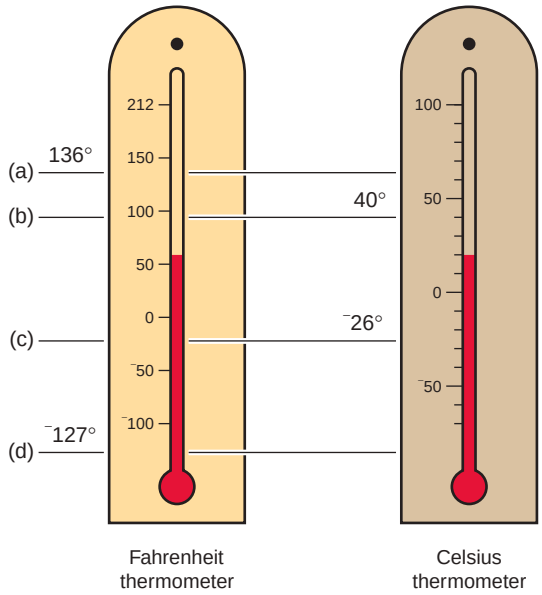
Four temperatures are described in parts a through d below and are indicated on the following thermometers. Convert each temperature to Fahrenheit or Celsius (to the nearest .1°).

a. Highest recorded temperature, Libya, 1922.

b. At this body temperature, see a doctor.

c. At this temperature, check your car's antifreeze.

d. Lowest recorded temperature, Antarctica, 1960.



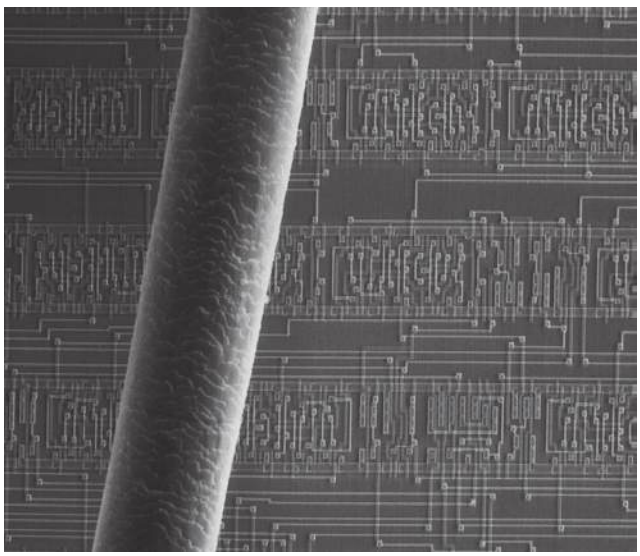
21. The strand of hair in the following photograph has been magnified 200 times.

a. Measure the thickness of the magnified strand of hair to the nearest millimeter.

b. Use the results of part a to determine the thickness of a strand of human hair.

c. How thick is a strand of human hair in micrometers? (1000 micrometers = 1 millimeter)

d. Some of the wires in a microcircuit have a thickness (diameter) of 1 micrometer. A human hair is how many times the thickness of a microcircuit wire?



A scanning electron micrograph magnified 200 times shows a microcircuit and a human hair

22.

Ready or not – metric system is coming

LOS ANGELES (AP) – The mile run and the 100-yard dash, two of track's glamor races, may soon join the horse-drawn carriage and the five-cent beer as relics of days gone by.

The United States soon will be forced to switch from measuring track meets in yards to measuring them in meters. The Amateur Athletic Union and the National Collegiate Athletic Association have long fought such a switch, but both agree it is becoming mandatory.

Under an international rule which went into effect on June 1, an athlete who runs a race in yards may not qualify for the Olympics, whether he sets a record or not.

As a result of an international rule that went into effect in 1975, track events traditionally measured in yards and miles are now measured in meters, as shown in the table below. Using the fact that 1 yard $\approx$ 91.5 centimeters, determine whether the metric event is longer or shorter, and compute the difference in meters to the nearest tenth.

	Old Race	New Race	Difference
a.	100 yards	100 meters	_____
b.	220 yards	200 meters	_____
c.	440 yards	400 meters	_____
d.	880 yards	800 meters	_____
e.	1 mile	1500 meters	_____

23. A measurement given to a certain unit may be off by as much as $\pm\frac{1}{2}$ of that unit. For example, a can of pineapple juice labeled 1.32 liters is measured to the nearest .01 liter. Its volume is greater than the minimum of 1.315 liters and less than the maximum of 1.325 liters. Find the minimum and maximum numbers associated with each of the following measurements.

a. A two-speed heavy-duty washing machine with mass of 112 kilograms (to the nearest kilogram)

b. A patient's temperature of 38.2°C (to the nearest .1°C)

c. A stereo speaker with a width of 48.3 centimeters (to the nearest .1 centimeter)

d. A 3-day-old baby has a mass of 3.46 kilograms (to the nearest .01 kilogram)

Reasoning and Problem Solving

24. Each of the grocery store items in the photograph at the top of the next column is measured either by mass or by volume. Determine which of the following measures goes with each item: 946 milliliters, 4.54 kilograms, 567 grams, 354 milliliters, 40 grams, 118 milliliters.



25. A shopper purchased the following items: tomatoes, 754 grams; soup, 772 grams; potatoes, 3.45 kilograms; sugar, 4.62 kilograms; raisins, 425 grams; vegetable shortening, 1.361 kilograms; and baking powder, 218 grams. What was the total mass of this purchase in kilograms?

26. A curtain for a single window can be made from a piece of material that is 1 meter wide and 120 centimeters long. Suppose you need two curtains per window and have six windows. If the curtain material comes in rolls 1 meter wide, how many meters of length will be needed to make curtains for all six windows?

27. The following amounts of gasoline have been charged on a credit card: 38.2, 26.8, 54.3, 44.7, and 34.0 liters. The price of gasoline is 81 cents per liter.

a. Use estimation techniques and mental arithmetic to approximate the total cost of the gasoline.

b. Use a calculator to compute the exact cost.

28. A car owner has her tank filled and notices that the odometer reads 14,368.7 kilometers. After a trip in the country, it takes 34.5 liters to fill the tank, and the odometer reads 14,651.6. How many kilometers per liter is this car getting?

29. A 24-kilogram bag of birdseed is priced at $16.88. If 75 grams of this feed is put in a bird feeder each day, how many days will it be before the bag of seed is empty? Rounded to the nearest penny, how much does it cost to feed the birds each day?

30. The recipe for a fruit punch calls for these ingredients: 3.5 liters of unsweetened pineapple juice, 400 milliliters of orange juice, 300 milliliters of lemon juice, 4 liters of ginger ale, 2.5 liters of soda water, 500 milliliters of mashed strawberries, and a base of sugar,

mint leaves, and water that has a total volume of 800 milliliters.

a. How much punch will the recipe make in liters?

b. If you serve the punch at a party of 30 people, how many milliliters of punch will there be per person?

c. This punch was sold at a fair, and each drink of 80 milliliters cost 25 cents. What was the profit on the sale of this punch if the ingredients cost $12.50 and all the punch was sold?

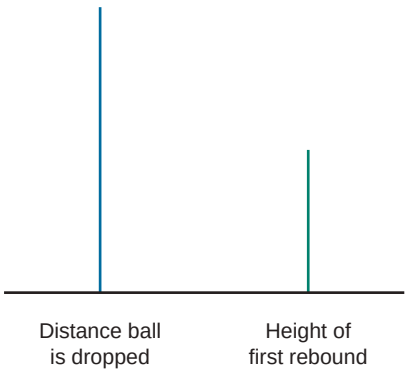
31. Prescription dosages of the antibiotic Garamycin vary from 20 milligrams for a child to 80 milligrams for an adult. The Garamycin is contained in a vial that has a volume of 2 cubic centimeters (2 milliliters). The Garamycin in each vial has a mass of 80 milligrams.

a. How many cubic centimeters of Garamycin are needed for 12 injections of 24 milligrams each?

b. How many injections of 60 milligrams each can be obtained from 24 vials?

32. Featured Strategies: Making a Drawing and Working Backward. When a special ball is dropped perpendicular to the floor, it rebounds to one-half its previous height on each bounce for five bounces. On its fifth bounce, it rebounds to a height of 6 centimeters. What is the total distance the ball has traveled after it bounces the fifth time and falls to the floor. Before reading further, make a drawing and try to work backward to solve this problem.

a. Understanding the Problem. A diagram will help you to visualize the problem. The height of each rebound can be represented by a vertical line that is one-half the height of the preceding line. Here is a diagram of the original distance the ball is dropped (blue line) and the height of its first rebound (green line). Explain why the height of each rebound must be doubled when the total distance that the ball travels is computed. Draw the complete diagram for this problem.



b. Devising a Plan. Working backward is a natural strategy for solving this problem. After its fourth bounce, how high does the ball rebound?

c. Carrying Out the Plan. Continue to work backward to get the height of each rebound and the original distance the ball is dropped. What is the total distance the ball has traveled when it falls to the floor after its fifth bounce?

d. Looking Back. For this problem, the total distance the ball has traveled is the height from which it was dropped, plus 2 times the height of each rebound. Will this statement be true if the problem is changed so that the ball rebounds to one-third its previous height on each bounce?

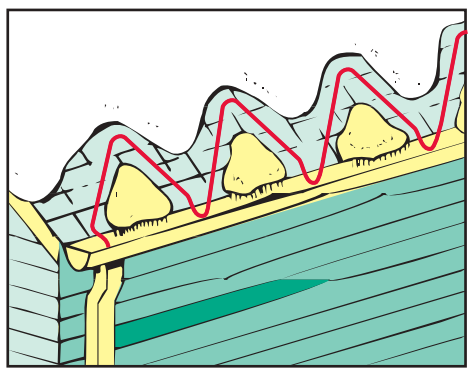
33. In 1983, at its General Conference on Weights and Measures, the National Institute of Standards and Technology used the speed of light to define the length of a meter. One meter is the distance light travels in $\frac{1}{299{,}792{,}458}$ second.

a. How many meters does light travel in 1 second?

b. What is the speed of light in kilometers per second?

c. In England in 1956, the speed of light was measured as $299{,}792.4 \pm .11$ kilometers per second. Is the speed of light that was used by the General Conference on Weights and Measures within this range?

34. Roof deicers are designed to prevent ice dams from building up on roofs and gutter pipes. An electric heating cable is clipped to the edge of the roof in a sawtooth pattern. In answering the following questions, assume that this pattern is to run along the edges of a roof and that the total length of the edges is 28 meters.



a. How many meters of heating cable will be needed for the edges of the roof if each 1 meter of roof edge requires 2 meters of cable?

b. In addition, heating cable is placed in gutter pipes and downspouts. Two gutter pipes run along the edges of the roof. Each has a length of 14 meters. There are two downspouts, one from each gutter pipe to the ground. Each downspout has a length of 3.2 meters. How many meters of cable will be

required to go along the gutter pipes and the downspouts if each 1 meter of gutter pipe or downspout requires 1 meter of cable?

c. The heating cable sells for $1.20 per meter. What is the total cost of the cable for the roof, gutter pipes, and downspouts?

35. The distance around the Earth's equator is 40,077 kilometers, and the population of the United States in 2010 was approximately 309,000,000 people. If this many people were spaced equally around the equator, what would be the length of the space each person would have, to the nearest centimeter?

Teaching Questions

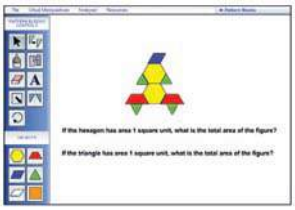
1. How would you reply to a student who said, "Why do we need to study the metric system when it is harder and nobody uses it?"
2. Read the **Research** statement on page 657. If you were asked to teach a lesson about measuring with a standard inch ruler, describe what you would do to help students understand how to read the ruler and measure using fractional parts of an inch.
3. One way to introduce students to the idea of precision is to use cardboard strips to form three types of rulers: one marked off only in inches, one marked off in inches and half inches, and one marked off in inches and quarter inches. Give an example of how you can use these rulers to illustrate the idea of precision of measurements.
4. The Celsius scale for temperature is used around the world. As a teacher, explain how you can help your students become as familiar with this scale as they are with the Fahrenheit scale for temperature.

Classroom Connections

1. The weight of a seemingly small amount of water may amaze you. Use the approximate conversions between metric units and English units discussed in this section to determine the approximate weight in pounds of a cubic decimeter of water—see the diagram on page 662. Then, use your results to find the weight of a cubic meter of water. Explain your thinking.
2. In elementary schools, children first learn about measuring concepts using nonstandard units. The **Standards** statements on pages 654, 655, and 660 all stress the importance of using nonstandard units in a variety of measuring experiences. Explain how having students measure objects with nonstandard units (paper clips, pencils, etc.) helps them learn about measuring as opposed to just starting with a normal ruler.
3. Look through the measurement recommendations for **PreK–2 Standards—Measurement** (see inside front cover) and explain what is meant by the last recommendation about developing common referents.
4. Use the library or Internet to make a list of metric prefixes other than the six standard metric prefixes used in this section. Describe the relationship of the prefixes you found to the six standard metric prefixes. For each prefix give an example of how and where a measure with that prefix is used.
5. The **Historical Highlight** on page 665 mentions that the French government had to pass a law forbidding the use of any system other than the metric system. The United States passed the Metric Conversion Act in 1975 with voluntary conversion and no timetable. Look at some elementary textbooks to see what effect voluntary conversion has had on the inclusion of the metric system in sections on measurement.

MATH ACTIVITY 10.2

Virtual Manipulatives



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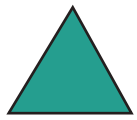
Areas of Pattern Blocks Using Different Units

Purpose: Explore areas of pattern block pieces using different units of area.

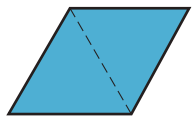
Materials: Pattern Blocks in the Manipulative Kit or Virtual Manipulatives.

***1.** Two of the pattern block triangles cover the blue parallelogram. So, if the triangle is the unit of area, then the parallelogram has an area of 2 triangular units.

a. Use your pattern blocks to find the areas of the trapezoid and the hexagon if the triangle is the unit of area.



Area of 1 triangular unit



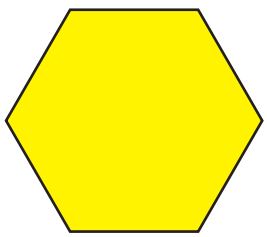
Area of 2 triangular units

b. Using the triangle as the unit of area, approximate the area of the square and the tan parallelogram. Draw sketches and explain your reasoning. Will the area of the square be greater or less than 2 triangular units? Will the area of the tan parallelogram be greater or less than 1 triangular unit?

2. Suppose the pattern block hexagon is the unit of area.

a. What are the areas of the trapezoid, blue parallelogram, and triangle?

b. What are the approximate areas of the square and tan parallelogram? Draw diagrams to support your conclusions.



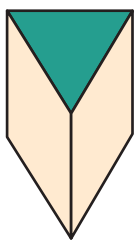
Area of 1 hexagonal unit

3. Normally a square is used for the unit of area.

a. Trace a pattern block hexagon on paper, and show that its area is approximately $2\frac{2}{3}$ times the area of the square.

b. If the square is used as the unit of area and the area of the hexagon is $2\frac{2}{3}$ times the area of the square, find the areas of the trapezoid, blue parallelogram, and triangle in terms of square units.

c. Use your pattern blocks and the diagram below to show the area of a tan parallelogram is one-half the area of the square pattern block. Use diagrams to support your reasoning.



Research Statement

The 7th national mathematics assessment found that students performed better on questions that were accompanied by manipulatives than on items that asked them to outline figures on a grid.

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Section 10.2 AREA AND PERIMETER



PROBLEM OPENER

Each of the 10 equilateral triangles in the following figure has sides of length 1 unit, and the perimeter of the entire figure is 12 units. What will the perimeter of the figure be if it is extended to include 50 such triangles?



The Marquette Plaza in Minneapolis (see the preceding photo) was designed to fulfill some unusual zoning restrictions. One of these was that the *coverage,* or ground area occupied by the building, could be only 2.5 percent of the area of the city block on which the building was to be built. To satisfy this condition, the 10 floors of the building are supported by two towers and cables that form a mathematical curve called a **catenary.**

In fact, this building can be thought of as a bridge that is 10 stories deep. There was no prototype for its structural design. Each floor has an unobstructed area of 60 feet $\times$ 275 feet. No other building had ever included floors that spanned such a length without internal columns.

NONSTANDARD UNITS OF AREA

To measure the sizes of plots of land, panes of glass, floors, walls, and other such surfaces, we need a new type of unit, one that can be used to cover a surface. The number of units it takes to cover a surface is called its **area.** Squares have been found to be the most convenient shape for measuring area. If we use the square region in part a of Figure 10.24 as the

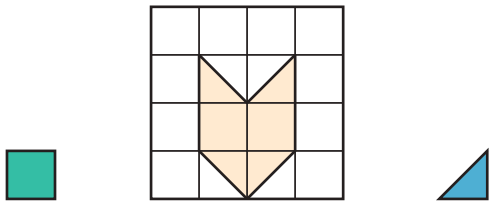


Figure 10.24

unit square, the area of the colored region in part b is 4 square units, because it can be covered by 2 squares and 4 half-squares.

The basic concept involved in calculating area—determining the number of units required to cover a region or surface—is often poorly understood by schoolchildren. A Michigan State assessment found that fewer than one-half of the seventh-graders examined could calculate the area of the region in Figure 10.24b by using the square in part a as the unit of area.* Nineteen percent of them thought the area was 6. Can you see why they might have obtained this answer?

Theoretically, the unit for measuring area can have any shape. It can be rectangular, triangular, etc. The only requirement is that the figure used for the unit area must tessellate (cover a region without gaps or overlapping).

EXAMPLE A

What is the area of the colored region in Figure 10.24b if the blue triangular region in part c is the unit for measuring area?

Solution Eight triangles are required to cover the colored region, so the area is 8 triangular units.

NCTM Standards

Teachers should provide many hands-on opportunities for students to choose [measurement] tools. . . . Although for many measurement tasks students will use nonstandard units, it is appropriate for them to experiment with and use standard measures such as centimeters and meters and inches and feet by the end of grade 2. p. 105

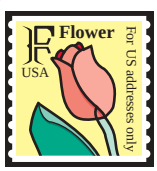
The earliest units for measuring area were associated with agriculture. The amount of land that could be plowed in a day with the aid of a team of oxen was called an **acre.** In Germany, a scheffel was a volume of seed, and the amount of land that could be sown with this volume of seed became known as a **scheffel of land.**

Just as nonstandard units for length help children in learning about linear measure, nonstandard units for area help them acquire an understanding of the concept of area.

EXAMPLE B

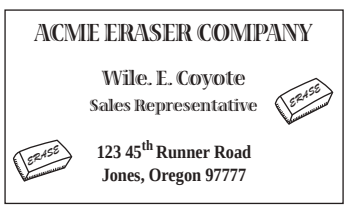
Trace each of the following regions. Then use each region as a unit of area to determine the approximate area of the rectangle on page 678.

(1)



Stamp

(2)



Business card

(3)



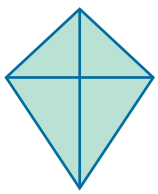
Eraser

*T. G. Coburn, Leah M. Beardsley, and Joseph Payne, *Michigan Educational Assessment Program, Mathematics Interpretive Report, Grade 4 and 7 Tests.* Guidelines for Quality Mathematics Teaching Monograph Series no. 7.

Technology Connection

Area Relationships

Did you know you can find the area of a kite by just knowing the lengths of its diagonals? What is this formula and can it be used to find the areas of other quadrilaterals? Use Geometer's Sketchpad® student modules available at the companion website to explore this and similar questions in this investigation.



Mathematics Investigation
Chapter 10, Section 2
www.mhhe.com/bbn

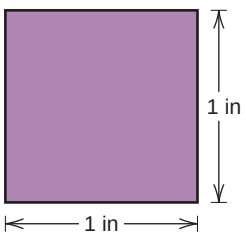
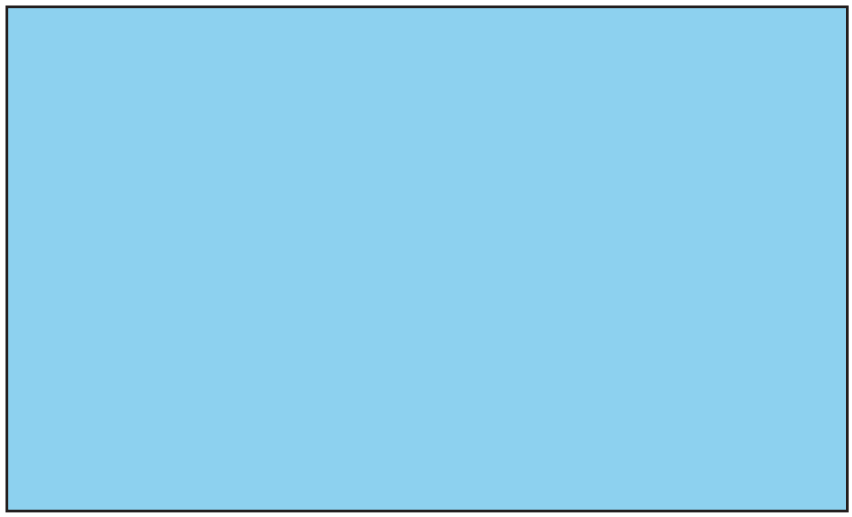


Figure 10.25



Solution **1.** Approximately 18 to 19 stamp units. **2.** Approximately 6 to 7 business card units. **3.** Approximately 23 to 24 eraser units.

STANDARD UNITS OF AREA

To standardize area measurement, a square became the accepted area unit shape. However, the size of the unit square differed in the two predominant systems of measurement: the English system and the metric system.

English Units In the English system, area is measured by using squares whose sides have lengths of 1 inch, 1 foot, 1 yard, or 1 mile. Each square unit is named according to the length of its sides. The 1 inch $\times$ 1 inch square in Figure 10.25 has an area of **1 square inch,** abbreviated 1 sq in or 1 in^2 (think of the exponent 2 as indicating a square).

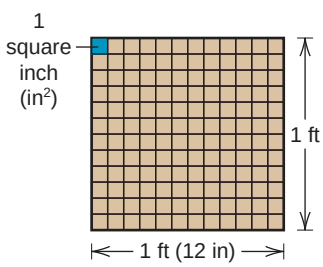
Similarly, we can measure larger areas by using a **square foot** (1 ft^2), a **square yard** (1 yd^2), and a **square mile** (1 mi^2).

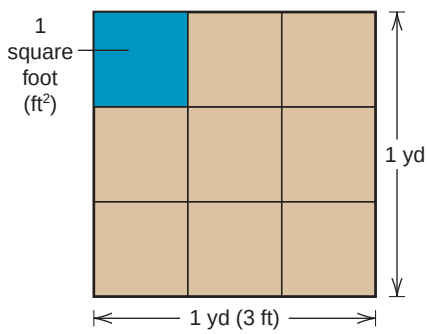
EXAMPLE C

The different square units are related to one another.

1. How many square inches equal 1 square foot?
2. How many square feet equal 1 square yard?

Solution **1.** 144 square inches. **2.** 9 square feet.





(*Note:* The squares are not drawn to scale.)

The common units for measuring area in the English system are shown in Figure 10.26.

Figure 10.26

English Units for Area

Square inch	in^2	$\frac{1}{144}$ square foot
Square foot	ft^2	144 square inches
Square yard	yd^2	9 square feet
Acre	ac	43,560 square feet
Square mile	mi^2	27,878,400 square feet

NCTM Standards

In grades 3–5, students should learn about area more thoroughly, as well as perimeter, volume, temperature, and angle measure. In these grades they learn that measurements can be computed using formulas and need not always be taken directly with a measuring tool. p. 44

Metric Units In the metric system there is a square unit for area corresponding to each unit for length. For example, a **square meter** (1 m^2) is a square whose sides have a length of 1 meter (shown in Figure 10.27, although not to scale). Square meters are used for measuring the areas of rugs, floors, swimming pool covers, and other such intermediate-size regions. Smaller areas are measured in square centimeters. A **square centimeter** (1 cm^2) is a square whose sides have lengths of 1 centimeter. Even smaller areas are measured with the **square millimeter,** a square whose sides have lengths of 1 millimeter. The actual sizes of the square centimeter and the square millimeter are shown in Figure 10.27.

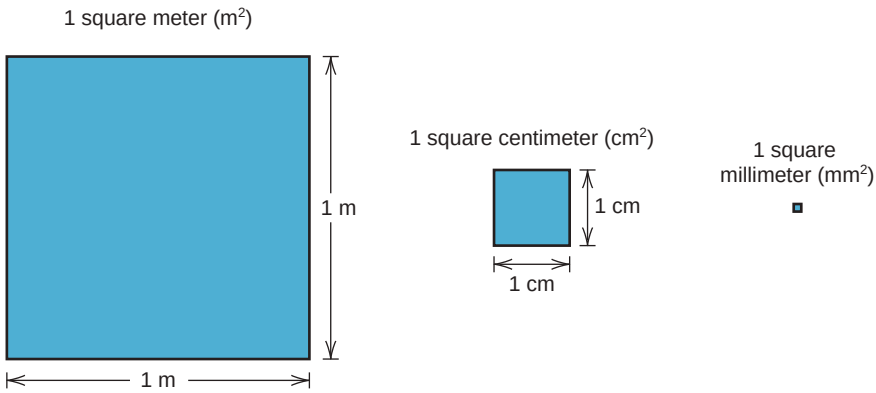


Figure 10.27

(*Note:* The square meter is not drawn to scale.)

EXAMPLE D Determine the following relationships between the metric units for area.

1. How many square millimeters are there in 1 square centimeter?
2. How many square centimeters are there in 1 square meter?

Solution **1.** Since each side of a square centimeter [see figure (1) on page 680] has a length of 1 centimeter, and 1 centimeter = 10 millimeters, a square centimeter can be covered by $10 \times 10 = 100$ square millimeters. **2.** Since 1 meter equals 100 centimeters, 100 square centimeters can be

Research Statement

Results involving questions on perimeter from the 7th national mathematics assessment reveal that fourth-grade students have difficulty with perimeter concepts, and eighth-grade students show a lack of understanding in more complex contexts that require a conceptual understanding of perimeter.

Martin and Strutchens

placed along each side of a square with dimensions of 1 meter × 1 meter [see figure (2) here], and so 10,000 square centimeters will cover the square meter.

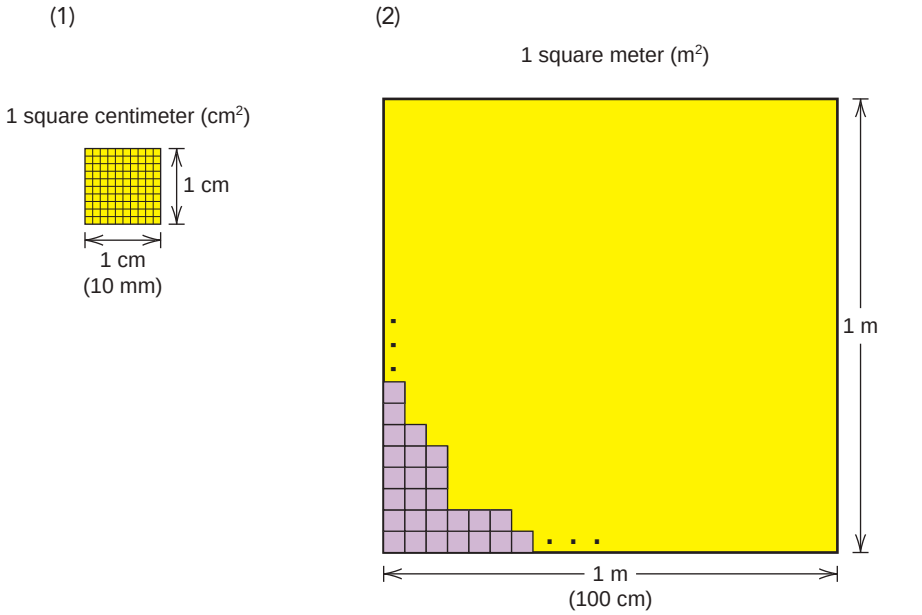


Figure 10.28

The areas of countries, national forests, oceans, and other such large surfaces are measured with the **square kilometer,** a square whose sides each have a length of 1 kilometer.

Some metric units for area and their relationships are shown in Figure 10.28.

Metric Units for Area

Square millimeter	mm^2	$\frac{1}{100}$ square centimeter
Square centimeter	cm^2	100 square millimeters
Square meter	m^2	10,000 square centimeters
Square kilometer	km^2	1,000,000 square meters

PERIMETER

Another measure associated with a region is its **perimeter**—the length of its boundary. The perimeter of Figure 10.29 is 23 centimeters, which is greater than the width of this page.

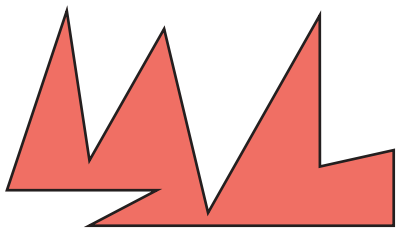
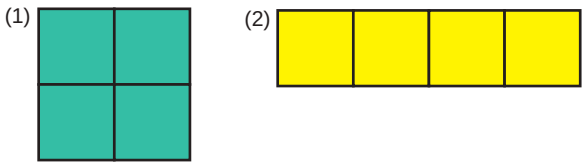


Figure 10.29

Intuitively, it may seem that the area of a region should depend on its perimeter. For example, if one person uses more fence to close in a piece of land than another person, it is tempting to assume the first person has enclosed the greater amount of land. However, this is not necessarily true.

EXAMPLE E

Each of the following figures has an area of 4 square centimeters. What is the perimeter of each figure?

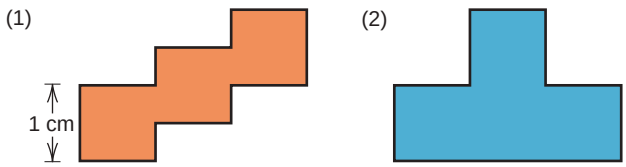


Solution **1.** The perimeter is 8 centimeters. **2.** The perimeter is 10 centimeters.

Example E shows that it is possible for two figures to have the same area but different perimeters. It is also possible for two figures to have the same perimeter but different areas.

EXAMPLE F

For each figure determine the area in square centimeters and the perimeter in centimeters.



Solution **1.** The area is 3 square centimeters, and the perimeter is 10 centimeters. **2.** The area is 4 square centimeters, and the perimeter is 10 centimeters.

NCTM Standards

Students should begin to develop formulas for perimeter and area in the elementary grades. Middle-grade students should formalize these techniques, as well as develop formulas for the volume and surface area of objects like prisms and cylinders. p. 46

AREAS OF POLYGONS

Rectangles Rectangles have right angles and pairs of opposite parallel sides, so unit squares fit onto them quite easily. The rectangle in Figure 10.30 can be covered by 24 whole squares and 6 half-squares. Its area is 27 square units. This area can be obtained from the product 6×4.5, because there are $4\frac{1}{2}$ squares in each of 6 columns. In general, if a rectangle has a length l and a width w, the **area of the rectangle** is the product of its length and its width.

Area of rectangle $= l \times w = lw$

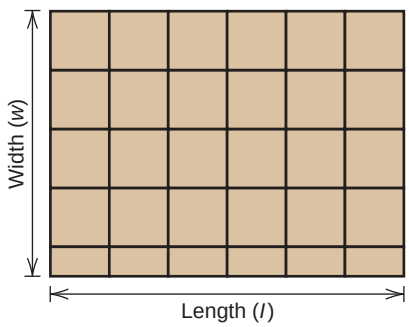


Figure 10.30

For a given perimeter, the dimensions of a rectangle affect its area. In the photographs in Figure 10.31 on the next page, the same knotted piece of string has been formed into three different rectangles. Using the lengths and widths of the rectangles, you can calculate that each of their perimeters is 36 centimeters. Yet, the area decreases from 80 square centimeters (8×10) to 72 square centimeters (6×12) to 32 square centimeters (16×2), as the

shape of the rectangle changes. If we continue to decrease the width of the rectangle, we can make its area as small as we please, although the perimeter will remain 36 centimeters.

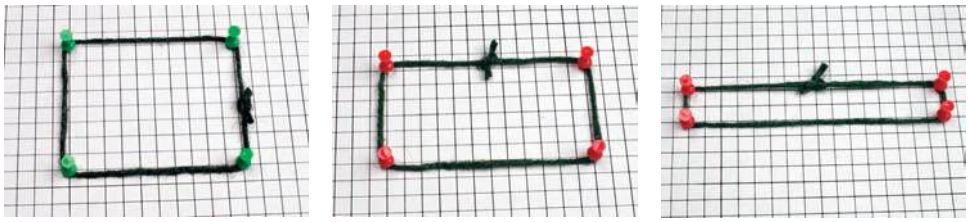
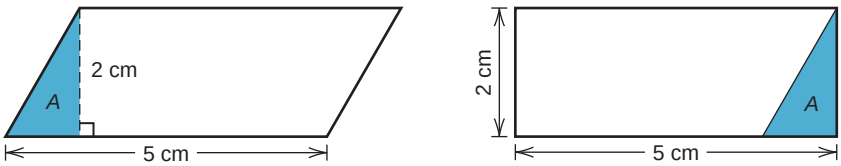


Figure 10.31
8 cm × 10 cm (left)
6 cm × 12 cm (middle)
16 cm × 2 cm (right)

Parallelograms Fitting unit squares onto a figure is a good way for schoolchildren to acquire an understanding of the concept of area. However, actually placing squares on a region is usually difficult because of the shape of the boundary (see part a of Figure 10.32).

One of the basic principles in finding area is that *a region can be cut into parts and reassembled without changing its area.* This principle is useful in developing a formula for the area of a parallelogram. The rectangle in part b of Figure 10.32 has been obtained from the figure in part a by cutting triangle A from the left side of the parallelogram and moving it to the right side to create a rectangle. The base of the rectangle is 5 centimeters, and its height is 2 centimeters, so its area is 10 square centimeters. Since the rectangle was obtained by rearranging the parts of the parallelogram, the area of the parallelogram is also 10 square centimeters. Notice that the base of the parallelogram is 5 centimeters and its **height,** or **altitude** (the perpendicular distance between opposite parallel sides), is 2 centimeters. This suggests that the *area of a parallelogram is the product of its base and its height.*

Area of parallelogram = $b \times h = bh$



(a) (b)

Figure 10.32

NCTM Standards

The notion that shapes that look different can have equal areas is a powerful one that leads eventually to the development of general methods (formulas) for finding the area of a particular shape, such as a parallelogram. p. 166

For a given perimeter, the area of a parallelogram depends on its shape. The two parallelograms formed by the inside edges of the linkages in the photograph in Figure 10.33 both have the same perimeter; but as the parallelogram is skewed more to the right, its height decreases. Since the base of both parallelograms is the same and the area of a parallelogram is the base times the height, the parallelogram with the smaller height has the smaller area. The area of the first parallelogram is approximately 72 square centimeters (9 × 8), and the area of the second one is approximately 45 square centimeters (9 × 5). The height of the parallelogram, and consequently its area, can be made arbitrarily small by further skewing the linkages while the perimeter stays constant.

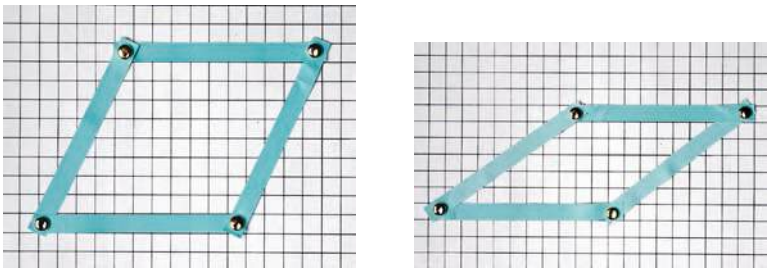


Figure 10.33

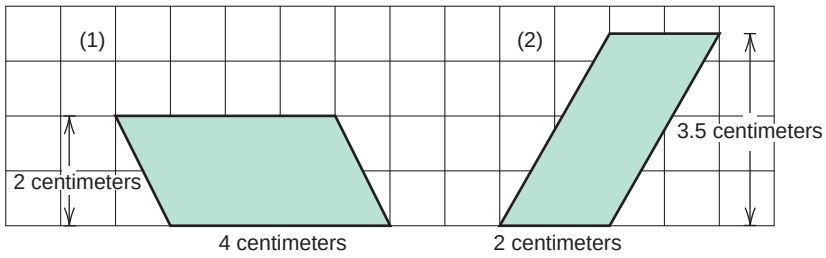
EXAMPLE G

Estimate the area of each parallelogram by visualizing the number of 1 centimeter × 1 centimeter squares needed to cover the figure. Then compute the area. Was your estimate within 1 square centimeter of the correct area?

Research Statement

The 7th national mathematics assessment found that only 18 percent of the fourth-grade students could draw a geometric shape with two side lengths given.

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Solution **1.** 8 square centimeters ($4 \times 2 = 8$). **2.** 7 square centimeters ($2 \times 3.5 = 7$). Were your estimates close to the areas of these figures?

Triangles The triangle in Figure 10.34a is covered with 1 centimeter × 1 centimeter squares and parts of squares. Can you see why this shows that the area of the triangle is more than 4 square centimeters? Since it is inconvenient to cover the triangle with squares, we will use a different approach to find its area. Two copies of a triangle can be placed together to form a parallelogram, as shown in part b. This can be accomplished by rotating the triangle in part a about the midpoint of side $\overline{AB}$. Since the parallelogram has a base of 5 centimeters and a height of 2 centimeters, its area is 10 square centimeters. Thus, the area of the triangle is one-half as much, or 5 square centimeters.

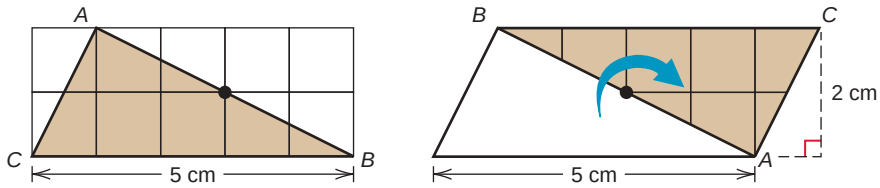


Figure 10.34

The preceding example suggests a general approach to finding the area of a triangle: Place two copies of the triangle together to form a parallelogram, and then find the area of the parallelogram. If the length of the base of the triangle is b and its **height,** or **altitude** (the perpendicular distance to its base from the opposite vertex), is h, then the base and altitude of the parallelogram are also b and h (Figure 10.35). So, the area of the parallelogram is $b \times h$, and since the parallelogram is formed from two triangles,

$$\textbf{Area of triangle} = \tfrac{1}{2} \times b \times h = \tfrac{1}{2}bh$$

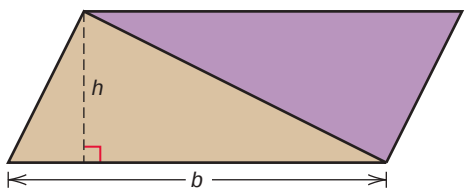
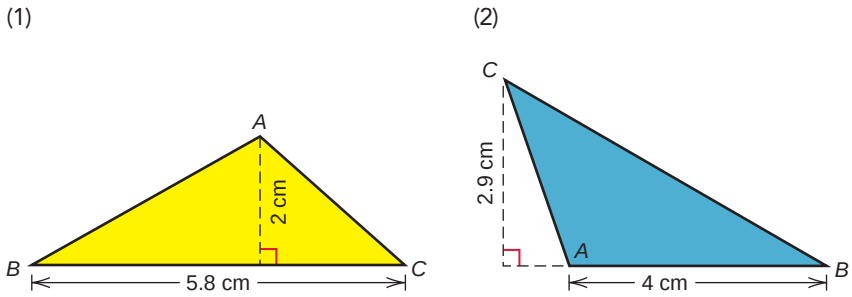


Figure 10.35

Any side of a triangle may be considered the base, and each base has its corresponding altitude. Regardless of the base and altitude chosen, as shown in Example H on the next page, the triangle will have the same area.

EXAMPLE H

Triangle *ABC* is shown below in two positions, with two different bases and altitudes. One of these altitudes falls outside the triangle. Determine the area of each triangle.



Solution **1.** The area is 5.8 square centimeters: $\frac{1}{2} \times 5.8 \times 2 = 5.8$. **2.** The area is 5.8 square centimeters: $\frac{1}{2} \times 4 \times 2.9 = 5.8$.

NCTM Standards

Students can develop formulas for parallelograms, triangles, and trapezoids using what they have previously learned about how to find the area of a rectangle, along with an understanding that decomposing a shape and rearranging its component parts without overlapping does not affect the area of the shape. p. 244

Trapezoids It is inconvenient to cover a trapezoid with square units because of its sloping sides. However, as with a triangle, we can obtain a parallelogram by placing two trapezoids together.

The trapezoid in Figure 10.36a has a *lower base* of length b and an *upper base* of length u, and its **height,** or **altitude** (the perpendicular distance between its bases), is h. The parallelogram in part b was obtained by placing two copies of the trapezoid side by side. The parallelogram has a base of $b + u$ and a height of h, so its area is $(b + u) \times h$. Since the parallelogram is formed from two trapezoids, this example suggests a general approach for finding the area of a trapezoid:

Area of trapezoid $= \frac{1}{2} \times (b + u) \times h = \frac{1}{2}(b + u)h$

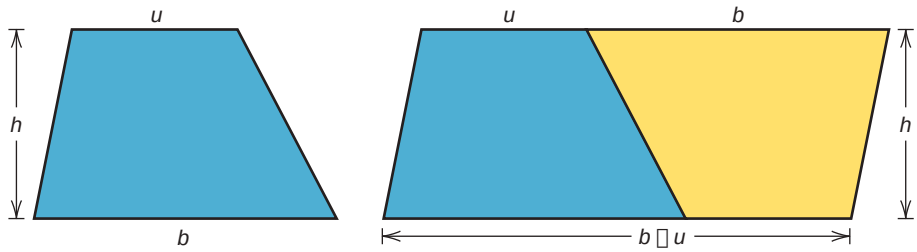


Figure 10.36

EXAMPLE I

If the upper and lower bases of the trapezoid in Figure 10.36 are 2 centimeters and 4 centimeters, respectively, and the height is 2.5 centimeters, try to estimate the area to see if you can come within 1 square centimeter of the correct area. Then use the formula for the area of a trapezoid to exactly compute the area.

Solution The area of the trapezoid is 8.75 square centimeters: $\frac{1}{2} \times (4 + 2) \times 2.5 = 7.5$. Was your estimate between 6 and 9 square centimeters?

CIRCUMFERENCES AND AREAS OF CIRCLES

Circles are part of our natural environment. The Sun, the Moon, flowers, whirlpools, and cross sections of some trees have circular shapes. School children see many everyday examples of circles, such as bottle caps, tops of cans, lamp shades, and toys like the hula hoop.



Figure 10.37
Hula hoops can be formed by joining the ends of tubing

Circumference The perimeter or distance around a circle is called the **circumference.** There is something deceptive about trying to estimate the circumference of a circle, as illustrated in Example J.

EXAMPLE J

Obtain a circular object near you (a cup, glass, etc.). Estimate how many diameter lengths it would take to wrap around the circumference of the object. Use string or the edge of a piece of paper to measure the circumference and the diameter of the object, and then compare the diameter to the circumference to check your estimate.

Solution Were you surprised that it took a little more than 3 diameters to reach around the circumference?

There is a tendency to underestimate the circumferences of circles. The hula hoop in the photograph above has a diameter of 3 feet and it took between 9 and 10 feet of tubing to make it. Often people estimate the circumference of a circular object by doubling the diameter of the object. Actually, the circumference is a little greater than 3 times the diameter.

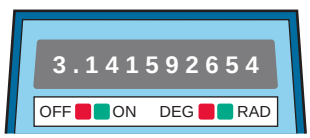


Figure 10.38

The exact ratio of the circumference of a circle to its diameter is the irrational number π **(pi),** which is 3.1416 rounded to four decimal places. This ratio is expressed in the following equations, where C is the circumference of a circle, d is the diameter, and r is the radius.

$$\frac{C}{d} = \pi \quad \text{or} \quad C = \pi d \quad \text{or} \quad C = 2\pi r$$

Technology Connection

Some calculators have a key for π. Pressing $\boxed{\pi}$ on a calculator with 10 places for digits will give the number shown in the display in Figure 10.38. However, since π is an irrational number, the number in this display is only a rational number approximation of π. For most purposes it is sufficient to approximate π by 3.1416. *Note:* For the exercises and problems in this text, use a calculator value of π or approximate π by 3.1416.

EXAMPLE K The photograph shows a piece of string being stretched around a tennis ball can.

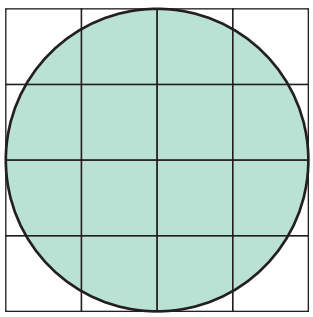


Predict how the length of the string will compare to the height of the can. The can has a diameter of approximately 7 centimeters and a height of approximately 20 centimeters. Compute the length of the string, and compare it to the height of the can.

Solution It is common for people to predict that the length of the string is less than the height of the can. However, since 3.1416×7 is approximately 22, the length of the string is approximately 22 centimeters, which is 2 centimeters greater than the height of the can.

Areas The area of a circle can be approximated by counting unit squares and parts of squares that cover the circle.

EXAMPLE L The following circle has been drawn on a centimeter grid. Approximate its area in square centimeters.



Research Statement

Research shows that additional attention needs to be given to geometry and measurement across the curriculum, as students consistently score poorly in these areas.

Strutchens and Blume; Kenney and Kouba; Lindquist and Kouba

Solution A first step might be to note that the circle is contained inside a 4×4 grid, which shows that its area is less than 16 square centimeters. Next, we can count the squares inside the circle and then combine the parts of the remaining squares, or we can estimate the parts of the squares outside the circle and subtract their area from 16. A reasonable estimate for the area of the circle is 12 square centimeters.

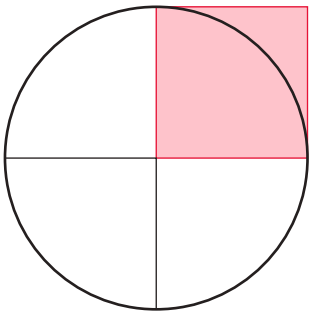


Figure 10.39

You might have noticed in Example L, on the previous page, that one-quarter of the circle is contained in a square whose side has a length that is equal to the radius of the circle. This is illustrated in Figure 10.39, which shows that the area of one-quarter of the circle with radius r is less than the radius times itself, or r^2. Thus, the area of the whole circle is less than $4 \times r^2$.

Let's consider a method for determining the area of a circle. The circle in part a of Figure 10.40 has been divided into 16 *sectors*. When these 16 sectors are rearranged and placed together, as shown in part b, they form a figure whose shape is close to that of a parallelogram. The length of the base of the parallelogram-like figure is one-half the circumference of the circle, and the height of the figure is approximately the radius of the circle. If the circle is cut into a greater number of sectors, the shape of the resulting parallelogram-like figure will be even closer to that of a parallelogram. Using the formula for the area of a parallelogram, we can approximate the area of the figure in part b:

$$\textbf{Area of parallelogram} = b \times h \approx \tfrac{1}{2} C \times r = \tfrac{1}{2}(2\pi r) \times r = \pi r^2$$

which is the formula for the area of a circle.

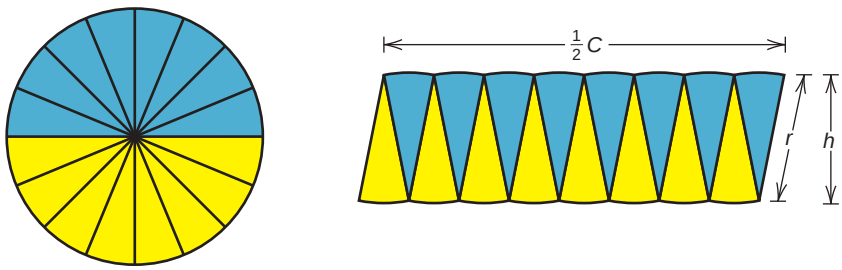


Figure 10.40

(a) (b)

$$\textbf{Area of circle} = \pi r^2$$

Notice that since $\pi \approx 3.1416$, the area of a circle is a little more than 3 times the square of the radius of the circle (see Figure 10.39).

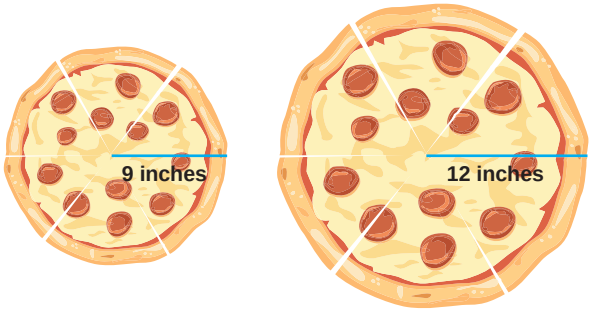
NCTM Standards

The *Curriculum and Evaluation Standards for School Mathematics* (p. 221) suggest that students construct such a model to develop the formula for the area of a circle:

> Students who can use the relationship between the shape of the "parallelogram" and its area and the circumference of the circle to develop the formula for the area of the circle are demonstrating plausible and deductive reasoning.

EXAMPLE M

Approximate the areas of 9-inch and 12-inch pizzas, using a value of 3 for π. If a 9-inch pizza costs \$10.80 and the unit cost per square inch of a 9-inch and a 12-inch pizza is the same, what is the cost of the 12-inch pizza?



Solution The area of a 9-inch pizza is approximately 243 square inches (3×9^2), and the area of a 12-inch pizza is approximately 432 square inches (3×12^2). The cost per square inch of the 9-inch pizza is $4.\overline{4}$ cents ($1080 \div 243$), so the cost of the 12-inch pizza is \$19.20 ($4.\overline{4} \times 432$).

HISTORICAL HIGHLIGHT

π

Pi has had a long and interesting history. In the ancient Orient, π was frequently taken to be 3. This value also occurs in the King James Bible in nearly identical verses (1 Kings 7:23 and 2 Chronicles 4:2) that describe the circumference of a circular container as being 3 times its diameter.

There have been many attempts to compute π. Archimedes computed π to the equivalent of two decimal places, and in 1841 Zacharias Dase computed π to 200 places. In 1873, William Shanks of England computed π to 707 places. In 1946, D. F. Ferguson of England discovered errors starting with the 528th place in Shanks' value for π, and a year later he gave a corrected value of π to 710 places. In recent years electronic computers have calculated π to hundreds of thousands of decimal places. Among the curiosities connected with π are the word devices for remembering the first few decimal places. In the following sentence, the number of letters in each word is a digit in π.

3. 1 4 1 5 9 2 6

May I have a large container of coffee?*

*This mnemonic and others are given by H. W. Eves, *An Introduction to the History of Mathematics,* p. 94.

PROBLEM-SOLVING APPLICATION

Sometimes it is necessary to find the areas of irregular or nonpolygonal shapes. Some of the water supplied to a leaf by its system of tiny veins is lost through small openings called **stomates.** Botanists collect the water by tying a plastic bag around a branch (Figure 10.41). Then they compute the areas of leaves to determine the amount of water lost for each square centimeter of surface area.

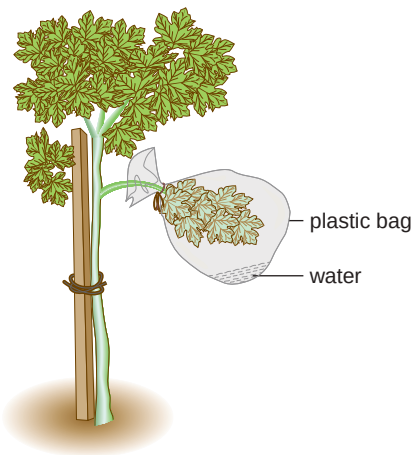


Figure 10.41

Problem

If a certain leaf loses 2 milliliters of water in a 24-hour period, how much water does it lose for each square centimeter of surface area?

Understanding the Problem First, it is necessary to determine the area of the leaf. **Question 1:** If the given leaf has an area of 10 square centimeters, how much water does it lose for each square centimeter?

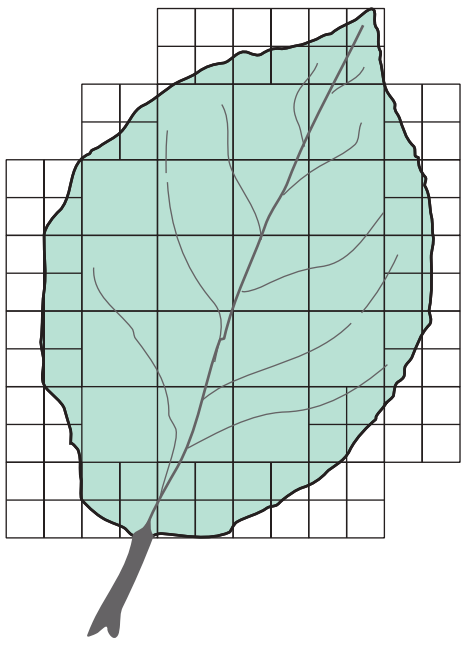
Devising a Plan One approach to finding the area of a leaf is to trace the leaf on cardboard and cut it out. Comparing the weight of the cut-out leaf to the weight and area of the original piece of cardboard by using ratios will give an approximation of the leaf's area. Another approach is to trace the leaf on grid paper and count the number of squares that fall inside the boundary of the leaf. **Question 2:** If this approach is used, what can be done with the squares that lie on the boundary?

Carrying Out the Plan The given leaf has been traced on a centimeter grid in the figure. There are 18 squares, each 1 centimeter $\times$ 1 centimeter, that fall inside the boundary. So the area of the leaf is at least 18 square centimeters. To permit a more accurate estimate for the remainder of the leaf's area, each square on the boundary has been divided into four smaller squares. One approach is to count the number of small squares that are half or more than half covered by the leaf. There appear to be 32 such squares. Using the 18 large interior squares and the 32 small boundary squares, we can estimate the area of the leaf. **Question 3:** What is the approximate area, and how much water is lost for each square centimeter?

Research Statement

The 7th national mathematics assessment results indicate that most students primarily experience the concept of area through the memorization of formulas, and that they may never really understand what the concept means in terms of determining the size of a region.

Martin and Strutchens

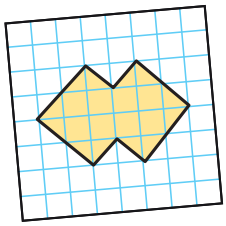


Looking Back The 32 quarter-squares we counted in the preceding step represent 8 square centimeters of leaf area. The process of subdividing boundary squares can be continued to obtain more accurate estimates. **Question 4:** For example, if each quarter-square is divided into 4 tiny squares and 136 of these tiny squares are half or more than half covered by the leaf, what is the new estimate of the leaf's area?

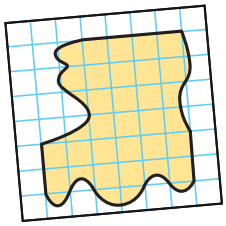
CHECK What You Know

Estimate the area of each figure. Each square represents 1 square centimeter. See Examples 1-3 (pp. 612–613)

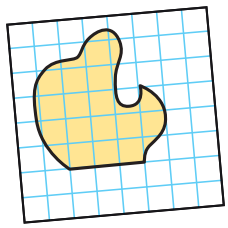
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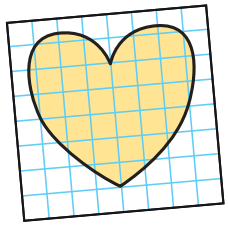
2.



3.



4. A cake decorator is drawing a heart on a cake. Each square represents 1 square inch. Estimate the area of the heart.



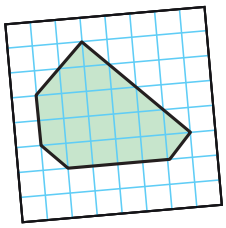
5. Talk About It Describe one way to estimate the area of an irregular figure that is drawn on grid paper.

Practice and Problem Solving

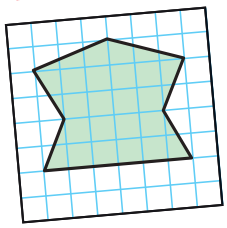
EXTRA PRACTICE See page R37.

Estimate the area of each figure. Each square represents 1 square centimeter. See Examples 1-3 (pp. 612–613)

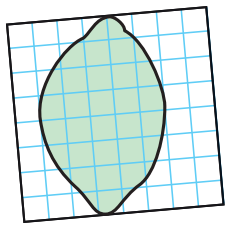
6.



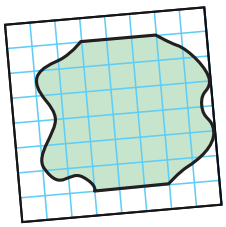
7.



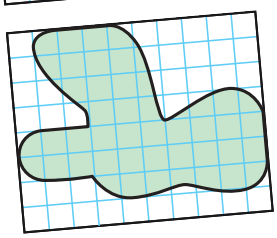
8.



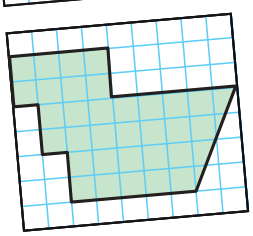
9.



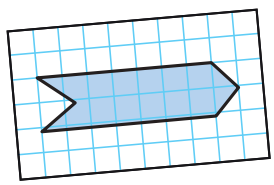
10.



11.



12. Isaiah made a sign for his yard sale shown at the right. If each square represents 1 square inch, estimate the area of the sign.



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Answers to Questions 1–4 **1.** .2 milliliter. **2.** One approach is to combine parts of squares to obtain whole squares. Another is to subdivide the boundary squares. **3.** 26 square centimeters; approximately .08 milliliter of water is lost per square centimeter ($2 \div 26 \approx .08$). **4.** Since 16 of the tiny squares have an area of 1 square centimeter, 136 of these tiny squares have an area of 8.5 square centimeters ($136 \div 16 = 8.5$). So, the new estimate of the area of the leaf is 26.5 square centimeters ($18 + 8.5 = 26.5$).

Exercises and Problems 10.2*

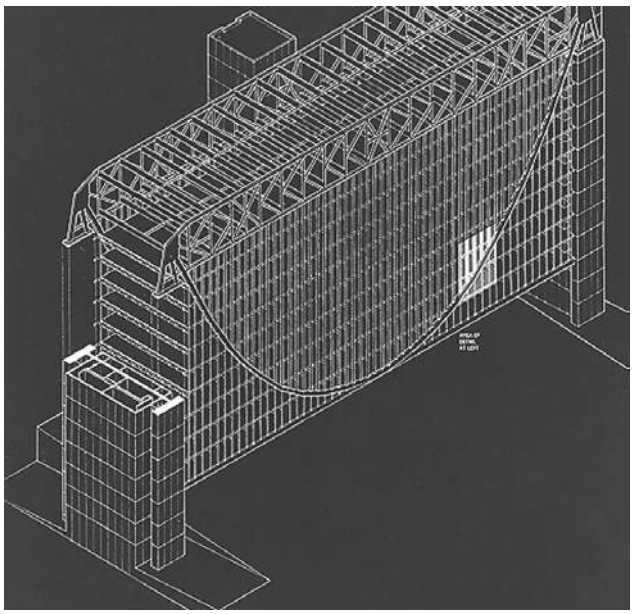
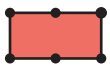


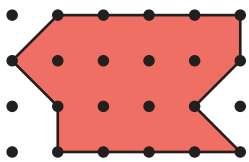
Diagram of the skeletal structure of Marquette Plaza in Minneapolis

1. The drawing above shows the skeletal structure of the Marquette Plaza in Minneapolis. Its 10 floors and the vertical beams partition the front of the building into congruent rectangles. The dimensions of these rectangles are approximately 2 meters × 4 meters.
   a. How does the width and height of these rectangles compare to the width of your outstretched arms and the height of the average room?
   b. There are 53 rectangles across the front face of the building. What is the width of the front face of the building?
   c. There are 10 rectangles running from the bottom to the top of the front face of the building (one for each floor). What is the height of the front face of the building?
   d. What is the area of the front face?
   e. Each floor has dimensions 60 feet × 275 feet. What is the total area of the 10 floors to the nearest square foot? acre?

The basic unit for measuring area does not have to be a square. Measure the area of the concave octagon on the grid that follows by using the different area units given in exercises 2 and 3.

2. a.  **b.** 

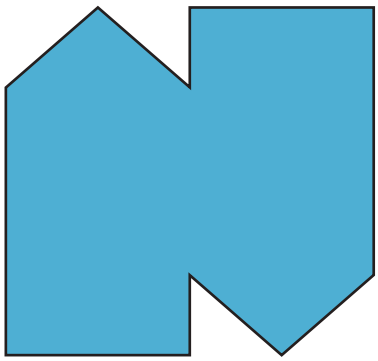
3. a.  **b.** 



Octagon

*For computations involving π, use the value on your calculator or approximate π by 3.1416.

Determine the approximate area of the following figure, using the nonstandard area units in exercises 4 and 5.



4.



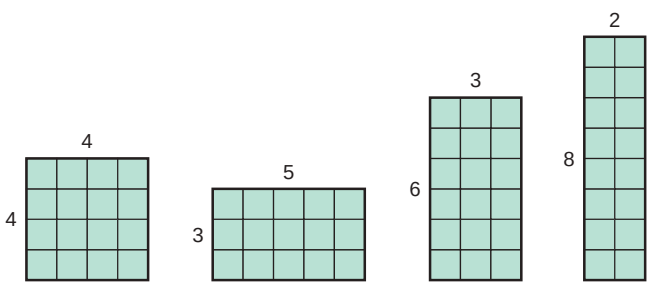
Gum wrapper

5.

Plastic fastener

6. The following question is taken from a mathematics test given to 9-year-olds by the National Assessment of Educational Progress (NAEP).

a. Which of the following figures has the same area as the 4×4 square?



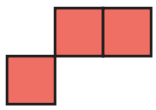
b. Only 44 percent of the students who took the test chose the 8×2 rectangle, and almost as many selected the 3×5 rectangle. The selection of the 3×5 rectangle may indicate confusion about which two concepts of measurement? Explain why students might have made this choice.

7. a. How many square feet are there in 1 square mile?
b. How many acres are there in 1 square mile?
c. Rhode Island has the least land area of the 50 states. Its area is 1049 square miles. How many acres is this?

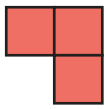
8. a. How many square millimeters are there in 1 square centimeter?
b. How many square centimeters are in 1 square meter?
c. A standard sheet of paper has dimensions of approximately 28 centimeters $\times$ 21.5 centimeters. What is the area of one side of such a sheet of paper?

Each of the figures in exercises 9 and 10 has an area of 3 square units. Using the length of the side of one of these squares as the unit of measure, calculate the perimeter of each figure.

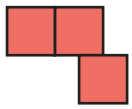
9. a.



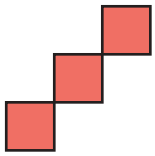
b.



10. a.



b.



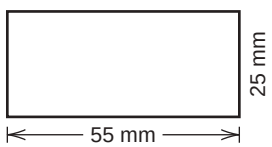
Exercises 11 and 12 use the metric units of *are* and *hectare*. The **are** (pronounced as "air") is the unit for measuring the area of house lots, gardens, and other medium-size regions. The *are* is equal to the area of a square whose sides measure 10 meters each. The **hectare** is a unit of area for measuring larger regions. It is the area of a square whose sides measure 100 meters each.

11. a. How many ares equal 1 square kilometer?
b. How many hectares equal 1 square kilometer?

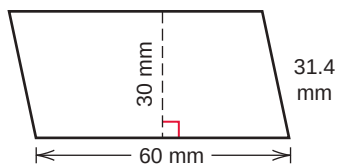
12. a. How many ares equal 1 hectare?
b. How many hectares equal the area of a 2.4-kilometer $\times$ 3.5-kilometer rectangle?

Use the value of π from your calculator or $\pi \approx 3.14$ to compute the perimeter or circumference to the nearest millimeter and the area to the nearest square millimeter of each figure in exercises 13 through 18. Then determine each area to the nearest .01 square centimeter. (*Hint:* In some cases the Pythagorean theorem will be needed to find the length of a side or hypotenuse of a right triangle; see section 6.4.)

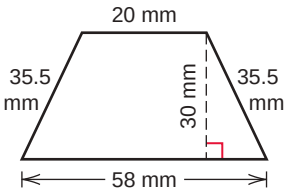
13. a. Rectangle



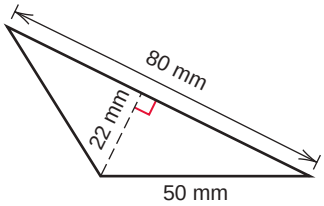
b. Parallelogram



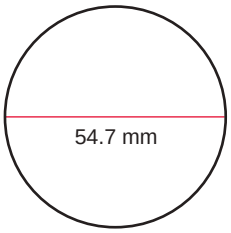
14. a. Trapezoid



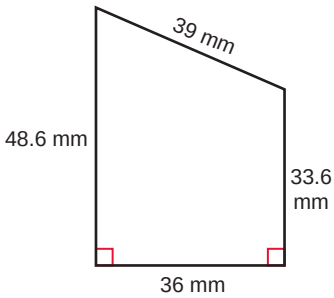
b. Scalene triangle



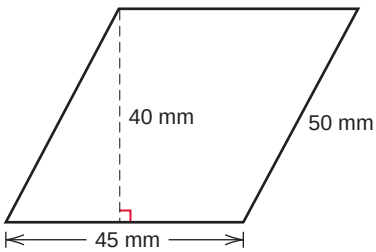
15. a. Circle



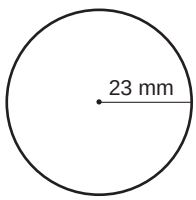
b. Trapezoid



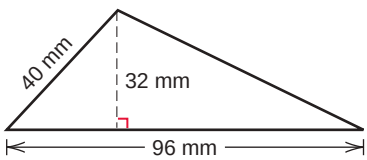
16. a. Parallelogram



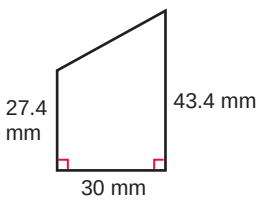
b. Circle



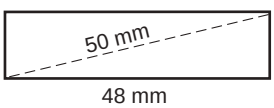
17. a. Triangle



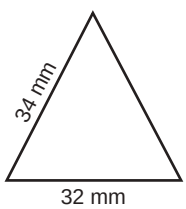
b. Trapezoid



18. a. Rectangle

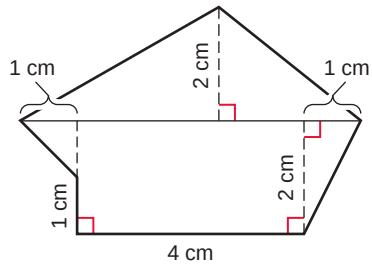


b. Isosceles triangle

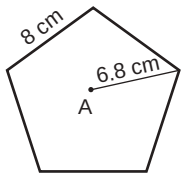


The area of a polygon can be found by subdividing it into smaller regions. Use this principle to find the area of the polygons in exercises 19 and 20 to the nearest .1 square centimeter.

19.

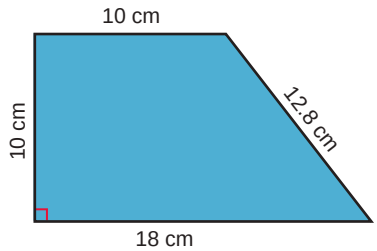


20. Regular pentagon with center *A*

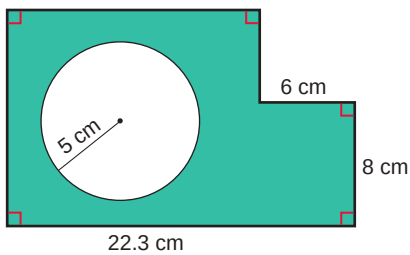


Determine the area to the nearest square centimeter of the shaded region of each figure in exercises 21 and 22. (Use the value of π from your calculator or $\pi \approx 3.14$.)

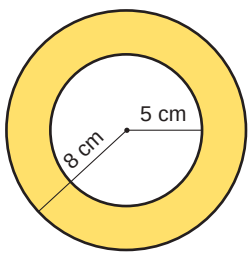
21. a. Trapezoid



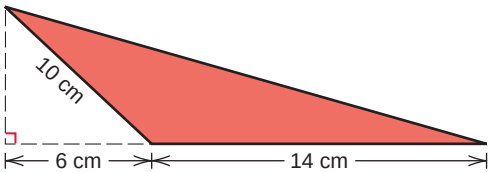
b. Rectangle with circle and square removed



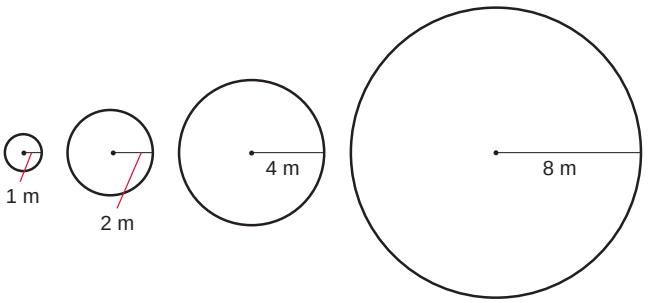
22. a. Two circles with the same center (**concentric circles**)



b. Triangle



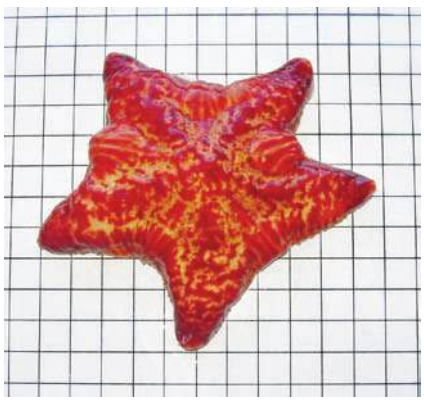
Use these circles in exercises 23 and 24 and the value of π from your calculator or $\pi \approx 3.1416$.



23. Determine the circumference of each circle. What happens to the circumference when the radius is doubled? What happens to the circumference when the radius is tripled? You may wish to sketch several more circles to help see the pattern.

24. Determine the area of each circle. What happens to the area of a circle when the radius is doubled? What happens to the area of a circle when the radius is tripled? You may wish to sketch several more circles to help see the pattern.

25. The common starfish has five arms. Some species grow as large as 20 to 30 centimeters in diameter, but other species reach only 1 centimeter. The diameter is determined from the circular disk, not including the arms. This starfish is on a centimeter grid and has a diameter of approximately 6 centimeters. What is the approximate area of the underside of the starfish, including arms, in square centimeters? (*Hint:* Enclose the starfish in a large square, and approximate the area that is not covered.)



26. Following is a brief chronology of some early approximations for π. Compare the decimals for these fractions with the value of π to 15 decimal places:

$$\pi \approx 3.141592653589793$$

Which one of the following fractions is closest to the value of π? Which two of these fractions are equal?

a. Archimedes (240 B.C.E.), $\frac{223}{71}$

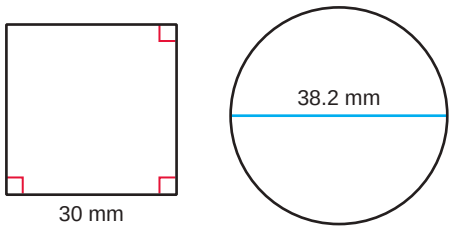
b. Claudius Ptolemy (A.D. 150), $\frac{377}{120}$

c. Tsu Ch'ung-chih (A.D. 480), $\frac{355}{113}$

d. Aryabhata (A.D. 530), $\frac{62{,}832}{20{,}000}$

e. Bhaskara (A.D. 1150), $\frac{3927}{1250}$

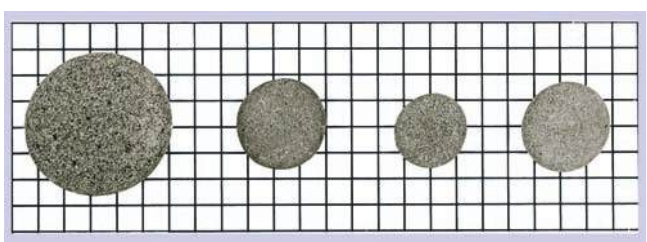
27. Of all simple closed curves of equal length, the circle encloses the largest area. Consider the following square and circle.



a. What is the perimeter of the square and the circumference of the circle to the nearest millimeter?

b. How much greater (to the nearest square millimeter) is the area of the circle than the area of the square?

28. Rocks are sometimes shaped into disks by the tumbling action of ocean waves. The photograph shows four such rocks on a centimeter grid. Consider the area of the portion of the grid that is occupied by each rock.

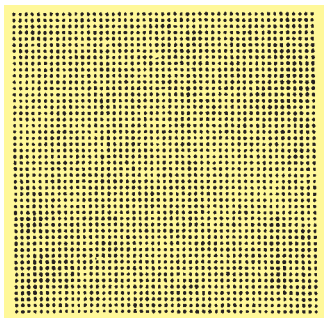


a. Estimate the diameter of the large rock. Use this number to find the area occupied by the rock to the nearest square centimeter.

b. What is the difference between the area occupied by the largest rock and the total area occupied by the three smaller rocks?

29. The 1998–99 Manhattan telephone directory had approximately 3400 pages, each with an 8-inch × 10-inch printed surface.

a. If every square inch of printed surface on these pages contained a 50 × 50 array of dots, as shown here, how many dots would there be in this directory?



b. In 2010 the world's population was approximately 7 billion. If each person were represented by one dot, about how many of these telephone directories, to the nearest tenth, would be required to represent everyone in the world?

Reasoning and Problem Solving

30. A pane of antique stained glass has dimensions of 30 centimeters × 58 centimeters. If the glass sells for 25 cents per square centimeter, what is the cost of the pane?

31. A store sells two types of Christmas paper. Type A has 4 rolls per package and costs $2.99, and each roll is 75 centimeters × 150 centimeters. Type B has a single roll, costs $3.19, and is 88 centimeters × 500 centimeters. Which type gives you more paper for your money?

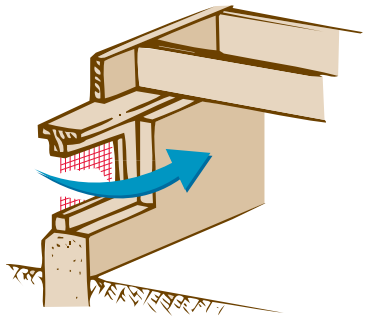
32. All-purpose carpeting costs $27.50 per square meter. What is the cost of carpeting a room from wall to wall whose dimensions are 360 centimeters × 400 centimeters?

33. Glass for picture frames sells for $20.00 per square meter. What is the total cost of the glass in two picture frames, of which one is 58 centimeters × 30 centimeters and the other 40 centimeters × 60 centimeters?

34. The length of a kitchen cupboard is 3.5 meters. There are three shelves in the cupboard, each 30 centimeters wide. How many rolls of shelf paper will be needed to cover these shelves if each roll is 30 centimeters × 3 meters?

35. The instructions on a bag of lawn fertilizer recommend that 35 grams of fertilizer be used for each square meter of lawn. How many square meters of lawn can be fertilized with a 50-kilogram bag?

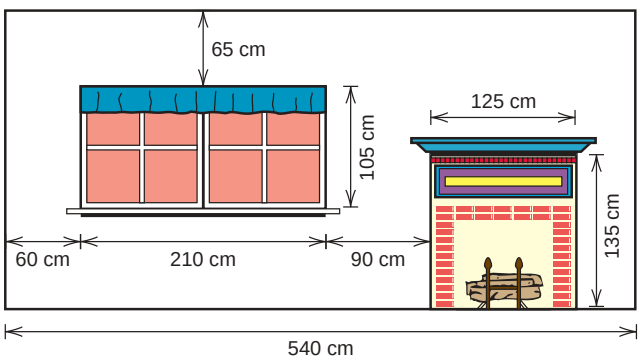
36. Some humidity is necessary in homes for comfort, but too much can cause mold and peeling paint. Paint-destroying moisture can come from walls, crawl spaces, and attics.



a. An attic should have 900 square centimeters of ventilation for each 27 square meters of floor area. How many square centimeters of ventilation are needed for an attic with a 4-meter × 12-meter floor?

b. A crawl space should have 900 square centimeters of ventilation for each 27 square meters of ceiling area, plus 1800 square centimeters for each 30 meters of perimeter around the crawl space. How many square centimeters of ventilation are needed for a 10-meter × 15-meter crawl space?

37. The wall shown in the following figure has a length of 540 centimeters and a height of 240 centimeters. A few dimensions are also given around the window and fireplace.



a. How many square centimeters of wallpaper will it take to paper the wall?

b. A standard roll of wallpaper is 12.8 meters × 53 centimeters. A store will not sell partial rolls. How many rolls must be purchased to cover this wall?

Many factors go into assessing the value of a house for tax purposes. Once the proper category has been determined, the assessment rate is per square foot or square meter. Compute the value of the one-floor dwellings in exercises 38 and 39 if the assessment rate is \$389 per square meter for the base of the house.

38. a. Ranch-style house with a rectangular base 8 meters × 13.75 meters.

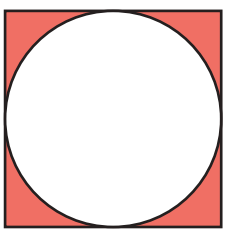
b. How much tax must be paid on this house if the tax rate is \$52 on every \$1000 of assessed value?

39. a. L-shaped house consisting of two parts with rectangular bases: 8.7 meters × 11 meters and 8 meters × 9.5 meters.

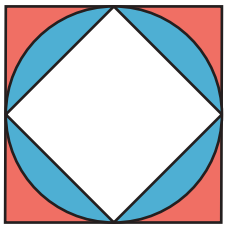
b. How much tax must be paid on the L-shaped house if the tax rate is \$73 on every \$1000 of assessed value?

40. Featured Strategy: Making a Drawing. Draw the largest circle possible on a square piece of paper. Cut out the circle and discard the trimmings. Inside the circle, draw the largest square possible. Cut out the square and discard the trimmings. What fraction of the original square piece of paper has been cut off and thrown away?

a. Understanding the Problem. The shaded portion of this diagram shows the trimmings that will be thrown away in the first step of the paper-cutting process. If the length of the side of the square is 2 centimeters, what percentage of the area of the square is the area of the shaded region?

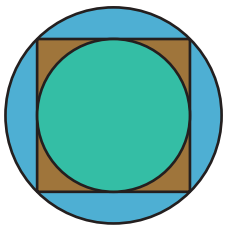


b. Devising a Plan. One approach to the problem is to compute the total area of the trimmings from the two steps separately. A different approach is suggested by inscribing the second square inside the circle. Describe the total region of the following figure that represents the amount that will be cut off from the original square.

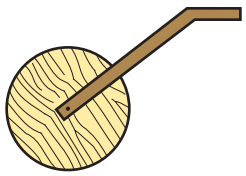


c. Carrying Out the Plan. Choose a plan from part b, or devise one of your own; and use it to determine what fraction of the original square piece of paper is cut off.

d. Looking Back. Suppose we begin with a circle, inscribe a square, and then inscribe a smaller circle in the square. How does the area of the small circle compare with the area of the large circle? Will the answer be the same as the answer to the original problem?

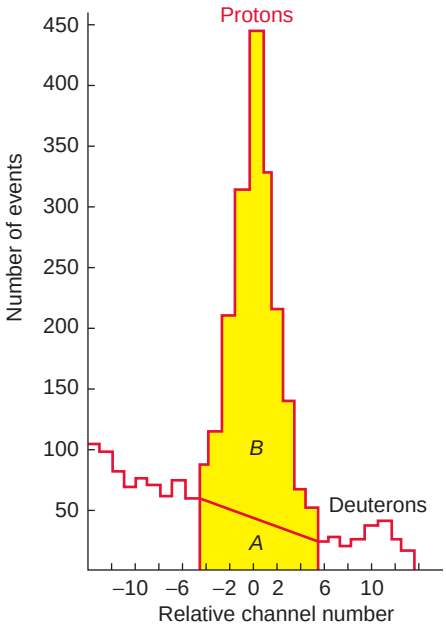


41. A meter trundle wheel is a convenient device for measuring distances along the ground. Every time the wheel makes 1 complete revolution, it has moved forward 1 meter. If you were to cut this wheel from a square piece of plywood, what would be the dimensions of the smallest square you could use? (Use the value of π from your calculator or $\pi \approx 3.1416$.)



42. Physicists study cosmic radiation to learn about properties of our galaxy and levels of Sun activity. The proton histogram on the next page contains information on the intensity level of cosmic rays. Region A is called the **background area,** and is compared with the total area of regions A and B.

a. What is the approximate area of region A under the Deuterons histogram in square millimeters?



b. What is the approximate area of region B under the Protons histogram?

c. What is the ratio of the area of region A to the total area of regions A and B to the nearest percent?

43. The 73-story cylindrical building shown in the photograph at the top of the next column is the Westin Peachtree Plaza Hotel in Atlanta, Georgia. It contains a seven-story central court with a small pond and over 100 trees. The diameter of this building is 35.36 meters (116 feet). According to its architect, John Portman, cylindrical walls were chosen rather than the more common rectangular walls because a circle encloses more area with less perimeter than any other shape. (Use the value of π from your calculator or $\pi \approx 3.1416$.)

a. What is the area of a horizontal cross section of the cylindrical building (the area of a floor) to the nearest square meter?

b. What is the perimeter, to the nearest meter, of a square that encloses the same area as you found in part a? (*Hint:* First find the square root of the area in part a.)

c. What is the circumference of the cross section in part a to the nearest meter?

d. How many meters longer is the perimeter of the square in part b than the circumference of the hotel in part c?

e. The Westin Peachtree Plaza Hotel is 220 meters (723 feet) tall. This number multiplied by the correct answer for part d will give the additional wall area that would be needed to enclose the same space if the hotel had a square base rather than a circular one. What is this area?



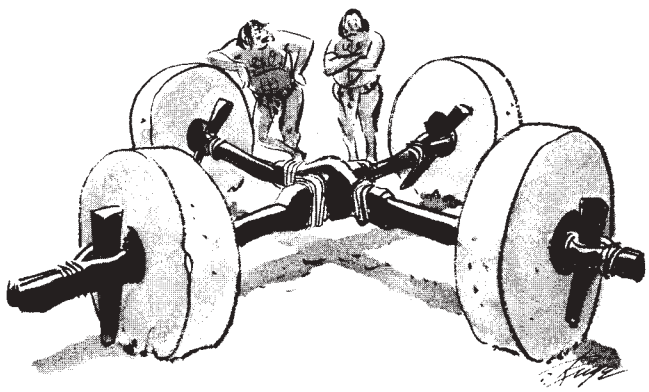
Westin Peachtree Plaza Hotel in Atlanta, Georgia—Tallest hotel in the Western Hemisphere

44. Egyptian scrolls dating from the period between 1850 and 1650 B.C.E. show many formulas for computing land areas for the purposes of taxation. Does the following formula produce the correct area for a quadrilateral, with successive side lengths of a, b, c, and d?

$$Area = \frac{(a + c) \times (b + d)}{4}$$

If not, is the result too large or too small? (*Hint:* Try this formula on some figures.)

45. The wheel has been called the most important invention of all time. Assume that the diameter of each wheel in this cartoon is 75 centimeters and that the distance between opposite pairs of wheels is 300 centimeters. How many revolutions of each wheel will it take to turn this contraption in one complete circle?



"Just because you invented the wheel, *it doesn't follow logically that you can go on to the* wagon."

46. The height of a tennis ball can is approximately equal to the circumference of a tennis ball. This can be illustrated by rolling a tennis ball along the edge of a can. The ball will make one complete revolution in rolling from one end of the can to the other. Since a can holds three tennis balls, what does this demonstration imply about the diameter of a ball compared to its circumference?



Teaching Questions

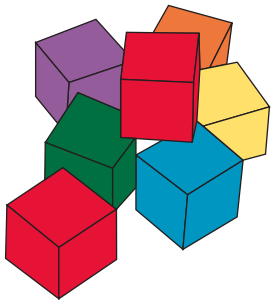
1. Design an activity for elementary students that uses manipulatives (colored tiles, string, grid paper, etc.) to help them discover that figures with the same areas can have different perimeters. Describe the activity and include the necessary diagrams.
2. Andrew's dad showed him how to draw a regular hexagon as follows: Draw a circle with a compass; use the same compass setting to "walk the compass" around the circle marking six points on the circle; join the six points consecutively around the circle to form the hexagon. Is his dad's method correct? Draw diagrams as you explain why this method does or does not create a regular hexagon.
3. Frank is in sixth grade and loves math. One day he told his teacher that a figure with an area of 10 square units could have a perimeter as long as 100 units or 1000 units or even longer. His teacher asked Frank to explain his thinking. How would you explain that the perimeter of a figure with an area of 10 square units could be as large as one wishes?
4. Design an activity for students that illustrates that figures with the same perimeter can have different areas. Describe the activity and supply any necessary diagrams.

Classroom Connections

1. The **Standards** statement on page 684 notes that formulas for the area of parallelograms, triangles, and trapezoids can be developed by using the formula for the area of a rectangle, along with an understanding that decomposing a shape and rearranging its component parts without overlapping does not affect the area of a shape. Show with diagrams how you can obtain the formula for the area of any triangle with height h and base b by knowing only that the area of a rectangle is length times width.
2. After completing questions 3a–c in the one-page **Math Activity** at the beginning of this section, assign the value of one unit of area to the tan parallelogram. Then, using the diagram in part c and your pattern blocks, determine the area of the square pattern block. Explain your thinking and include sketches of pattern blocks.
3. Read the **Elementary School Text** on page 690. Assume that the small squares in the grids measure one unit of area. Choose one of the irregular areas with one or more of your classmates. To estimate the area, count the number of whole squares and then count and group the partial squares. Have each person compute the approximate area independently and then compare answers. Explain how the slightly different approximate answers obtained by your group can be used to obtain one measure for the area of an irregular figure.
4. The **Research** statement on page 686 indicates there is poor performance in measurement and geometry across the curriculum. Read the **Research** statements that address assessment on pages 680, 683, and 689. Then, reflecting on your own school experiences, describe why you think students achieve poorly in these areas.
5. Examine the **Standards** expectations **Measurement** (see inside front cover) for both **PreK–2** and **Grades 3–5** and describe what they suggest regarding the learning and use of the metric system.

MATH ACTIVITY 10.3

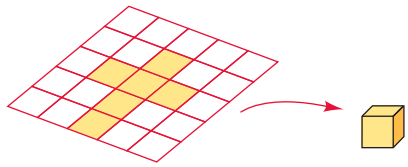
Surface Area and Volume for Three-Dimensional Figures



Purpose: Make nets and explore surface area and volume of cube figures.

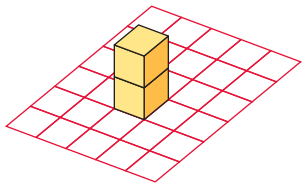
Materials: Sheets of 2-Centimeter Grid Paper (copy from the website) and scissors. The 2-centimeter cubes for building figures are optional.

***1.** A cube can be formed by creasing and folding along the lines of the pattern shown below. Use grid paper to form and cut out several different types of patterns that will each fold into a cube with no overlaps. Sketch your patterns. Two-dimensional patterns for three-dimensional figures are called **nets.**



Net for a cube

2. a. Form and cut out a net that will fold into the column of two cubes shown here. (*Hint:* One way is to imagine this column sitting on a square of the grid and visualize the squares that would need to be folded up to cover the column.) Sketch your net.



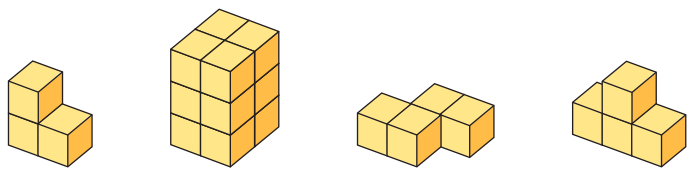
b. The number of cubes in a figure is the volume of the figure (in cubic units), and the number of squares in a net for a figure is the surface area of the figure (in square units). Determine the volume and surface area of a two-cube column.

3. a. Visualize a column of n cubes, and describe a net of squares that will fold and cover this column.

b. Write an algebraic expression for the surface area of a column of n cubes.

c. Write an algebraic expression for the volume of a column of n cubes.

4. Select at least two of the following figures, and sketch their nets on grid paper. Determine the surface area and the volume of each figure you select.



Section 10.3 VOLUME AND SURFACE AREA



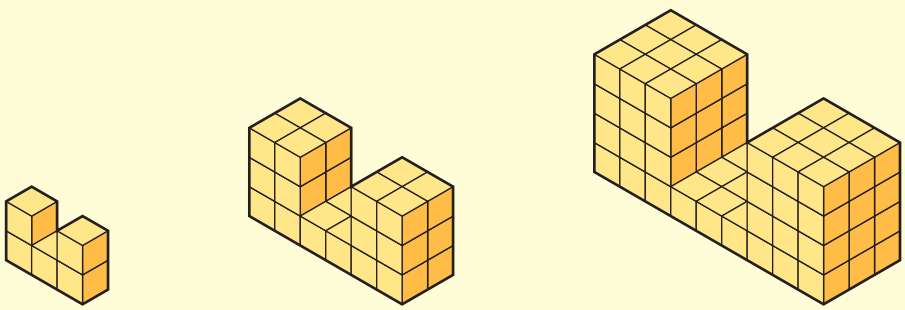
Liquefied natural gas tanker *Aquarius*

PROBLEM OPENER

The numbers of cubes in these U-shaped figures are the beginning of the sequence

5, 28, 81, . . .

If this geometric pattern is continued, how many cubes will there be in the 10th figure?



The huge ship in the photograph is the *Aquarius,* one of 12 liquefied natural gas tankers built by the Quincy Shipbuilding Division of General Dynamics. The *Aquarius* is longer than three football fields (285 meters) and carries five spherical aluminum tanks, each with a diameter of 36.58 meters. To appreciate the size of one of these spheres, consider the fact

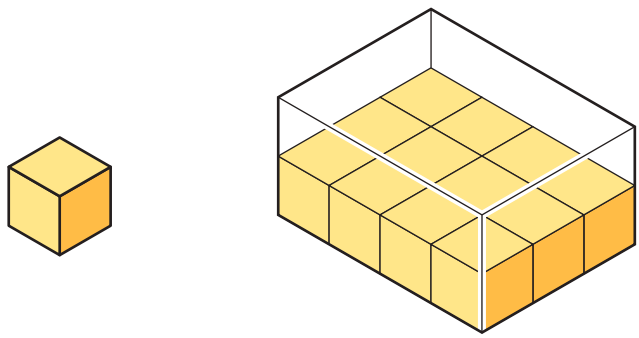
that its diameter is greater than the diameter of one of the floors in the Westin Hotel (page 697) and greater than the length of a professional basketball court. The space inside these spheres is measured by the number of *unit cubes* required to fill it. Each sphere holds the equivalent of about 25,000 cubes, each with dimensions of 1 meter × 1 meter × 1 meter.

NONSTANDARD UNITS OF VOLUME

NCTM Standards

The need for a standard three-dimensional unit to measure volume grows out of initial experiences filling containers with items such as rice or packing pieces. p. 172

To measure the amount of space in tanks, buildings, refrigerators, cars, and other three-dimensional figures, we need units of measure that are also three-dimensional figures. The number of such units needed to fill a figure is its **volume.** Cubes are convenient because they pack together without gaps or overlapping. Using the cube in part a of Figure 10.42 as the unit cube, we can determine that the volume of the box in part b is 24 **cubic units.** The figure shows 12 cubes on the base of the box, and 12 more cubes can be placed above these to fill the box.



(a) (b)

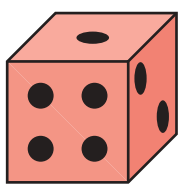
Figure 10.42

The first school experiences with measuring volume should involve nonstandard units of measure. Before introducing units of length for the dimensions of a cube, teachers should involve students in activities that require stacking, building, and counting cubes.

EXAMPLE A

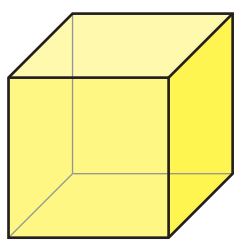
1. Suppose the rectangle at the top of page 702 is the base of a box. Determine the approximate number of each of the following units of volume (the die and the cube) that will fit onto this base. (*Hint:* Trace the front face of each volume unit.)

(1)



Die

(2)



Cube



Research Statement

Students have difficulty in determining the volume of a three-dimensional array of cubes that is presented as a diagram. One study showed that fewer than 25 percent of fifth-grade students could solve such a problem.

Ben-Haim, Lappan, and Houang

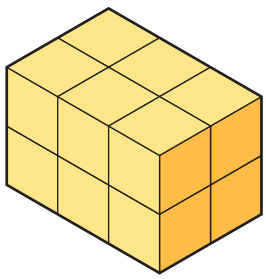
2. If 6 layers of the die on page 701 fill the box, what is the volume of the box?

3. If 4.5 layers of the cube on page 701 fill the box, what is the volume of the box?

Solution **1.** Approximately 33 dice will cover the base, and approximately 18 of the larger cubes will cover the base. **2.** The volume of the box is approximately 198 dice units. **3.** The volume of the box is approximately 81 cube units.

The difficulty that children at all grade levels have in understanding the concept of volume is indicated in the next example. Every 4 years the NAEP (National Assessment of Educational Progress) administers mathematics tests in schools throughout the United States. Example B contains a question on volume from one of these tests.

EXAMPLE B



In a national assessment on mathematics, students were shown the figure at the left and asked how many cubes the box contained. What is the correct answer, and what do you think was the incorrect answer most commonly given by students?

Solution The box contains 12 cubes. Only 6 percent of the 9-year-olds, 21 percent of the 13-year-olds, and 43 percent of the 17-year-olds answered the question correctly.* The most common incorrect answer was 16. Can you see how students might have obtained this answer?

STANDARD UNITS OF VOLUME

For each English unit of length (inch, foot, etc.) and each metric unit of length (centimeter, meter, etc.) there is a corresponding unit of volume, a cube whose three dimensions are the given length.

English Units Cubic units are named according to the length of their edges. The most commonly used units for measuring nonliquid volume in the English system are the **cubic inch,** the **cubic foot,** and the **cubic yard.** The cubes for these units are illustrated in Figure 10.43 on the next page. These cubes are not drawn to scale.

*T. P. Carpenter, T. G. Coburn, R. E. Reys, and J. W. Wilson, "Results and Implications of the NAEP Mathematics Assessment: Elementary School," *Arithmetic Teacher*: 438–450.

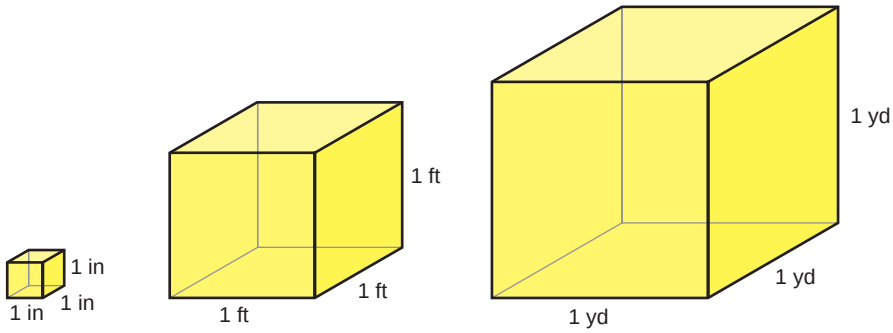


Figure 10.43

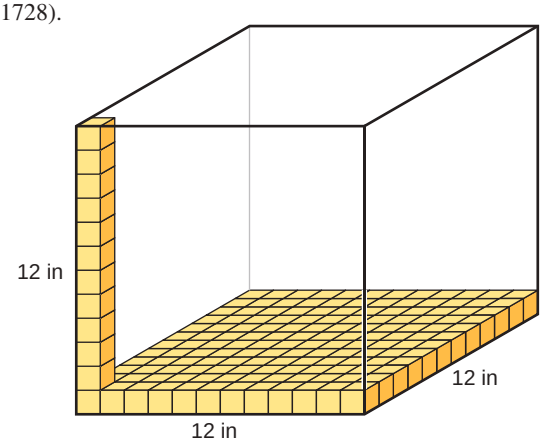
The volume of a microwave oven might be measured in cubic inches. A larger volume, such as that of a room or freezer, might be measured in cubic feet. Larger volumes, such as the volume of a truckload of gravel or crushed rock, are measured by the cubic yard.

EXAMPLE C

Determine the following English unit relationships.

1. How many cubic inches equal 1 cubic foot?
2. How many cubic feet equal 1 cubic yard?
3. How many cubic inches equal 1.4 cubic feet?

Solution **1.** Imagine a box in the shape of a cube whose dimensions are each 1 foot. The base of the box is 12 inches $\times$ 12 inches, so the floor of the box can be covered by 144 cubes, each of which is 1 inch $\times$ 1 inch $\times$ 1 inch. Since 12 such layers will fill the box, its volume is 1728 cubic inches ($12 \times 144 = 1728$).



2. Similarly, the floor of a cube-shaped box whose dimensions are each 1 yard can be covered with 9 cubes, each of which is 1 foot $\times$ 1 foot $\times$ 1 foot. Three such layers will fill the box, so its volume is 27 cubic feet. **3.** Since 1 cubic foot = 1728 cubic inches, 1.4 cubic feet = 2419.2 cubic inches ($1.4 \times 1728 = 2419.2$).

The English units for volume and their relationships are summarized in Figure 10.44.

English Units for Volume

Cubic inch	in^3	$\frac{1}{1728}$ cubic foot
Cubic foot	ft^3	1728 cubic inches
Cubic yard	yd^3	27 cubic feet

Figure 10.44

Metric Units The common metric units for measuring nonliquid volume are the **cubic millimeter,** the **cubic centimeter,** and the **cubic meter.** Each unit is named according to the length of the edges of its cube. For example, the edges of the cube for the cubic centimeter each have a length of 1 centimeter. The cubes for these three metric units are shown in Figure 10.45. These cubes are not drawn to scale.

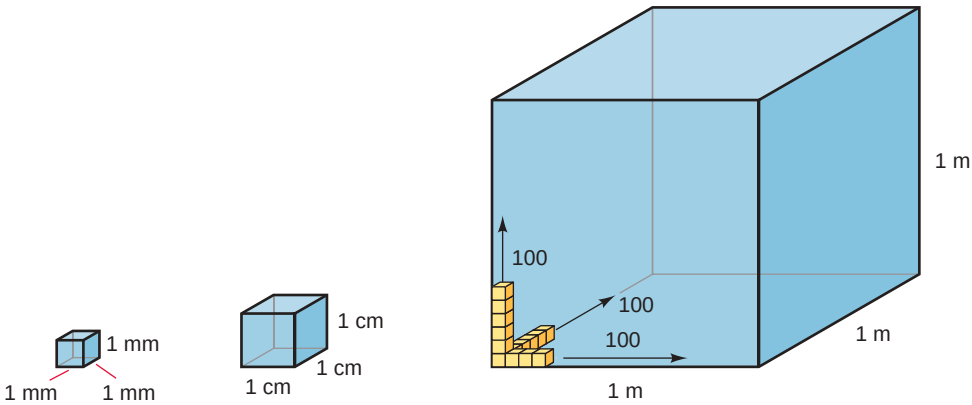


Figure 10.45

EXAMPLE D

Determine the following relationships.

1. How many cubic centimeters equal 1 cubic meter?
2. How many cubic millimeters equal 1 cubic centimeter?
3. How many cubic millimeters equal 3.4 cubic centimeters?

Solution **1.** Visualize a cube-shaped box that measures 1 meter on each edge (see part c of Figure 10.45), and imagine filling it with cubes whose edges measure 1 centimeter. The floor of the large cube is 100 centimeters $\times$ 100 centimeters, so it can be covered by 10,000 of the smaller cubes. Since there are 100 such layers, the volume of the box is 1,000,000 cubic centimeters (100 $\times$ 10,000). **2.** A cube whose edges each have a length of 1 centimeter has dimensions of 10 millimeters $\times$ 10 millimeters $\times$ 10 millimeters. So, its volume is 1000 cubic millimeters (10 $\times$ 10 $\times$ 10). **3.** Since 1 cubic centimeter = 1000 cubic millimeters, 3.4 cubic centimeters = 3400 cubic millimeters.

Some of the metric units for volume and their relationships are shown in the table in Figure 10.46.

Metric Units for Volume

Cubic millimeter	mm^3	$\frac{1}{1000}$ cubic centimeter
Cubic centimeter	cm^3	1,000 cubic millimeters
Cubic meter	m^3	1,000,000 cubic centimeters

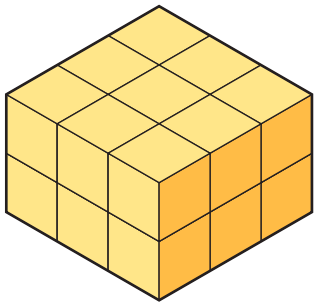
Figure 10.46

SURFACE AREA

Another important measure associated with objects in space is their amount of surface. **Surface area** is expressed as the number of unit squares needed to cover the surface of a three-dimensional figure.

EXAMPLE E

The area of the top of the box in the figure is 9 square centimeters. What is the total surface area, including the base of the box?

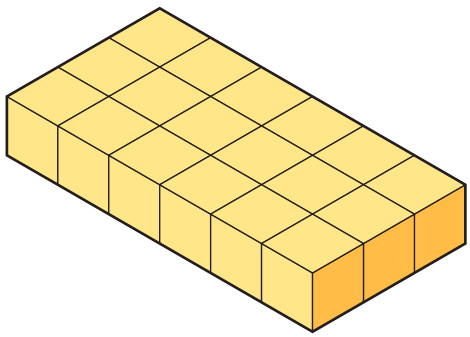


Solution The top and bottom faces of the box each have an area of 9 square centimeters. The right and left faces each have an area of 6 square centimeters, and the front and back faces each have an area of 6 square centimeters. So, the total surface area is 42 square centimeters ($2 \times 9 + 2 \times 6 + 2 \times 6$).

The surface area of an object cannot be predicted on the basis of its volume, any more than the perimeter of a figure is determined by the area of the figure.

EXAMPLE F

The box in Example E and the box below each have a volume of 18 cubic centimeters. How do the surface areas of the two boxes compare?



Solution The top and bottom faces of the box in this figure each have an area of 18 square centimeters, the right and left sides each have an area of 3 square centimeters, and the front and back faces each have an area of 6 square centimeters. So the total surface area is 54 square centimeters. This is 12 square centimeters greater than the surface area of the box in Example E.

Examples E and F show that figures in space can have the same volume but different surface areas. The amount of material you would need to build the box in Example F is about 130 percent of the amount of material you would need to build the box in Example E (130 percent of $42 = 1.3 \times 42 \approx 54$), although both have a volume of 18 cubic centimeters.

VOLUMES AND SURFACE AREAS OF SPACE FIGURES

Prisms In Figure 10.47, on the next page, the length (20) times the width (10) gives the number of cubes (200) on the floor of the box (or base of the rectangular prism). Since the box can be filled with 6 levels of cubes, it will hold 1200 cubes. This volume of 1200 cubic centimeters

NCTM Standards

Students in grades 3–5 should develop strategies for determining surface area and volume on the basis of concrete experiences. They should measure various rectangular solids using objects such as tiles and cubes, organize the information, look for patterns, and then make generalizations. p. 175

can be obtained by multiplying the three dimensions of the box: length × width × height. In general, a rectangular prism with length l, width w, and height h has the following volume:

Volume of rectangular prism = length × width × height

$$V = lwh$$

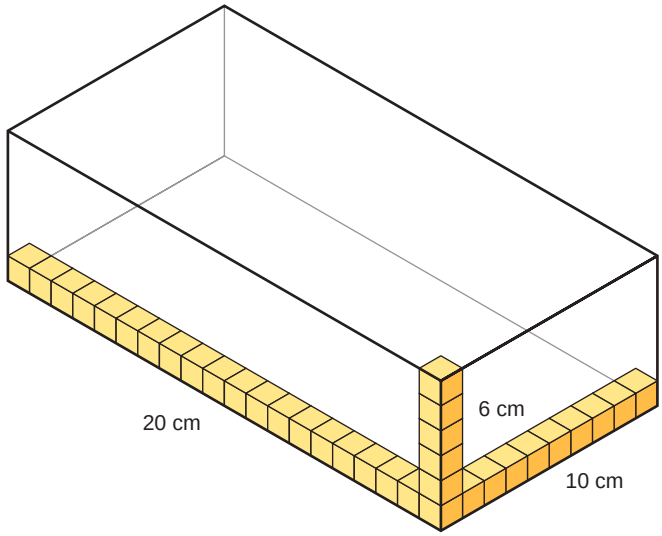


Figure 10.47

The volume of any prism can be found in a similar way. The base of the prism in Figure 10.48 is a right triangle, which is covered by $4\frac{1}{2}$ cubes. Since 6 levels each with $4\frac{1}{2}$ cubes fill the prism, its volume is 27 cubic centimeters (6 × 4.5).

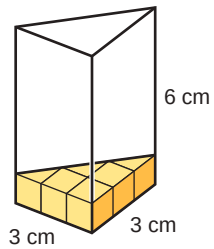


Figure 10.48

The number of cubes that cover the base of this prism is the same as the area of the base. Therefore, the volume of the prism can be computed by multiplying the area of the base by the height, or altitude, of the prism. In general, the volume of any right prism having a base of area B and a height of h can be computed by the formula:

Volume of prism = area of base × height

$$V = Bh$$

The formula for the volume of an oblique prism is suggested by beginning with a stack of cards, as in part a of Figure 10.49, and then pushing them sideways to form an oblique prism, as in part b on page 708. If each card in part a is 12.5 centimeters × 7.5 centimeters and the stack is 5 centimeters high, the volume of the right prism in Figure 10.49a is 12.5 centimeters × 7.5 centimeters × 5 centimeters, or 468.75 cubic centimeters. The base of the oblique prism in part b is also 12.5 centimeters × 7.5 centimeters, and its **height,** or **altitude** (the perpendicular distance between its upper and lower bases), is also 5 centimeters. Since both stacks contain the same number of cards, their volumes are both

Explore

Measurement Activity for 10-7

Volume

Volume is the number of cubic units needed to fill a space. Volume is measured in **cubic units**.



MAIN IDEA

I will use models to explore volume.

You Will Need

cubes, small containers

New Vocabulary

volume

cubic unit

Math Online

macmillanmh.com

- Extra Examples
- Personal Tutor
- Self-Check Quiz

ACTIVITY

1 Estimate and measure volume.

Step 1 **Estimate the volume.**

A small paper clip box is shown. How many cubes do you think will fit into the box?



Step 2 **Fill the box.**

Place the cubes in the box in rows as shown until it is full.



Step 3 **Find the volume.**

Remove the cubes. Count to find how many filled the box. This is the volume. Compare this to your estimate.





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468.75 cubic centimeters. This means that the volume of the oblique prism can be computed by multiplying the area of its base by its height. In general, *the volume of any prism, right or oblique, is the area of its base times its height.*

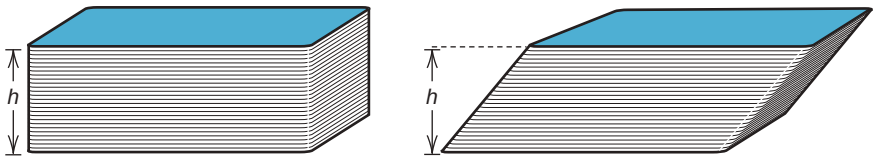
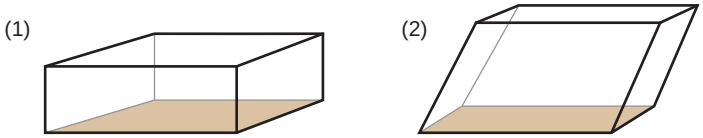


Figure 10.49

EXAMPLE G

Sometimes a prism will have more than one pair of bases, as shown in the following figures. Figure (1) is a right prism whose base is a parallelogram. By turning it so that one of its lateral faces becomes the base, as in figure (2), we can classify the prism as an oblique prism. Determine the volume of each prism, given the following dimensions:

1. Figure (1): area of base, 280 square centimeters; altitude, 6 centimeters
2. Figure (2): area of base, 120 square centimeters; altitude, 14 centimeters



Solution **1.** 1680 cubic centimeters. **2.** 1680 cubic centimeters.

The faces of right prisms are rectangles; and the faces of oblique prisms are rectangles and parallelograms.

The **surface area of a prism** is the sum of the areas of its bases and faces.

Cylinders The cylindrical buildings in Figure 10.50 are skyscrapers in Los Angeles. Just as with conventional rectangular buildings, architects need to know the volumes and surface areas of these glass-walled cylinders.



Figure 10.50
Los Angeles skyscrapers and street lights

NCTM Standards

Although [middle-grade] students may have developed an initial understanding of area and volume, . . . some measurement of area and volume by actually covering shapes and filling objects can be worthwhile for many students. p. 242

To compute the volumes of cylinders, we continue to use unit cubes even though they do not conveniently fit into a cylinder. More than 33 cubes are needed to cover the base of the cylinder in Figure 10.51. Furthermore, since the cylinder has a height of 12 centimeters, it will take *at least* 12×33, or 396, cubes to fill the cylinder.

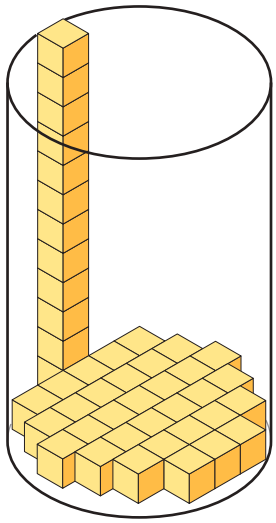


Figure 10.51

If we used smaller cubes in Figure 10.51, they could be packed closer to the boundary of the base and a better approximation would be obtained for the volume of the cylinder. This suggests that the formula for the volume of a cylinder is the same as that for the volume of a prism. For a cylinder with a base of area B and a height h,

Volume of cylinder = area of base × height

$$V = Bh$$

A right cylinder without bases can be formed by joining the opposite edges of a rectangular sheet of paper (Figure 10.52). The circumference of the base of the cylinder is the length of the rectangle, and the height of the cylinder is the height of the rectangle. Therefore, the surface area of the sides of a cylinder is the circumference of the base of the cylinder times its height.

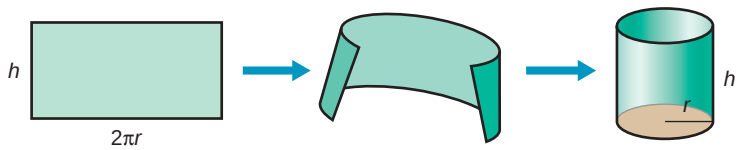


Figure 10.52

For any right cylinder whose base has a radius r and whose height is h, the base has a circumference of $2\pi r$, and the surface area of the side of the cylinder is $2\pi r \times h$. Adding the area of both bases, $2\pi r^2$, to the area of the side of the cylinder produces the total surface area:

Surface area of cylinder = $2\pi rh + 2\pi r^2$

EXAMPLE H

Compute the volume and surface area of a tennis ball can if the diameter of its base is 7 centimeters and the height of the can is 20 centimeters.

Solution Surface area: The radius of the base is 3.5 centimeters, so the top and the base of the can each have an area of $\pi(3.5)^2$ square centimeters, which to two decimal places is 38.48 square centimeters. The circumference of the can is 7π centimeters, which to two decimal places is 21.99 centimeters; so the lateral surface of the can has an area of approximately 21.99×20 square centimeters, which to two decimal places is 439.80 square centimeters. Thus, the total surface area of the can is approximately 516.76 square centimeters, or 517 square centimeters when rounded to the nearest whole number. Volume: $20 \times \pi(3.5)^2$ cubic centimeters, which to the nearest whole number is 770 cubic centimeters.

Research Statement

The 7th national mathematics assessment found that questions assessing familiarity with volume and surface area were difficult for fourth and eighth-grade students.

Martin and Strutchens

Pyramids The Pyramid of Cheops, also known as the Great Pyramid of Egypt, was built about 2600 B.C.E. and is one of the seven wonders of the ancient world. It has a height of 148 meters. The Transamerica Pyramid in San Francisco (Figure 10.53) has a height of 260 meters and was built in 1972. Even though the Egyptian pyramid is the shorter of these giant pyramids, its volume is several times the volume of the taller pyramid. (See exercise 23 in Exercises and Problems 10.3.)



Figure 10.53
Transamerica Pyramid in San Francisco

The red pyramid in Figure 10.54d on the next page is inside the cube in part b, which has a square base *EFGH* and height *GC*. The red pyramid, together with the green pyramid in part a and the blue pyramid in part c divide the cube into three congruent pyramids. (Can you see how these fit together?) Therefore, the volume of each pyramid is one-third of the volume of the cube. Since the volume of the cube is the area of its base *EFGH* times the height of the cube, the volume of each pyramid is one-third of the area of the base of the cube times the height of the cube.

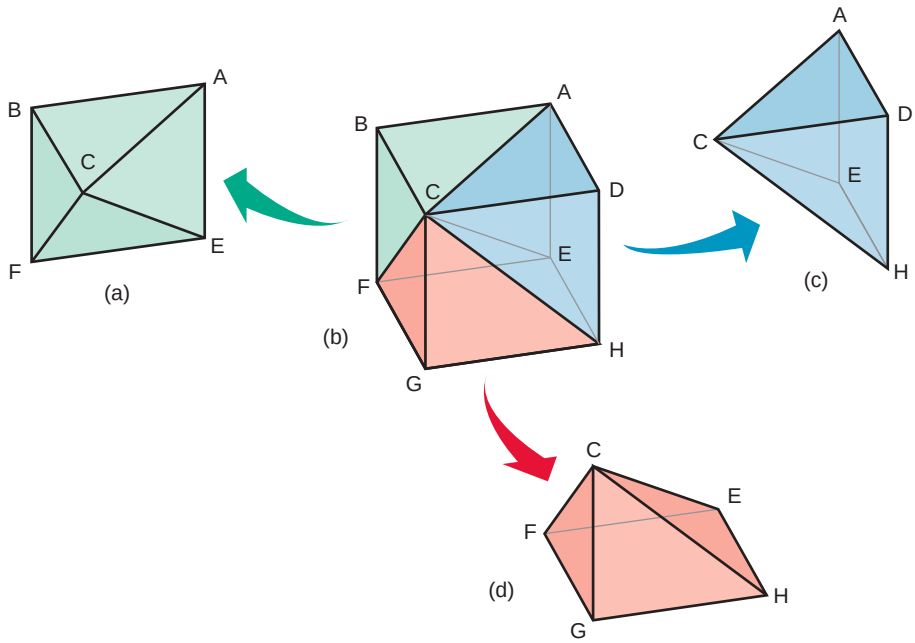


Figure 10.54

In general, if B is the area of the base of a pyramid and h is the **height,** or **altitude** (perpendicular distance from the apex to the base), of the pyramid, then

$$\textbf{Volume of pyramid} = \frac{1}{3} \times \textbf{area of base} \times \textbf{height}$$

$$V = \frac{1}{3}Bh$$

The faces of a pyramid are its base and triangular lateral sides. The surface area of the pyramid is the sum of the areas of the faces. Exercises and Problems 10.3 has several questions involving the surface area of pyramids.

The **surface area of a pyramid** is the sum of the area of its base and its triangular lateral sides.

EXAMPLE 1

Determine the volume of the Pyramid of Cheops to the nearest cubic meter. Its square base has sides of length 232.5 meters, and its height is 148 meters.

Solution The area of the base is 54,056.25 square meters (232.5 × 232.5), and

$$\frac{1}{3} \times 54{,}056.25 \times 148 = 2{,}666{,}775$$

So, the volume of the pyramid is 2,666,775 cubic meters.

Cones The pile of crude salt in Figure 10.55, on the next page, has the shape of a cone with a circular base. The conical shape forms as salt is poured from the conveyor belt at the top left. The salt has been evaporated from ocean water and awaits further purification.



Figure 10.55

The volume of a cone can be approximated by the volume of a pyramid inscribed in the cone. The hexagonal pyramid in Figure 10.56 has a volume of approximately 430 cubic centimeters, which is slightly less than the volume of the cone. As the number of sides in the base of the pyramid increases, the volume of the pyramid becomes closer to the volume of the cone. This intuitively indicates that the pyramid and cone formulas are the same. Since the volume of the pyramid is one-third the area of its base times its height, we can use the same formula to calculate the volume of a cone. In general, for any cone whose base has area B and whose height is h,

$$\textbf{Volume of cone} = \frac{1}{3} \times \textbf{area of base} \times \textbf{height}$$

$$V = \frac{1}{3}Bh$$

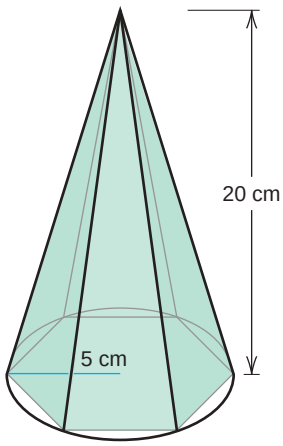


Figure 10.56

The surface area of the cone in Figure 10.56 is the area of its base, $\pi(5)^2$, plus the area of its lateral surface. The area of the lateral surface can be approximated by the area of the triangular sides of the hexagonal pyramid. The altitude of one of these triangular sides (see Figure 10.57 on the next page) is approximately equal to the slant height of the cone, which by the Pythagorean theorem is $\sqrt{5^2 + 20^2}$. If s is the length

of one side of the hexagonal base, the area of one triangle is $\frac{1}{2}s\sqrt{5^2 + 20^2}$, and the area of the lateral surface is $6\left(\frac{1}{2}s\sqrt{5^2 + 20^2}\right)$. Notice in Figure 10.57 that $6s$ is the perimeter of the base of the pyramid. Closer approximations can be found by increasing the number of sides of the polygonal base of the pyramid. As the number of sides of the polygon increases, its perimeter $6s$ gets closer to the circumference of the circle, $2\pi(5)$, and the area of the lateral sides of the pyramid gets closer to

$$\frac{1}{2}(2\pi)(5)\sqrt{5^2 + 20^2} = \pi(5)\sqrt{5^2 + 20^2}$$

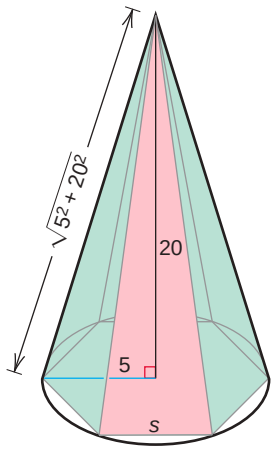


Figure 10.57

In general, for a cone of radius r and altitude h, the area of the lateral surface is $\pi r\sqrt{r^2 + h^2}$ and the surface area of the cone is the sum of the area of its base and its lateral face.

Surface area of cone = $\pi r\sqrt{r^2 + h^2} + \pi r^2$

EXAMPLE J

The height of the cone of salt, in Figure 10.55 on the previous page, is 12 meters, and the diameter of its base is 32 meters.

1. Determine the number of railroad boxcars this salt will fill if each boxcar has a volume of 80 cubic meters.
2. Determine the number of square meters of plastic sheet needed to cover the lateral surface of this cone of salt.

Solution **1.** The area of the base of the cone of salt is approximately 804 square meters. Since

$$\frac{1}{3} \times 804 \times 12 = 3216$$

the volume of the cone to the nearest cubic meter is 3216 cubic meters. This amount of salt will fill about 40 boxcars. **2.** The area of the lateral surface of the cone is approximately

$$\pi(16)\sqrt{16^2 + 12^2} \approx 1005.312$$

So, to the nearest square meter, 1005 square meters of plastic sheet is needed to cover the lateral surface of the cone.

Spheres Figure 10.58, on the next page, shows a view of Earth as seen from the *Galileo* spacecraft. Earth, the planets, and their moons are all spherical shapes, spinning and orbiting about a spherical Sun. There is considerable variation in the volumes of these objects. Earth has about 18 times the volume of the smallest planet, Mercury. The volume of the largest planet, Jupiter, is 10,900 times the volume of Earth.



Figure 10.58
The *Galileo* spacecraft returned these images of the Earth and Moon and they were combined to generate this view. The west coast of South America can be seen as well as the Caribbean, and the swirling white cloud patterns indicate storms in the southern Pacific. The lunar dark areas are lava-rock-filled impact basins. This picture contains same-scale and relative color/albedo images of the Earth and Moon.

The formulas for the volume of a sphere and the surface area of a sphere were known by the ancient Greeks. In fact, Archimedes (ca. 287 to 212 B.C.E.) discovered some remarkable relationships between a sphere and the smallest cylinder containing it. *The volume of the sphere is two-thirds the volume of the cylinder,* and *the surface area of the sphere is two-thirds the surface area of the cylinder.*

NCTM Standards

Whenever possible, students should develop formulas and procedures meaningfully through investigation rather than memorize them. Even formulas that are difficult to justify rigorously in the middle grades, . . . , should be treated in ways that help students develop an intuitive sense of their reasonableness. p. 244

Figure 10.59 shows a sphere of radius r. The smallest cylinder that contains the sphere has a height of $2r$. The volume of this cylinder is

$$\pi r^2 \times 2r = 2\pi r^3$$

Using the relationship discovered by Archimedes, we know that two-thirds of this volume is the volume of the sphere. So,

$$\textbf{Volume of sphere} = \frac{2}{3} \times 2\pi r^3 = \frac{4}{3}\pi r^3$$

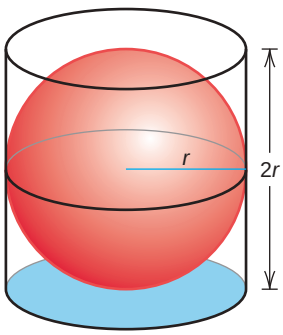


Figure 10.59

The surface area of the cylinder in Figure 10.59 is the sum of the areas of the two bases and the lateral surface of the cylinder. One base has an area of πr^2, and the two bases together have an area of $2\pi r^2$. Since the circumference of the cylinder is $2\pi r$ and the height of this cylinder is $2r$, the lateral surface area is $2\pi r \times 2r = 4\pi r^2$. So the total surface area of the cylinder that just contains a sphere is

$$2\pi r^2 + 4\pi r^2 = 6\pi r^2$$

Using another of Archimedes' discoveries, we know that two-thirds of the surface area of the cylinder containing the sphere is the surface area of the sphere. So,

$$\textbf{Surface area of sphere} = \frac{2}{3} \times 6\pi r^2 = 4\pi r^2$$

Thus, the surface area of a sphere is *exactly 4 times* the area of a great circle of the sphere.

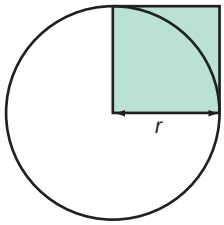
The area of a circle and the volume of a sphere can be nicely approximated if we think of π as approximately equal to 3. Then the approximate area of the circle is 3 times the square of the circle's radius,

$$\pi r^2 \approx 3r^2$$

and the volume of the sphere is approximately 4 times the cube of the sphere's radius,

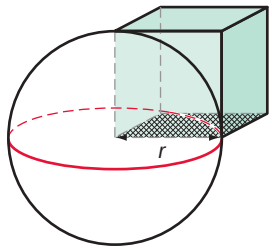
$$\text{Volume of sphere} = \frac{4}{3}\pi r^3 \approx \frac{4}{3}3r^3 = 4r^3$$

The square on the radius of a circle (a) and the cube on the radius of a sphere (b) are shown in Figure 10.60.



The area of a circle is approximately 3 times the area of the square on its radius

(a)



The volume of a sphere is approximately 4 times the volume of the cube on its radius

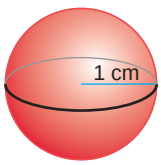
(b)

Figure 10.60

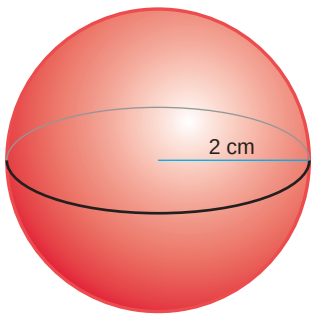
EXAMPLE K

Compare the volumes and surface areas of the following two spheres, using the estimations suggested in Figure 10.60.

(1)



(2)



Solution The volume of sphere (1) is approximately 4 cubic centimeters (4×1^3), and the volume of sphere (2) is approximately 32 cubic centimeters (4×2^3). The sphere whose radius is twice as large has a volume that is 8 times the smaller volume.

The surface area of sphere (1) is approximately 12 square centimeters ($4 \times 3 \times 1^2$), and the surface area of sphere (2) is approximately 48 square centimeters ($4 \times 3 \times 2^2$). The sphere whose radius is twice as large has a surface area that is 4 times the smaller surface area.

NCTM Standards

The *Curriculum and Evaluation Standards for School Mathematics,* grades 5–8, *Geometry* (p. 118), recommends that formulas for area and volume be developed gradually:

> As students progress through grades 5–8, they should develop more efficient procedures and, ultimately, formulas for finding measures. Length, area, and volume of one-, two-, and three-dimensional figures are especially important over these grade levels. For example, once students have discovered that it is possible to find the area of a rectangle by covering a figure with squares and then counting, they are ready to explore the relationship between areas of rectangles and areas of other geometric figures.

HISTORICAL HIGHLIGHT



Archimedes,
ca. 287–212 B.C.E.

Archimedes is considered the greatest creative genius of the ancient world. He earned great renown for his mathematical writings and his mechanical inventions. One familiar legend concerns his launching a large ship using pulleys. Archimedes is reported to have boasted that if he had a fixed fulcrum with which to work, he could move anything: "Give me a place to stand and I will move the Earth." Archimedes requested that his tomb be inscribed with a figure of a sphere and a cylinder to commemorate his discovery that the volume of a sphere is two-thirds the volume of the circumscribed cylinder. Many centuries later, the Roman orator Cicero discovered the tomb of Archimedes by identifying the inscription honoring Archimedes' request.*

*D. M. Burton, *The History of Mathematics,* 7th ed. (New York: McGraw-Hill, 2010), pp. 196–211.

IRREGULAR SHAPES

The volumes of some figures with irregular shapes can be determined quite easily by submerging them in water and measuring the volume of the water that is displaced. To illustrate this method, we will find the volume in cubic centimeters of the miniature statue of a bowler in Figure 10.61. Before submerging the statue, we fill the cylinder with water to a height of 900 milliliters. When the statue is placed in the cylinder, the water level rises to

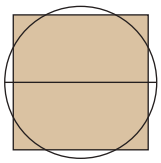


Figure 10.61

Technology Connection

Areas and Volumes

What fraction of the diameter of a circle can be used as the side of a square so that the area of the square will be approximately equal to the area of the circle? This is how the ancient Egyptians (1650 B.C.E.) found the area of a circle. Use Geometer's Sketchpad® to explore this and similar questions in this investigation.



Mathematics Investigation
Chapter 10, Section 3
www.mhhe.com/bbn

the 1000-milliliter level. This means that the volume of the statue is approximately equal to the volume of 100 milliliters of water. Since each milliliter equals 1 cubic centimeter, the volume of the statue is 100 cubic centimeters. A company producing such statues for bowling trophies would need this information to order the amount of metal required.

CREATING SURFACE AREA

A potato can be cooked in a shorter time if it is cut into pieces, ice will melt faster if it is crushed, and coffee beans will give a richer flavor if they are ground before they are steeped. The purpose of crushing, grinding, cutting, or, in general, subdividing is to increase the surface area of a substance. You may be aware of this principle and yet be surprised at the rate at which additional surface area is produced.

To illustrate how rapidly surface area can be created, consider a cube that is 2 centimeters on each edge (part a of Figure 10.62). Its volume is 8 cubic centimeters, and its surface area is 24 square centimeters. If this cube is cut into 8 smaller cubes, as in part b, the total volume is still 8 cubic centimeters, but the surface area is doubled, to 48 square centimeters. This can be easily seen by looking at the small cube in the front corner of part b. Faces *a*, *b*, and *c* contributed 3 square centimeters to the surface area of the original cube; after the cut, 3 more faces of the small cube are exposed, contributing 3 more square centimeters of area. Since this is true for each of the 8 smaller cubes in part b, the total increase in surface area is 8×3, or 24 square centimeters. Following this reasoning, when all the small cubes in part b are separated from the large cube in part a, the surface area will be doubled.

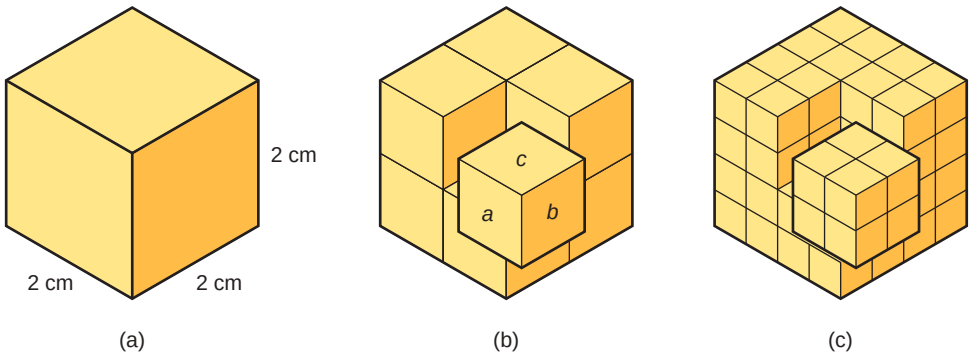


Figure 10.62

If we continue the process, cutting each of the centimeter cubes in Figure 10.62b into 8 smaller cubes, we have 64 cubes whose edges have lengths of $\frac{1}{2}$ centimeter (part c). The total volume of these cubes is still 8 cubic centimeters, but the second cut has again doubled the surface area, increasing it to 96 square centimeters. If the process of halving the dimensions of each small cube is continued, the third set of cuts produces a surface area of $2^3 \times 24 = 192$ square centimeters; after the 12th set of cuts, the surface area has increased to $2^{12} \times 24 = 98{,}304$ square centimeters! During this splitting process *the volume has remained the same*—8 cubic centimeters.

This process of subdividing can also be used to double the surface area of a sphere. If, for example, a sphere of radius 2 centimeters is formed into 8 smaller spheres, each with a radius of 1 centimeter, the total volume will remain the same, but the surface area will double (see Example K on page 715). As in the case of the cubes, if we continue this subdividing process with the smaller spheres, the surface area will double for each subdivision. Consider the effect when water is sprayed into the air in a fine mist, as from snow-making machines. The surface area of each drop of water is increased many times, allowing the small particles of water to freeze quickly in midair into snowflakes.

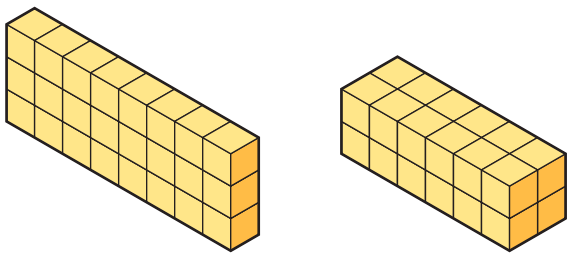
PROBLEM-SOLVING APPLICATION

Examples at the beginning of this section showed that, for a given volume, the surface area of a figure can vary. We also saw in the preceding paragraphs how the surface area can become arbitrarily large while the volume remains constant. These examples suggest the following question: For a given volume, is there a shape that has the least surface area, and if so, what is this shape?

Problem

If a rectangular prism has a volume of 24 cubic centimeters, what is the smallest surface area it can have?

Understanding the Problem Figures made of 24 cubes can help us consider different shapes. The following rectangular prisms are two possibilities. **Question 1:** What is the surface area of each, and which has less surface area?



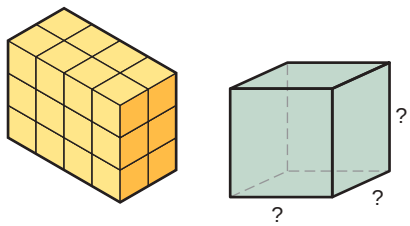
(a) (b)

NCTM Standards

If students move rapidly to using formulas without adequate conceptual foundation in area and volume, many could have underlying confusion. . . . For example, some students may hold the misconception that if the volume of a three-dimensional shape is known, then its surface area can be determined. p. 242

Devising a Plan One approach is to build (or sketch) figures and compute their surface areas. **Question 2:** How many different rectangular prisms can be built using 24 whole cubes, and which has the least surface area?

Carrying Out the Plan A $2 \times 3 \times 4$ prism [figure (c)] has the least amount of surface area of all the figures that can be built from 24 whole cubes. Imagine that the 24 cubes are made of clay that can be molded into one large cube [figure (d)]. **Question 3:** What is the length of a side of this cube, and what is the cube's surface area?



(c) (d)

Looking Back Each edge of the cube in figure (d) has a length of $\sqrt[3]{24}$ centimeters, or approximately 2.88 centimeters. Thus, the area of 1 face is approximately 8.29 square centimeters ($2.88 \times 2.88 \approx 8.29$), and the total surface area of the cube is approximately 50 square centimeters ($6 \times 8.29 \approx 50$). This is 2 square centimeters less than the area of figure (c). Now imagine the 24 cubes of clay being molded into a sphere of volume 24 cubic centimeters. **Question 4:** What is the surface area of a sphere that has a volume of 24 cubic centimeters?

Answers to Questions 1–4 **1.** Prism (a) has a surface area of 70 square centimeters and prism (b) has a surface area of 56 square centimeters. **2.** Six such prisms can be built; the $2 \times 3 \times 4$ prism has the least surface area, 52 square centimeters. **3.** The length of a side is $\sqrt[3]{24}$ centimeters, or approximately 2.88 centimeters; the cube's surface area is approximately 50 square centimeters. **4.** A sphere with a volume of 24 cubic centimeters has a radius of approximately 1.8 centimeters.

$$\frac{4}{3}\pi(1.8)^3 \approx 24.43 \approx 24$$

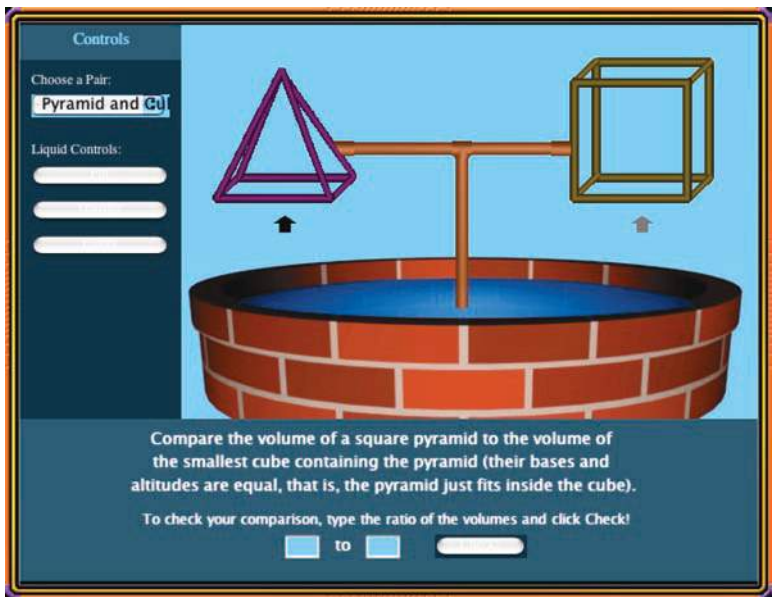
A sphere with a radius of 1.8 centimeters has a surface area of approximately 41 square centimeters.

$$4\pi(1.8)^2 \approx 40.72 \approx 41$$

Notice that the surface area of the sphere is approximately 9 square centimeters less than the surface area of the cube in figure (d). In general, for a given volume, *the sphere is the shape with the least surface area.*

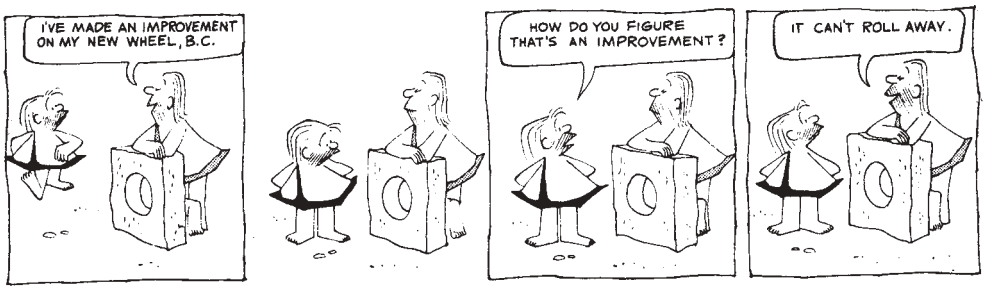
Technology Connection

How does the volume of a cube compare with the volume of a square pyramid that just fits into the cube? (Their bases have the same area and their altitudes are equal.) This applet enables you to experiment by pumping water from one figure to another to discover relationships. In a similar manner, you can compare the volumes of other solids.



Filling 3-D Shapes Applet, Chapter 10, Section 3
www.mhhe.com/bbn

Exercises and Problems 10.3*



1. a. In the cartoon, what is the volume of the wheel, to the nearest cubic centimeter, if the wheel has a length and height of 1 meter, a thickness of 20 centimeters, and an inner diameter of 46 centimeters?

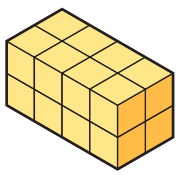
b. If this wheel is made of stone that has a mass of 7 grams per cubic centimeter, what is the mass of the wheel to the nearest kilogram?

The measures of volume and surface area depend on the size of the unit for measuring. Use each of the following cubic units to find the surface area and volume of each figure in 2 and 3.

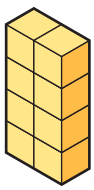

Cubic unit (i)

Cubic unit (ii)

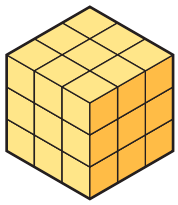
2. a.



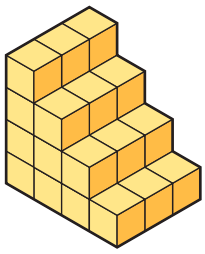
b.



3. a.



b.



4. a. How many cubic feet equal 5.2 cubic yards?

b. How many cubic centimeters equal .3 cubic meter?

*For computations involving π, use the value on your calculator or approximate π by 3.1416.

5. a. How many cubic inches equal 1 cubic yard?

b. How many cubic millimeters equal 1 cubic meter?

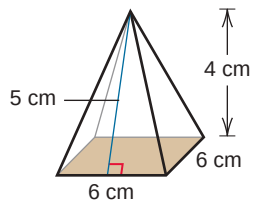
6. The liter is a metric unit approximately equal to 1 quart. A 1-quart milk carton has a square base of 7 centimeters $\times$ 7 centimeters and vertical sides of height 19.3 centimeters.

a. What is its volume?

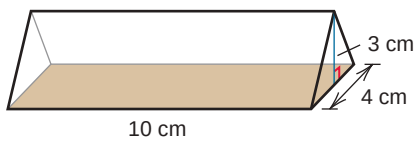
b. Which has a greater volume, a quart or a liter?

Compute the volumes of the figures in 7 through 12 to the nearest cubic centimeter, and compute their surface areas to the nearest square centimeter. (*Hint:* The Pythagorean theorem is needed in exercises 7b, 8b, 11b, 12b, 13a, 14a, and 14b.)

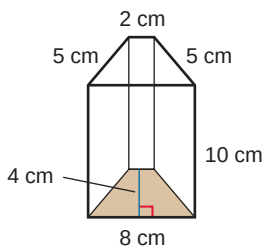
7. a. Square pyramid



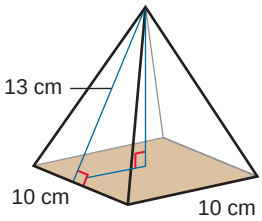
b. Triangular isosceles prism



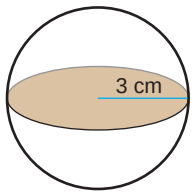
8. a. Trapezoidal prism



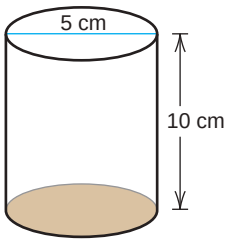
b. Square pyramid



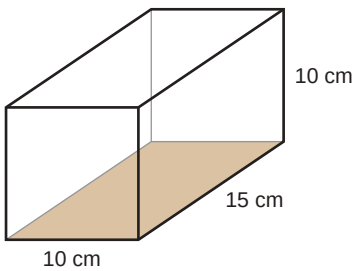
9. a. Sphere



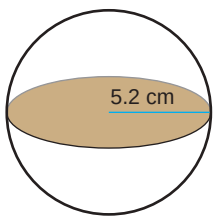
b. Cylinder



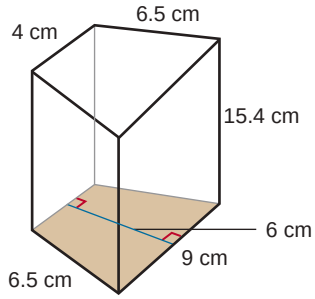
10. a. Rectangular prism



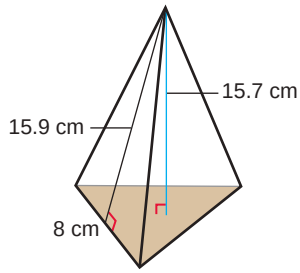
b. Sphere



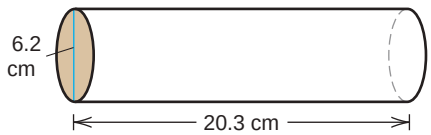
11. a. Trapezoidal prism



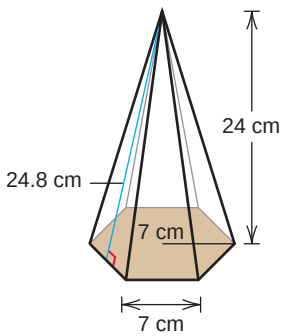
b. Equilateral triangular pyramid



12. a. Cylinder

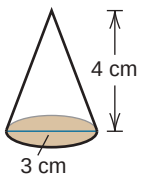


b. Hexagonal pyramid

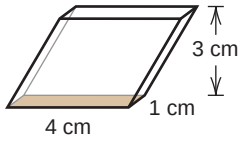


Compute the volumes to the nearest .1 cubic centimeter for the figures in exercises 13 and 14.

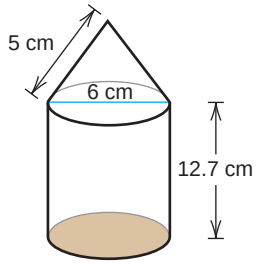
13. a. Cone



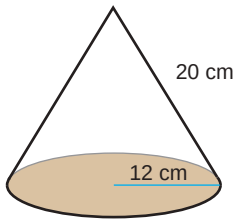
b. Oblique rectangular prism



14. a. Cone on a cylinder

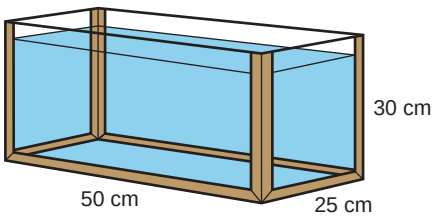


b. Cone



Reasoning and Problem Solving

15. The number of fish that can be put in an aquarium depends on the amount of water the tank holds, the size of the fish, and the capacity of the pump and filter system.



a. How many liters of water will this tank hold?
b. The recommended number of tropical fish for this tank is 30. How many cubic centimeters of space would each fish have?
c. Goldfish need more space and oxygen than tropical fish. Goldfish that are about 5 centimeters long require 3000 cubic centimeters of water. How many goldfish could live in this tank?
d. How many square centimeters of glass are needed for this tank if there is glass on all sides except the top?

16. The concrete foundation for the office building on the corner of Congress and State streets in Boston required 496 truckloads of concrete and was formed in one continuous pouring carried on over a 30-hour period.
a. The concrete was poured to a depth of 1.8 meters and covered an area of 2420 square meters. How many cubic meters of concrete were used?
b. If each truckload was the same size, what was the volume of each load of concrete to the nearest .1 cubic meter?

17. A house with ceilings that are 2.4 meters high has five rectangular rooms with the following dimensions: 4 meters $\times$ 5 meters; 4 meters $\times$ 4 meters; 6 meters $\times$ 4 meters; 6 meters $\times$ 6 meters; and 6 meters $\times$ 5.5 meters. Which of the following air conditioners will be adequate to cool this house: an 18,000-Btu unit that will cool 280 cubic meters; a 21,000-Btu unit that will cool 340 cubic meters; or a 24,000-Btu unit that will cool 400 cubic meters?

18. A woodshed is 3 meters $\times$ 2 meters $\times$ 2 meters. If each 1.5 cubic meters of firewood sells for \$25, how much will it cost to fill the shed with wood?

19. A catalog describes two types of upright freezers. Type *A* has a storage capacity of 60 centimeters $\times$ 60 centimeters $\times$ 150 centimeters and costs \$339; type *B* has a storage capacity of 55 centimeters $\times$ 72 centimeters $\times$ 160 centimeters and costs \$379. Which freezer gives you more cubic centimeters per dollar?

20. A drugstore sells the same brand of talcum powder in two types of cylindrical cans: Can *A* has a diameter of 5.4 centimeters and a height of 9 centimeters and sells for \$1.59; can *B* has a diameter of 6.2 centimeters and a height of 12.4 centimeters and sells for \$2.99. Which can is the better buy?

21. An auditorium has 20 large cylindrical columns. Each column has a height of 22 feet from the floor to the ceiling and a diameter of 2.5 feet. How many gallons of paint, to the nearest whole number, must be purchased to paint the lateral surface area of the columns if each gallon of paint covers 350 square feet?

22. A cubic box with no top is to be made of plywood. How many square feet of plywood are needed if the box is to hold 64 cubic feet?

23. The Great Pyramid of Egypt has a height of 148 meters and a square base with a perimeter of 930 meters. The Transamerica Pyramid in San Francisco has a height of 260 meters and a square base with a perimeter of 140 meters.
a. The volume of the Great Pyramid is how many times the volume of the Transamerica Pyramid?

b. The heights (altitudes) of the triangular faces of the Great Pyramid and Transamerica Pyramid are 188 and 261 meters, respectively. The bases of these triangles are 232.5 and 35 meters, respectively. The surface area of the four faces of the Great Pyramid is about how many times the surface area of the four faces of the Transamerica Pyramid?

24. Swimming pools must be tested daily to determine the pH factor and the chlorine content. Pumps and filters are also necessary, and some pools have heating systems.

a. What is the depth of a 6-meter × 12-meter pool to the nearest .01 meter that contains 193 kiloliters of water?

b. If this pool requires 112 grams of chlorine every 2 days, how many kilograms of chlorine should be purchased for a 90-day period?

c. The Alcoa Solar Heating System for pools has 32 square panels, each 120 centimeters × 120 centimeters. Will this heating system fit onto a 5-meter × 8-meter roof?

d. Each panel for this system holds 5.68 liters of water. What is the total mass of the water in 32 panels to the nearest kilogram?

25. One of the silos pictured below holds corn, and the other holds hay. Chopped corn and hay are blown into the tops of the silos through pipes running up from the ground.

a. The silos have a radius of 3 meters and a height of 18 meters. What is the volume, to the nearest cubic meter, of one of these silos?

b. If a blower can load 1 cubic meter of hay in 3 minutes, how many hours (to the nearest .1 hour) will it take to fill one of these silos?



26. The sphere shown in the photograph at the top of the next column was constructed in Charleston, South Carolina, and then towed by tug to Quincy, Massachusetts. Each aluminum sphere for a liquefied natural gas tanker has a diameter of approximately 36.6 meters and a mass of 725,750 kilograms (800 tons).



Sphere for storage of liquefied natural gas

a. What is the volume, to the nearest cubic meter, of this sphere?

b. A heavy external coating of insulation on the surface of the sphere enables the sphere to maintain liquefied natural gas at $^{-}165°C$. How many square meters of insulation, to the nearest whole number, are needed for one sphere?

27. This art form is Alex Lieberman's *Argo,* which is at the Walker Art Center in Minneapolis. The entire display has a mass of about 4535 kilograms.



a. The cylinder shown in front is 2 meters tall and 1 meter in diameter. What is its surface area (including the bases) to the nearest .01 square meter?

b. If each square meter of metal in this cylinder has a mass of 92 kilograms, what is the mass of the cylinder to the nearest .1 kilogram?

28. Featured Strategies: Making a Drawing and Making a Table. An open-top box is to be formed by cutting out squares from the corners of a 50-centimeter × 30-centimeter rectangular sheet of material. The height of the box must be a whole number of centimeters. What size squares should be cut out to obtain the box with maximum volume?

a. Understanding the Problem. This diagram shows how the box is to be formed. If 6-centimeter × 6-centimeter squares are cut from the corners, the height of the box will be 6 centimeters. In this case, what will the width and length of the box be?



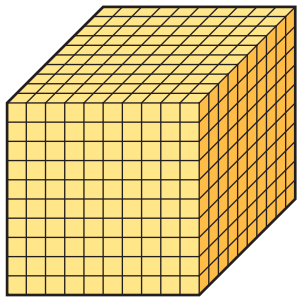
b. Devising a Plan. One plan for solving this problem is to systematically consider corner squares of increasing size. What is the largest square with whole-number dimensions that can be cut from the corners and still produce a box?

c. Carrying Out the Plan. Complete the following table, and use inductive reasoning to predict the size of the corner squares needed to obtain the box of maximum volume.

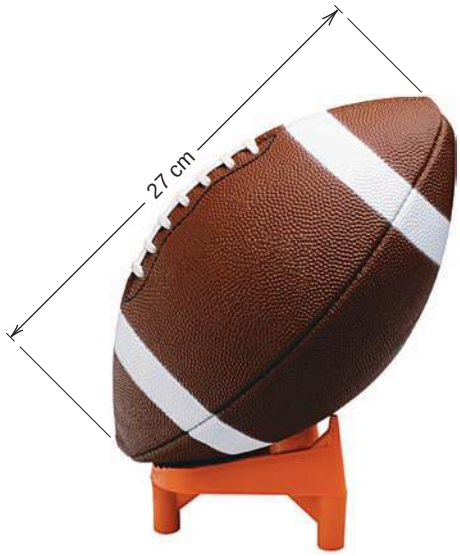
Size of Square (square centimeters)	Volume of Box (cubic centimeters)
2 × 2	
4 × 4	
6 × 6	
8 × 8	
10 × 10	
12 × 12	
14 × 14	

d. Looking Back. The preceding table shows that as the size of the squares at the corners increases, the volume of the box increases for a while and then decreases. Try a few more sizes for the squares, using whole numbers for dimensions, to see if you can obtain a greater volume for the box.

29. Suppose a large cube is built from 1000 small cubes and then painted on all six faces. When the large cube is disassembled, how many of the small cubes will be unpainted? Separate the remaining small cubes into groups according to the number of faces painted, and compute the number in each group. Generalize this result for an $n \times n \times n$ cube.



30. A regulation football has a length of approximately 27 centimeters and a diameter of approximately 16 centimeters. Describe two different methods of approximating its volume in cubic centimeters.



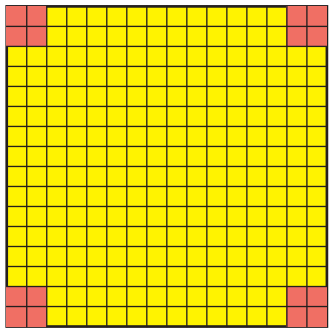
31. Assume that a drop of unvaporized gasoline is a sphere with a diameter of 4 millimeters.

a. If this drop is divided into 8 smaller drops, each with a diameter of 2 millimeters, the total surface area of the 8 drops is how many times the surface area of the original drop?

b. If each drop with a diameter of 2 millimeters is divided into 8 smaller drops, each with a diameter of 1 millimeter, the total surface area of the 64 drops is how many times the surface area of the original drop?

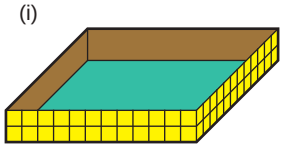
c. If the vaporizing mechanism in a car's engine carries out this splitting process 20 times, how many times is the surface area of the original drop of gasoline increased?

32. An open-top box is to be formed from a sheet of material 16 inches by 16 inches by cutting out the squares with whole-number dimensions from the corners and folding up the edges.

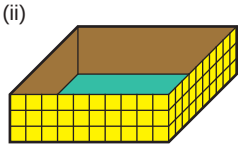


Two such boxes are shown here.

(i)



(ii)



a. What is the volume of the box in (i)?
b. What is the volume of the box in (ii)?
c. What are the dimensions and volume of the box having the greatest volume that can be made from the original sheet of material?

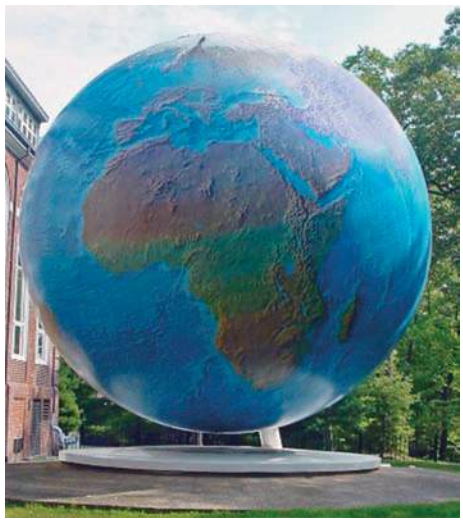
33. A cubic block of cement is tossed into a cylindrical tank of water with a diameter of 2 feet, causing the water to rise 1.5 inches.
a. What is the volume of the cube to the nearest cubic inch?
b. What is the length of the edge of the cube to the nearest .1 inch?

34. A bank's monthly rental fees for safe deposit boxes with various dimensions are listed here.

	Box Size (cubic inches)	Fee ($)
(1)	$12\frac{3}{4} \times 4\frac{1}{2} \times 1\frac{1}{2}$	6.80
(2)	$22 \times 4\frac{3}{4} \times 1\frac{1}{2}$	10.75
(3)	$23\frac{3}{4} \times 4\frac{3}{4} \times 2\frac{1}{2}$	18.40
(4)	$21\frac{1}{2} \times 3\frac{5}{8} \times 5$	23.45
(5)	$21\frac{1}{4} \times 5\frac{1}{2} \times 4\frac{3}{4}$	32.00
(6)	$23\frac{3}{4} \times 10 \times 2\frac{3}{4}$	36.00
(7)	$21\frac{1}{4} \times 10\frac{3}{4} \times 3\frac{1}{4}$	38.60
(8)	$21\frac{1}{4} \times 10\frac{1}{2} \times 4\frac{1}{2}$	50.00

a. Find the volume of each box to the nearest .01 cubic inch. (*Suggestion:* Change the fractions to decimals.)
b. What is the cost per cubic inch to the nearest .1 cent for renting each box?
c. What happens to the cost per cubic inch as the size of the boxes increase?

This is the Babson World Globe on the campus of Babson College in Wellesley, Massachusetts. The original globe was completed in the 1950s, but the new surface shown here was completed in 1993. The scale for the globe is 1 inch to 24 miles. The globe is 28 feet in diameter, weighs 41 tons, and originally was capable of rotating. (Use $\pi = 3.14$ for problems 35 and 36.)



35. **a.** What is the circumference of the globe to the nearest foot?
b. What is the surface area of the globe to the nearest square foot?
c. The inner surface of the globe is covered with 28 curved steel plates, each weighing approximately one ton. If each plate is the same size, what is the surface area of each to the nearest square foot?
d. Using the scale factor for the globe and the fact that the Himalayan mountains have an altitude of approximately 6 miles, what would be the height of these mountains on the Babson World Globe?

36. **a.** The cylindrical steel shaft for the globe was built by Bethlehem Steel and has a diameter of approximately 2 feet and a length of 25 feet. What is the volume to the nearest cubic foot of this shaft? If each cubic foot of this steel weighs .2 ton, what is the weight of the shaft to the nearest ton?

b. The outer surface is formed with 584 plates that are bolted to the sphere. If each of these plates is the same size, what is the area of each plate to the nearest square foot?

c. Using the circumference of the globe and the scale from the globe to the Earth, what is the distance around the Earth to the nearest thousand miles?

d. The distance from New York City to San Francisco is approximately 2900 miles. Using the scale for the Babson World Globe, what is the distance to the nearest inch on the surface of the globe between these two locations?

Teaching Questions

1. Two students were discussing the meaning of 3 m^3. One said that it meant 3 cubes, each with side lengths of 1 meter, and the other said that it meant it was a larger cube whose side lengths were 3 meters. Explain whether or not these students are expressing the same thing? Explain how you would resolve the difference of opinion.

2. A teacher gave her students a box that is 10 cm wide, 12 cm long, and 5 cm high. She asked them to construct a box that is twice as large. Give examples of different possible meanings of her request and the dimensions of the box for each example.

3. Describe, in a series of steps, how you would help a student "see" how many cubic centimeters are in a cubic meter. Assume that you have a meterstick and centimeter cubes.

4. For a class demonstration you need paper models of a right cylinder and a right circular cone that has the same base and height as the cylinder. Make paper models and then describe your construction so the reader can follow your directions.

Classroom Connections

1. Under *"Apply appropriate techniques, . . ."* in the **Grades 3–5 Standards—Measurement** (see inside front cover), read the second expectation that refers to using appropriate units. What are the appropriate units for each of the measurements mentioned? Explain why you would or would not include metric units.

2. Archimedes discovered that the volume of a sphere is $\frac{2}{3}$ the volume of the circumscribed cylinder and that the surface area of a sphere is $\frac{2}{3}$ the surface area of the circumscribed cylinder. As the **Historical Highlight** in this section relates, he was so proud of these discoveries he had a sphere and cylinder inscribed on his tomb. Describe a way you could demonstrate each of these relationships by using physical materials.

3. Read the **Standards** statement on page 718. Describe an activity to help dispel the misconception that the volume of a figure determines its surface area. Design this activity so students use cubes to form three-dimensional figures, determine the volumes, and determine the surface areas.

4. Design activities for school students that involve models and concrete objects that can be filled, submerged, etc., to illustrate the following formulas. Then explain whether or not you believe your activities satisfy the suggestion of the **Standards** statement on page 714.

- Volume of a rectangular prism (box) = (length) × (width) (height)
- Volume of a pyramid = $\frac{1}{3}$(area of the base)(height of pyramid)
- Volume of cone = $\frac{1}{3}$(area of base)(height of cone)

5. Design an activity to follow the **Elementary School Text** on page 707 that introduces the concept of surface area. First explain how you would introduce the idea of surface area. Then design your activity to have students build figures with cubes and determine the surface areas of the figures.

6. Read the **Spotlight on Teaching** at the beginning of this chapter. For the towers of cubes at the bottom of that page, determine the surface area of a tower 30 cubes high. If a similar tower had a surface area of 242 square units, how tall would it be? Write an algebraic expression for the height of a tower in terms of the surface area *S*. Explain your thinking.

CHAPTER 10 REVIEW

1. **Systems and units of measurement**
   a. A **unit of measure** is any reproducible unit that can be used to measure a physical property.
   b. **Span, foot, hand, cubit, carat, grain, stone,** and **scheffel of land** are historic units of measurement.
   c. **Nonstandard units** of length, area, and volume provide background for understanding standard units of measure.
   d. The **English system** arose from natural nonstandard units of measure such as the length of a foot. It is used in the United States.
   e. The **metric system** and the **International System of Units** (SI) are based on the meter and are used in almost all countries.
   f. Comparing a unit to an object to determine the number of units is called **measurement.**
   g. The **precision** of a measurement is to within one-half of the smallest unit of measure used.
   h. The number of units it takes to cover a surface is called its **area.**
   i. The number of three-dimensional units needed to fill a figure is its **volume.**
   j. **Surface area** is the number of unit squares needed to cover the surface of a three-dimensional object.
   k. **Mass** is a measure of a quantity of matter and is not affected by the force of gravity.
   l. **Weight** is a measure of the force of gravitational pull on a body.

2. **English system**
   a. **Units for length: inch** (in), **foot** (ft), **yard** (yd), and **mile** (mi) are defined.
   b. **Units for volume: ounce** (oz), **cup** (c), **pint** (pt), **quart** (qt), and **gallon** (gal) are defined.
   c. **Units for weight: ounce** (oz), **pound** (lb), **ton** (tn), **troy unit,** and **avoirdupois unit** are defined.
   d. **Temperature** is measured in degrees on the **Fahrenheit scale.**

3. **Metric system**
   a. The **metric prefixes** are related by powers of 10. (Those in the following list with asterisks are the most common.)

kilo*	1000
hecto	100
deka	10
deci	$\frac{1}{10}$
centi*	$\frac{1}{100}$
milli*	$\frac{1}{1000}$

   b. **Units for length: millimeter** (mm), **centimeter** (cm), **meter** (m), and **kilometer** (km) are defined.
   c. **Units for volume: milliliter** (mL), **liter** (L), and **kiloliter** (kL) are defined.
   d. **Units for mass: milligram** (mg), **gram** (g), and **kilogram** (kg) are defined.
   e. **Temperature** is measured in degrees on the **Celsius scale.**
   f. Other units are given for: time **(second),** electric current **(ampere),** light intensity **(candela),** and the molecular weight of a substance **(mole).**

4. **Area and perimeter**
   a. The common **English units for area: square inch** (in^2), **square foot** (ft^2), **square yard** (yd^2), **acre,** and **square mile** (mi^2)
   b. The common **metric units for area: square millimeter** (mm^2), **square centimeter** (cm^2), **square meter** (m^2), and **square kilometer** (km^2)
   c. **Perimeter** is a measure of the length of the boundary of a region.
   d. **Rectangle:** $A = l \times w$, where l is the length and w is the width of the rectangle.
   e. **Parallelogram:** $A = b \times h$, where b is the length of the base and h is the altitude to the base.
   f. **Triangle:** $A = \frac{1}{2} \times bh$, where b is the length of the base and h is the altitude to that base.
   g. **Trapezoid:** $A = \frac{1}{2}(b + u) \times h$, where b and u are the lengths of the bases and h is the altitude between the bases.
   h. **Circle:** $A = \pi r^2$, where r is the radius of the circle.
   i. The **circumference** C of a circle with diameter d and radius r is $C = \pi d = 2\pi r$.

5. **Volume and surface area**
   a. The common nonliquid **English units for volume: cubic inch** (in^3), **cubic foot** (ft^3), and **cubic yard** (yd^3)
   b. The common nonliquid **metric units for volume: cubic millimeter** (mm^3), **cubic centimeter** (cm^3), and **cubic meter** (m^3)
   c. A 10 cm × 10 cm × 10 cm cube is a **cubic decimeter,** which has volume of 1 liter.
   d. **Prism and cylinder:** $V = B \times h$, where B is the area of the base of the prism and h is the altitude.
   e. **Pyramid and cone:** $V = \frac{1}{3} \times Bh$, where B is the area of the base and h is the altitude.
   f. **Sphere:** $V = \frac{4}{3} \times \pi r^3$, where r is the radius of the sphere.

g. The **surface area of a prism** or **pyramid** is the total area of the faces of these polyhedra.
h. The **surface area of a right cylinder** is $2\pi rh + 2\pi r^2$, where r is the radius of the base and h is the altitude of the cylinder.
i. The **surface area of a sphere** is $4\pi r^2$, where r is the radius of the sphere.
j. The **surface area of a right cone** is $\pi r\sqrt{r^2 + h^2} + \pi r^2$ where r is the radius of the base of the cone.

CHAPTER 10 TEST

(For all questions involving π, use the value on your calculator or approximate π by 3.1416.)

1. Indicate the most appropriate metric unit to use in measuring each item.
a. Mass of a bar of soap
b. Volume of a bottle of eyedrops
c. Length of a house
d. Mass of a person
e. Area of a football field
f. Volume of a truckload of gravel

2. Complete each equality below.
a. 3.5 feet = _____ inches
b. 1 square yard = _____ square feet
c. 3.4 gallons = _____ quarts
d. 2.5 quarts = _____ ounces
e. 2.5 cubic yards = _____ cubic feet
f. 2.75 pounds = _____ ounces

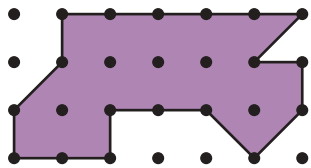
3. Complete each equality below.
a. 1.6 grams = _____ milligrams
b. 4.7 meters = _____ centimeters
c. 5.2 kilometers = _____ meters
d. 2500 milliliters = _____ liters
e. 1.6 square centimeters = _____ square millimeters
f. 1 cubic meter = _____ cubic centimeters

4. Complete each statement below.
a. 1.6 liters of water has a mass of _____ grams
b. 32° Fahrenheit equals _____ Celsius
c. 55 cubic centimeters has a volume of _____ milliliters
d. 2 cubic decimeters of water has a mass of _____ kilograms
e. 1 kilometer equals approximately _____ miles
f. 1 kilogram of water weighs approximately _____ pounds

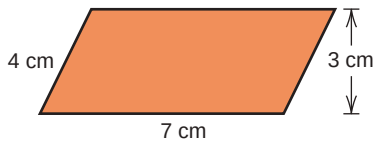
5. Precision is determined by the smallest unit used for a given measurement. Determine the minimum and maximum measurement for each of the following.
a. A 5.3-kilogram bag of dog food
b. An 85-gram tube of toothpaste
c. A 4.12-ounce box of cake mix

6. Find the area of the dodecagon below, using each of the given area units.

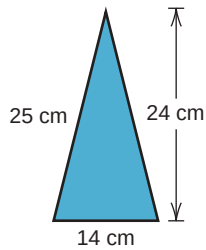




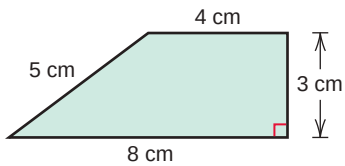
7. Find the area and perimeter or circumference of each figure.
a. Parallelogram



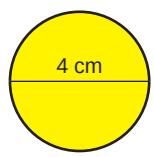
b. Isosceles triangle



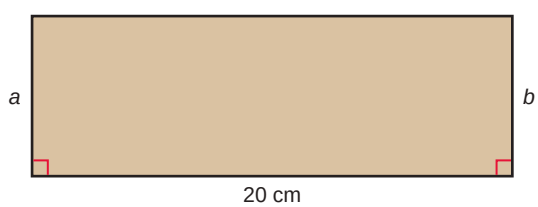
c. Trapezoid



d. Circle with diameter

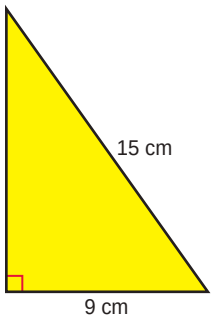


8. Edges a and b of a rectangular sheet of paper are taped together to form a cylinder without bases. What is the diameter of the cylinder to the nearest .1 centimeter?

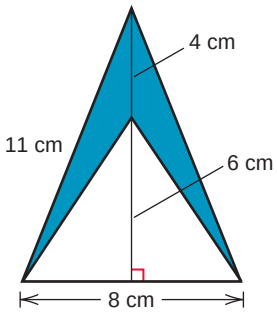


9. Find the area of each shaded region.

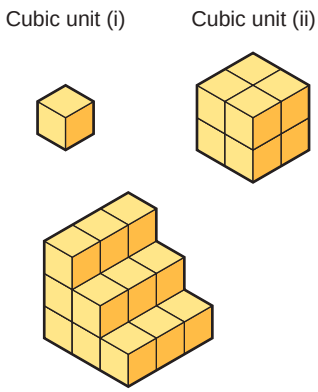
a.



b.

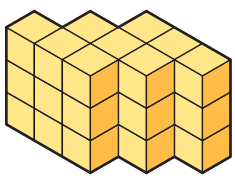


10. Find the volume and surface area of the staircase figure below, using each of the given cubic units. The unit of area is the face of each unit cube.

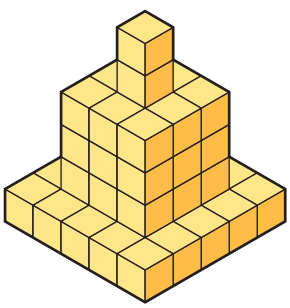


11. Find the volume and surface area of each figure if the figures are solid (that is, no missing cubes) and each single cube has a volume of 1 cubic centimeter.

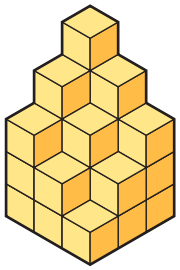
a.



b.

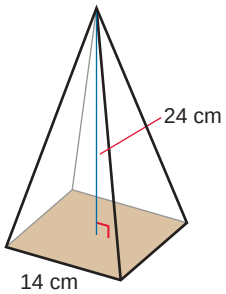


c.

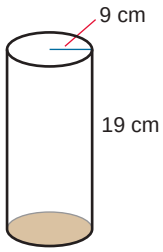


12. Find the volume of each figure to the nearest .1 cubic centimeter.

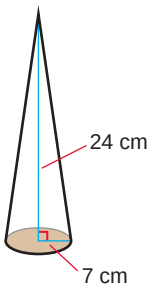
a. Square pyramid



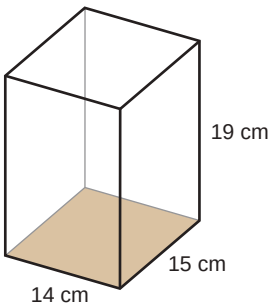
b. Right cylinder



c. Right cone

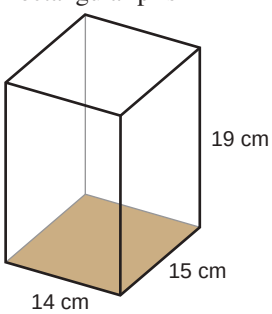


d. Rectangular prism

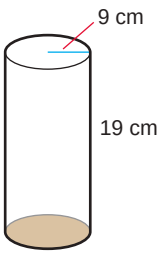


13. Find the surface area of each figure to the nearest .1 square centimeter.

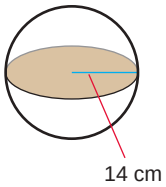
a. Rectangular prism



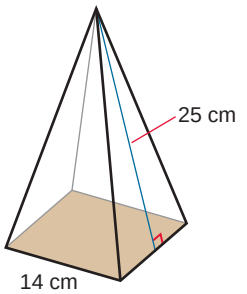
b. Right cylinder



c. Sphere



d. Square pyramid



14. How many cubic yards of concrete (to the nearest .1 cubic yard) are needed to make a base for a square patio if each edge of the square has a length of 11 feet and the cement is poured to a depth of .8 foot?

15. A store sells two types of shelf paper. Type *A* has dimensions of 5 meters $\times$ 30 centimeters and costs \$3.70. Type *B* has dimensions of 4 meters $\times$ 35 centimeters and costs \$3.50. Which type is the better buy?

16. The cost of a rental car for a two-week 1600-kilometer trip across northern Spain is \$414. The cost does not include gasoline, which is \$2.08 cents per liter. If the car uses 1 liter of gasoline per 13 kilometers, what is the total cost for the rental fee plus the gasoline?

CHAPTER

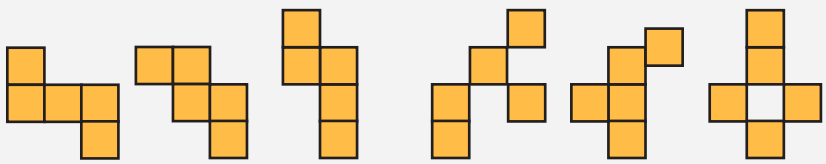
11 Motions in Geometry

Spotlight on Teaching

Excerpts from NCTM's Standards for School Mathematics Prekindergarten through Grade 2*

Students can naturally use their own physical experiences with shapes to learn about transformations such as slides (translations), turns (rotations), and flips (reflections). They use these movements intuitively when they solve puzzles, turning the pieces, flipping them over, and experimenting with new arrangements. . . .

Teachers should choose geometric tasks that are accessible to all students and sufficiently open-ended to engage students with a range of interests. For example, a second-grade teacher might instruct the class to find all the different ways to put five squares together so that one edge of each square coincides with an edge of at least one other square (see Figure 4.15). The task should include keeping a record of the pentominoes that are identified and developing a strategy for recognizing when they are transformations of another pentomino. Teachers can encourage students to develop strategies for being systematic by asking, "How will you know if each pentomino is different from all the others? Are you certain you have identified all the possibilities?"



Pentominoes Not Pentominoes

Figure 4.15

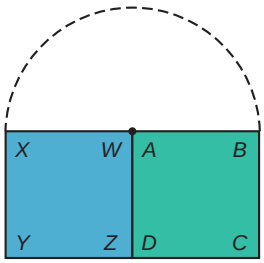
**Principles and Standards for School Mathematics*, pp. 99–100.

MATH ACTIVITY 11.1

Tracing Figures from Motions with Tiles

Purpose: Rotate color tiles to study the motion of points in the plane.

Materials: Color Tiles in Manipulative Kit.



1. The two tiles shown here are a stationary tile (blue) and a moving tile (green). Imagine rotating the green tile about point A, keeping point A fixed, until side $\overline{AB}$ is next to side $\overline{XW}$; then rotate the green tile by keeping point B fixed until side $\overline{CB}$ is next to side $\overline{XY}$; then rotate the green tile by keeping point C fixed so that side $\overline{DC}$ is next to side $\overline{YZ}$; finally, rotate the green tile by keeping point D fixed so that the green tile is back in its starting position.

a. For the first part of the motion of the green tile about the blue tile, point B traces out the dotted semicircle shown in the figure at the left. Try to visualize the figure traced out by point B for the complete motion of the green tile about the blue tile, and make a prediction about its shape.

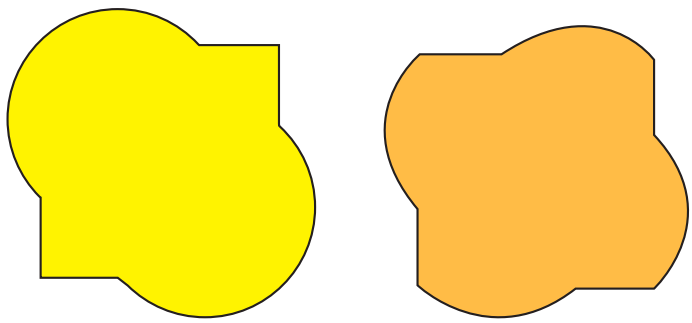
b. Carry out the motion of the green tile about the blue tile, and sketch the path traced by B. You may want to have a classmate help you hold and move the tiles.

***c.** Remove the green tile and trace around the blue tile to mark its position relative to the figure traced by point B. The figure traced by point B can be subdivided into a large right triangle with three semicircles on its legs. Write about relationships between the area of the blue tile and the area of the right triangle; the area of the blue tile and the total area of the two small semicircles; and the total area of the two small semicircles and the area of the large semicircle.

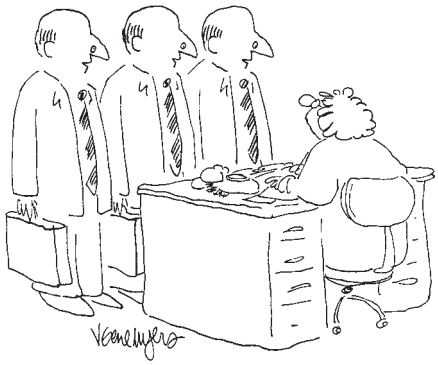
2. Each of the figures shown below is created by the path of a point on the green tile as it is moved about the blue tile. (*Note:* These figures are smaller than the ones that will be reproduced by your tiles.)

a. Write detailed directions for moving the green tile about the blue tile so that some point on the green tile traces out each figure. Include in your description the location of the blue tile and the point on the green tile that traces out the figure.

b. Using the blue tile as the unit of area, determine the area of each figure traced out in part a. Explain with a diagram how you obtained each area.



Section 11.1 CONGRUENCE AND CONSTRUCTIONS



"WE'RE HERE TO FIX THE COPIER."

PROBLEM OPENER

Cut a 3×8 rectangle into two congruent parts, and form a 2×12 rectangle.



There is an old belief that everyone has a "double"—someone who looks exactly like him or her—somewhere in the world. Two-dimensional and three-dimensional objects often do look exactly alike. Copy machines are able to make reproductions of two-dimensional figures that have the same *size* and *shape* as the originals. The reproductions are said to be *congruent* to the original figures. Intuitively, we think of two plane figures as **congruent** if one can be moved onto the other so that they coincide. The idea of motion or movement is an important concept in mathematics and will be explored in this chapter.

MAPPINGS

If triangle ABC in Figure 11.1 is traced on paper and flipped over, it can be placed on triangle RST so that the points of each triangle coincide. The correspondence of point A with R, B with S, and C with T is indicated by

$$A \leftrightarrow R \qquad B \leftrightarrow S \qquad C \leftrightarrow T$$

We say that A corresponds to R, B corresponds to S, and C corresponds to T. These pairs of vertices are called **corresponding vertices.**

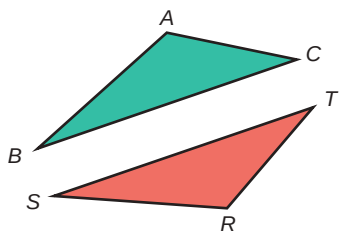


Figure 11.1

If triangle ABC is placed onto triangle RST, each point on the first triangle corresponds to exactly one point on the second triangle. This one-to-one correspondence of points is a special type of *function.* In Section 2.2 we discussed functions that assign numbers to numbers. In geometry, there are functions that assign points to points, such that to each point in one set there corresponds a unique point, called the **image,** in a second set. Such functions are called **mappings.** In the mapping of $\triangle ABC$ to $\triangle RST$ in Figure 11.1, the following sides and angles are matched with one another:

Corresponding sides	*Corresponding angles*
$\overline{AB} \leftrightarrow \overline{RS}$	$\measuredangle B \leftrightarrow \measuredangle S$
$\overline{BC} \leftrightarrow \overline{ST}$	$\measuredangle C \leftrightarrow \measuredangle T$
$\overline{AC} \leftrightarrow \overline{RT}$	$\measuredangle A \leftrightarrow \measuredangle R$

Such pairs of sides and angles are called **corresponding sides** and **corresponding angles;** these concepts are used in the following definition.

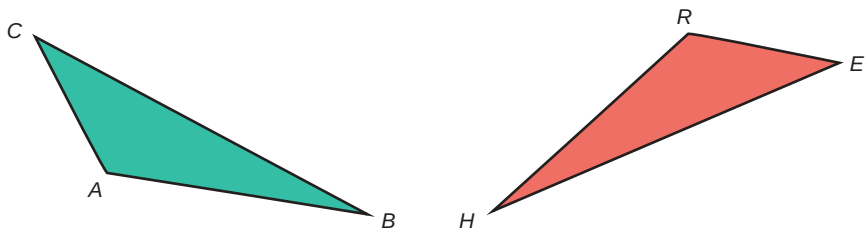
Congruent Polygons Two polygons are **congruent** if and only if there is a mapping from one to the other such that:

1. Corresponding sides are congruent;
2. Corresponding angles are congruent.

In Figure 11.1 on the preceding page, triangle ABC and triangle RST are congruent because their corresponding sides and angles are congruent. This congruence is indicated by writing $\triangle ABC \cong \triangle RST$.

EXAMPLE A

The following triangles are congruent. Complete the congruence statement $\triangle$ _____ $\cong$ $\triangle$ _____, and list the pairs of corresponding vertices, sides, and angles.



Solution $\triangle ABC \cong \triangle RHE$

Corresponding vertices: $A \leftrightarrow R, B \leftrightarrow H, C \leftrightarrow E$

Corresponding sides: $\overline{AB} \leftrightarrow \overline{RH}, \overline{BC} \leftrightarrow \overline{HE}, \overline{CA} \leftrightarrow \overline{ER}$

Corresponding angles: $\measuredangle A \leftrightarrow \measuredangle R, \measuredangle B \leftrightarrow \measuredangle H, \measuredangle C \leftrightarrow \measuredangle E$

Notice that the order of the letters in the statement of congruence in Example A indicates which pairs of vertices, sides, and angles correspond.

$$\triangle ABC \cong \triangle RHE$$

NCTM Standards

The study of geometry in grades 3–5 requires thinking *and* doing. As students sort, build, draw, model, trace, measure, and construct, their capacity to visualize geometric relationships will develop. At the same time they are learning to reason and to make, test, and justify conjectures about these relationships. p. 165

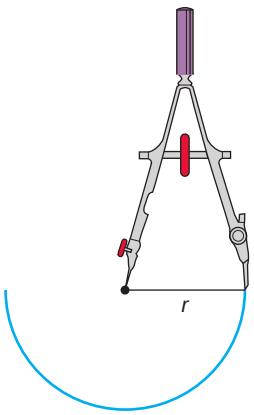


Figure 11.2

To determine if *two* polygons are congruent, we set up a correspondence between their vertices and check to see if their corresponding sides and corresponding angles are congruent. To construct a figure that is congruent to a given figure, we construct corresponding sides and corresponding angles that are congruent to those given. This section introduces techniques for constructing congruent figures.

CONSTRUCTING SEGMENTS AND ANGLES

Geometric figures can be constructed in many ways. One of the more common methods in recent years is the drawing and shading of figures by computers. In Section 9.1 we used paper folding to form perpendicular and parallel lines, angle and line bisectors, and various types of angles and polygons. The Mira, which was introduced in Section 9.4 for locating lines of symmetry, is also useful for certain constructions. A few examples of constructions with the Mira are given at the end of this section and in Exercises and Problems 11.1. Historically, a geometric figure that is produced with only a straightedge and compass is called a **construction.** A **straightedge** is used to draw lines, and unlike a ruler, it has no markings. The compass was introduced in Section 9.2 for constructing circles and regular polygons. A compass opening of length r can be used to draw a circle of radius r (Figure 11.2).

To carry out the constructions in this section, you will need a straightedge (or a ruler) and a compass. Try each construction on your own before reading the steps that are given. Attempting each construction will help you to think about the steps that are given, and you may discover a method of your own.

Constructing Segments Two **line segments** are **congruent** if they have the same length. For example, if $\overline{AB}$ and $\overline{CD}$ have the same length, then $\overline{AB}$ is congruent to $\overline{CD}$, and we write $\overline{AB} \cong \overline{CD}$. The common method of obtaining a line segment that is congruent to a given segment is to measure the given segment with a ruler and then mark off this length on a line. Example B shows how congruent segments can be constructed by using a straightedge and compass.

EXAMPLE B

Construct a line segment that is congruent to segment $\overline{AB}$.

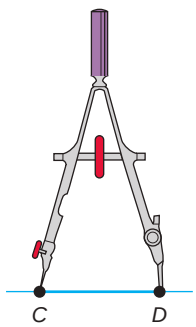


Original segment

Solution **Step 1** Use a straightedge to draw a line segment that is longer than $\overline{AB}$; label point C.



Step 2 Open the compass to span $\overline{AB}$. Place one end of the compass at point C, and mark point D. Then $\overline{AB} \cong \overline{CD}$.



If $\overline{AB}$ in Example B is longer than the opening of the compass, intermediate points can be marked off on $\overline{AB}$. Two such intermediate points are shown in Figure 11.3. The parts of $\overline{AB}$ can then be transferred to the new line by using the compass, as in Example B.

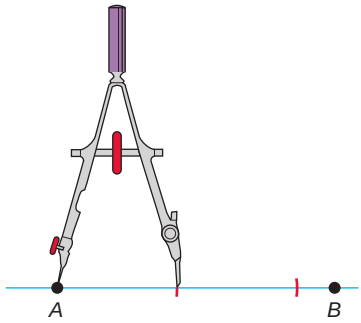
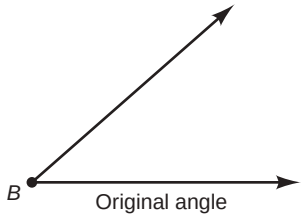


Figure 11.3

Constructing Angles Two **angles** are **congruent** if they have the same measure. For example, if $\measuredangle ABC$ and $\measuredangle DEF$ have the same measure, then $\measuredangle ABC$ is congruent to $\measuredangle DEF$, and we write $\measuredangle ABC \cong \measuredangle DEF$. In Section 9.2 we constructed angles of a given number of degrees by measuring the angles with a protractor. Angles can also be reproduced by using a straightedge and compass.

EXAMPLE C

Construct an angle that is congruent to angle B.

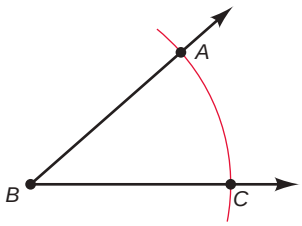


Solution

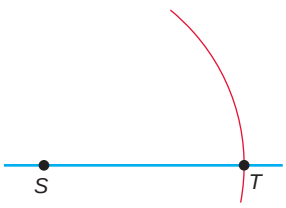
Step 1 Use a straightedge to draw a new line segment and label point S.



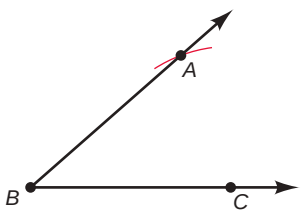
Step 2 Place the end of the compass at point B of the original angle (see above), and draw an arc. Label points A and C on the sides of the angle, as shown.



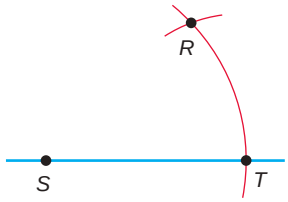
Step 3 Using the same compass opening, place the compass at S from step 1 and draw an arc. Label point T as shown.



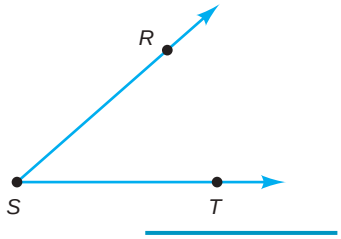
Step 4 Place the compass at point C of the original angle and adjust the opening to produce an arc through point A.



Step 5 Using the same opening, place the compass at point T on the new line segment and draw an arc to locate point R.



Step 6 Use a straightedge to connect point R to point S. Then $\measuredangle ABC \cong \measuredangle RST$.



CONSTRUCTING TRIANGLES

Figure 11.4 shows two congruent triangles: $\triangle ABC \cong \triangle DEF$.

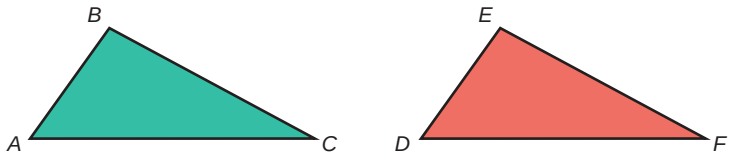
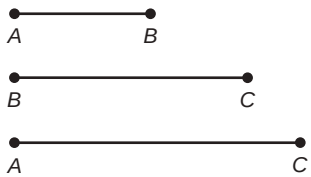


Figure 11.4

The congruence of the triangles implies the correspondence $A \leftrightarrow D$, $B \leftrightarrow E$, and $C \leftrightarrow F$, such that the corresponding sides and corresponding angles of the triangles are congruent. However, to construct a triangle that is congruent to a given triangle, it is not necessary to construct three congruent sides and three congruent angles separately. Example D shows that it is only necessary to construct three congruent sides. The sides of $\triangle ABC$ are used in this example.

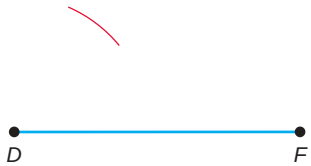
EXAMPLE D Construct a triangle whose sides are congruent to the three given line segments:



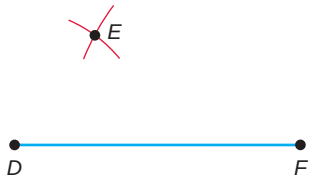
Solution **Step 1** Use a straightedge and compass to construct a new line segment $\overline{DF}$, which is congruent to $\overline{AC}$.



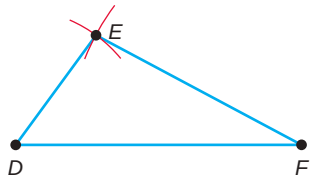
Step 2 Place the ends of the compass on points A and B of $\overline{AB}$. Then place one end of the compass at point D of the new line segment and draw an arc. All the points on this arc are the distance AB from point D.



Step 3 Place the ends of the compass on points B and C of $\overline{BC}$. Then place one end of the compass on F of the new line segment and draw an arc. The intersection of the two arcs is the third vertex point, E.



Step 4 Draw segments $\overline{DE}$ and $\overline{EF}$ to form $\triangle DEF$.



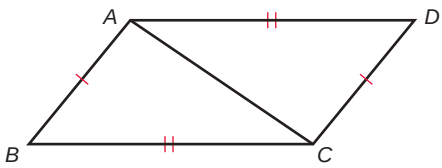
The construction in Example D could have been varied in several ways. For example, we could have begun in step 1 by constructing a segment congruent to either $\overline{AB}$ or $\overline{BC}$. Or in step 2 we could have constructed an arc whose points were the distance BC from point D. However, these variations will all result in a triangle that is congruent to $\triangle DEF$. The following *congruence property of triangles* states that constructing a triangle with three sides that are congruent to three sides of another triangle results in two congruent triangles.

Side-Side-Side (SSS) Congruence Property If three sides of one triangle are congruent to three sides of another triangle, the two triangles are congruent.

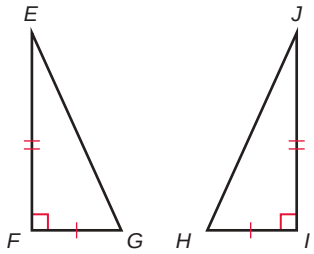
EXAMPLE E

Use the SSS congruence property to show that the following pairs of triangles are congruent. (*Note:* Small slash marks are used to denote congruent segments on geometric figures. For example, in figure (1), $\overline{AD} \cong \overline{CB}$ and $\overline{AB} \cong \overline{CD}$.)

(1)



(2)



Solution **1.** Since the third side of both triangles, $\overline{AC}$, is congruent to itself, the two triangles are congruent. **2.** Since the legs of the right triangles are congruent ($\overline{EF} \cong \overline{JI}$ and $\overline{FG} \cong \overline{IH}$), the Pythagorean theorem guarantees that hypotenuse $\overline{EG}$ is congruent to hypotenuse $\overline{JH}$. Therefore, the two triangles are congruent.

The fact that three line segments determine the *shape and size* of a triangle is of major importance in the construction of a wide range of objects, from bridges and buildings to playground equipment and furniture. Because of this fact, triangular supports are more rigid than supports having other polygonal shapes. This can be illustrated by linkages such as those shown in Figure 11.5 on the next page. The shapes of all these polygons can be changed, except for that of the triangle. For example, the pentagon can be made concave by pushing one of its vertex points toward the interior of the polygon, or the hexagon can be reshaped into a convex hexagon. The triangle is the only linkage whose shape cannot be changed.

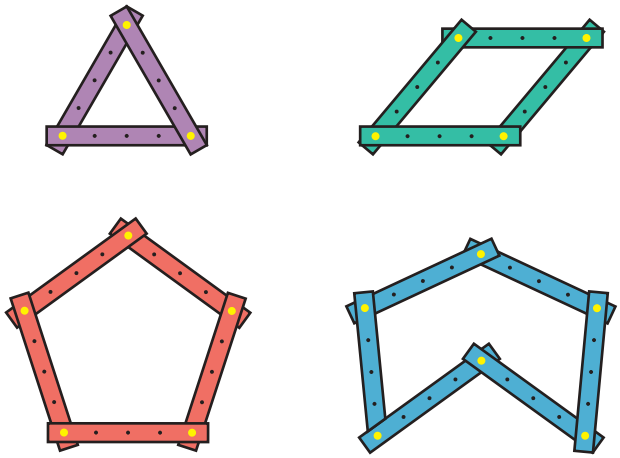


Figure 11.5

Notice the use of triangles for structural support in the cranes in Figure 11.6.



Figure 11.6
Construction showing cranes with triangles

Example D *should not* lead you to conclude that it is possible to construct a triangle whose sides are congruent to any three given line segments. Consider using the three line segments in part a of Figure 11.7. First, set the compass opening by placing it on points C and D, and then draw an arc on $\overline{AB}$ with center A, as shown in part b. Next, set the compass opening by placing it on points E and F, and then draw an arc on $\overline{AB}$ with center B. Since the arcs do not intersect, a triangle cannot be constructed from these three segments.

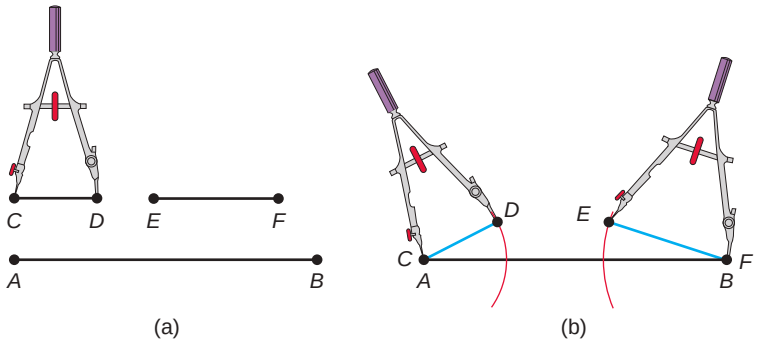


Figure 11.7

The preceding demonstration illustrates a property of triangles known as the **triangle inequality.**

Triangle Inequality The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

EXAMPLE F

Can a triangle be constructed whose sides are congruent to the given segments?

(1) a b c

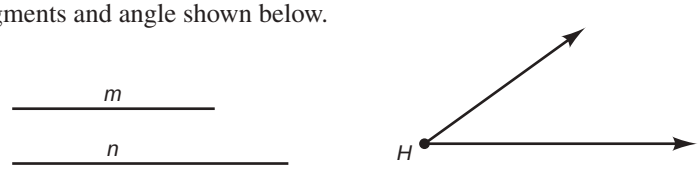
(2) d e f

Solution **1.** No, because $a + b < c$. Notice that $b + c > a$ and $c + a > b$, but these conditions are not sufficient for a triangle to be constructed. **2.** Yes, because the sum of the lengths of any pair of segments is greater than the length of the third segment.

We have seen that if three sides of one triangle are congruent to three sides of another, the triangles are congruent. That is, if three line segments can be used to form a triangle, they determine a unique triangle. Are there other conditions that can be used to determine a unique triangle? Example G focuses on two sides of a triangle and the **included angle,** that is, the angle formed by the two sides.

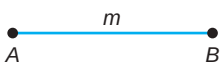
EXAMPLE G

Construct a triangle such that two of its sides and the included angle are congruent to the two line segments and angle shown below.

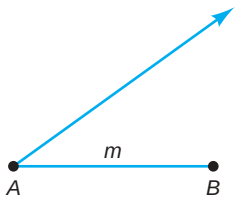


Original segments and angle

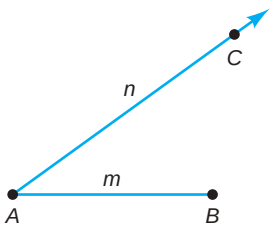
Solution **Step 1** Construct new line segment $\overline{AB}$ with length m.



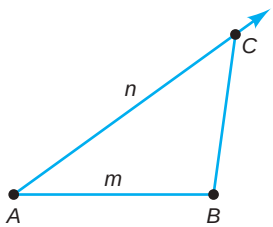
Step 2 Construct an angle congruent to the original $\angle H$ with A as a vertex.



Step 3 Use a compass to locate point C as shown so that $AC = n$.



Step 4 Draw segment $\overline{CB}$.



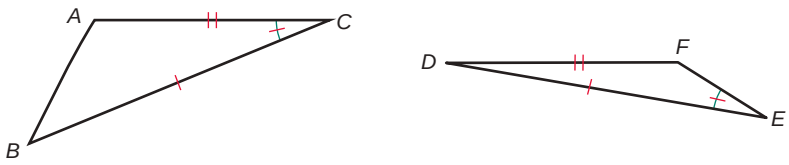
Since the locations of points C and B on the sides of the angle in steps 1 and 3 of Example G on the previous page are determined by the lengths n and m, and since points C and B determine a unique line segment $\overline{CB}$, it seems reasonable to expect that any triangle constructed by placing $\measuredangle H$ between the two given segments will be congruent to $\triangle ABC$. This result is summarized by the following *congruence property of triangles.*

Side-Angle-Side (SAS) Congruence Property If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, the two triangles are congruent.

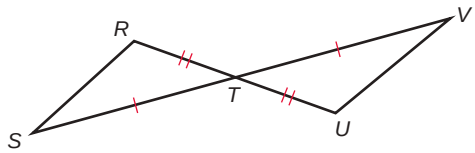
EXAMPLE H

Use the SAS congruence property to determine whether the following pairs of triangles are congruent. (*Note:* Small slash marks are used to denote congruent angles. For example, $\measuredangle C \cong \measuredangle E$.)

(1)



(2)



Solution **1.** The triangles are not congruent, because $\measuredangle C$ and $\measuredangle D$, which are included between the pairs of congruent sides, are not known to be congruent. **2.** The triangles are congruent, because $\measuredangle RTS$ and $\measuredangle UTV$, which are included between the pairs of congruent sides, are vertical angles, and vertical angles are congruent.

There is one other property of triangles that is useful for showing that two triangles are congruent. This property involves the **included side** of two angles, that is, the side that is common to two angles. For example, $\overline{AB}$ is the included side for $\measuredangle A$ and $\measuredangle B$ in Figure 11.8.

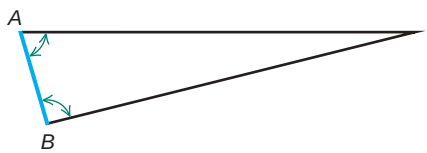


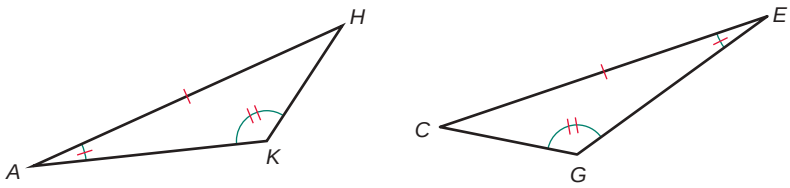
Figure 11.8

Angle-Side-Angle (ASA) Congruence Property If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, the two triangles are congruent.

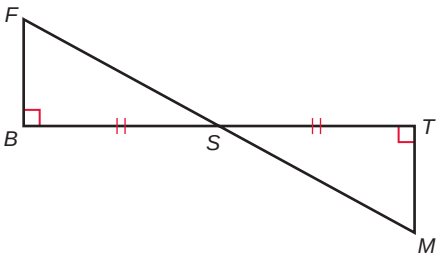
EXAMPLE I

Use the ASA congruence property to show that the following pairs of triangles are congruent. (Remember, slash marks are used to show congruent segments and congruent angles. So, $\overline{AH} \cong \overline{EC}$, $\measuredangle A \cong \measuredangle E$, and $\measuredangle K \cong \measuredangle G$.)

(1)



(2)



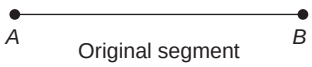
Solution **1.** Since there are 180° in every triangle, $\measuredangle H$ must be congruent to $\measuredangle C$. Therefore, $\triangle AKH \cong \triangle EGC$ by the ASA congruence property. **2.** Since vertical angles are congruent, $\triangle FSB \cong \triangle MST$ by the ASA congruence property.

CONSTRUCTING BISECTORS

Bisecting Segments In the paper-folding examples in Section 9.1, a line segment was bisected by folding the segment onto itself so that the endpoints coincided. In addition to locating the point that bisects the segment, the crease of the paper produces a line perpendicular to the given segment. A line that is perpendicular to a segment at its midpoint is called the **perpendicular bisector** of the segment.

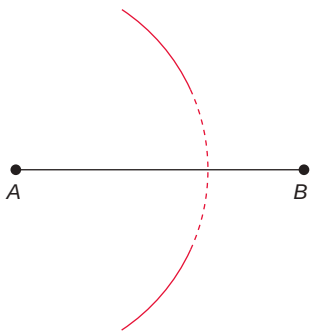
EXAMPLE J

Construct the perpendicular bisector of the line segment $\overline{AB}$.

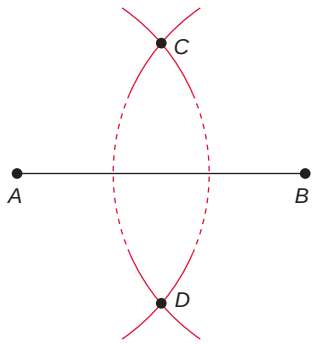


Original segment

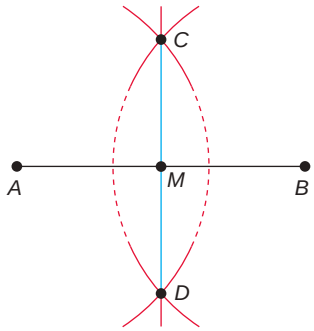
Solution **Step 1** Open the compass to span more than one-half the distance from A to B. Then, with one end of the compass at A, draw an arc that intersects $\overline{AB}$ as shown.



Step 2 With the same compass opening, place the end of the compass at B and draw arcs that intersect the arcs created in step 1. Label the points of intersection of the arcs as C and D.



Step 3 Use a straightedge to draw $\overleftrightarrow{CD}$, and label its intersection with $\overline{AB}$ as point M. Then $\overleftrightarrow{CD}$ is the perpendicular bisector of $\overline{AB}$, and M is the midpoint of $\overline{AB}$.

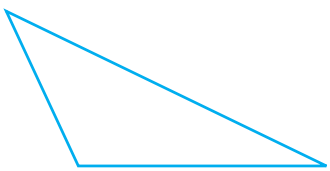


Justification: On the figure in step 3, draw segments $\overline{AC}$, $\overline{BC}$, $\overline{AD}$, and $\overline{BD}$ (see following figure). We know that these segments are all congruent because arcs of the same size were used to locate points C and D; that is, these segments are radii of circles that are the same size. So, by the SSS congruence property, $\triangle ACD \cong \triangle BCD$. Then $\measuredangle ACD \cong \measuredangle BCD$, because these angles are corresponding parts of congruent triangles. Thus, $\triangle ACM \cong \triangle BCM$ by the SAS congruence property. Finally, the parts of $\triangle ACM$ and $\triangle BCM$ correspond: $\overline{AM} \cong \overline{BM}$, so M is the midpoint of $\overline{AB}$; $\measuredangle CMA \cong \measuredangle CMB$, and since these angles are supplementary angles, each must be a right angle. So $\overline{CD}$ is the perpendicular bisector of $\overline{AB}$.

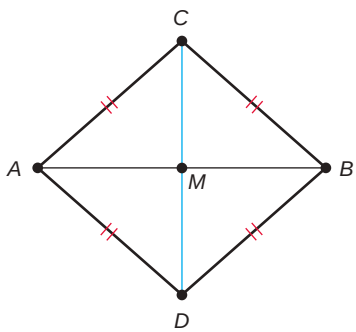
Laboratory Connection

Paper Folding

Using paper folding or other methods, obtain the perpendicular bisectors of the three sides of this obtuse triangle. What conjecture can you make about these lines and will it be true for an acute triangle? Explore this and related questions in using Geometer's Sketchpad® student modules available at the companion website.



Mathematics Investigation
Chapter 11, Section 1
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The perpendicular bisector in Example J was constructed by locating two points C and D that were equidistant from the endpoints of $\overline{AB}$. Since this distance was chosen arbitrarily, the justification for this construction proves that any point that is equidistant from the endpoints of a segment will be on the perpendicular bisector of the segment. Conversely, an arbitrary point P on the perpendicular bisector of $\overline{AB}$ is equidistant from the endpoints A and B. These facts are summarized in the following theorem.

Perpendicular Bisector Theorem A point is on the perpendicular bisector of a line segment if and only if it is equidistant from the endpoints of the segment.

Bisecting Angles Since $\measuredangle ABD$ in Figure 11.9 (see page 744) is congruent to $\measuredangle DBC$, the ray $\overrightarrow{BD}$ is called the **angle bisector** of $\measuredangle ABC$. If an angle is drawn on paper, the bisector of the angle can be formed by folding one side of the angle onto the other. The crease is the bisector of the angle. Try it.

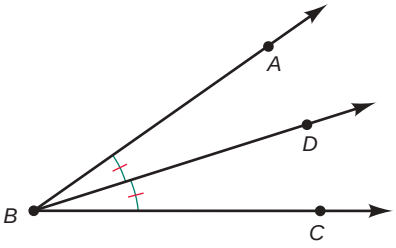
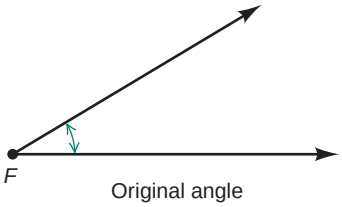
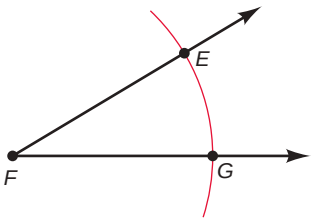


Figure 11.9

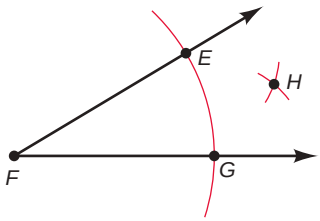
EXAMPLE K Construct a bisector for the angle shown here.



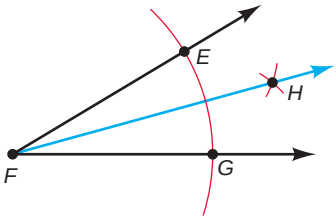
Solution **Step 1** Place the end of a compass at point F and draw an arc. Label the points where the arc intersects the sides of the angle as E and G.



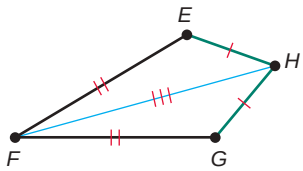
Step 2 Place the end of the compass at point E and draw an arc. Repeat this step, using point G and the same compass opening. Label the intersection of the two arcs as H.



Step 3 Draw $\overrightarrow{FH}$, which is the angle bisector of $\measuredangle EFG$.



Justification: On the figure in step 3, draw segments $\overline{EH}$ and $\overline{GH}$. Since $\overline{FE} \cong \overline{FG}$ (they were constructed with the same compass opening) and $\overline{EH} \cong \overline{GH}$ (they also were constructed with the same compass opening), $\triangle FEH \cong \triangle FGH$ by the SSS congruence property. Therefore, $\measuredangle EFH \cong \measuredangle GFH$, because these angles are corresponding parts of congruent triangles.



HISTORICAL HIGHLIGHT



Evariste Galois, 1811–1832

For 2000 years, beginning with the ancient Greeks, mathematicians sought to solve the following construction problems, using only a straightedge and compass.

1. **Squaring a circle:** Constructing a square whose area equals that of a given circle.
2. **Duplicating a cube:** Constructing a cube whose volume is twice that of a given cube.
3. **Trisecting an angle:** Constructing rays that trisect a given angle.

The results of algebraic developments by the young French mathematician Evariste Galois have been used to prove that these constructions cannot be done by using only a straightedge and compass. Galois died at the age of 20 in a duel, and it wasn't until after his death that his contributions to mathematics were recognized. One important branch of algebra currently bears his name, *Galois theory.**

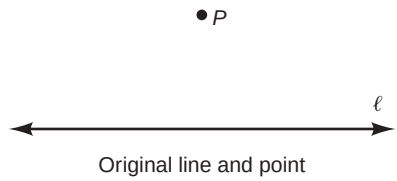
*E. T. Bell, *Men of Mathematics* (New York: Simon and Schuster, 1965), pp. 362–377.

CONSTRUCTING PERPENDICULAR AND PARALLEL LINES

Construction of a perpendicular bisector, as in Example J on page 742, accomplishes two purposes: It locates the midpoint of a segment, and it creates a right angle. Before you carry out the steps in Example L, think about how the steps in constructing the perpendicular bisector can be used to accomplish the construction.

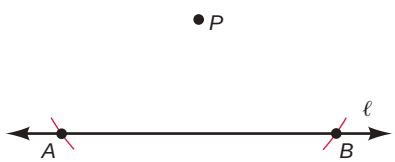
EXAMPLE L

Construct a perpendicular to a line through a point that is not on the line.

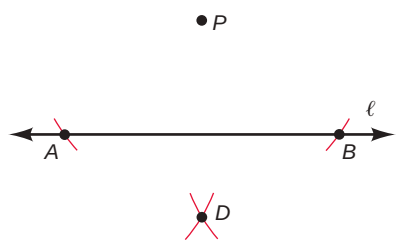


Original line and point

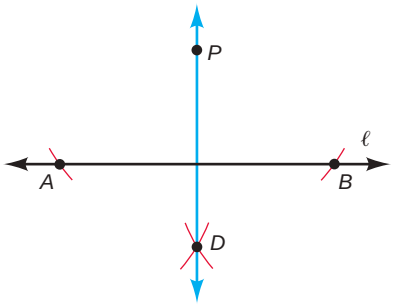
Solution **Step 1** Place one end of a compass at point P, and draw an arc that intersects line ℓ in two points. Label these points A and B.



Step 2 Place the compass at point A, and draw an arc in the half-plane not containing P. With the same compass opening, place the compass at point B and draw an arc in the same half-plane. Label the intersection of these arcs D.



Step 3 Use a straightedge to draw $\overleftrightarrow{PD}$. Line $\overleftrightarrow{PD}$ is perpendicular to line ℓ.



Justification: Since P is the same distance from A as from B (they were constructed with the same compass opening), we know by the perpendicular bisector theorem (page 743) that P is on the perpendicular bisector of $\overline{AB}$. Also, D is the same distance from A and B, so it is on the perpendicular bisector of $\overline{AB}$. Since the two points P and D determine a line, line $\overleftrightarrow{PD}$ is the perpendicular bisector of $\overline{AB}$.

EXAMPLE M

Construct a line parallel to a given line ℓ and through a point K that is not on ℓ.



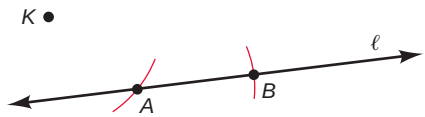
Original line and point

Solution

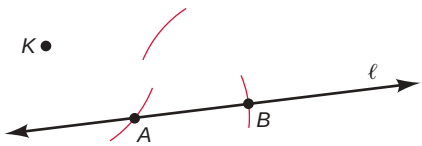
Step 1 With one end of a compass on point K, draw an arc that intersects ℓ at point A.



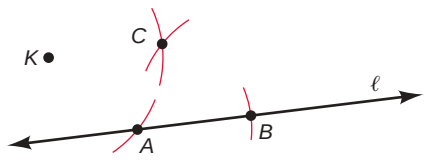
Step 2 With the same compass opening and A as center, draw an arc that intersects ℓ at point B.



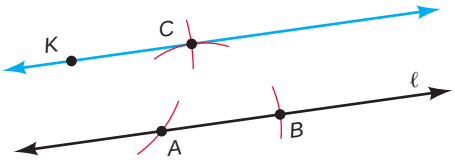
Step 3 With the same compass opening and B as center, draw an arc in the same half-plane as K.



Step 4 With the same compass opening and K as center, draw an arc that intersects the arc drawn in step 3. Label the intersection of the arcs C.



Step 5 Draw line $\overleftrightarrow{KC}$. This line is parallel to ℓ.



Justification: On the figure in step 5, draw segments $\overline{KA}$, $\overline{CB}$, and $\overline{KB}$. Since $\overline{KA}$, $\overline{AB}$, $\overline{BC}$, and $\overline{KC}$ were constructed as congruent segments, $\triangle KAB \cong \triangle BCK$ by the SSS congruence property. Therefore, $\measuredangle KBA \cong \measuredangle BKC$. Thus, since $\measuredangle KBA$ and $\measuredangle BKC$ are congruent alternate interior angles ($\overleftrightarrow{KB}$ is a transversal intersecting $\overleftrightarrow{KC}$ and $\overleftrightarrow{AB}$), line $\overleftrightarrow{KC}$ is parallel to line $\overleftrightarrow{AB}$.

CIRCUMSCRIBING CIRCLES ABOUT TRIANGLES

Section 9.2 illustrated two methods of constructing regular polygons. One method involved marking off equal lengths on a circle to obtain an **inscribed polygon** (a polygon whose vertices are points of the circle). The circle for an inscribed polygon is called a **circumscribed circle.** A circumscribed circle for a pentagon is shown at the left in Figure 11.10. The sides of the pentagon, $\overline{AB}$, $\overline{BC}$, etc., are chords of the circle.

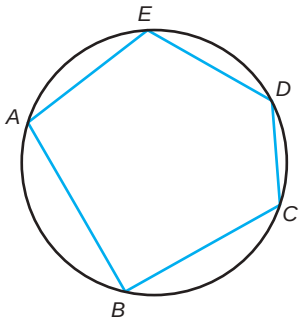


Figure 11.10

Not all polygons will have a circumscribed circle. However, a circumscribed circle can be constructed for any triangle. Consider the triangle in Figure 11.11.

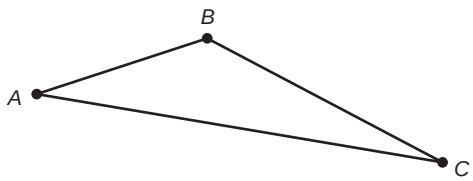


Figure 11.11

If points A and B are to be on a circle with center O, then $\overline{OA}$ must have the same length as $\overline{OB}$ (see Figure 11.12). Since the length of a line segment can be indicated by removing the bar above the letters, we can write $OA = OB$. Thus, by the Perpendicular Bisector Theorem, on page 743, the center of the circumscribed circle must be on the perpendicular bisector of $\overline{AB}$. Similarly, if points B and C are on the circle, then $OB = OC$, and the center of the circle is also on the perpendicular bisector of $\overline{BC}$. So, the center of the circle containing A, B, and C can be located by constructing the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$, as shown in Figure 11.12. The intersection of the perpendicular bisectors is the center of the circumscribed circle about $\triangle ABC$.

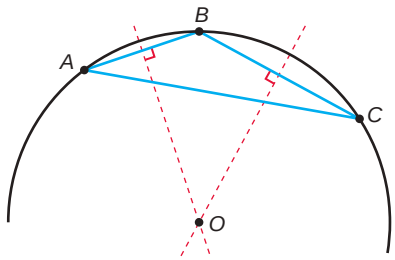


Figure 11.12

The preceding construction can be checked by using a compass to draw a circle with center O and radius $\overline{OA}$, to see if points B and C lie on the circle, or by constructing the perpendicular bisector of the third chord $\overline{AC}$ to determine if it passes through point O. Notice that we now have a method for locating the center of any circle if three points of the circle are given: *The center of a circle is the intersection of the perpendicular bisectors of two chords of the circle.*

CONSTRUCTIONS WITH A MIRA

The Mira is made of red Plexiglas, which reflects images. It is transparent so that the person looking through the front of the Mira can see the reflected image that can be traced on the paper by reaching behind the Mira. Figure 11.13, on the next page, shows the sketch of

$\triangle A'B'C'$, which is congruent to $\triangle ABC$. To obtain $\triangle A'B'C'$, the person looks through the front of the Mira while reaching behind the Mira to mark the images of points A, B, C and then connects these points by drawing along a straightedge or the edge of a Mira. (*Note:* The Mira has a beveled edge that should be placed on the paper, facing the viewer.)

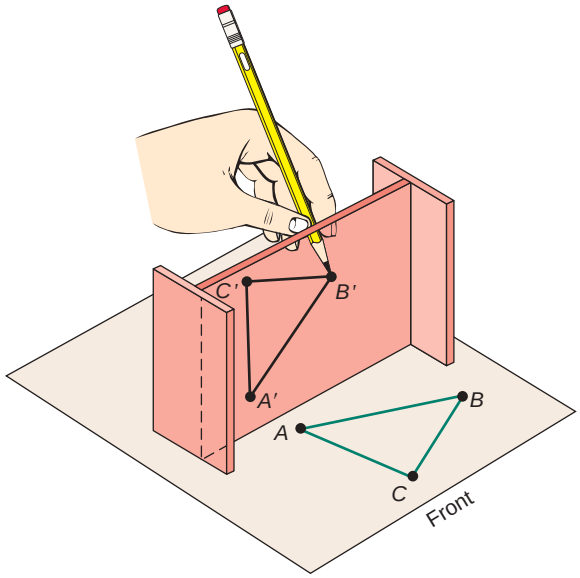


Figure 11.13

Bisecting Segments and Angles with a Mira To bisect $\overline{AB}$ in Figure 11.14a, place the Mira across $\overline{AB}$ (see dotted line) so that the image of point A is point B and draw along the beveled edge. This will produce line ℓ, that is the perpendicular bisector of $\overline{AB}$.

To bisect $\measuredangle CDE$ in Figure 11.14b, place the beveled edge of the Mira between the two rays of the angle (see dotted line) so that the image of $\overleftrightarrow{DE}$ is $\overleftrightarrow{DC}$. Drawing along the beveled edge will produce ray k, which bisects $\measuredangle CDE$.

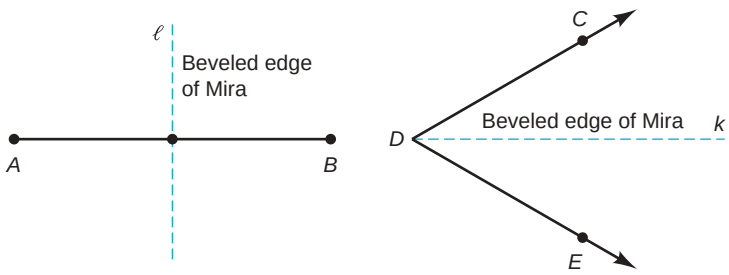


Figure 11.14

Constructing Perpendicular and Parallel Lines with a Mira To construct a line perpendicular to line ℓ at point P in Figure 11.15a, on the next page, place the beveled edge of the Mira on P and across ℓ (see dotted line) so that the image of the half-line in front of the Mira falls on the half-line behind the Mira. Drawing along the beveled edge will produce the line perpendicular to ℓ at point P.

To construct a line through point Q that is parallel to line k in Figure 11.15b, place the beveled edge of the Mira perpendicular to line k (see dotted line), and mark the image of Q as Q'. Then use a straightedge or the edge of a Mira to draw QQ', which is parallel to line k.

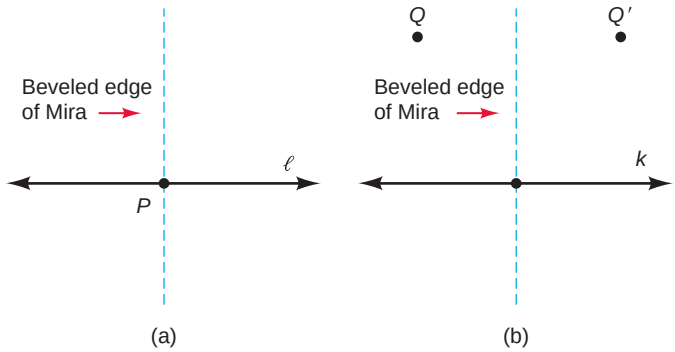
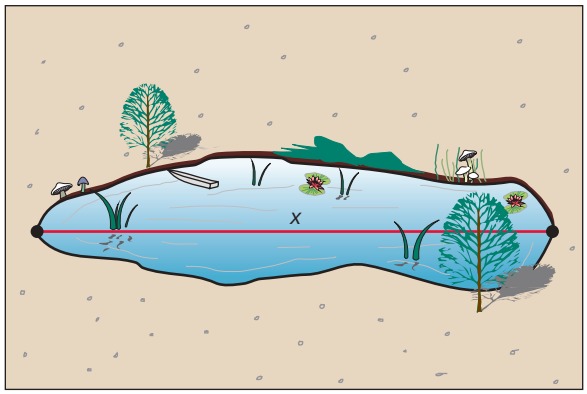


Figure 11.15

PROBLEM-SOLVING APPLICATION

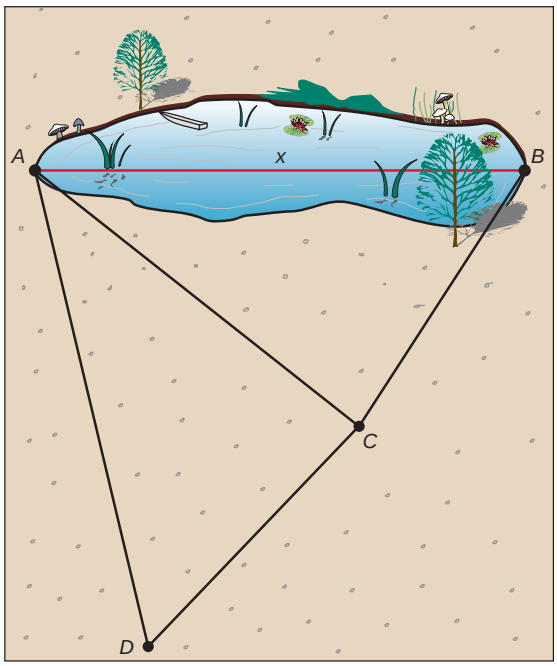
Problem

A math club is spending the day at a pond (see the figure here), and the question of the length of the pond arises. One of the club members claims the length can be found by using congruent triangles. How can this be done?



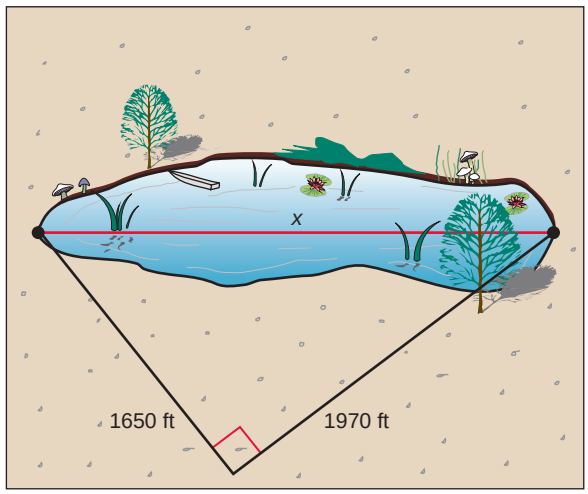
Understanding the Problem The problem is to determine the distance x by using congruent triangles.

Devising a Plan One possibility is to select a point C so that points A, B, and C form a triangle in which $\measuredangle ACB$ and side $\overline{BC}$ can be measured (see the next figure). Then select a point D so that $\overline{CD} \cong \overline{CB}$ and $\measuredangle ACD \cong \measuredangle ACB$. **Question 1:** Why is $\triangle ACD \cong \triangle ACB$?



Carrying Out the Plan Suppose you find that the sides of $\triangle ACD$ have the following lengths: $AC = 2400$ feet, $CD = 1630$ feet, and $AD = 2570$ feet. Because of what we know about corresponding parts of congruent triangles, we know that one of these is the length AB. **Question 2:** Which one?

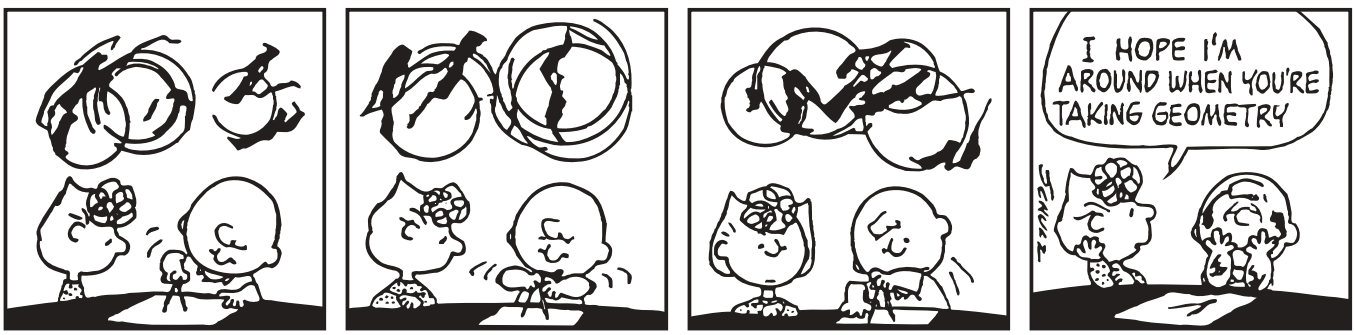
Looking Back The length of the pond can also be found by using a right triangle, as shown in the next figure. **Question 3:** If the triangle has legs of 1650 and 1970 feet, what is the length of the pond?



Answers to Questions 1–3 **1.** Since $\overline{AC}$ is congruent to itself, $\triangle ACB \cong \triangle ACD$ by the SAS congruence property. **2.** Since $\overline{AD}$ corresponds to $\overline{AB}$, $AB = 2570$ feet. **3.** By the Pythagorean theorem,

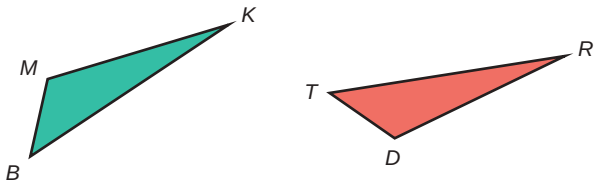
$$\begin{aligned} 1650^2 + 1970^2 &= x^2 \\ 2{,}722{,}500 + 3{,}880{,}900 &= x^2 \\ 6{,}603{,}400 &= x^2 \\ 2570 &\approx x \end{aligned}$$

Exercises and Problems 11.1



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1. The following two triangles are congruent. Determine the corresponding angles and sides in a–f for the congruence, and complete the congruence statement in g.



a. $\measuredangle B \leftrightarrow$ _____ **b.** $\measuredangle M \leftrightarrow$ _____

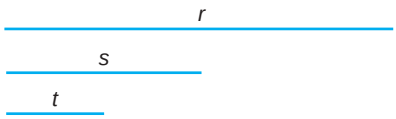
c. $\measuredangle K \leftrightarrow$ _____ **d.** $\overline{MB} \leftrightarrow$ _____

e. $\overline{BK} \leftrightarrow$ _____ **f.** $\overline{MK} \leftrightarrow$ _____

g. Complete the statement: $\triangle BMK \cong \triangle$ _____.

2. If $\triangle ABC \cong \triangle DEF$, list the three pairs of corresponding congruent sides and the three pairs of corresponding congruent angles.

The ancient Greeks represented numbers by the lengths of line segments. Addition was represented by placing two line segments end to end and subtraction by comparing the difference between the lengths of two line segments. Use the following segments and a straightedge and compass to construct line segments having the lengths specified in exercises 3 and 4.

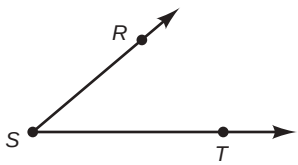


3. a. $r + s$ **b.** $r - t$ **c.** $r + (s - t)$

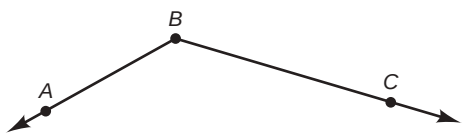
4. a. $2r - s$ **b.** $(r - s) + t$ **c.** $3t + s$

Show and explain how a straightedge and compass can be used to construct angles that are congruent to the angles in exercises 5 and 6.

5.



6.

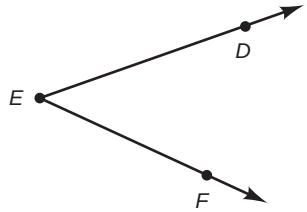


Trace each figure in exercises 7 through 11 on a separate sheet of paper. Then show and explain how a straightedge and compass can be used to obtain the construction listed.

7. The perpendicular bisector of $\overline{AB}$.



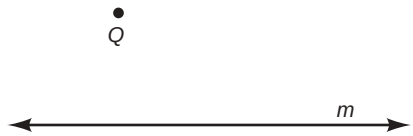
8. The bisector of $\measuredangle DEF$.



9. A line through K that is perpendicular to line n.



10. A line through Q that is parallel to line m.



11. A line through S that is perpendicular to $\overline{RS}$.

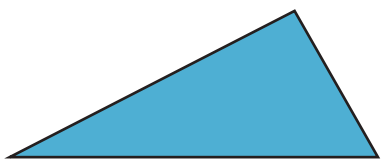


12. Draw a sketch to show how the Mira can be placed to obtain the constructions in exercises 7 and 8. Explain your reasoning.

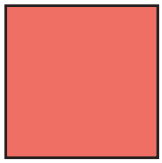
13. Draw a sketch to show how the Mira can be placed to obtain the constructions in exercises 9 and 10. Explain your reasoning.

Trace the figures in exercises 14 and 15 on a piece of paper and construct a circumscribed circle for each figure. Explain the steps in each construction.

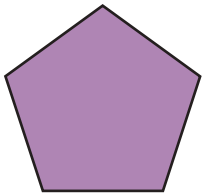
14. a. Scalene triangle



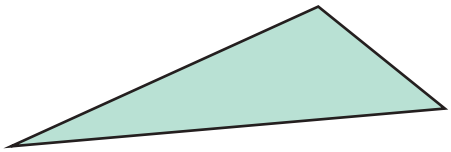
b. Square



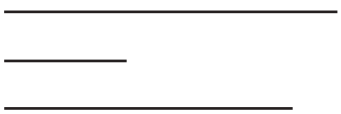
15. a. Regular pentagon



b. Scalene triangle



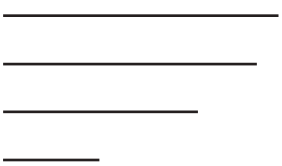
16. a. Construct a triangle whose sides are congruent to these line segments.



b. Using the line segments in part a, can you construct another triangle that is not congruent to your first triangle? Explain why or why not.

17. The importance of triangles to architecture is due to a basic mathematical fact that is not true of polygons with more than three sides.

a. Construct two noncongruent quadrilaterals whose sides are congruent to the line segments shown here.



b. How many noncongruent quadrilaterals can be constructed whose sides are congruent to the four line segments used in part a? (*Hint:* Imagine changing the shape of a quadrilateral formed by linkages.)

If a construction is possible, construct triangles in exercises 18 and 19 with the given characteristics. (Use a ruler, compass, or protractor as needed.)

18. a. Three sides of lengths 5, 6, and 7 centimeters

b. Three sides of lengths 5, 6, and 10 centimeters

c. Three sides that are congruent to $\overline{AB}$ shown here

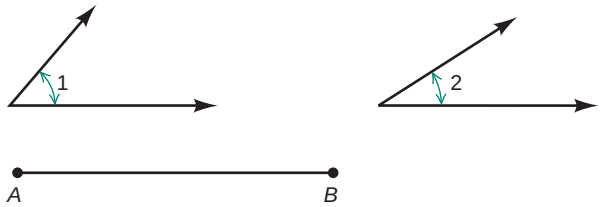


19. a. Two sides of lengths 2 and 3 centimeters and a nonincluded angle with a measure of 60°

b. Two sides of length 5 centimeters and an included angle with a measure of 45°

c. A right triangle with one leg of length 7 centimeters and one angle of measure 35°

20. In parts a and b, construct a triangle in which one side and two angles are congruent to the line segment and angles below.



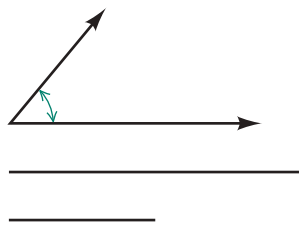
a. Construct the triangle so that the given side is included between the two given angles.

b. Construct the triangle so that the given side is not included between the two given angles.

c. Compare the two triangles in parts a and b. Are they congruent?

d. What conclusion can you draw from your answer in part c?

21. In parts a and b, construct triangles that have one angle and two sides congruent to the angle and segments shown below.



a. Construct the triangle so that the given angle is included between the two given sides.

b. Construct the triangle so that the given angle is not included between the two given sides.

c. Compare the triangles in parts a and b. Are they congruent?

d. What conclusion can you draw from part c?

For each case in exercises 22 and 23, determine if the given conditions are sufficient to conclude that $\triangle ABC$ is congruent to $\triangle HMS$. Draw diagrams and justify your answers.

22. a. $\overline{AB} \cong \overline{HM}$, $\overline{BC} \cong \overline{MS}$, $\overline{AC} \cong \overline{HS}$

b. $\measuredangle A \cong \measuredangle H$, $\measuredangle B \cong \measuredangle M$, $\measuredangle C \cong \measuredangle S$

23. a. $\overline{AB} \cong \overline{HM}$, $\overline{BC} \cong \overline{MS}$, $\measuredangle B \cong \measuredangle M$

b. $\overline{AB} \cong \overline{HM}$, $\overline{BC} \cong \overline{MS}$, $\measuredangle A \cong \measuredangle H$

Use a compass and straightedge to construct each of the polygons in exercises 24 through 26 having sides congruent to segment $\overline{AB}$. Explain the steps of each construction.



24. Square

25. Equilateral triangle

26. Nonsquare rhombus

27. There are an infinite number of line segments that can be drawn to a line from a point that is not on the line.

a. Trace line ℓ and point P on a sheet of paper. Use a ruler and compass to locate points on line ℓ that are the following distances from point P: 3, 2, and 1.5 centimeters.

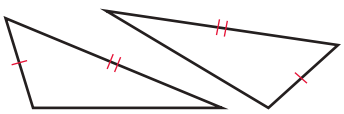


b. To the nearest .1 centimeter, what is the shortest distance from P to ℓ?

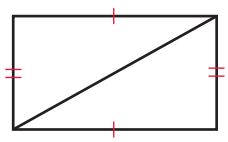
c. Form a conjecture about the shortest distance to a line from a point that is not on the line.

Each pair of triangles in exercises 28 through 33 has corresponding parts (marked by small red slashes) that are congruent. Determine whether or not the given information is sufficient to conclude that the triangles are congruent by the SSS, SAS, or ASA congruence properties. If congruent, state the property that shows the triangles are congruent. If not necessarily congruent, explain why not.

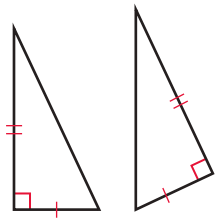
28. a.



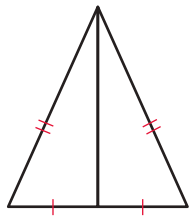
b.



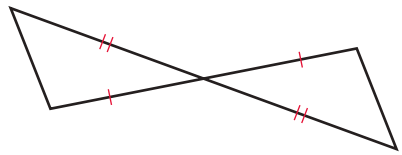
29. a.



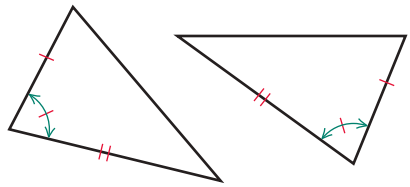
b.



30. a.



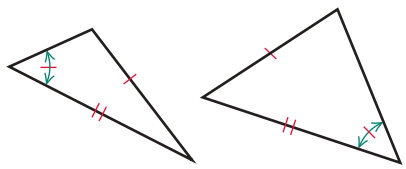
b.



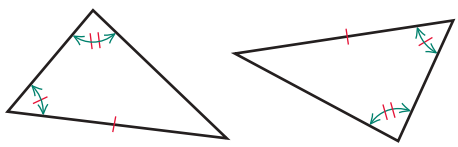
31. a.



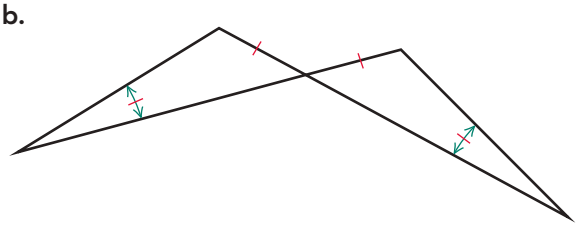
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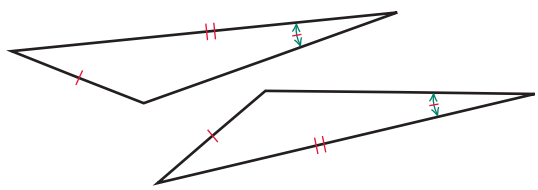
32. a.



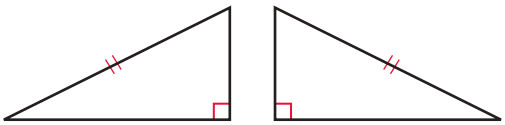
b.



33. a.



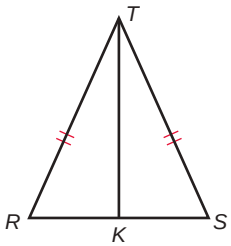
b.



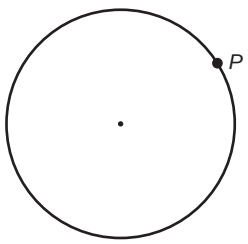
34. Construct an isosceles triangle with two sides congruent to $\overline{AB}$ and one side congruent to $\overline{CD}$. Form a conjecture about the angles that are opposite the two congruent sides of an isosceles triangle.



35. In the isosceles triangle below, point K is the midpoint of $\overline{RS}$. What congruence property of triangles can be used to show that $\triangle RKT \cong \triangle SKT$? How can this congruence be used to show that $\measuredangle R \cong \measuredangle S$?



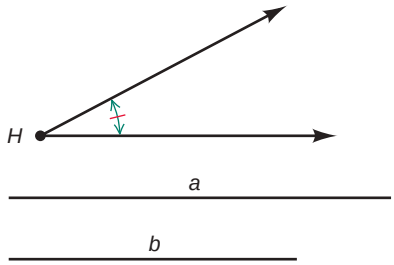
36. A **tangent** to a point P on a circle is a line through P that is perpendicular to the radius from the center of the circle to point P. Construct a circle and label a point P on its circumference. Using only a straightedge and compass, construct the tangent to the circle at point P. Show and explain the steps of your construction.



Reasoning and Problem Solving

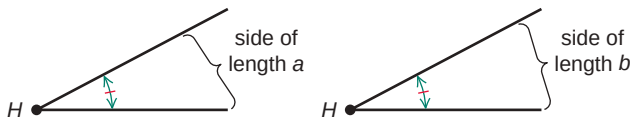
37. It has been proven that it is impossible to trisect every angle by using only a straightedge and compass (see Historical Highlight, page 745). However, trisections of certain angles can be constructed. Diagram and list the steps in trisecting a right angle, using only a straightedge and compass.

38. Featured Strategy: Making a Drawing. How many different noncongruent triangles can be formed that have two sides and one angle congruent to the line segments and angle here?



a. Understanding the Problem. The third side of the triangle can be any length needed to form a triangle with the given two sides and angle. Construct a triangle with two sides and the included angle congruent to the given segments and angle. If another triangle is constructed using the given segments and the given angle as the included angle, will it be congruent to the triangle you constructed?

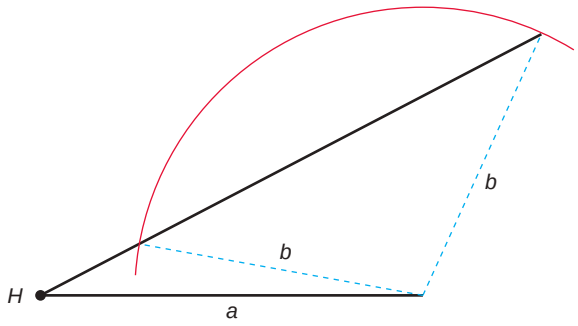
b. Devising a Plan. One approach is to *make drawings* that will help you consider different combinations of the given segments and angle systematically. For example, suppose the segment of length b is opposite $\measuredangle H$ in one triangle and the segment of length a is opposite $\measuredangle H$ in the second triangle. Are these two triangles congruent?



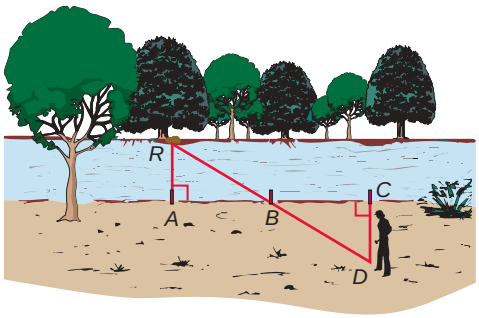
c. Carrying Out the Plan. Construct triangles, using different configurations of the given line segments and angle, and determine if they are congruent. How many noncongruent triangles can be constructed?

d. Looking Back. Draw a segment of length a, and place the vertex of $\measuredangle H$ at one endpoint, as shown in the figure at the top of the next page. With the point of the compass on the other endpoint of the line segment, draw an arc of length b. This arc intersects two points on one side of $\measuredangle H$, and these points determine two noncongruent triangles, as shown. Suppose a line segment of length b is drawn and $\measuredangle H$ is placed at one endpoint. Will

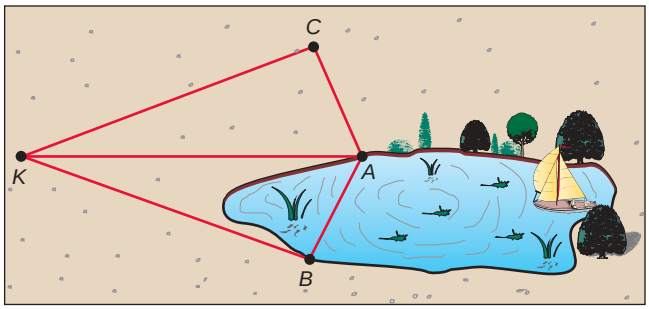
drawing an arc of length a determine two noncongruent triangles?



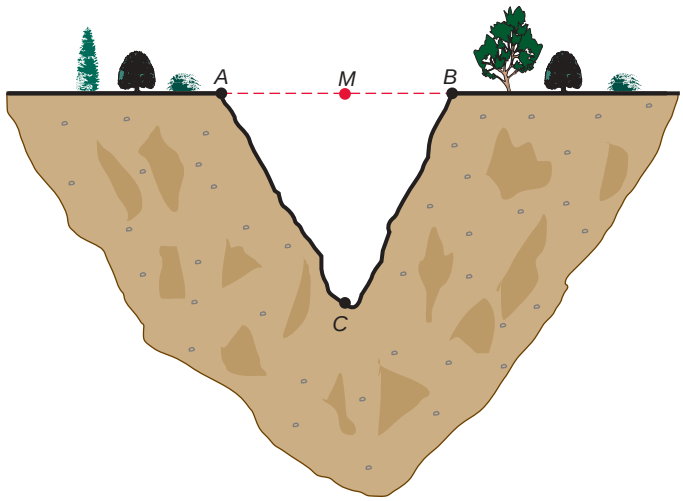
39. To measure the width of a river, a ranger stands across the river directly opposite rock R and places a stake at point A on the river's edge. Then she measures off equal distances and places stakes at points B and C so that $AB = BC$. Finally, the ranger moves directly away from the river to a point D, at which stake B and rock R are in a straight line. The diagram illustrates this information. How can this information be used to measure the width of the river?



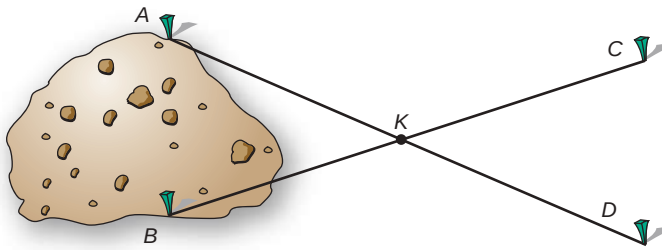
40. Jeff and Vonda want to swim from point A to point B on a lake, and they are concerned about the distance (see figure below). Vonda suggests laying out triangles ABK and ACK such that $\measuredangle CKA$ is congruent to $\measuredangle BKA$. They pace off $\overline{KC}$ and $\overline{KB}$ to be 380 feet each, and they find that $\overline{CA}$ is 205 feet. Explain how this information can be used to find the distance from A to B. What is this distance?



41. An outing club wants to build a rope footbridge across a deep gorge and needs to determine the distance from point A to point B (see the figure). They stretch a line from A to C and from B to C and find that each distance equals 120 feet. They use an altimeter to find that A and B have the same altitudes and that MC equals 96 feet with point M directly above point C. How can they use this information to find the distance from A to B? What is this distance?



42. There is a mound of dirt and rocks behind the Palumbo house that is to be removed. The Palumbos must supply measurements to determine the number of truckloads needed to remove the mound. Ms. Palumbo puts a stake at point A (see the figure) and moves in a straight line from A through point K, putting a stake at point D so that $AK = KD$. Then she places a stake at point B and moves in a straight line from B through K, placing a stake at point C so that $BK = KC$. Explain how she can find the distance at the base of the mound from A to B. Include any additional information she may need.



Teaching Questions

1. It is common knowledge from geometry that two triangles are congruent if they satisfy the SSS, SAS, or ASA property. One student suggested another property for congruence, the OSS property. That is, if an obtuse angle, the side opposite the obtuse angle, and one other side of one triangle are congruent to an obtuse angle, the side opposite the obtuse angle, and one other side of another triangle, the two triangles are congruent. Draw some sketches to see whether or not this conjecture seems to be true. If not, find a counterexample. Include sketches with your conclusions.
2. If a line is drawn on paper, other lines that are perpendicular or parallel to the original line can be formed by paper folding. Explain how you can illustrate each of the following by paper folding: perpendicular bisector of a segment, angle bisector, parallel lines, right angle, acute angle, and obtuse angle.
3. A student suggests that if you know four sides and one angle of a quadrilateral, that is enough to determine a unique quadrilateral. Draw some sketches to see whether or not this conjecture seems to be true. If not, find a counterexample. Include sketches with your conclusions.
4. Write a paragraph to explain how you would respond to a student who asked you why it is important to use a straightedge and compass to learn how to construct triangles given the sides and angles.

Classroom Connections

1. Read the **Standards** quote on page 735. Devise an activity with a Mira appropriate for students in the third to the fifth grade and explain how successful completion of your activity meets both the "thinking" and "doing" components of the quotation.
2. The **Spotlight on Teaching** at the beginning of this chapter describes a geometric activity using pentominoes. Do the same activity with six squares to show examples of hexominoes and nonhexominoes. Find some of the hexominoes and record them on grid paper. When finished, explain what strategies you used for finding the hexominoes and what techniques you used to ensure that each of your hexominoes is different from the others.
3. The **Cartoon** at the beginning of this exercise set indicates that Charlie Brown does not yet have the motor skills to use a compass to draw figures. Look through an elementary math textbook series (grades 1 to 6) and summarize the types of geometry activities that are done at each grade level and which manipulatives they suggest using. Is the use of a compass or ruler for constructions suggested at any of these levels?
4. Examine the **PreK–2, Grades 3–5,** and **Grades 6–8 Standards—Geometry** (see inside front and back covers) and list the levels and expectations that involve the notion of congruence. Also list any reference to geometric constructions.

MATH ACTIVITY 11.2

Rotating, Reflecting, and Translating Figures on Grids

Virtual Manipulatives



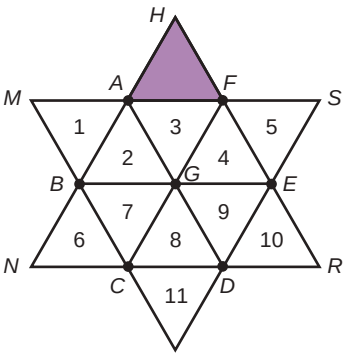
www.mhhe.com/bbn

Purpose: Visualize rotations, reflections, and translations of plane figures on grids.

Materials: Tracing paper (suggested) and Pattern Blocks in the Manipulative Kit or Pattern Block Virtual Manipulatives.

***1.** The shaded triangle shown below on the left can be turned (*rotated*) about either point F or point G so that it coincides with (fits exactly on top of) triangle 5. It can also be rotated about the lettered points so that it coincides with all the other numbered triangles except one. Record the letters of the points about which the shaded triangle can be rotated to coincide with these other triangles.

Triangle	1	2	3	4	5	6	7	8	9	10	11
Center of rotation					F, G						

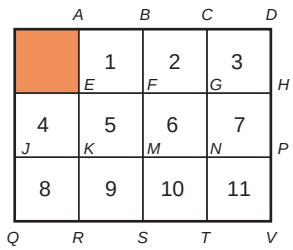


2. The shaded triangle at the left can be flipped (*reflected*) about line $\overleftrightarrow{AG}$ (think of extending $\overline{AG}$) to coincide with triangle 1. Find the seven other triangles to which the shaded triangle can be reflected. Record each triangle that the shaded triangle can be reflected onto, and the line about which it is reflected.

Triangle	1	2	3	4	5	6	7	8	9	10	11
Line of reflection	$\overleftrightarrow{AG}$										

3. The shaded triangle at the left can be slid without any turning motion (*translated*) to coincide with triangle 2. This can be visualized by imagining point H sliding down line $\overleftrightarrow{HB}$ to point A, point F sliding down line $\overleftrightarrow{FG}$ to coincide with point G, and point A sliding down line $\overleftrightarrow{AB}$ to point B. Find the four other triangles that the shaded triangle can be translated onto.

***4.** In activity 1, the shaded triangle could not be rotated onto triangle 8. However, it can be rotated about point G onto triangle 11 and then reflected about line $\overleftrightarrow{CD}$ to coincide with triangle 8. Find three other triangles that the shaded triangle can be rotated to and then reflected onto triangle 8.

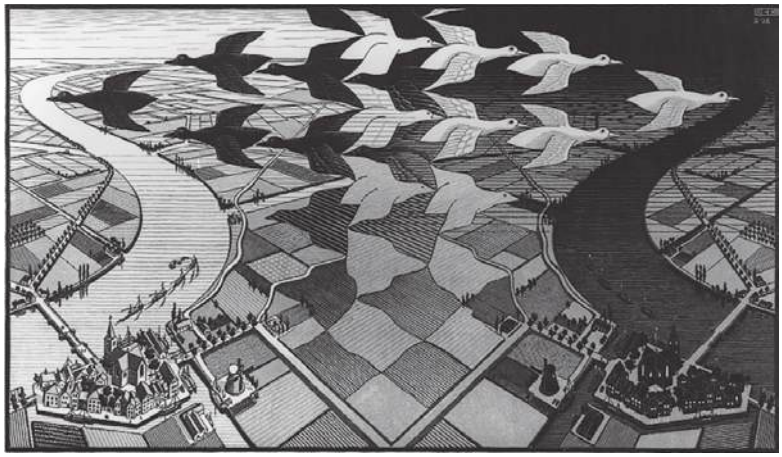


5. a. The shaded square shown at the left can be rotated about a lettered point to coincide with eight of the 11 numbered squares. Find each square to which it can be rotated and the point about which it is rotated.

b. The shaded square can be reflected onto five of the numbered squares by reflections about lines through the lettered points. Find each of these squares and the lines about which the shaded square can be reflected.

c. The shaded square cannot be rotated onto square 2 by a rotation about a lettered point, but it can be reflected onto square 1 and then rotated onto square 2. Describe how the shaded square can be reflected and then rotated onto squares 7, 8, 10, and 11. List the lines of reflection and points of rotation.

Section 11.2 CONGRUENCE MAPPINGS

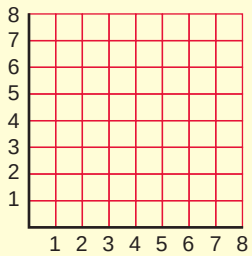


M.C. Escher's "Day and Night"

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PROBLEM OPENER

A hidden rectangle whose area is 12 square units is formed on the grid shown here. The coordinates of each vertex of the rectangle are whole numbers whose sum is either 8 or 12, and the first coordinate is not 8. What is the location of this rectangle?



NCTM Standards

Students in grades 3–5 should consider three important kinds of transformations: reflections, translations, and rotations (flips, slides, and turns). p. 167

In designing the woodcut *Day and Night,* M.C. Escher used geometric mappings. The particular mappings associated with congruence—translations, reflections, and rotations—will be studied in this section, and examples of how Escher used these mappings to create tessellations will be introduced.

TRANSLATIONS

A **translation** is a special kind of mapping that can be described as a sliding motion. Each point is moved the same distance and in the same direction. The translation in Figure 11.16, on the next page, maps A to A', B to B', C to C', $\overline{BC}$ to $\overline{B'C'}$, and pentagon K to pentagon K'. This translation is completely determined by point A and its image A'. That is, given any point X in this figure, we can find its image X' by moving in the *same direction* as from A to A' and the *same distance* as AA'. Or, stated another way, X' can be located by constructing $\overline{XX'}$ so that it is parallel to $\overline{AA'}$ and the distance from X to X' equals the distance from A to A'.

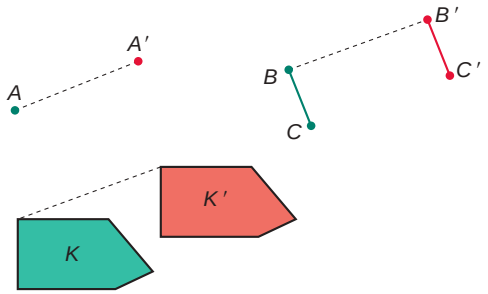


Figure 11.16

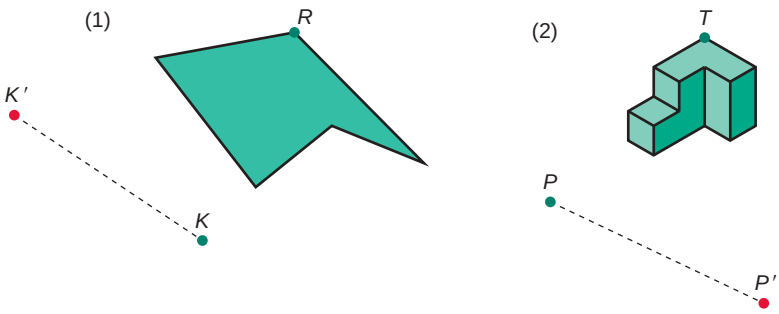
Translations occur with space figures as well as with plane figures. Just as in the case of two-dimensional figures, a translation in three dimensions is described as a sliding motion of points in space in the same direction and for the same distance. For example, a box moving on a conveyor belt or a child going down a slide. The photograph in Figure 11.17 shows the results of a sliding motion of the Earth's crust, which geologists call a *block fault.* The arrow points to one side of the fault along which the Earth's crust has been displaced.



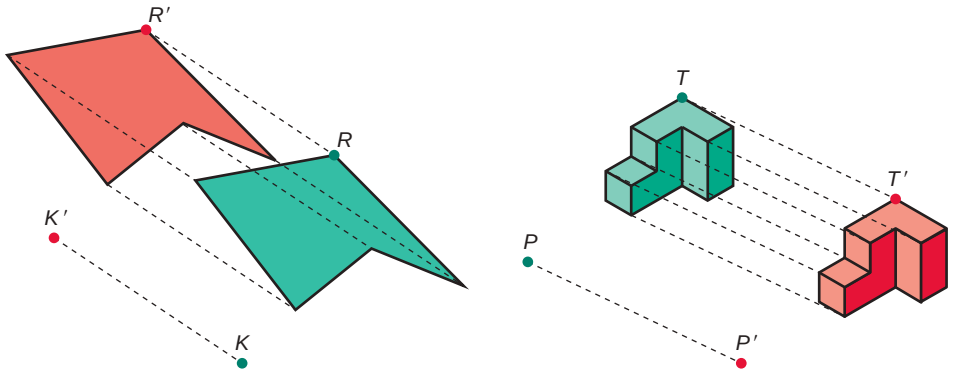
Figure 11.17
Fault line caused by an earthquake along the Motagua Fault in Guatemala showing sliding (shearing) motion of Earth's surface. Aftershocks continuing for several months increased the total horizontal displacement to approximately 2.8 feet.

EXAMPLE A

Trace figure (1) and determine the image of the pentagon obtained by a translation that maps K to K'. Trace figure (2) and determine the image of the three-dimensional figure obtained from a translation that maps P to P'.



Solution One method of locating the images of these figures is to use constructions to first find the images of the vertex points. The images of the vertex points can then be connected to obtain the image of the figure. For example, using vertex point R on the pentagon, construct a line through R that is parallel to $\overline{K'K}$. Then, moving in the direction from K to K', locate the image R' of R so that $RR' = KK'$. Similar constructions can be carried out for the remaining vertex points as shown.



Another method of locating the images is to trace the pentagon and the three-dimensional figure on separate sheets of paper. Then slide K along $\overline{KK'}$ so that it coincides with K' to locate the image of the pentagon, and slide P along $\overline{PP'}$ so that it coincides with P' to locate the image of the three-dimensional figure.

Research Statement

The 7th national mathematics assessment found that less than one-half of the eighth-graders could identify the image of a point when folded over an oblique line of reflection.

Martin and Strutchens

REFLECTIONS

A **reflection** about a line is a mapping that can be described by folding. If this page is folded about line ℓ in Figure 11.18, each point will coincide with its image. Point E will be mapped to E', F to F', $\overline{EF}$ to $\overline{E'F'}$, and figure M to figure M'. Line ℓ is called a **line of reflection.** Since point S is on ℓ, it does not move for this mapping. Point S and all other points on ℓ are called **fixed points** for the reflection about ℓ, because each point coincides with (is the same as) its image.

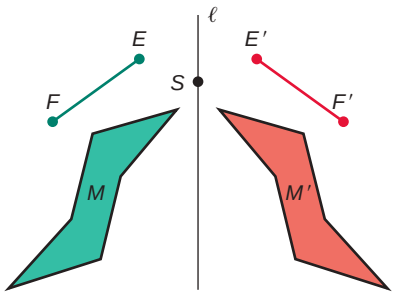


Figure 11.18

Reflections in space take place about planes. Each point to the left of plane P in Figure 11.19 on the next page has a unique image on the right side of P. The sphere is mapped to the sphere, point K to K', and tetrahedron T to tetrahedron T'. Point R and all other points on the plane are fixed points of the mapping. That is, each point on the plane is its own image.

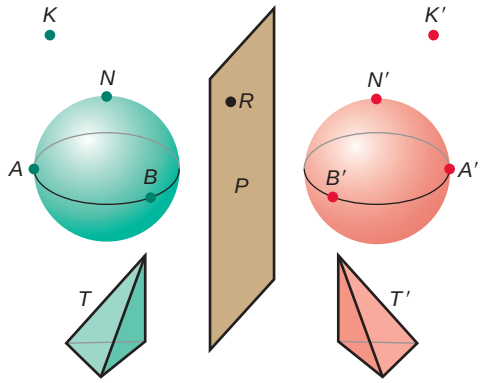


Figure 11.19

Reflections in space can be illustrated by mirrors. If plane P in Figure 11.19 is replaced by a mirror so that the figures to the left of the mirror are reflected, their images will appear to be in the positions of the figures on the right side of the mirror. Plane P is called the **plane of reflection.**

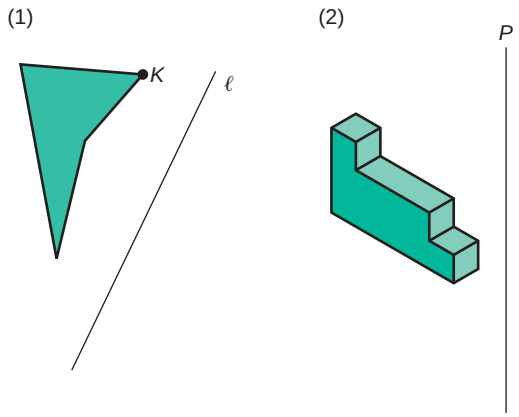
Surprisingly clear images can be created by reflections in water. Pick out some points on the mountains and their images in the photograph in Figure 11.20.



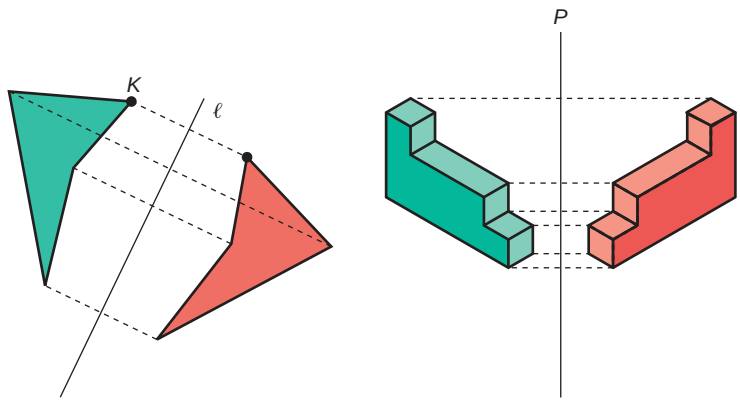
Figure 11.20
Grand Teton Mountain Range in Grand Teton National Park, Wyoming

In the mappings about line ℓ (see Figure 11.18 on page 760) and plane P (Figure 11.19, above), each point and its image are on lines that are perpendicular to the line or plane of reflection. For example, $\overline{EE'}$ in Figure 11.18 is perpendicular to line ℓ, and $\overline{NN'}$ in Figure 11.19 is perpendicular to plane P. Furthermore, each point is the same distance from the line or plane as its image. These two conditions hold for all reflections.

EXAMPLE B Trace figure (1) and determine the image of the quadrilateral obtained from a reflection about line ℓ. Trace figure (2) and determine the image of the "three-dimensional figure" obtained from a reflection about plane P, which is perpendicular to this page.



Solution The image of a polygon can be located by first constructing the images of its vertex points. For example, using vertex point K on the quadrilateral, construct a line through K that is perpendicular to ℓ. Then locate the image K' of K so that K' is on the perpendicular line and the distances from K' to ℓ and from K to ℓ are equal. The images of the vertex points of figure (2) can be located in a similar manner by constructing lines perpendicular to "plane" P.



Another method is to fold a paper with the traced figures so that line ℓ coincides with itself (or "plane" P coincides with itself) and trace the image in the opposite half-plane.

ROTATIONS

The third type of mapping is a **rotation,** which for plane figures can be described as turning about a point. As an example, consider a 90° rotation about point O in Figure 11.21, on the next page. Place a piece of paper on this figure, and trace $\overline{FG}$ and quadrilateral $ABCD$. Hold a pencil at point O, and rotate the paper 90° in a clockwise direction. (A 90° rotation

can be determined by beginning with the edges of the paper parallel to the edges of this page.) Each of the points you trace will coincide with its image after this rotation. Quadrilateral $ABCD$ is mapped to quadrilateral $A'B'C'D'$, and $\overline{FG}$ is mapped to $\overline{F'G'}$. Point O is called the **center of rotation** and is the only fixed point for this mapping. Notice that each point from the quadrilateral and the line segment has been rotated 90° about point O to its image. That is, if a line is drawn from any point to the center O, and from the image of the point to the center O, the resulting angle will be a right angle. For example, both $\measuredangle GOG'$ and $\measuredangle COC'$ are 90° angles.

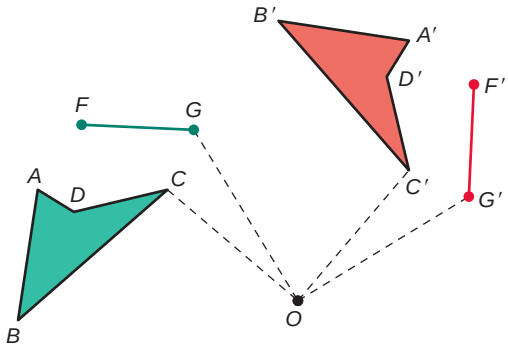


Figure 11.21

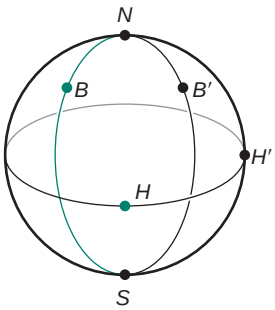


Figure 11.22

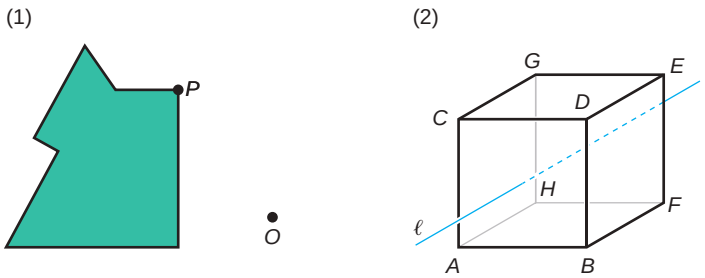
Figures in space are rotated about lines. If the sphere shown at the left in Figure 11.22 is rotated 90° about the vertical axis through N and S, point H will be mapped to H' and point B to B'. Each point will be mapped to a new location except for points N and S, which remain fixed.

The Earth and other spinning objects such as toy tops rotate about axes. The restaurant and observation deck at the top of the 60-story Space Needle in Seattle, Washington (Figure 11.23), rotate once every 60 minutes about a vertical shaft. Each point on this moving structure traces out a circular path during one complete revolution. These moving points are constantly changing their locations and being mapped to one another.

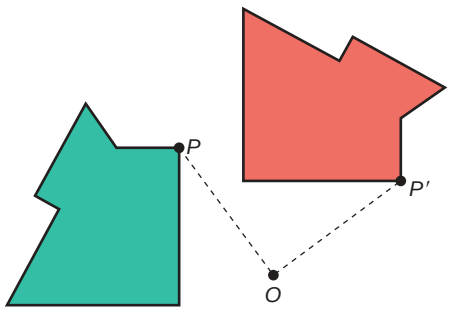


Figure 11.23
Space Needle in Seattle, Washington, with Mt. Rainier in the background

EXAMPLE C Trace the polygon in figure (1) on a piece of paper, and determine its image for a 90° clockwise rotation about point O. Line ℓ in figure (2) passes through the centers of the opposite faces $ABDC$ and $HFEG$ of the cube. Determine the image of each vertex of the cube for a 90° clockwise rotation about line ℓ.



Solution **1.** One method of locating the image of the polygon in figure (1) is to use constructions to first find the images of the vertex points. Using vertex point P of the heptagon, construct a right angle having O as the vertex of the angle and $\overrightarrow{OP}$ as one side. Then locate the image P' of P on the other ray of the angle so that $OP = OP'$. A similar construction can be repeated for the other vertex points.



Another method of finding the image of the polygon is to place a second piece of paper on the traced figure and trace the figure onto this top sheet. Then hold a pencil at point O and rotate the paper 90° clockwise. The figure will be rotated to its new location, where it can be imprinted onto the bottom sheet by pressing on the boundary lines of the figure with a ballpoint pen. Sometimes shading the back side of the paper with a pencil will help to show the imprinted image on the bottom sheet. **2.** The rotation of the cube about line ℓ maps the vertices as follows: $A \to C$, $C \to D$, $D \to B$, $B \to A$, $H \to G$, $G \to E$, $E \to F$, and $F \to H$. In fact, line ℓ is an *axis of symmetry* for the cube.

Research Statement

The ability to visualize geometric figures and operations on them has been recognized as an important component of mathematical thinking.

Wheatley

COMPOSITION OF MAPPINGS

The wood engraving by M.C. Escher (Figure 11.24, on the next page) combines translations and reflections. The white swan W can be mapped onto the black swan B by a translation followed by a reflection. First, trace swan W and then translate the traced figure vertically (on line ℓ) so it is level with swan B. The traced swan can now be made to coincide with swan B by a reflection about line ℓ. A *translation* followed by a *reflection* about a line that is parallel to the line of translation is called a **glide reflection.**

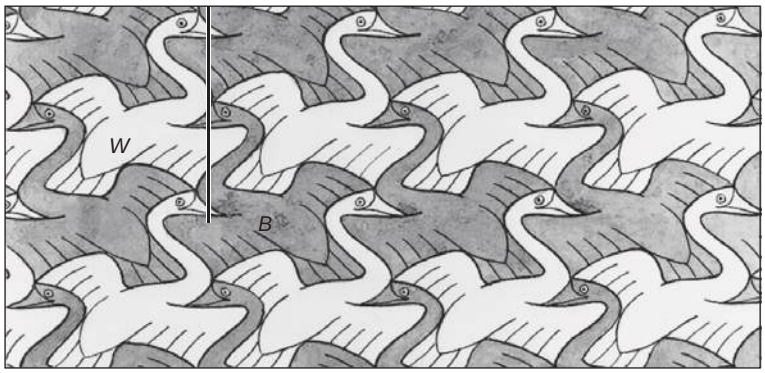
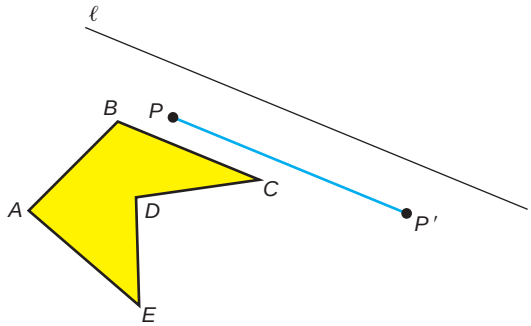


Figure 11.24
M.C. Escher's "Swans"

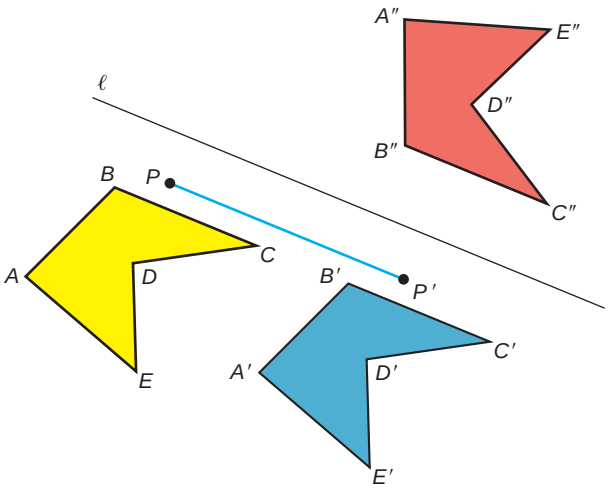
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EXAMPLE D

1. Trace pentagon $ABCDE$ and determine the glide reflection image of the figure obtained by the translation that maps P to P' followed by a reflection about line ℓ.
2. If $ABCDE$ is first reflected about line ℓ, and then followed by a translation that maps P to P', will the result be the same?



Solution 1. Pentagon $A'B'C'D'E'$ (blue) is the image of the pentagon $ABCDE$ (yellow) by the translation that maps P to P', and pentagon $A''B''C''D''E''$ (red) is the image of the blue pentagon reflected about line ℓ, as well as the glide reflection image of the yellow pentagon.

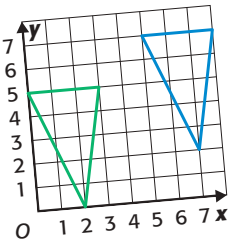


2. Yes, the final image will be the same. The order in which you glide and reflect in a glide reflection does not matter.

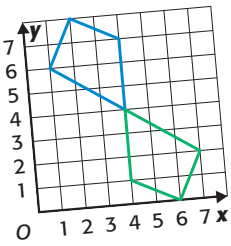
13-9 Identify Transformations

Determine whether each transformation is a *translation, reflection,* or *rotation*. See Examples 1, 2 (p. 591)

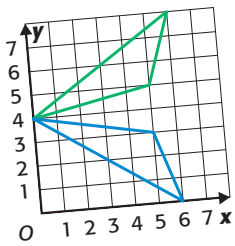
1.



2.



3.



4. Which transformations appear in the pattern of bricks shown at the right?



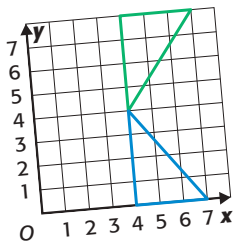
5. Talk About It — Describe how you could use a symmetrical shape on grid paper to show a translation, reflection, and rotation.

Practice and Problem Solving

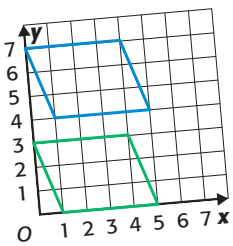
EXTRA PRACTICE See page R37.

Determine whether each transformation is a *translation, reflection,* or *rotation*. See Examples 1, 2 (p. 591)

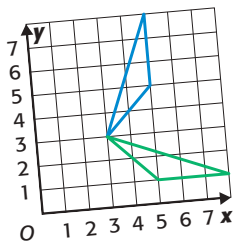
6.



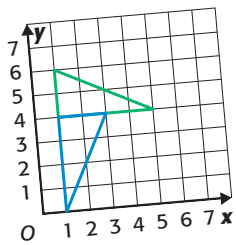
7.



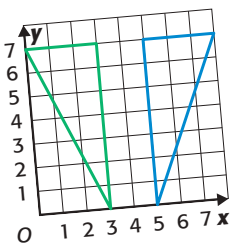
8.



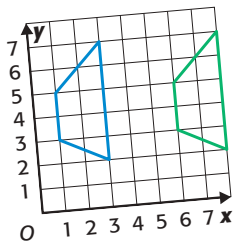
9.



10.



11.



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NCTM Standards

In grades 3–5 students can investigate the effects of transformations and begin to describe them in mathematical terms. p. 43

When one mapping is followed by another, the combination is called a **composition of mappings.** The glide reflection in Example D, on page 765, is a special kind of a composition of mappings. Any combination of translations, reflections, rotations, or glide reflections can be used. In Figure 11.25, a 90° clockwise rotation about point O is followed by a translation of each point 3 spaces to the right. Triangle ABC is mapped to triangle $A'B'C'$ by the rotation, and then the translation maps triangle $A'B'C'$ to triangle $A''B''C''$. The composition of the rotation and translation is the single mapping that takes triangle ABC to triangle $A''B''C''$. In this case, the composition is a 90° rotation about point X. Try it.

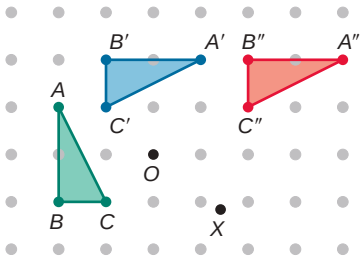


Figure 11.25

In general, composition of mappings is non-commutative; the order in which the mappings are composed matters. Where is the image of triangle $A''B''C''$ if triangle ABC is first translated and then rotated?

NCTM Standards

Students need to learn to physically and mentally change the position, orientation, and size of objects in systematic ways as they develop their understandings about congruence, similarity, and transformations. p. 43

Composing mappings is similar to performing an operation on numbers. For example, the product of two integers is always another integer (closure property), and the composition of two congruence mappings is always another congruence mapping. The composition of some pairs of mappings is quite easy to determine. A 30° rotation followed by a 45° rotation can be replaced by a 75° rotation, and two translations can always be replaced by one translation.

The situation for reflections is more interesting, as shown in Figure 11.26. For a reflection about line m, figure H is mapped to H'. Then H' is mapped to H'' by a reflection about line n. These two reflections can be replaced by a single rotation about point O that maps H to H''.

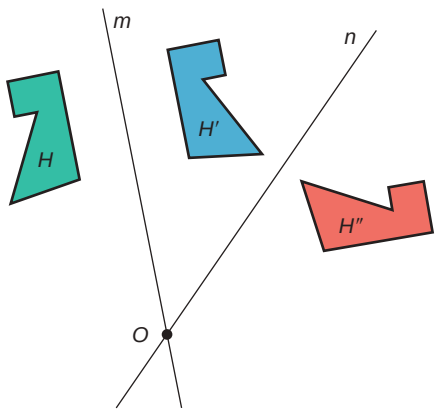


Figure 11.26

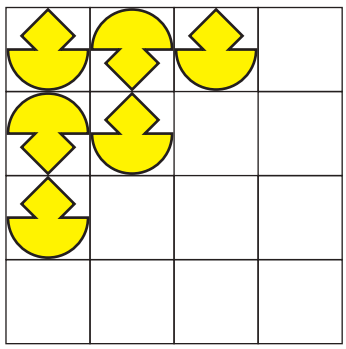
Compositions of different types of mappings, such as a rotation followed by a translation or a reflection followed by a rotation, can also be replaced by a single mapping. It can be proved that the composition of any two of the four mappings (rotation, translation, reflection, or glide reflection) is also a rotation, translation, reflection, or glide reflection. In other words, the set of congruence mappings is **closed with respect to composition.**

WALLPAPER PATTERNS

Two-dimensional patterns such as those on wallpaper and tiled floors are created by systematic translations, reflections, and rotations of a basic figure, as illustrated by Example E.

EXAMPLE E

The basic figure in the upper left corner of the following grid has the shape of a mushroom. This first column of the grid was obtained by repeatedly reflecting about the common edge of adjacent squares.



Determine a mapping (translation, reflection, or rotation) that produces each of the following.

1. The top row **2.** The second column **3.** The second row

4. The diagonal from upper left to lower right

Solution **1.** Rotations of 180° about the midpoints of the edges of adjacent squares in the top row. **2.** Reflections about the common edge of adjacent squares in the second column or a 180° rotation about the midpoint of these edges. **3.** Rotations of 180° about the midpoints of the edges of adjacent squares in the second row. **4.** Translations along the diagonal.

CONGRUENCE

Translations, reflections, rotations, and glide reflections all have something very important in common. If A and B are any two points and A' and B' are their respective images, then the distance between A and B is the same as the distance between A' and B' (Figure 11.27). That is, for these mappings the lengths of the line segments are the same as the lengths of their images. Such mappings are called **distance-preserving mappings.** This property reflects the simple intuitive notion that as figures are rotated, translated, or reflected, their size and shape do not change.

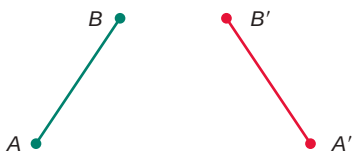


Figure 11.27

Up to this point we have thought of two figures as being congruent if they have the same size and shape, or if one can be made to coincide with the other. The need for a

more careful definition of congruence becomes evident when we consider congruence in three dimensions. For example, we need a way of defining congruence for the two ice cream machines in Figure 11.28, and it does not make sense to say that they coincide.



Figure 11.28

A suitable definition of congruence for both plane and space figures can be given in terms of mappings.

Congruent Figures Two geometric figures are **congruent** if and only if there exists a translation, reflection, rotation, or glide reflection of one figure onto the other.

This definition says that for each of these four mappings, a figure is congruent to its image. Conversely, if two figures are congruent, one can always be mapped to the other by one of these mappings. This definition gives us a way of viewing congruence of both plane and space figures. The plane figures H and H'' in Figure 11.26 on page 767, are congruent because a rotation maps one to the other. The ice cream machines in Figure 11.28 are congruent because a translation maps one to the other. Each point on the left ice cream machine is mapped to a corresponding point on the right ice cream machine. The distances between any two points on the left ice cream machine, such as points A and B, and between their images, A' and B', are equal. Defining congruence in terms of distance-preserving mappings is the mathematical way of saying that two objects have the same size and shape.

MAPPING FIGURES ONTO THEMSELVES

In their book *Let's Play Math,* Michael Holt and Zoltan Dienes illustrate a scheme for coloring pictures of a house. Cut out a square and color the corners four different colors (Figure 11.29 on page 770). Both the front and back sides of each corner should be the same color. Place the square on a piece of paper and draw a frame around it (Figure 11.29). At each corner of the frame, write (or draw pictures for) one of the words *wall, roof, door,* and *window.* The entire configuration is called the Rainbow Toy.*

*M. Holt and Z. Dienes, *Let's Play Math,* pp. 88–94.

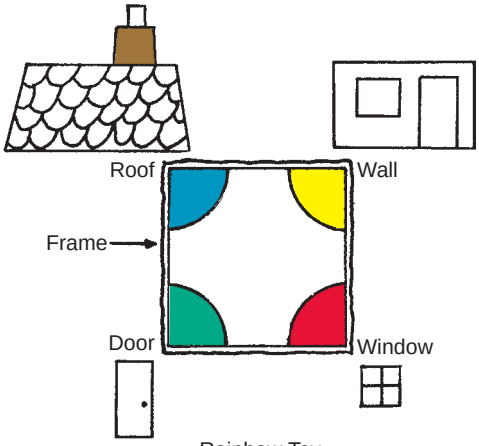


Figure 11.29

The different positions in which the square can be placed on the frame determine different arrangements of colors for the wall, roof, door, and window of the house. With the square in the position shown in Figure 11.29, we get the colors for house (a) in Figure 11.30. Color schemes for houses (b), (c), and (d) are obtained by rotating the square into three different positions; by flipping the square over, we get four more positions for the color schemes for houses (e) through (h).

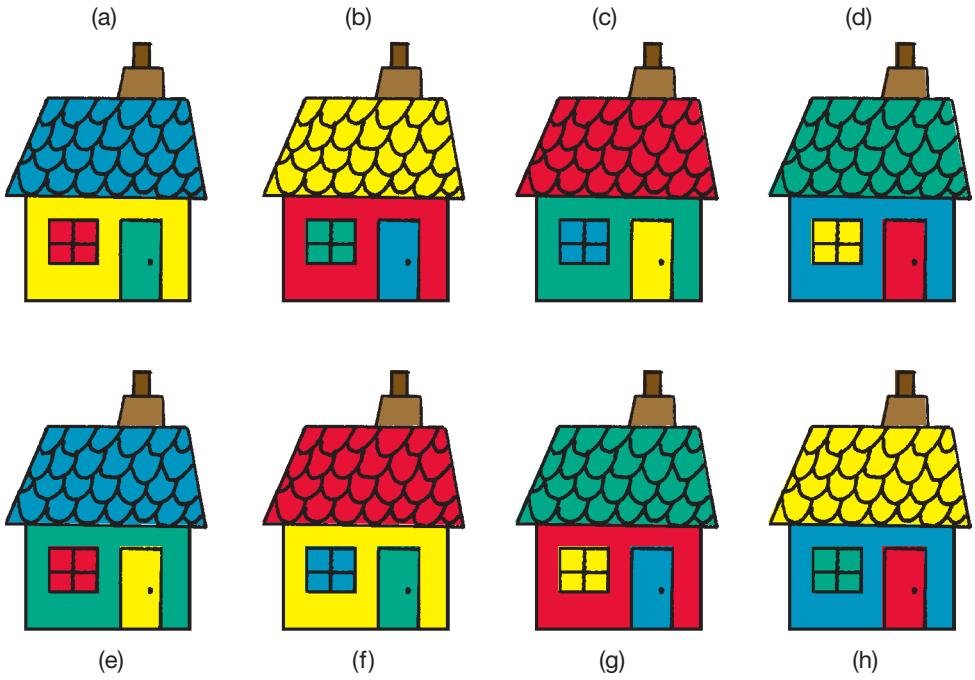


Figure 11.30

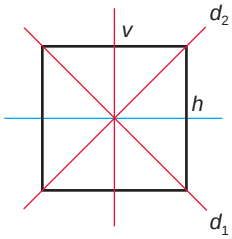


Figure 11.31

The Rainbow Toy provides an elementary way of illustrating the mappings of a square onto itself. Remember from Section 9.4 that a square has four rotational symmetries: 90°, 180°, 270°, and 360°. It also has four lines of symmetry (Figure 11.31): two diagonal lines d_1 and d_2, a horizontal line h, and a vertical line v. Thus, there are eight mappings of the square onto itself, and these mappings produce the eight color combinations for the houses in Figure 11.30.

In general, the number of mappings of a plane figure onto itself is the total number of lines of symmetry (four in Figure 11.31 on the previous page) and rotation symmetries (four in Figure 11.31).

EXAMPLE F Determine the number of mappings of each figure onto itself.

1. A rectangle **2.** The letter S **3.** A regular hexagon

Solution **1.** Four, because it has two lines of symmetry and two rotation symmetries. **2.** Two, because it has two rotation symmetries. **3.** Twelve, because it has six lines of symmetry and six rotation symmetries.

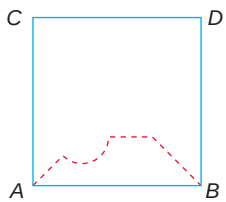
ESCHER-TYPE TESSELLATIONS

The Dutch artist M.C. Escher visited the Alhambra in Spain in the 1930s and was inspired by the geometric tilings of the walls and ceilings to create some very unusual tessellations. He used translations, rotations, and reflections to reshape polygons such as squares, equilateral triangles, and regular hexagons, which are known to tessellate, into nonpolygonal figures that also tessellate. In the following paragraphs, we will examine the use of mappings in creating tessellations.

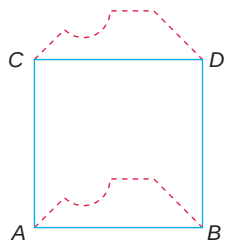
Translation Tessellations One of the simplest ways to create a tessellation is to begin with a square (or a rectangle, a rhombus, or other parallelograms) and change its opposite sides. This technique is illustrated in Figure 11.32.

Figure 11.32

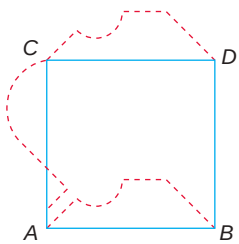
Step 1 Draw a curve from A to B.



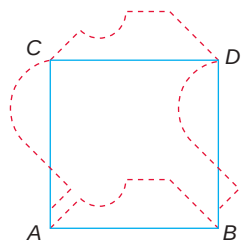
Step 2 Translate the curve to the opposite side of the square so that A maps to C and B maps to D.



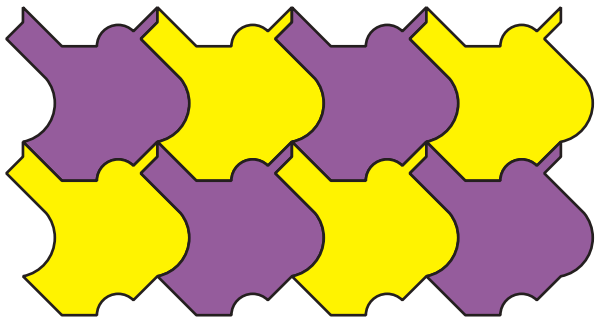
Step 3 Draw a curve from C to A. (Notice that the curve does not have to be within the original figure.)



Step 4 Translate the curve to the opposite side of the square so that A maps to B and C maps to D.



Step 5 Erase the lines of the original square that are not part of the curve to obtain a non-polygonal figure, and tessellate with this figure.

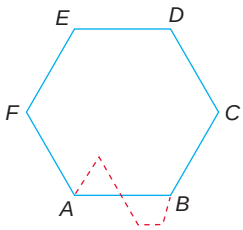


The tessellation in step 5 of Figure 11.32 can be translated onto itself in many ways: moving left or right; up or down; or diagonally up or down.

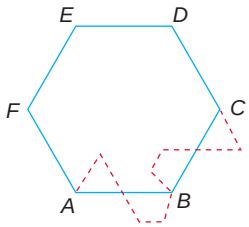
Rotation Tessellations A rotation tessellation is created next by beginning with a regular hexagon (Figure 11.33, step 1). Rotations are used to alter the sides of the original polygon.

Figure 11.33

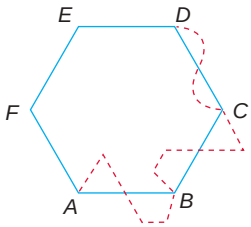
Step 1 Draw a curve from A to B.



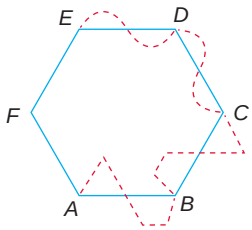
Step 2 Rotate the curve about point B so that A maps to C.



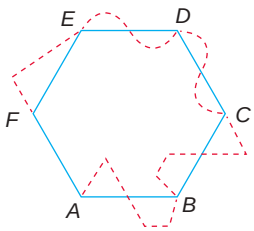
Step 3 Draw a curve from C to D.



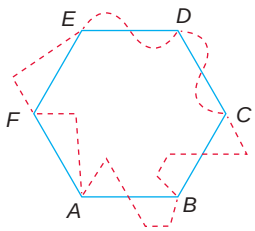
Step 4 Rotate the curve about point D so that C maps to E.



Step 5 Draw a curve from E to F.



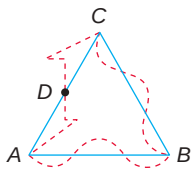
Step 6 Rotate the curve about point F so that E maps to A.



Laboratory Connection

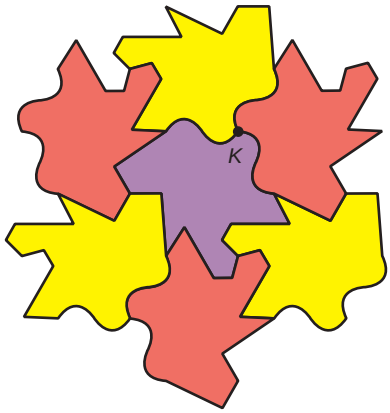
Tessellations

How were the sides of this equilateral triangle altered and can the resulting figure be used as a template for an Escher-type tessellation? Explore this and related questions using Geometer's Sketchpad® student modules available at the companion website.



Mathematics Investigation
Chapter 11, Section 2
www.mhhe.com/bbn

Step 7 Erase the lines of the original hexagon, and form a tessellation with the resulting figure.

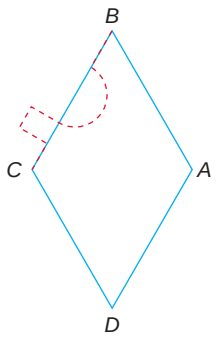


The tessellation in step 7 of Figure 11.33 can be rotated on itself. For example, a rotation of 120° or 240° about point K maps the tessellation onto itself.

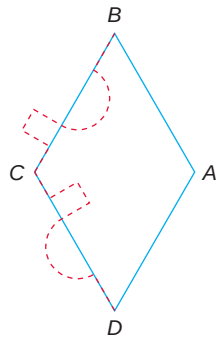
Reflection Tessellations The steps for creating a reflection tessellation are shown in Figure 11.34. The basic figure in this case is a rhombus.

Figure 11.34

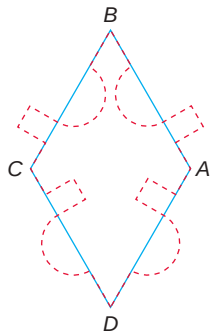
Step 1 Draw a curve from B to C.



Step 2 Rotate the curve from B to C about point C so that B maps to D.



Step 3 Reflect the curves from B to C and C to D about line $\overleftrightarrow{BD}$ so that C maps to A.



Step 4 Erase the lines of the original rhombus that are not part of the curve, and tessellate with the resulting figure.



Note that in step 4 of Figure 11.34, the tessellation has many lines about which it can be reflected onto itself. One such line passes through the highest and lowest points of the figure.

PROBLEM-SOLVING APPLICATION

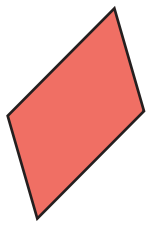
Problem

Given any plane figure and its image for a rotation, how can we determine the center of rotation?

Understanding the Problem The following parallelogram has been rotated a certain number of degrees to its image. The problem is to find the center of rotation.

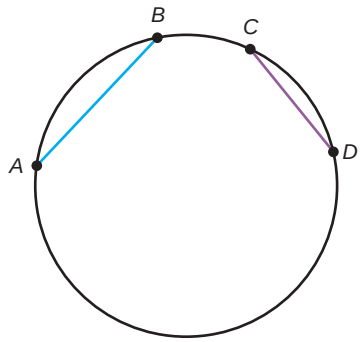


Figure

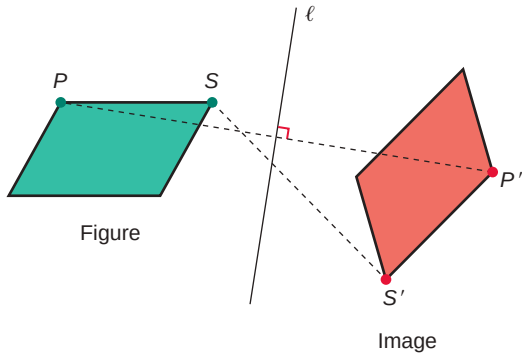


Image

Devising a Plan One approach is to trace the original figure and then to *guess and check* to locate a point about which the figure can be rotated to coincide with its image. Another approach is to use two chords of a circle. **Question 1:** How can chords $\overline{AB}$ and $\overline{CD}$ of the following circle be used to locate the center of the circle?

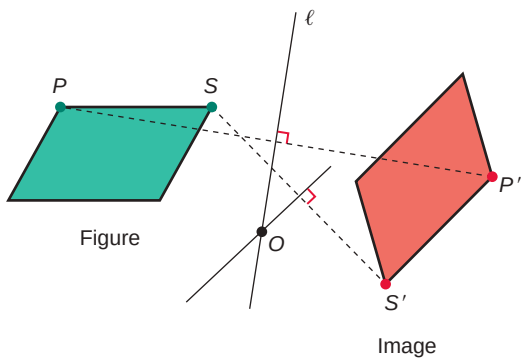


Carrying Out the Plan Each point of the original figure traces an arc as it is rotated to its image point. So, the point and its image lie on a circle whose center is the center of rotation. In the following figure, S maps to S', P maps to P', and the perpendicular bisector of $\overline{PP'}$ is ℓ. **Question 2:** How can the center of rotation be found?



Looking Back Check the location of the center of rotation by tracing the original figure and rotating it to its image. **Question 3:** How can the measure of the angle of rotation be determined?

Answers to Questions 1–3 **1.** Draw the perpendicular bisectors of $\overline{AB}$ and $\overline{CD}$. Their intersection is the center of the circle. **2.** The intersection of ℓ and the perpendicular bisector of $\overline{SS'}$ is the center of rotation.



3. With O as the center of rotation, use a protractor to measure the central angle $\measuredangle POP'$ or $\measuredangle SOS'$.

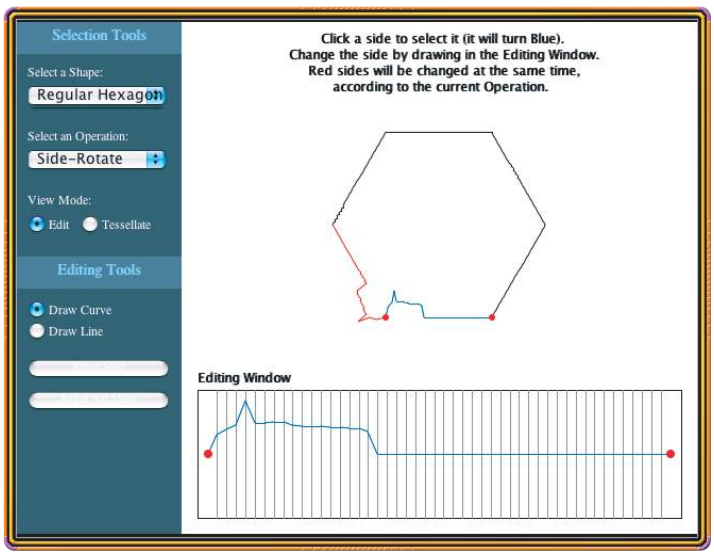
NCTM Standards

The *Curriculum and Evaluation Standards for School Mathematics,* grades 5–8, *Geometry* (p. 114), encourages the constructions and mappings of two- and three-dimensional shapes:

> Computer software allows students to construct two- and three-dimensional shapes on a screen and then flip, turn, or slide them to view them from a new perspective. Explorations of flips, slides, turns, stretchers, and shrinkers will illuminate the concepts of congruence and similarity. Observing and learning to represent two- and three-dimensional figures in various positions by drawing and construction also helps students develop spatial sense.

Technology Connection

If you have ever formed a tessellation by tracing figures on paper or at a window, you will be amazed at the power of this applet. By selecting one of several different polygons, you will be able to quickly modify its sides and then to click and see the resulting tessellation. You will be pleasantly surprised.



Tessellations Applet, Chapter 11, Section 2
www.mhhe.com/bbn

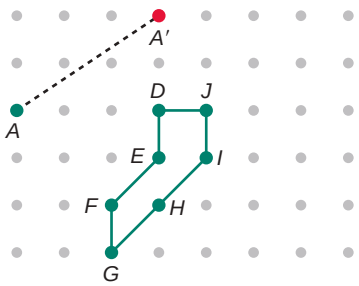
Exercises and Problems 11.2



1. This photo shows shopping carts queued up for customers. Their metal frames provide models for sets of parallel lines and angles.
   a. Describe two different models in this photo for sets of parallel lines.
   b. Describe two different models of congruent angles.
   c. What type of mapping is suggested by this photo?

Copy dot grid paper from the website to use in sketching the images in exercises 2 through 11.

2. For the translation that maps A to A', sketch the image of the hexagon.

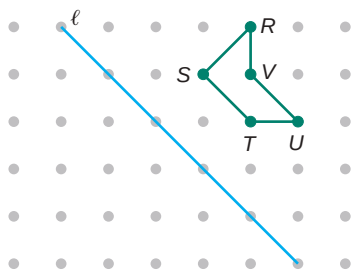


   a. The line through point D and its image is parallel to $\overleftrightarrow{AA'}$. Is this true for every point on the hexagon and its image?
   b. If E' and G' are the respective images of E and G, how does the length $E'G'$ compare with the length EG?
   c. Compare the area of the hexagon with the area of its image.

3. For a translation that maps F to I in exercise 2, sketch the image of the hexagon.
   a. How does the distance from F to I compare with the distance from G to its image for this mapping?
   b. How does the area of the original hexagon compare with the area of its image?

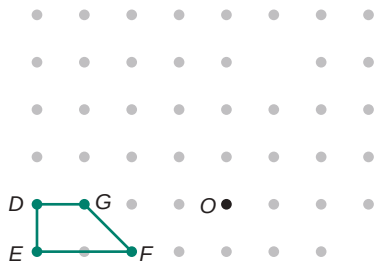
4. Sketch the image of the pentagon in exercise 5 for a reflection about line $\overleftrightarrow{TU}$.
   a. If S' is the image of S, what are the measures of the angles formed by the intersection of $\overleftrightarrow{SS'}$ and $\overleftrightarrow{TU}$?
   b. How does the distance from S to $\overleftrightarrow{TU}$ compare with the distance from S' to $\overleftrightarrow{TU}$?
   c. What are the fixed points of the original pentagon for this mapping?

5. Sketch the image of the pentagon for a reflection about the line ℓ.



   a. If R' is the image of R, what is the measure of the angles formed by the intersection of $\overleftrightarrow{RR'}$ and ℓ?
   b. If U' is the image of U, how does the distance from U to ℓ compare with the distance from U' to ℓ?
   c. What are the fixed points for this mapping?

6. Sketch the image of the quadrilateral for the 90° clockwise rotation about point O. (Trace the figure on a piece of paper and rotate the paper.)



a. If E' is the image of E, what is the measure of $\measuredangle EOE'$?

b. If G' is the image of G, how does the length EG compare with the length $E'G'$?

c. Are there any fixed points for this mapping?

7. Sketch the image of the quadrilateral in exercise 6 for the 90° counterclockwise rotation about point O.

a. If F' is the image of F for this mapping, what is the measure of $\measuredangle FOF'$?

b. If D' is the image of D, how does the length DF compare with the length $D'F'$?

c. How does the area of the original quadrilateral compare to the area of its image?

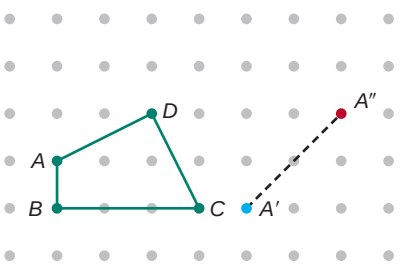
8. Map quadrilateral $ABCD$ in exercise 9 to quadrilateral $EFGH$ by a translation that maps B to D. Then map quadrilateral $EFGH$ to quadrilateral $IJKL$ by a translation that maps A to C.

a. The translation that maps B to D can be described as "right 2 and up 2." Describe the translation that maps A to C.

b. Describe the translation that maps C to K.

c. What is the image of quadrilateral $ABCD$ for the composition of the translation that maps B to D and the translation that maps A to C?

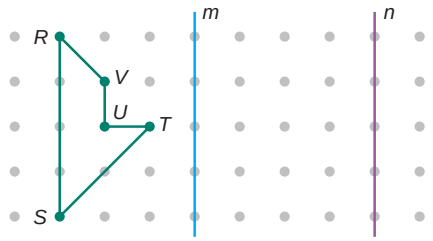
9. Map quadrilateral $ABCD$ to quadrilateral $A'B'C'D'$ by a translation that maps point A to A'. Then map quadrilateral $A'B'C'D'$ to quadrilateral $A''B''C''D''$ by a translation that maps A' to A''.



a. The translation that maps A to A' can be described as "right 4 and down 1." The translation that maps A' to A'' is "right 2 and up 2." Describe the translation that maps A to A''.

b. What is the image of quadrilateral $ABCD$ for the composition of the translation that takes A to A' followed by the translation that takes A' to A?

10. Reflect pentagon $RSTUV$ about line m and then reflect its image about line n.



a. What single mapping (rotation, translation, or reflection) is equal to the composition of these two reflections?

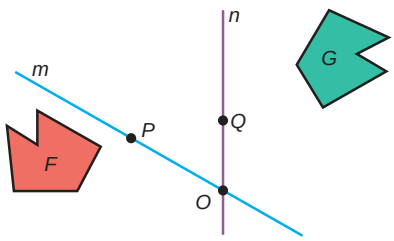
b. Let R' be the image of R for the reflection about m, and let R'' be the image of R' for the reflection about n. Compare the distance from R to R'' with the distance from line m to line n. What relationship do you find? Will this relationship hold for other points and their images?

11. Reflect pentagon $RSTUV$ in exercise 10 about line n, and then reflect its image about line m.

a. What single mapping (rotation, translation, or reflection) is equal to the composition of these two reflections?

b. Let R' be the image of R for the reflection about n, and let R'' be the image of R' for the reflection about m. Compare the distance from R to R'' with the distance from line m to line n. What relationship can you find? Will this relationship hold for other points and their images?

Use the following figure for exercises 12 and 13.



12. Sketch the image of hexagon G for a reflection about line n. Label this image G'.

a. What is the image of G' for a reflection about line m?

b. There is a counterclockwise rotation about point O that will map figure G to figure F. How is the number of degrees in the rotation related to $\measuredangle QOP$?

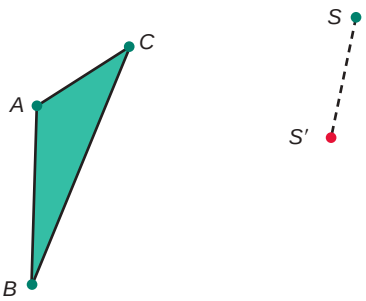
13. Sketch the image of hexagon F (on the previous page) for a reflection about line m.

a. What is the image of the image of F for a reflection about line n?

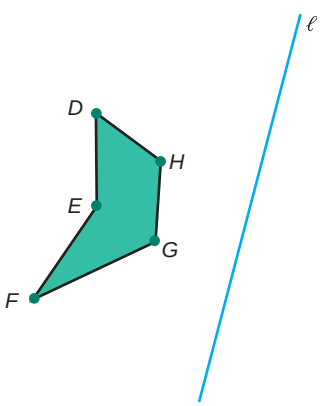
b. There is a clockwise rotation about point O that will map figure F to figure G. How is the number of degrees in this rotation related to $\measuredangle POQ$?

Trace each figure in exercises 14 and 15 on a sheet of paper. Find the image of the figure for the given transformation by constructing the images of the vertex points. Show each construction and give a brief explanation.

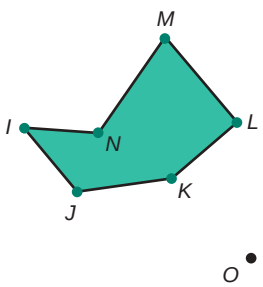
14. a. A translation that maps S to S'



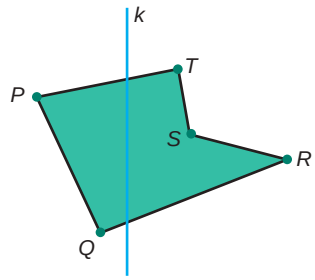
b. A reflection about line ℓ



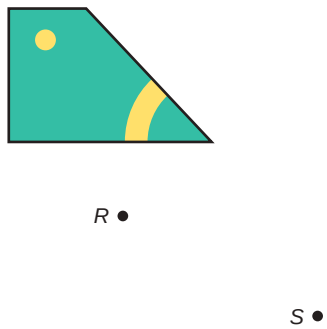
15. a. A 90° clockwise rotation about point O



b. A reflection about line k



Trace this figure and points R and S on a piece of paper for exercises 16 and 17.



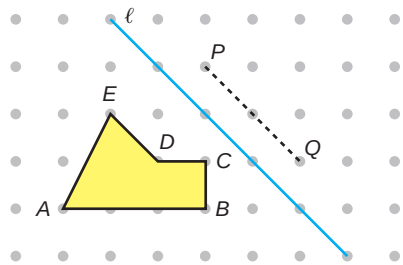
16. a. Locate the image of the figure for the composition of these two mappings: 180° rotation about R followed by a 180° rotation about S.

b. What single mapping (rotation, translation, reflection, or glide reflection) can be used to replace the two rotations in part a so that the figure is mapped to its image?

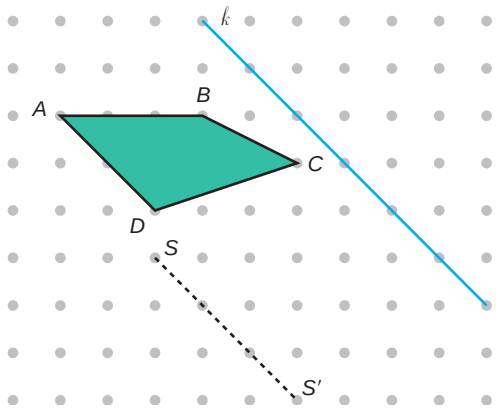
17. a. Locate the image of the figure for the composition of these two mappings: a 90° clockwise rotation about R followed by a 90° clockwise rotation about S.

b. What single mapping can be used to replace the two rotations in part a?

18. Map pentagon $ABCDE$ to pentagon $A''B''C''D''E''$ by the glide reflection that first translates P to Q and then reflects the translated image of $ABCDE$ over line ℓ. If you first reflect and then translate, do you obtain the same final image? Show each composition of mappings. (Copy dot paper from the website.)

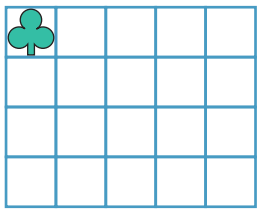


19. Map quadrilateral $ABCD$ to quadrilateral $A''B''C''D''$ by the glide reflection that first translates S to S' and then reflects the translated image of $ABCD$ over line k. If you first reflect and then translate, do you obtain the same final image? Show each composition of mappings. (Copy dot paper from the website.)

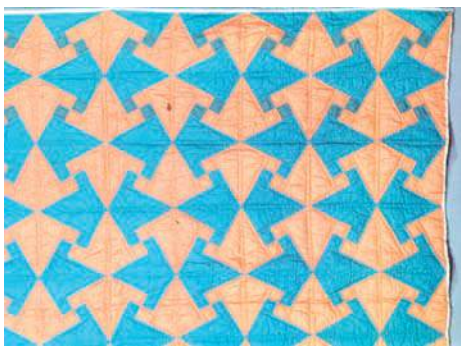


20. Complete the pattern in each grid square by carrying out the mappings on the basic figure in the small square in the upper left corner of the grid. (Copy rectangular grid paper from the website.)

Rows: Rotate 180° about the midpoints of the right sides of the squares. Columns: Reflect about the lower side of the squares.



21. The design on the nineteenth-century quilt shown here also occurs in the fourteenth-century Moorish palace, the Alhambra. The top row can be generated by a sequence of 180° rotations beginning with the orange figure in the upper left corner.

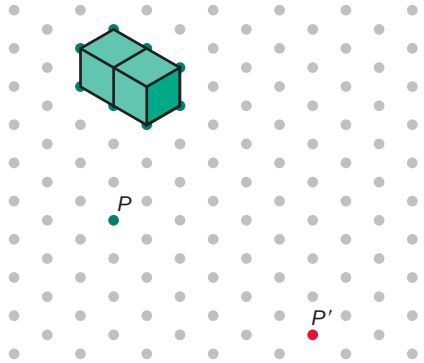


Patchwork quilt with Arabic lattice patterns, ca. 1850

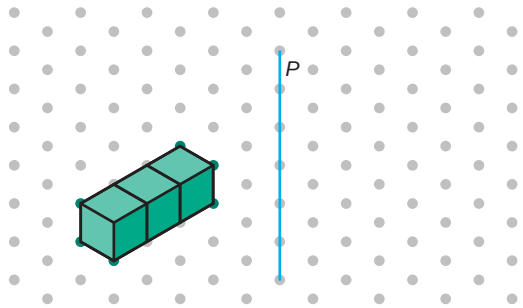
a. Locate the centers of rotation for these mappings.
b. What mapping can be carried out to generate the left column of figures, beginning with the figure in the upper left corner (the full first column of orange figures)?

Isometric grid paper, such as that shown in exercises 22 through 24, is helpful in drawing three-dimensional figures. Sketch the image of each figure for the given mapping. (Copy isometric grid paper from the website.)

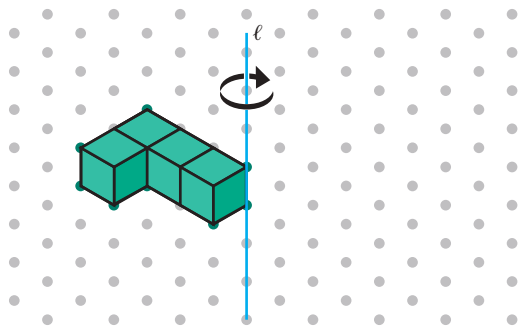
22. A translation that maps P to P'



23. A reflection about plane P that is perpendicular to this page



24. A 90° rotation in the indicated direction about line ℓ



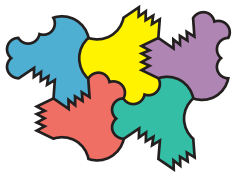
Determine the number of congruence mappings of each polygon onto itself in exercises 25 and 26, and indicate how you determined that number.

25. **a.** Rhombus **b.** Equilateral triangle
c. Regular octagon

26. **a.** Rectangle **b.** Parallelogram
c. Regular pentagon

Identify the basic figure (rectangle, hexagon, square, or parallelogram) that was used to create each Escher-type tessellation in exercises 27 through 29.

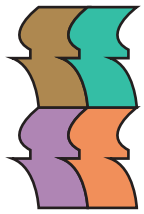
27.



28.



29.



Create an Escher-type tessellation for the techniques given in exercises 30 through 32.

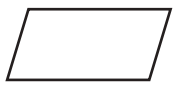
30. A rotation tessellation

31. A reflection tessellation

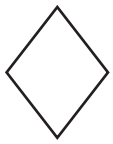
32. A translation tessellation

By altering the sides of each polygon in exercises 33 through 35, design a nonpolygonal figure that will tessellate. Sketch a few copies of your figures.

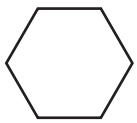
33.



34.

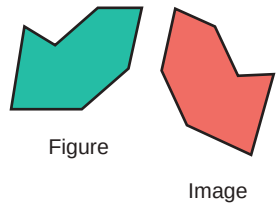


35.

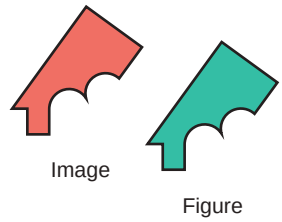


Determine the mapping (rotation, translation, or reflection) that maps each figure in exercises 36 and 37 to its image. Locate all lines of reflection, points of rotation, and line segments used for translation.

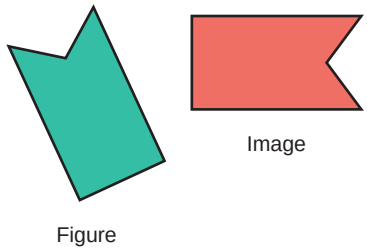
36. **a.**



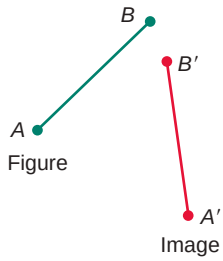
b.



37. **a.**

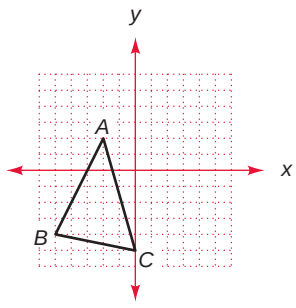


b.



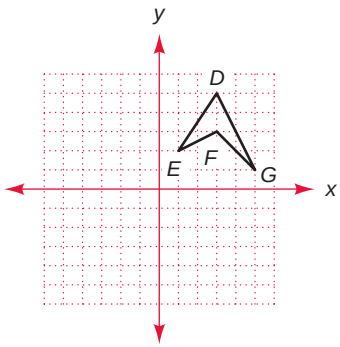
Reasoning and Problem Solving

38. A translation maps point P with coordinates $(^{-}2, 1)$ to P' with coordinates $(3, 2)$. Graph these points and draw an arrow from P to P' to indicate the length and direction of the translation. (Copy coordinate grid paper from the website.)



a. Sketch the image of $\triangle ABC$ for this translation.
b. If A', B', and C' are the images of A, B, and C, respectively, what are their coordinates?

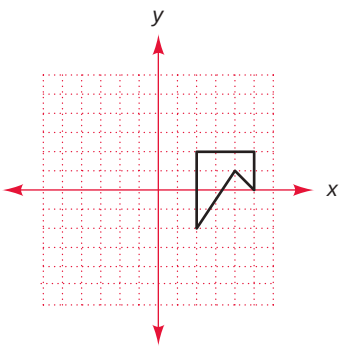
39. Draw the image of quadrilateral $DEFG$ for a reflection about the x axis. Label the images of these vertices D', E', F', and G'. (Copy coordinate grid paper from the website.)



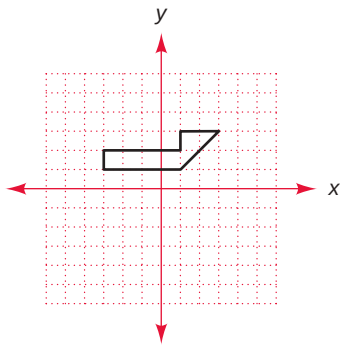
a. What are the coordinates of D', E', F', and G'?
b. Reflect quadrilateral $D'E'F'G'$ about the y axis and label its vertices D'', E'', F'', and G''. What are the coordinates of the vertices of this image?
c. What single rotation will map quadrilateral $DEFG$ to quadrilateral $D''E''F''G''$?

40. Mappings are sometimes given by describing what will happen to the coordinates of each point. For example, the mapping $(x, y) \rightarrow (x + 2, y - 3)$ is a translation that maps each point 2 units to the right and 3 units down. To find the image of a particular point, such as (1, 7), substitute these values for x and y into $(x + 2, y - 3)$. In this case (1, 7) maps to (3, 4). Find the image of each figure for the given mapping. Label each mapping as a translation, rotation, reflection, or glide reflection. (Copy coordinate grid paper from the website for the mappings.)

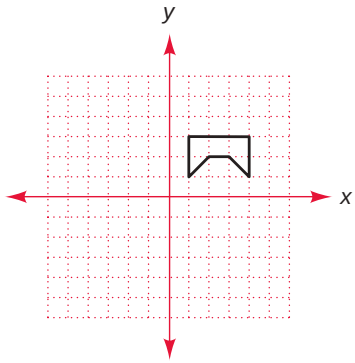
a. $(x, y) \rightarrow (x - 2, y + 1)$



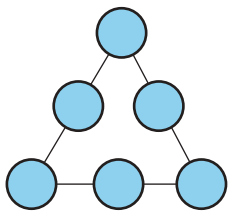
b. $(x, y) \rightarrow (x + 1, {}^{-}y)$



c. $(x, y) \rightarrow ({}^{-}x, {}^{-}y)$

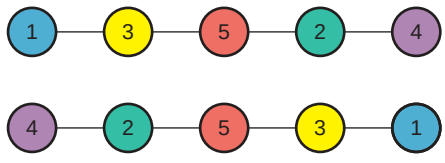


41. It is possible to place the numbers 1, 2, 3, 4, 5, and 6 in the circles of the following figure so that the sum along each side is the same. There are four solutions to this problem that are different, that is, that cannot be obtained from each other by rotations or reflections of the triangle. Find at least two of these solutions.



42. Featured Strategy: Making an Organized List. How many different ways can five consecutive whole numbers be placed in a row so that no two consecutive whole numbers are next to each other?

a. Understanding the Problem. Here is one solution for the first five consecutive whole numbers. If one arrangement can be obtained from another by a reflection, such as the one shown here, then we will consider the two arrangements to be the same. Find another solution.

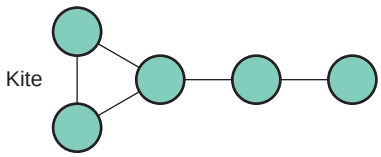


Five in a row

b. Devising a Plan. This type of problem can be solved by *making an organized list.* For example, you might begin by listing all the different arrangements with 1 in the first position; then those with 2 in the first position; etc. If you want to eliminate duplication due to reflections, what adjustment must you make in the total number of arrangements?

c. Carrying Out the Plan. Follow the system suggested in part b or one of your own to solve this problem.

d. Looking Back. There are several ways to extend this problem. For example, there are 45 different solutions for six consecutive whole numbers; you may want to see if you can find them. Another possibility is to use different configurations. How many different solutions are there for the following figure? (*Reminder:* Two arrangements are the same if they can be obtained from each other by a reflection.)



An ornamental design that extends to the right and left (around rooms, buildings, pottery, etc.) is called a **frieze.** The friezes in the following tile designs are mapped onto themselves by a translation.



Minton tiles in the United States Capitol

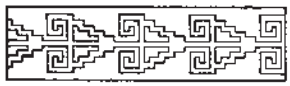
PS For each frieze pattern in exercises 43 and 44, which transformations will map the frieze pattern onto itself? (Consider these patterns as extending in both directions.)

43. a.



Chinese ornament painted on porcelain

b.



Masonry fret, Temple at Mitla, Mexico

44. a.



French renaissance ornament from casket

b.



Indian painted lacquer work

45. PS Design a frieze pattern (see exercises 43 and 44) that will be transformed onto itself by all of the following transformations: 180° rotation, translation, horizontal reflection, and vertical reflection.

46. PS Design a frieze pattern that will be transformed onto itself by a translation and a horizontal reflection, but not by a vertical reflection.

47. PS A square array of numbers in which the sum of numbers in any horizontal row, vertical column, or diagonal is always the same is called a **magic square.** The two magic squares shown here can be obtained from each other by a reflection about the diagonal from lower left to upper right. How many 3×3 magic squares with the numbers in different locations can be obtained from figure (a) by using rotations or reflections of the entire square?

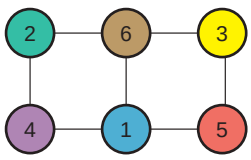
4	3	8
9	5	1
2	7	6

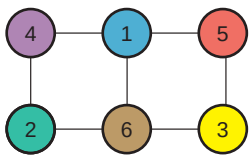
(a)

6	1	8
7	5	3
2	9	4

(b)

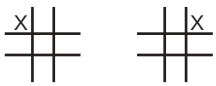
48. PS How many ways can the numbers 1, 2, 3, 4, 5, and 6 be placed in a 2 by 3 array of circles so that no two consecutive numbers are next to each other? Two arrangements, such as those shown below, are considered equivalent if one can be obtained from the other by a rotation or reflection.



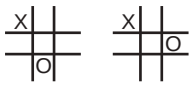


49. The first player to place three X's or three O's in a row, column, or diagonal of a 3×3 grid wins the game of Tic-Tac-Toe.

a. There are nine ways to place the first X (or O) on the grid. The following two grids are congruent because a vertical reflection maps one onto the other. How many noncongruent ways are there of making the first move? Sketch them.



b. After the first move (an X in the upper left corner in this example) there are only eight ways to place the O, but some of these, such as the ones shown in the next two grids, are congruent. Can you see what transformation maps one onto the other? Sketch all the noncongruent moves for the second turn.



c. Only one of the moves in part b can prevent the player using X's from winning. Which one?

Teaching Questions

1. One effective way to have students draw an Escher-type tessellation is by making a template out of heavy stock paper or cardstock and repeatedly tracing the pattern. For example, a student can start with a square about 2 inches on a side and follow directions similar to those in Figure 11.32 on page 771. Draw a curve on one side, cut it out and slide it to the other side and carefully tape it on. Repeat for the other two sides. Prepare a set of directions for school students to make their own template using a shape other than a square. Choose one of the three types of tessellations shown in the text.

2. Two students were discussing the four types of transformations they had learned: rotations, reflections, translations, and glide reflections. They noticed that reflections about two parallel lines seemed to produce a translation. Explain how any translation can be replaced by a reflection about two parallel lines. Use diagrams to explain how you can locate two lines of reflection for a given translation.

3. Research has shown that students were more successful in understanding motions in geometry when taught by methods that actually show the movements. Devise and describe one activity involving students moving objects to help school students understand the motions (translation, rotation, reflection, and glide reflection).

4. A middle-school math book described translations, rotations, and reflections by illustrating slides, turns, and flips. One student asked if the word "reflection" was the same as a reflection in a mirror. How would you answer this question?

Classroom Connections

1. The **Elementary School Text** example on page 766 asks students to determine whether each transformation is a translation, reflection or rotation. (a) Sketch three figures and their images on grid paper that could be classified as, respectively, a translation, a rotation, and a reflection. In each case, describe the transformation. (b) The school page does not ask the students about *glide reflections.* Describe how you could help the students understand the meaning of glide reflection using the students themselves as objects—then write a sample question for students that uses grid paper and illustrates the idea of a glide reflection transformation.

2. The **Standards** quote on page 767 urges investigations of transformations in grades 3 to 5. The quilt pattern in exercise 21 of this section is composed of congruent orange and blue shapes. Using tracing paper, explain how you can demonstrate the type of mapping that generates the top row of blue figures and then the first column of the blue figures.

3. The **Research Statement** on page 760 indicates that more than 50 percent of eighth-graders have trouble with reflections about a line. To see the difficulty, draw a line on a piece of paper that is not parallel to either edge of the paper. Print the letter Z on one side of the line and then try to visualize the image of Z for a reflection about the line (see below). Devise and record two or three activities that you believe will help school students recognize and correctly draw the image for reflections about a line. Explain how your activities will help students learn to both "physically and mentally" change the position and orientation of a figure as suggested in the second **Standards** quote on page 767.

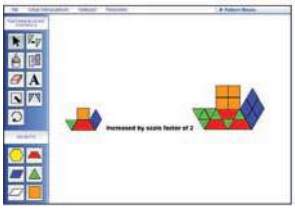


4. Look through the expectations in the **Geometry Standards** at each of the three levels (see inside front and back covers) and list any expectation that you think is met by students who successfully complete an Escher-type tessellation.

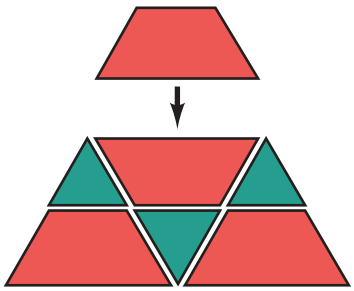
MATH ACTIVITY 11.3

Enlargements with Pattern Blocks

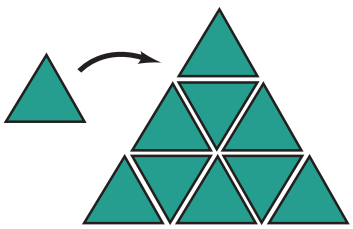
Virtual Manipulatives



www.mhhe.com/bbn



Enlargement by scale factor 2



Enlargement by scale factor 3

Purpose: Construct similar pattern block figures and explore the effects of scaling.

Materials: Pattern Blocks in the Manipulative Kit or Virtual Manipulatives.

1. The single trapezoid (red) at the left below is similar to the large trapezoid below it, which is formed with six pattern blocks.

a. One condition for similarity is that ratios of the lengths of corresponding sides of two figures are equal. What is the ratio of the lengths of the corresponding sides of these two figures?

b. The second condition for similarity is that the corresponding angles of the two figures have the same measure. Explain which angles in the two trapezoids are corresponding and how you know they have the same measure.

c. Because each side of the large trapezoid is 2 times the length of each corresponding side of the small trapezoid, the large trapezoid is an *enlargement* of the small trapezoid by a *scale factor* of 2. The area of the large trapezoid is how many times the area of the small trapezoid? Write a sentence or two to explain your reasoning.

d. Construct an enlargement by a scale factor of 2 for each of the other five pattern blocks. Sketch figures and describe how the area of a figure is related to the area of its enlargement.

***2.** The large triangle at the left is an enlargement of the small triangle by a scale factor of 3. Build and sketch an enlargement by a scale factor of 3 for each of the other five pattern blocks. (Trace pattern blocks, if there are not enough.) Explain how the area of each enlargement compares to the area of the single pattern block.

3. Build and sketch an enlargement of each of the following figures for the given scale factor. State a conjecture about how you think the area of an enlargement is related to the area of the smaller figure. Write an explanation to support your reasoning.

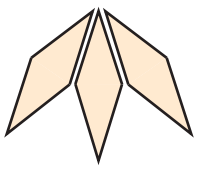
a. Enlarge by a scale factor of 3



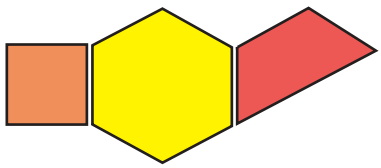
b. Enlarge by a scale factor of 4



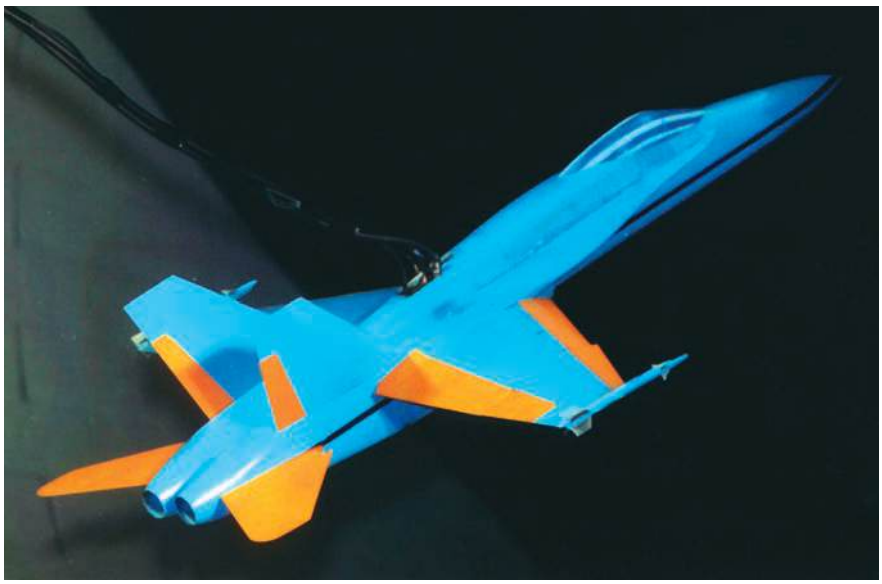
c. Enlarge by a scale factor of 3



d. Enlarge by a scale factor of 2



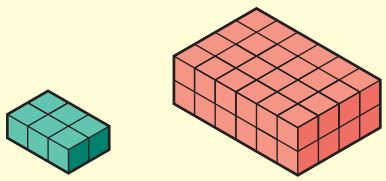
Section 11.3 SIMILARITY MAPPINGS



Model being tested in a 30-foot by 60-foot wind tunnel

PROBLEM OPENER

The following two figures are similar. Each dimension of the larger figure is twice the corresponding dimension of the smaller figure. If this doubling of dimensions is continued, how many cubes will there be in the fifth figure?



3 by 2 by 1 6 by 4 by 2

An indispensable phase in the design of large objects is the use of models and computer simulations. Research on ship and plane design, for example, is routinely carried out by testing models of ships in water and models of planes in wind tunnels. The photo above shows a model of a plane being tested in a vertical wind tunnel to study ways to decrease aircraft spin.

We say that two figures, such as a plane and its model, are *similar* if they have the same shape but not necessarily the same size. Most models of objects are similar but smaller than the actual object. However, when the object being modeled is very small, such as a computer chip, the model is similar but larger than the actual object.

SIMILARITY AND SCALE FACTORS

Similar figures can be created by lights and shadows. Hold a flat object perpendicular to a flashlight's rays, and the light will produce a shadow similar in shape to the original figure,

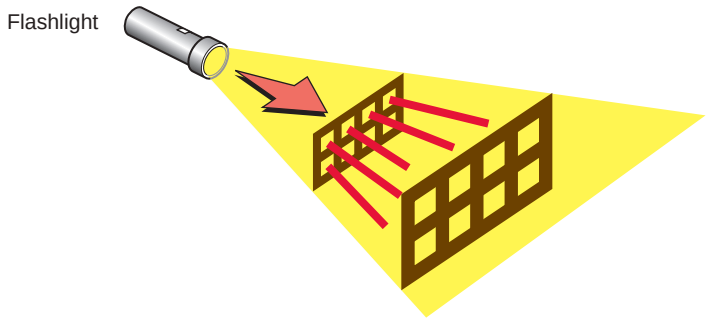


Figure 11.35

as in Figure 11.35. Because the light is from a small bulb and spreads out in the shape of a cone, the shadow is larger than the object.

Light rays and shadows are analogous to mappings and their images. For each point on the object there is a corresponding "shadow point," which is its image. The rays of light are like lines projecting from a central source to the object. This type of mapping is illustrated in Figure 11.36. Point O, which represents the light source, is called the **projection point,** and each point of quadrilateral $ABCD$ is mapped to exactly one point on quadrilateral $A'B'C'D'$. This type of mapping is called a **similarity mapping.** In this example, $ABCD$ is similar to $A'B'C'D'$, and we write $ABCD \sim A'B'C'D'$.

NCTM Standards

Although students will not develop a full understanding of similarity until the middle grades, when they focus on proportionality, in grades 3–5 they can begin to think about similarity in terms of figures that are related by the transformations of magnifying or shrinking. p. 166

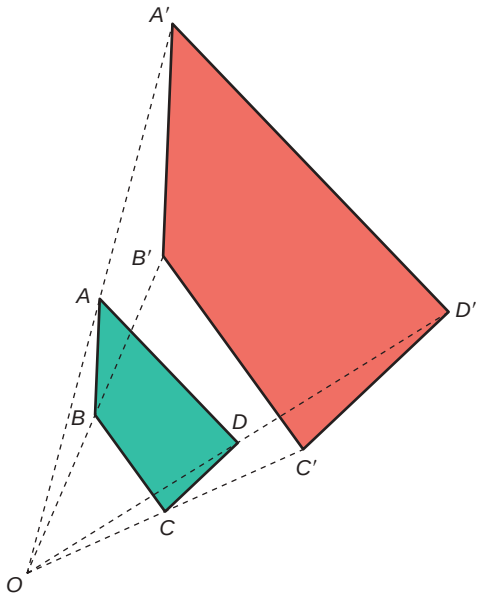


Figure 11.36

Each point on quadrilateral $A'B'C'D'$ in Figure 11.36 is twice the distance from point O as is its corresponding point on quadrilateral $ABCD$. For instance, distance OA' is twice OA, OB' is twice OB, etc. Because of this relationship between points and their images, this mapping is said to have a **scale factor** of 2.

If the scale factor for a similarity mapping is greater than 1, the image is an **enlargement** of the original figure. The mapping from figure XZW to figure $X'Z'W'$ in Figure 11.37 (see page 788) has a scale factor of 3. The distance of each image point of figure $X'Z'W'$ from O is 3 times the distance of its corresponding point on figure XZW from O. That is, OX' is 3 times OX, OZ' is 3 times OZ, and OW' is 3 times OW, etc.

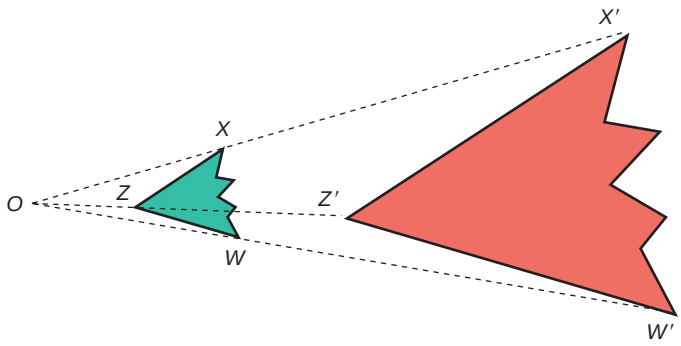


Figure 11.37

When the scale factor is less than 1, the image is a **reduction** of the original figure. In the similarity mapping in Figure 11.38, each of the distances from O to points A, B, C, and D has been multiplied by $\frac{1}{3}$ to get the image points A', B', C', and D'. That is, the larger figure has been reduced by a scale factor of $\frac{1}{3}$.

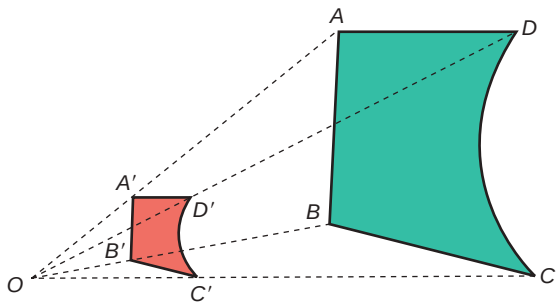


Figure 11.38

It is even possible for a scale factor to be negative. In that case, the original figure and its image are on opposite sides of the projection point, and the image is "upside down" in relation to the original figure. The larger flag shown in Figure 11.39 is projected through point O to the smaller flag using a scale factor of $\frac{-1}{2}$. In particular, A is mapped to A' and G is mapped to G'. As in the previous examples, the scale factor determines the size of the image. With a scale factor of $\frac{-1}{2}$, the distance of each image point from O is half the distance of its corresponding point on the original figure from O (and the point and its image are on opposite sides of the projection point). For example, OG' is half of OG, and OA' is half of OA.

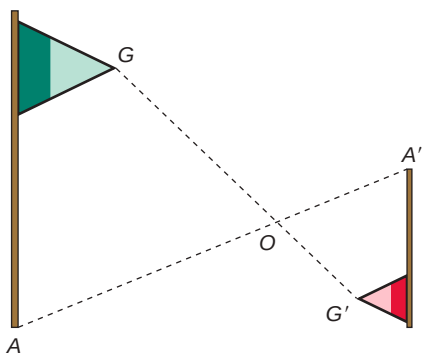


Figure 11.39

The lenses of our eyes and of cameras create inverted images of scenes, much like the images from a similarity mapping with a negative scale factor (see Figure 11.40 on page 789). Such lenses are like projection points, producing a scene upside down on the retinas of our eyes or the film of a camera.

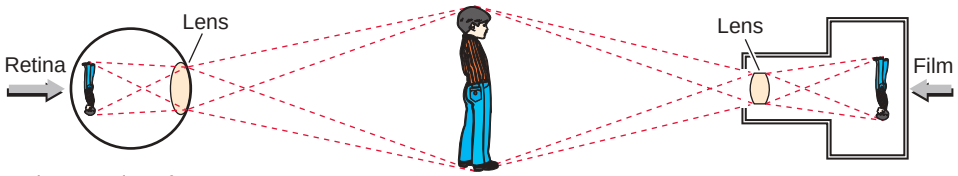


Figure 11.40

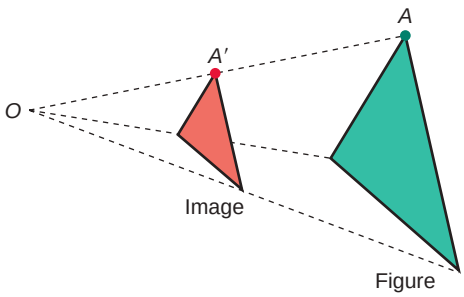
EXAMPLE A

Determine the scale factor for each mapping.

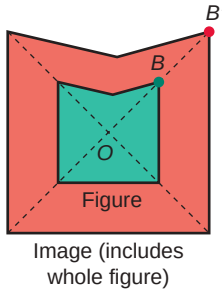
NCTM Standards

Problems that involve constructing or interpreting scale drawings offer students opportunities to use and increase their knowledge of similarity, ratio, and proportionality. Such problems can be created from many sources, such as maps, blueprints, science, and literature. p. 245

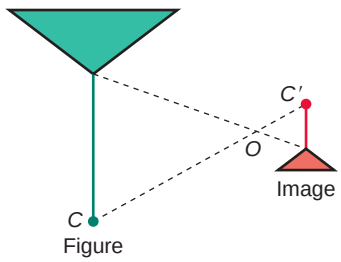
1.



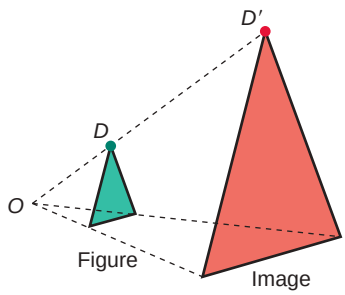
2.



3.



4.



Solution **1.** Each image point is one-half as far from O as is its corresponding point in the figure: $\frac{OA'}{OA} = \frac{1}{2}$. So the scale factor is $\frac{1}{2}$. **2.** Each image point is twice the distance from O as is its corresponding point on the figure: $\frac{OB'}{OB} = 2$. So, the scale factor is 2. **3.** $\frac{OC'}{OC} = \frac{1}{3}$, and since the figure and its image are on opposite sides of the projection point, the scale factor is $\frac{-1}{3}$. **4.** $\frac{OD'}{OD} = 3$, so, the scale factor is 3.

We have seen several examples of similarity mappings. For each mapping the original figure and its image are similar. We state this fact as the definition of similar figures.

Similar Figures Two geometric figures are **similar** if and only if there exists a similarity mapping of one figure onto the other.

The definition for similarity also holds for space figures. The two boxes in Figure 11.41 (see page 790) are similar because the smaller one can be mapped onto the larger one by using a projection from point O inside the small box. The scale factor for this similarity is 3.6. That is, the distance of each vertex of the larger box from O is 3.6 times the distance of its corresponding vertex of the smaller box from O. As with plane figures, a scale factor greater than 1 produces an enlargement of a three-dimensional figure, and a scale factor between 0 and 1 reduces the original figure.

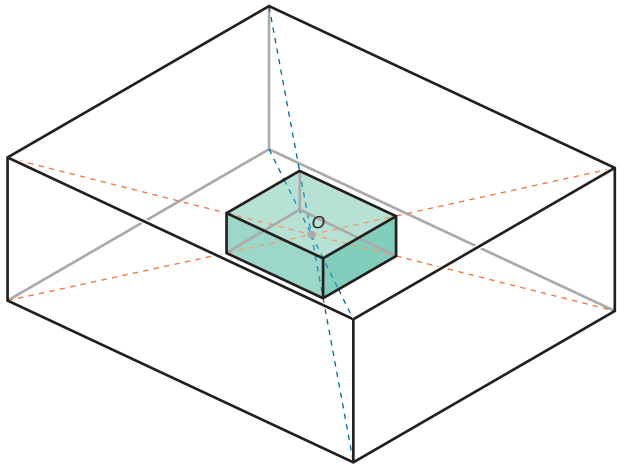


Figure 11.41

In our study of congruence, the mappings we used (rotations, translations, and reflections) *preserved size and shape.* Similarity mappings, on the other hand, except those whose scale factors are ± 1 and in which case the figure and its image are congruent, *do not preserve size* but *do preserve shape.*

SIMILAR POLYGONS

Similarity mappings have two important properties. First, the *measures of angles do not change.* Consider the similarity mapping in Figure 11.42, which maps $\triangle DEF$ to $\triangle D'E'F'$ with a scale factor of 2. In this case,

$$\measuredangle D \cong \measuredangle D' \qquad \measuredangle E \cong \measuredangle E' \qquad \measuredangle F \cong \measuredangle F'$$

Second, the lengths of line segments all *change by the same multiple,* which is the scale factor. The length of each side of $\triangle D'E'F'$ is twice the length of its corresponding side in $\triangle DEF$. That is,

$$D'E' = 2(DE) \qquad E'F' = 2(EF) \qquad D'F' = 2(DF)$$

Another way of stating this condition is to say that the ratios of the lengths of corresponding line segments are equal.

$$\frac{D'E'}{DE} = \frac{E'F'}{EF} = \frac{D'F'}{DF} = \frac{2}{1}$$

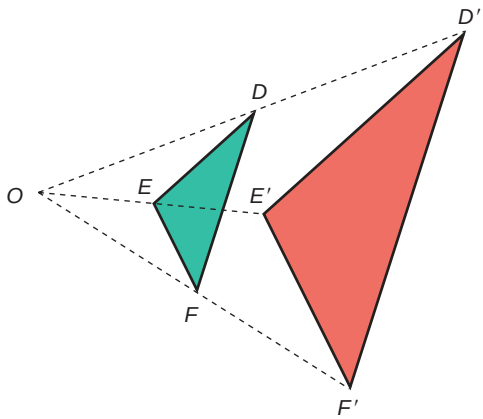


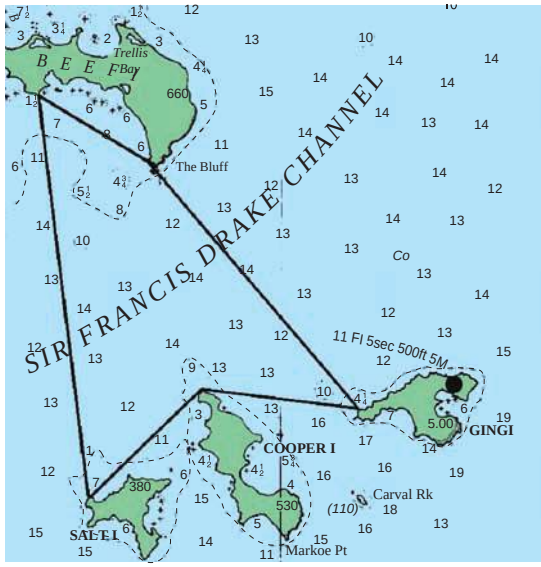
Figure 11.42

NCTM Standards

Students in grades 3–5 should have opportunities to use maps and make simple scale drawings. Grades 6–8 students should extend their understanding of scaling to solve problems involving scale factors. These problems can help students make sense of proportional relationships and develop an understanding of similarity. p. 47

In general, for a scale factor of k, where $k > 0$, each line segment will have an image whose length is k times the length of the original line segment.

There are many applications of similar figures in which it is inconvenient to set up similarity mappings by using projection points. A solution to this problem is to produce similar figures from measurements of angles and distances. The construction of maps and charts is an example. The polygon on the chart in Figure 11.43 connects five points and is approximately similar to the large imaginary polygon over the water that connects the actual landmarks. These positions on the chart were plotted by measuring the five vertex angles and five distances between these islands. The following theorem verifies that such measurements are all that is necessary to obtain similar polygons.



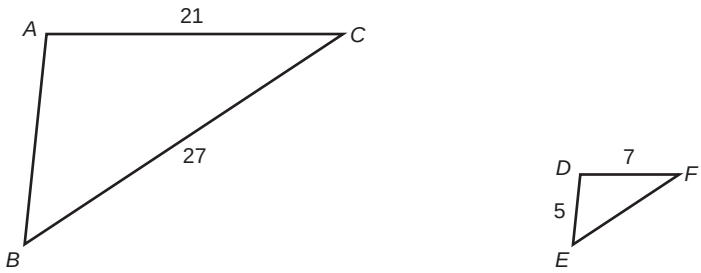
Virgin Islands, West Indies, scale factor 139,000

Figure 11.43

Similar Polygons Two polygons are similar if and only if there is a mapping from one to the other such that:

1. Their corresponding angles are congruent.
2. The lengths of their corresponding sides have the same ratio.

EXAMPLE B If $\triangle ABC$ is similar to $\triangle DEF$ ($\triangle ABC \sim \triangle DEF$), find the lengths of sides $\overline{AB}$ and $\overline{EF}$.



Solution Since $\frac{AC}{DF} = \frac{21}{7} = 3$, the ratio of the lengths of each pair of corresponding sides is 3. Thus,

$$\frac{AB}{DE} = \frac{AB}{5} = 3 \quad \text{so,} \quad AB = 15$$

$$\frac{BC}{EF} = \frac{27}{EF} = 3 \quad \text{so,} \quad EF = 9$$

Notice in Example B that the ratio of the lengths of any two sides of the first triangle is equal to the ratio of the lengths of the corresponding two sides of the second triangle. For example,

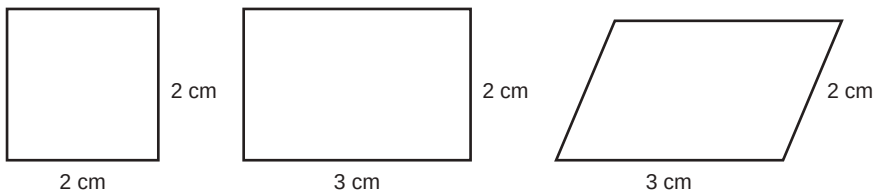
$$\frac{AC}{AB} = \frac{21}{15} \quad \text{and} \quad \frac{DF}{DE} = \frac{7}{5}$$

and these two ratios are equal. In general, if two polygons are similar, *the ratio of the lengths of any two sides of the first polygon is equal to the ratio of the lengths of the corresponding sides of the second polygon.*

To show that two polygons are similar, it is usually necessary to show that both conditions are satisfied: (1) corresponding angles are congruent, and (2) corresponding sides have the same ratio.

EXAMPLE C

1. Compare the following square and rectangle. Which condition for similarity do they satisfy? Are they similar?
2. Compare the rectangle and the parallelogram. Which condition for similarity do they satisfy? Are they similar?



Solution **1.** The square and the rectangle have congruent angles (all 90°), but they do not satisfy the second condition for similar polygons because the ratios of the lengths of their corresponding sides are not equal: $\frac{2}{2} \neq \frac{2}{3}$. So, these polygons are not similar. **2.** The rectangle and the parallelogram satisfy the second condition for similar polygons because the ratios of the lengths of their corresponding sides are equal, but the angles in the rectangle are not congruent to those in the parallelogram. Therefore, these polygons are not similar.

SIMILAR TRIANGLES

To determine if two triangles are similar, it is not necessary to check both conditions for similar polygons. Minimum conditions for similarity of triangles are examined in the following examples.

EXAMPLE D

Construct a triangle having one angle congruent to $\measuredangle A$ and another angle congruent to $\measuredangle B$.



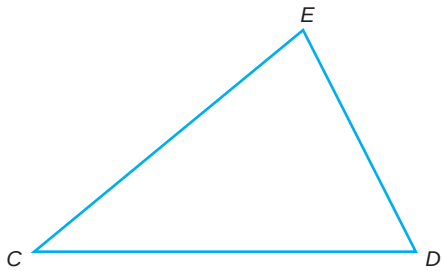
Solution Draw a line segment of arbitrary length, and label its endpoints C and D.



Then construct an angle at C that is congruent to $\measuredangle A$ and an angle at D that is congruent to $\measuredangle B$ (as shown).



Finally, extend the sides of $\measuredangle C$ and $\measuredangle D$, and label their intersection E.



Suppose that instead of beginning with line segment $\overline{CD}$, as in Example D, above, we begin with $\overline{FG}$, which is one-half as long, and as before construct two angles that are congruent to $\measuredangle A$ and $\measuredangle B$. The resulting triangle is shown in Figure 11.44 with $\measuredangle F \cong \measuredangle C$ and $\measuredangle G \cong \measuredangle D$.

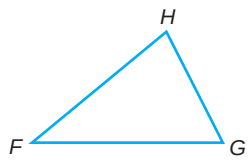


Figure 11.44

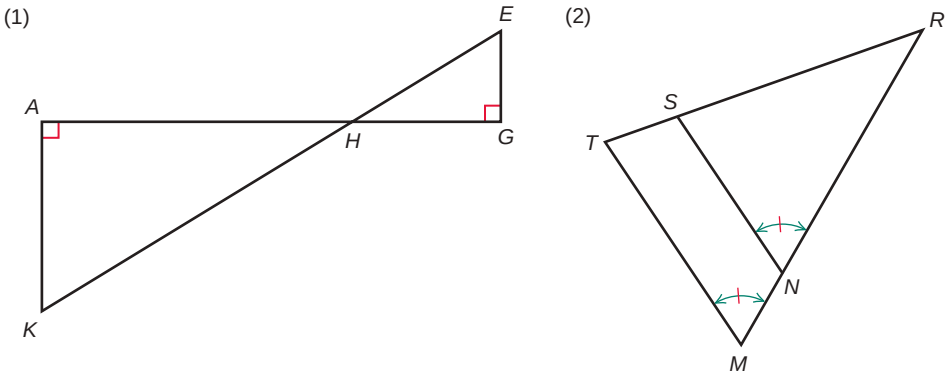
Comparing the lengths of the sides of $\triangle CDE$ and $\triangle FGH$, we see that FH is one-half of CE, GH is one-half of DE, and FG is one-half of CD. So, $\triangle CDE \sim \triangle FGH$. This result suggests the following *similarity property of triangles.*

Angle-Angle (AA) Similarity Property If two angles of one triangle are congruent to two angles of another triangle, then the two triangles are similar.

Notice that for two triangles to be similar, it is only necessary that two angles of one triangle be congruent to two angles of the other triangle, because if that is the case, the third angles of the triangles must be equal. Why?

EXAMPLE E

Find a congruence relationship between the angles of each pair of triangles in figure (1) to show that the triangles are similar. Explain why. Repeat for figure (2).

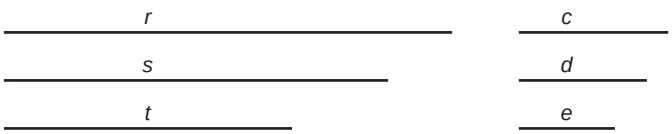


Solution **1.** $\triangle AKH \sim \triangle GEH$ by the AA similarity property because $\measuredangle A \cong \measuredangle G$ and $\measuredangle AHK \cong \measuredangle GHE$ (they are vertical angles). **2.** $\triangle RSN \sim \triangle RTM$ by the AA similarity property because $\measuredangle N \cong \measuredangle M$ and both triangles contain $\measuredangle R$.

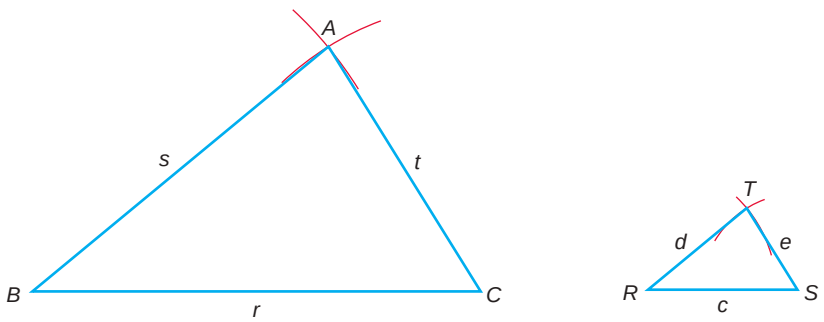
We have seen that in order to show that two triangles are similar, it is only necessary to check the measures of their angles. Example F, on the other hand, considers only the measures of the sides of two triangles.

EXAMPLE F

Two sets of three line segments are shown below. The lengths r, s, and t are 3 times the corresponding lengths c, d, and e. Construct a triangle from each set of segments, and determine if the triangles are similar.



Solution The following two triangles can be constructed by using the construction techniques described in Section 11.1. If the vertices of these triangles are matched so that $A \leftrightarrow T$, $B \leftrightarrow R$, and $C \leftrightarrow S$, the lengths of the corresponding sides have a ratio of 3. Measuring the corresponding angles of the triangles by using a protractor—or comparing the angles by tracing on paper—shows that angles A, B, and C are congruent, respectively, to angles T, R, and S. Thus, $\triangle ABC \sim \triangle TRS$.



Research Statement

Research suggests that students may regard orientation as a salient feature of a geometric figure, since they tend only to see figures in prototypical orientation—for example, with horizontal bases.

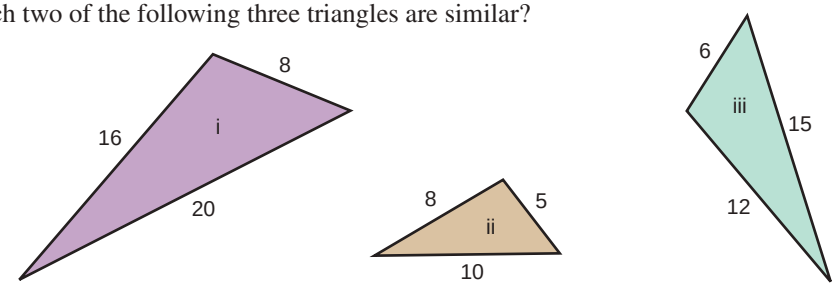
Vinner and Hershkowitz

Example F suggests the following *similarity property of triangles.*

Side-Side-Side (SSS) Similarity Property If the corresponding sides of two triangles are proportional, then the two triangles are similar.

EXAMPLE G

Which two of the following three triangles are similar?



Solution Triangle i is similar to triangle iii. The ratio of the lengths of their corresponding sides is 4 to 3.

$$\frac{8}{6} = \frac{20}{15} = \frac{16}{12} = \frac{4}{3}$$

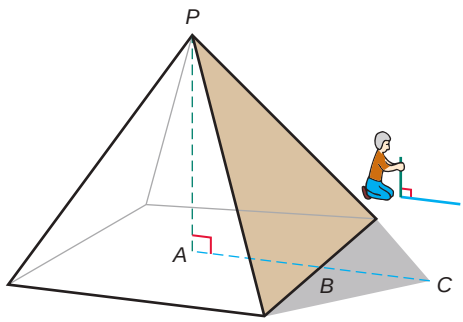
PROBLEM-SOLVING APPLICATION

One important application of similar triangles is in making indirect measurements. It is said that the Greek mathematician Thales computed the height of the Great Pyramid of Egypt through indirect measurements. One account describes his use of shadows and similar triangles. The sun is so far away that in a given vicinity the angles formed by its rays and the ground are approximately congruent. If a stick is held perpendicular to the ground, the stick and its shadow form a small right triangle that is similar to the right triangle formed by the pyramid and its shadow. The next problem shows how Thales might have computed the height of the Great Pyramid.

Problem

At a certain time on a sunny day, the shadow of the Great Pyramid might look as depicted in this figure. Suppose the tip of its shadow is 342 feet from the side of the pyramid (length *BC*). Also, assume that a 6-foot stick placed perpendicular to the ground casts a 9-foot shadow. If the pyramid has a square base with dimensions of 756 feet by 756 feet, what is the vertical height of the pyramid?

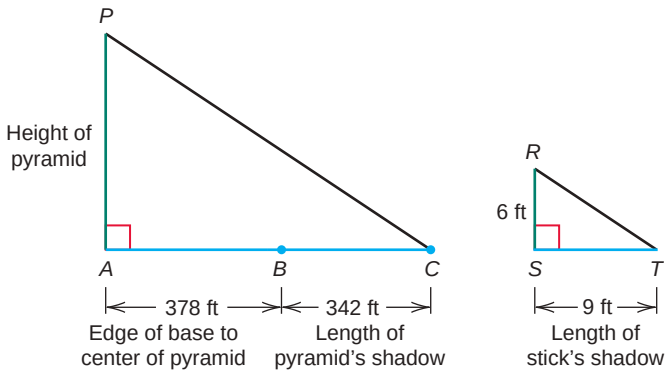
Understanding the Problem The following drawing shows the stick, the pyramid, and the shadows. Point *A* is the foot of the altitude of the pyramid. The distance from *B* to *C* is 342 feet. **Question 1:** What is the distance from *A* to *B*?



NCTM Standards

In the middle grades students should build on their formal and informal experiences with measurable attributes like length, area, and volume. . . . They should also become proficient at measuring angles and using ratio and proportion to solve problems involving scaling, similarity, and derived measures. p. 241

Devising a Plan The stick, the pyramid, and the sun's rays form two similar triangles, as shown in the next figure. Using the ratios of the lengths of corresponding sides, we can find the height of the pyramid. **Question 2:** Why is the large triangle similar to the small one?



Carrying Out the Plan **Question 3:** In light of the fact that the corresponding sides of similar triangles are proportional, what is the height of the pyramid?

Looking Back Once we have the height of the pyramid, the slant height along the face of the pyramid from B to P can be computed by using the Pythagorean theorem. **Question 4:** What is this distance?

Answers to Questions 1–4 **1.** This distance is one-half the width of the side of the pyramid: $\frac{756}{2} = 378$. **2.** $\measuredangle A$ and $\measuredangle S$ are both right angles. $\measuredangle C$ and $\measuredangle T$ are congruent angles that are formed by the sun's rays and the ground. Therefore, $\triangle PAC \sim \triangle RST$ by the AA similarity property of triangles.

3. $\frac{PA}{RS} = \frac{AC}{ST}$; $RS = 6$, $AC = 378 + 342 = 720$, and $ST = 9$. So,

$$\frac{PA}{6} = \frac{720}{9}$$

$$PA = \frac{6(720)}{9}$$

$$PA = 480 \text{ feet}$$

4. $\triangle PAB$ is a right triangle with legs $PA = 480$ and $AB = 378$. So, by the Pythagorean theorem,

$$480^2 + 378^2 = (PB)^2$$

$$230{,}400 + 142{,}884 = (PB)^2$$

$$373{,}284 = (PB)^2$$

$$611 \approx PB$$

So, the slant height of the face of the pyramid is 611 feet to the nearest foot.

HISTORICAL HIGHLIGHT

Thales (636–546 B.C.E.) was one of the earliest of many famous Greek mathematicians and is regarded by historians as the father of geometry. In his early years he traveled widely, learning geometry from the Egyptians and astronomy from the Babylonians. Thales is generally acknowledged as the first to introduce the use of logical proofs based on deductive reasoning, rather than experiments, to support conclusions. Thales was regarded as unusually shrewd in commerce and science, and many anecdotes are told about his

(continues)

cleverness. On one occasion, according to Aristotle, after several years in which the olive trees failed to produce, Thales suspected weather conditions would change and bought up all the olive presses. When the season came with its abundant crop, he was able to rent the presses for large profits. Thales was known as the first of the Seven Sages of Greece, the only mathematician to be so honored. He is supposed to have coined the maxim, "Know thyself."

SCALE FACTORS, AREA, AND VOLUME

The smaller of the two knives in Figure 11.45 is a regular-size Scout knife, whose length is about equal to the width of the palm of your hand. The newspaper clipping says that the bigger knife is "three times larger" than the conventional Scout knife. Does this mean that the length of the larger knife is 3 times greater, or that its surface area or volume is 3 times greater? Phrases such as *twice as large* and *3 times bigger* can be misleading. They often refer to a comparison of linear dimensions, as in the case of these knives (compare their lengths). The 3 in this example refers to the scale factor. It means that the length, width, and height of the big knife are 3 times the corresponding dimensions of the smaller knife. But what can be said about the relative sizes of the areas or volumes of these knives? In the following paragraphs you will see the effect of scale factors on area and volume.



Prepared for Anything

What could be the world's largest Scout-type knife is ready for the world's largest potato. Wayne Goddard, a professional knife-maker who works at his home at 473 Durham St., Eugene, turned this one out for Dennis and Raymond Ellingsen, Eugene knife collectors. Completely functional, the knife is $24\frac{1}{2}$ inches long when opened. It weighs $4\frac{1}{4}$ pounds and is three times larger than the conventional Scout knife.

Figure 11.45

Area The two rectangles in Figure 11.46 are similar. The scale factor from the smaller to the larger is 3. That is, the length and width of the larger rectangle are 3 times the length and width of the smaller rectangle. How do their areas compare? The area of the small rectangle is 4×2 square units. Because each of its linear dimensions is increased by a multiple of 3, the area of the larger rectangle is $(3 \times 4) \times (3 \times 2)$ square units. Using the commutative and associative properties for multiplication, we find that

$$(3 \times 4) \times (3 \times 2) = (3 \times 3) \times (4 \times 2)$$
$$= 3^2 \times (4 \times 2)$$

So, the area of the large rectangle is 9 (the square of the scale factor 3) times the area of the small rectangle. In general, the relationship between similarity and area is as follows:

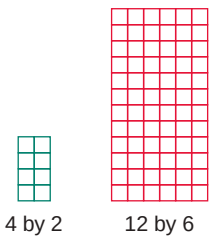


Figure 11.46

Similarity and Area If one plane figure is similar to another figure by a scale factor of k, where k is any positive real number, then the second figure will have an area k^2 times the area of the first figure.

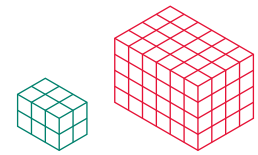


Figure 11.47

The relationship between scale factor and surface area for three-dimensional figures is the same as that for plane figures. Consider the two boxes shown in Figure 11.47. The scale factor from the small box to the large box is 2. That is, the length, width, and height of the large box are each 2 times the corresponding dimension of the small box. Let's compare the surface areas of the sides of these boxes. The front side of the small box has an area of 6, and the front side of the large box has an area of 24. The larger area is 2^2, or 4, times the smaller area. A similar comparison between each face of the small box and the corresponding face of the large box shows that the larger surface area is 4 times the smaller surface area. In general, the surface areas of two similar figures are related by the *square of their scale factor.* If the scale factor is k, where k is any positive real number, then one figure will have a surface area k^2 times the surface area of the other figure.

EXAMPLE H

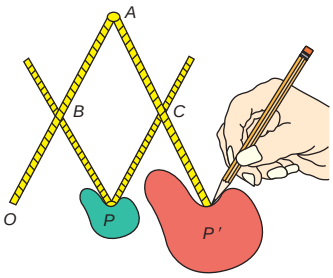
1. If the scale factor from a small photograph to its enlargement is 3 and the area of the small photograph is 15 square inches, what is the area of the large photograph?
2. If the surface area of a box is 52 square centimeters and the scale factor from the box to a larger similar box is 2, what is the surface area of the large box?
3. If the scale factor from a figure to its reduction is $\frac{1}{2}$ and the figure has a surface area of 76 square feet, what is the surface area of the smaller figure?

Solution **1.** $3^2 \times 15 = 135$, so the area of the large photograph is 135 square inches. **2.** $2^2 \times 52 = 208$, so the surface area of the large box is 208 square centimeters. **3.** $\left(\frac{1}{2}\right)^2 \times 76 = 19$, so the surface area of the smaller figure is 19 square feet.

Laboratory Connections

Pantographs

This device can be used for mechanically enlarging or reducing figures. The red figure being traced shows an enlargement of the green figure, but where should the pencil be placed for a reduction? Explore this and related questions in this investigation.



Mathematics Investigation
Chapter 11, Section 3
www.mhhe.com/bbn

Volume There is also a relationship between the volumes of two similar space figures. Consider the volumes of the boxes in Figure 11.47. The volume of the small box is $3 \times 2 \times 2$ cubic units. Because each of its linear dimensions is multiplied by a scale factor of 2, the volume of the large box is $(2 \times 3) \times (2 \times 2) \times (2 \times 2)$ cubic units. Using the commutative and associative properties for multiplication,

$$(2 \times 3) \times (2 \times 2) \times (2 \times 2) = (2 \times 2 \times 2) \times (3 \times 2 \times 2)$$
$$= 2^3 \times (3 \times 2 \times 2)$$

So, the volume of the large box is 8 (the cube of the scale factor 2) times the volume of the small box. In general, the relationship between similarity and volume is as follows:

Similarity and Volume If one space figure is similar to another figure by a scale factor of k, where k is any positive real number, then the second figure will have a volume k^3 times the volume of the first figure.

We are now prepared to examine the relationships between the areas and volumes of the knives in Figure 11.45, on page 797. The scale factor from the small knife to the large knife is 3. Therefore, the large knife has a surface area that is 3^2, or 9, times the surface area of the small knife. The volume of the large knife is 3^3, or 27, times the volume of the small knife.

Let's apply the relationships between length, area, and volume of similar figures to another example.

EXAMPLE I

The photograph on the next page shows a nineteenth-century scale model of a cookstove. This miniature stove was used by a traveling salesperson as a sample and has all the features of a life-size stove. The scale factor from this miniature stove to the life-size stove is 5.

1. If the small stove has a surface area of 300 square centimeters, what is the surface area of the life-size stove?
2. If the oven in the small stove has a volume of 1000 cubic centimeters, what is the volume of the oven in the life-size stove?

NCTM Standards

Geometric modeling and spatial reasoning offer ways to interpret and describe physical environments and can be important tools in problem solving. p. 41



Nineteenth-century scale model of a cookstove

Solution **1.** $5^2 \times 300 = 7500$, so the large stove has a surface area of 7500 square centimeters.
2. $5^3 \times 1000 = 125{,}000$, so the oven in the large stove has a volume of 125,000 cubic centimeters.

SIZES AND SHAPES OF LIVING THINGS

NCTM Standards

The *Curriculum and Evaluation Standards for School Mathematics* (p. 115) state that

> Investigations of two- and three-dimensional models foster an understanding of the different growth rates for linear measures, areas, and volumes of similar figures. These ideas are fundamental to measurement and critical to scientific applications.

One application of growth rates involves the various sizes of animals. For every type of animal there is a most convenient size and shape. One factor that governs the size and shape of a living thing is the ratio of its surface area to its volume. All warm-blooded animals at rest lose approximately the same amount of heat for each unit area of skin. Small animals have too much surface area for their volumes; a major reason they spend so much time eating is to keep warm. For example, 5000 mice may together weigh as much as a person, but their collective surface area and food consumption are each about 17 times greater! At the other end of the scale, large animals tend to overheat because they have too little surface area for their volumes.

Let's take a closer look at the relationship between surface area and volume numbers as the size of an object increases. The table in Figure 11.48, on the next page, lists the values of the surface areas and volumes of four different cubes (disregarding units). As the size of the cube increases, both the surface area and the volume increase, but the volume increases at a faster rate. One way of viewing this change is to form the ratio of surface area to volume. For a $2 \times 2 \times 2$ cube, the ratio is 3, and as the dimensions of the cube increase, the ratio decreases. For a $10 \times 10 \times 10$ cube, the ratio is less than 1.

Cube	Surface Area	Volume	Area/Volume
$2 \times 2 \times 2$	24	8	3
$3 \times 3 \times 3$	54	27	2
$4 \times 4 \times 4$	96	64	1.5
$10 \times 10 \times 10$	600	1000	.6

Figure 11.48

Another way to compare the changes in surface area and volume as the size of a cube increases is with graphs. If the length, width, and height of a cube are each x, the surface area of the cube is $6x^2$ and the volume of the cube is x^3. The graphs of $y = 6x^2$ and $y = x^3$ in Figure 11.49 show that for $x < 6$, the surface area is greater than the volume; for $x = 6$, the surface area equals the volume; and for $x > 6$, the volume is greater than the surface area. (*Note:* It is important to keep in mind that we are comparing only the numerical values of the surface areas and volumes without regard to units.)

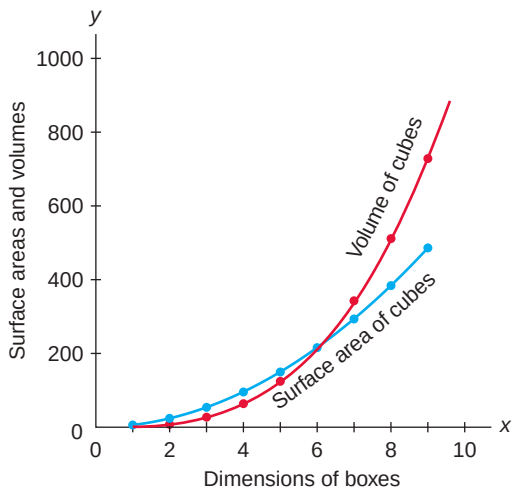


Figure 11.49

To relate these changes in surface area and volume to the problem of maintaining body temperatures, assume that the cubes shown in Figure 11.50 are animals and that the ideal ratio between surface area and volume is 2. In this case the $2 \times 2 \times 2$ animal has too much surface area (it would tend to be too cold), because its surface area-to-volume ratio is 3 $\left(\frac{24}{8}\right)$; the surface area-to-volume ratio for the $3 \times 3 \times 3$ animal is 2 $\left(\frac{54}{27}\right)$, which is just right; and the $4 \times 4 \times 4$ animal has too little surface area (it would tend to be too hot), because its ratio of surface area to volume is 1.5 $\left(\frac{96}{64}\right)$.

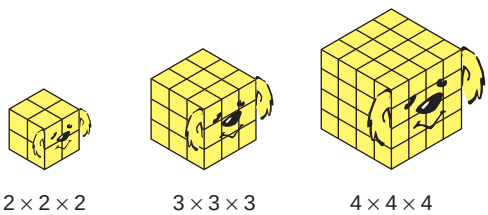
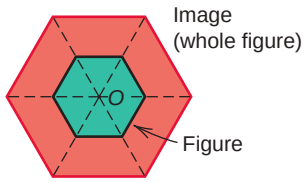


Figure 11.50

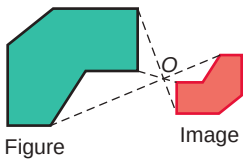
Exercises and Problems 11.3

1. Determine the scale factor for the mappings in parts a and b.

a.

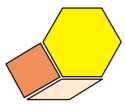


b.



Construct an enlargement by the given scale factor for the figures in parts c and d. Sketch your results.

c. Scale factor 2

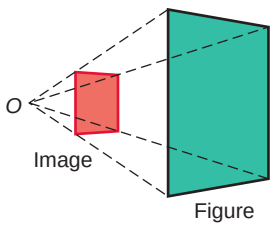


d. Scale factor 3

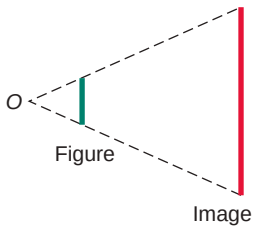


2. Determine the scale factor for the mappings in parts a and b.

a.



b.



Construct an enlargement by the given scale factor for the figures in parts c and d. Sketch your results.

c. Scale factor 2

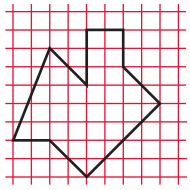


d. Scale factor 3

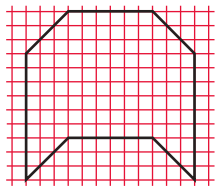


For each figure in exercises 3 and 4, sketch a similar figure using the given scale factor. (Copy rectangular grid paper from the website to use in your sketches.)

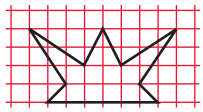
3. a. Scale factor 2



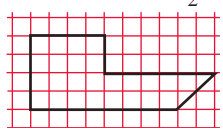
b. Scale factor $\frac{1}{3}$



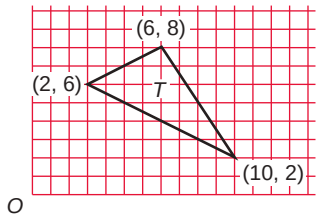
4. a. Scale factor 3



b. Scale factor $\frac{1}{2}$

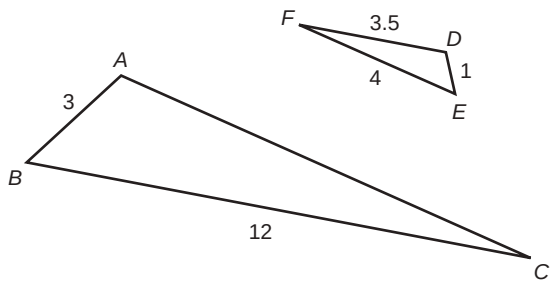


5. Use O as a projection point and find the images of triangle T on the following grid, using the three scale factors of 2, 3, and $\frac{1}{2}$. Look for a relationship between the coordinates of the vertices of T and the coordinates of their images. (Copy rectangular grid paper from the website.)

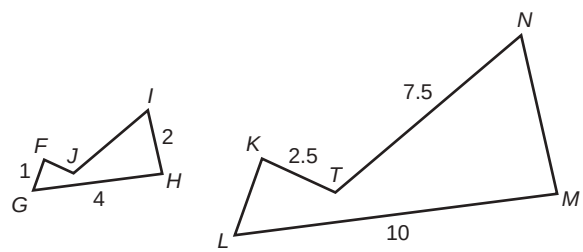


The pairs of polygons in exercises 6 and 7 are similar. Find the missing length for each side of the polygons.

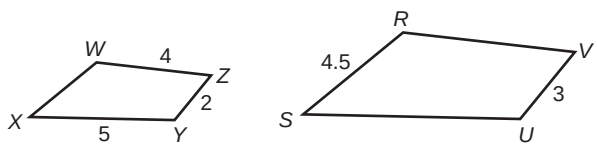
6. a. $\triangle ABC \sim \triangle DEF$



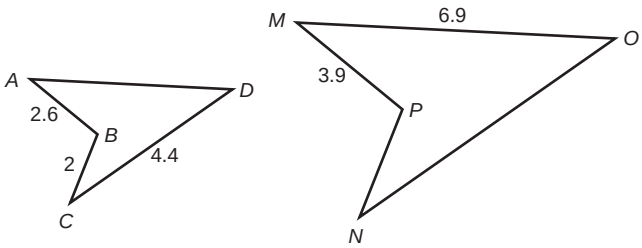
b. $FGHIJ \sim KLMNT$



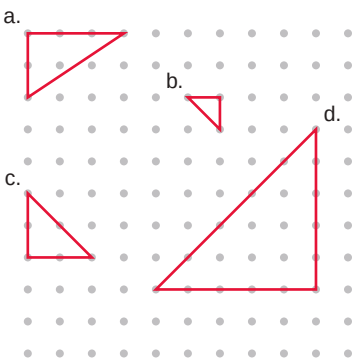
7. a. $WXYZ \sim RSUV$



b. $ADCB \sim MONP$

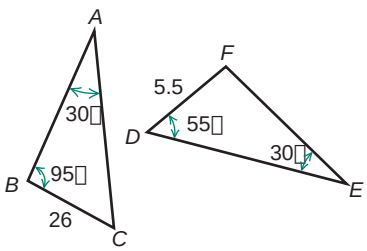


8. Which three of the following triangles are similar to each other? Which conditions are met and which conditions are not met?

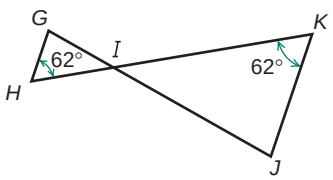


Determine whether the triangles in each pair in exercises 9 and 10 are similar. If so, give a reason and write the similarity correspondence.

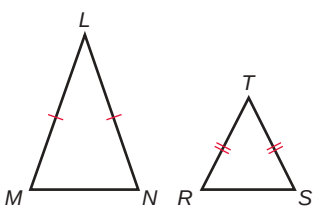
9. a.



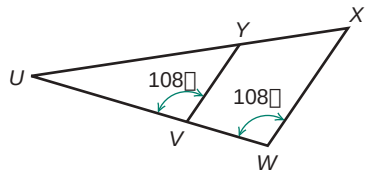
b.



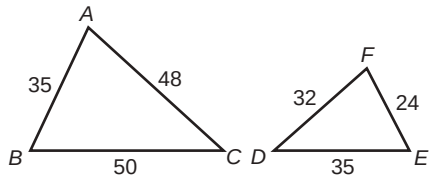
c.



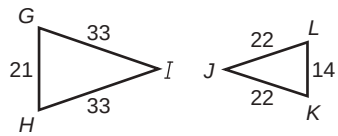
10. a.



b.



c.



Determine whether the figures given in exercises 11 and 12 are similar, and if so, explain why. If they are not similar, show a counterexample with measurements.

11. a. Any two squares
b. Any two isosceles triangles
c. Any two rhombuses
d. Any two regular octagons

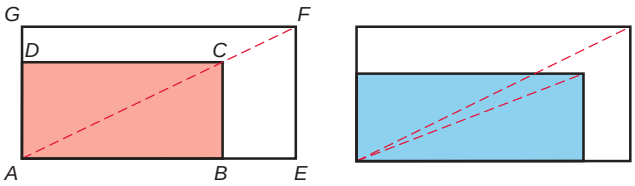
12. a. Any two equilateral triangles
b. Any two right triangles
c. Any two congruent polygons
d. Any two rectangles

13. Construct the triangles in parts a and b.
a. A triangle having angles of 42° and 70° and an included side of length 15
b. A triangle having angles of 42° and 70° and an included side of length 20
c. Are the triangles in parts a and b similar?
d. What conjecture about similar triangles is suggested by these constructions?

14. Construct the triangles in parts a and b.
a. A triangle having sides of 4 and 6 centimeters and an included angle of 45°
b. A triangle having sides of 2 and 3 centimeters and an included angle of 45°
c. Are the triangles in parts a and b similar?
d. What conjecture about similar triangles is suggested by these constructions?

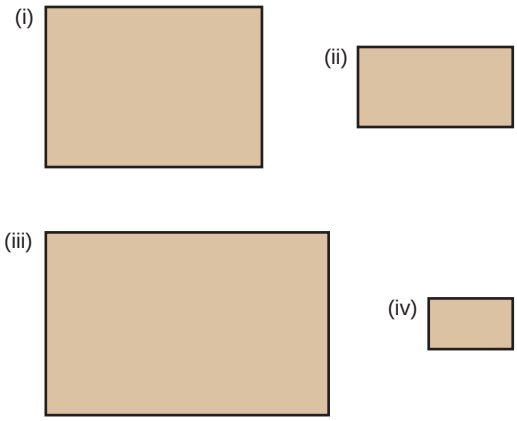
An easy way to test rectangles for similarity is illustrated in exercises 15 and 16. If one rectangle is placed on the other so that their right angles coincide, as shown on the following page, then the rectangles are similar if their diagonals lie

on the same line. For example, rectangle *AEFG* is similar to rectangle *ABCD*, but the two rectangles whose diagonals do not lie on the same line are not similar.



15. Explain why rectangle *ABCD* above is similar to rectangle *AEFG*. (*Hint:* To show the sides are proportional, use similar triangles.)

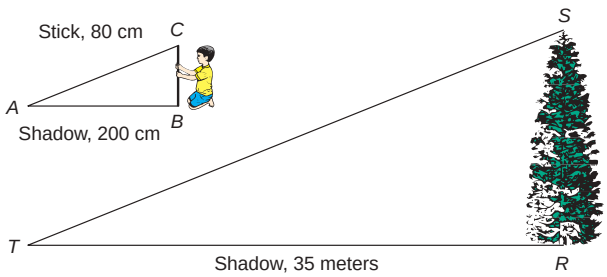
16. Trace the following rectangles onto another sheet, and use the *diagonal test* to find out which two are similar.



17. The stick and the tree in the following figure form right angles with the ground. Furthermore $\measuredangle CAB$ and $\measuredangle STR$ are congruent because they are formed by the sun's rays.

a. Why are the triangles in this figure similar?

b. Use the given information to find the height of the tree.



18. Use the figures in exercise 3 to answer these questions.

a. The area of the image of figure a is how many times the area of figure a?

b. The area of the image of figure b is what fraction of the area of the original figure b?

19. Use the figures in exercise 4 to answer these questions.

a. The area of the image of figure a is how many times the area of figure a?

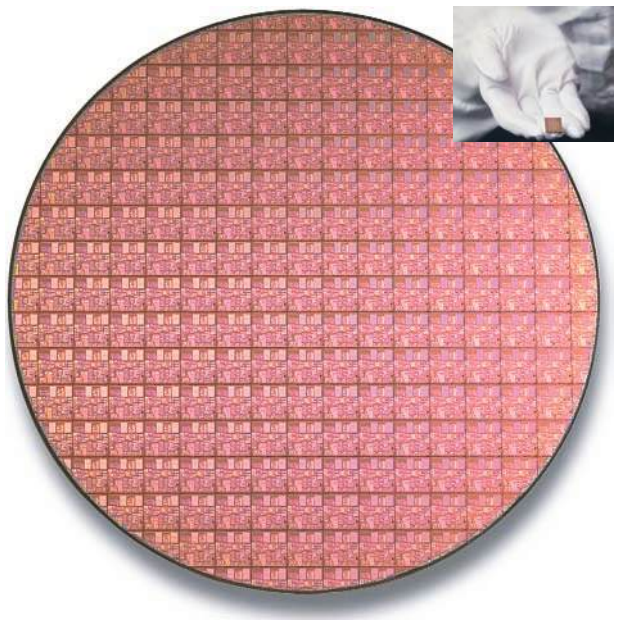
b. The area of the image of figure b is what fraction of the area of figure b?

Triangle *T* in exercise 5 has an area of 16 square units. What is the area of the enlargement or reduction of this triangle for a mapping with the scale factors given in exercises 20 and 21?

20. **a.** 2 **b.** $\frac{1}{3}$

21. **a.** $\frac{1}{2}$ **b.** 3

22. The computer wafer shown here contains a few hundred microchips. One such microchip is shown in the technician's hand. These new chips have wires with a diameter of .13 microns. This diameter is $\frac{1}{1000}$ the width of a human hair.



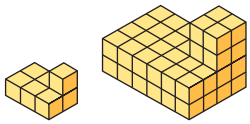
a. One company recently increased the size of its circular wafers from a diameter of 20 cm to a diameter of 30 cm. What is the scale factor from the smaller to the larger wafer?

b. The area of the 30-cm wafer is how many times the area of the 20-cm wafer?

c. If the 20-cm wafer contained 130 chips, approximately how many chips of the same size will the 30-cm wafer contain?

d. The area of the cross section of a human hair is how many times the area of the cross section of a .13-micron wire?

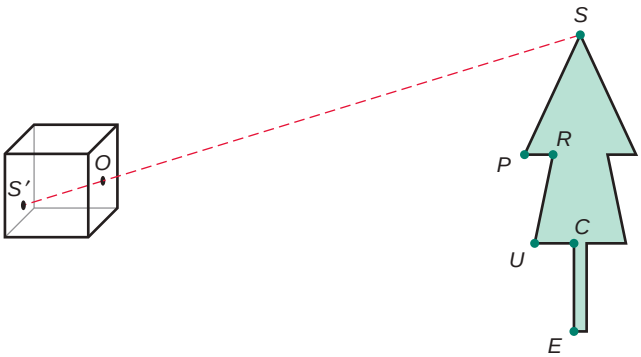
23. These two three-dimensional figures are similar. The scale factor from the small figure to the large figure is 2.



The surface area and volume of the smaller figure above is given in the top row of the following table. Complete the table for figures that are similar to the smaller figure, using the given scale factors.

Scale Factor	Surface Area (square units)	Volume (cubic units)
1	26	7
2	_____	_____
3	_____	_____
4	_____	_____
5	_____	_____

24. A pinhole camera can easily be made from a box. When the pinhole is uncovered, rays of light reflected from an object strike light-sensitive film. These rays of light travel in straight lines from the object to the pinhole of the camera, like lines through a projection point. Without a lens to gather light rays and increase their intensity, it takes from 60 to 75 seconds for enough light to pass through the pinhole for the image to be recorded.

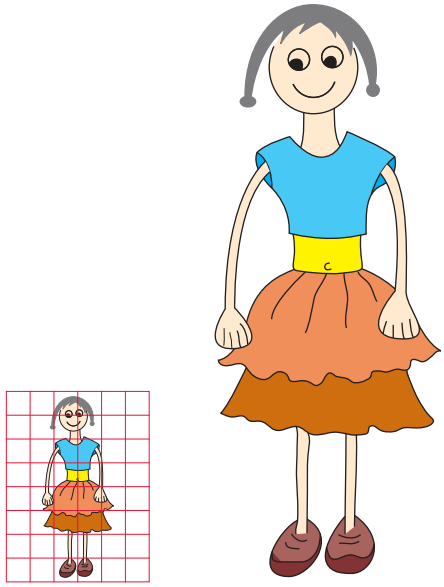


a. Draw lines from the lettered points on this tree through point O, and label their images on the back wall (film) of the camera.

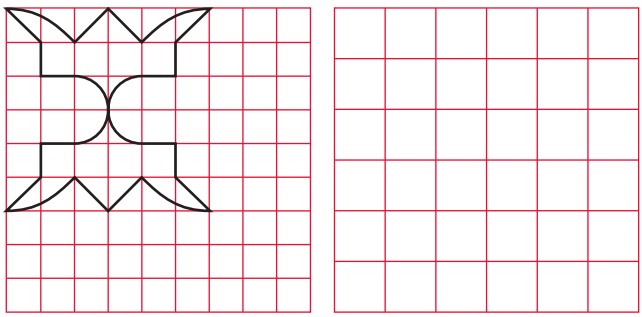
b. Sketch the complete image of the tree.

c. OS is approximately 66 millimeters and OS' is approximately 11 millimeters. What is the scale factor for this projection?

25. Similar figures can be obtained by reproducing a figure from one grid onto another grid of a different size. This procedure is commonly used for enlarging quilting and sewing patterns. What is the scale factor for the enlargement of the patchwork doll?

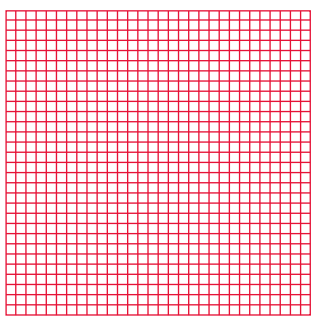


26. a. Reproduce the figure from grid A below onto grid B by copying one square at a time. What is the scale factor from grid A to grid B?



Grid A Grid B

b. Reproduce the figure from grid B onto grid C below. What is the scale factor from grid B to grid C?

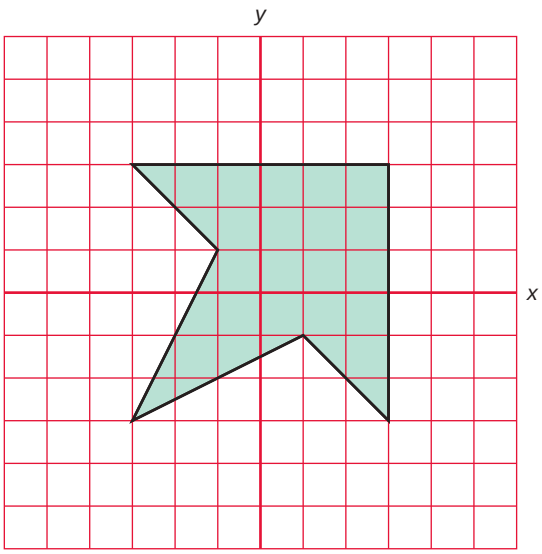


Grid C

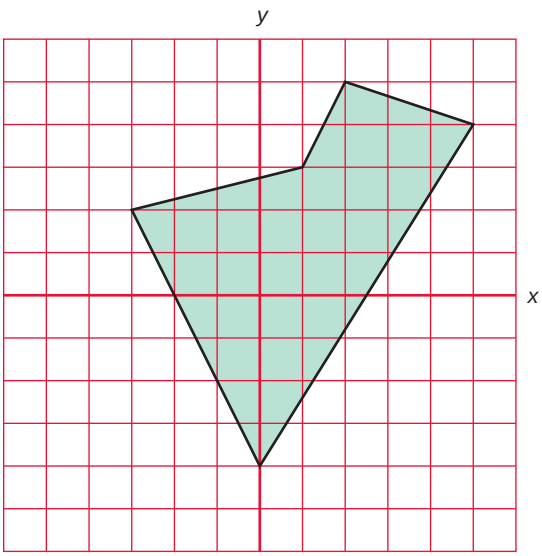
c. The scale factor from grid A to grid C is $\frac{3}{10}$. How can this scale factor be obtained from the two scale factors from grid A to grid B and from grid B to grid C?

Similarity mappings are indicated in exercises 27 through 29 by relating the coordinates of each point on the given figure to the coordinates of its image. For example, the mapping in 27 doubles the coordinates of each point: point $(^{-}1, 1)$ gets mapped to $(^{-}2, 2)$. Sketch the image of each figure. What is the scale factor for each mapping? (Copy coordinate grid paper from the website.)

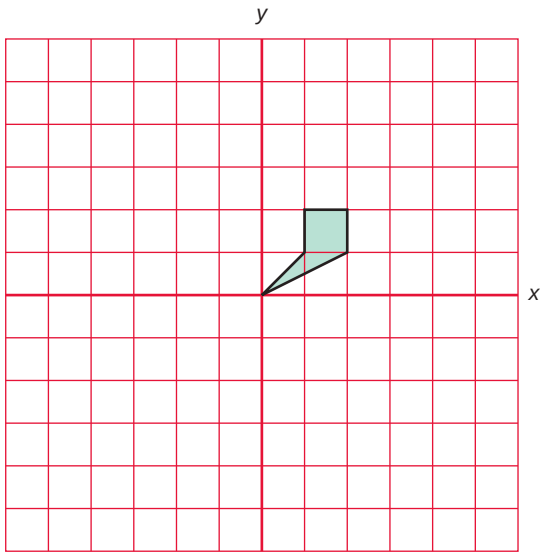
27. $(x, y) \rightarrow (2x, 2y)$



28. $(x, y) \rightarrow \left(\frac{x}{2}, \frac{y}{2}\right)$

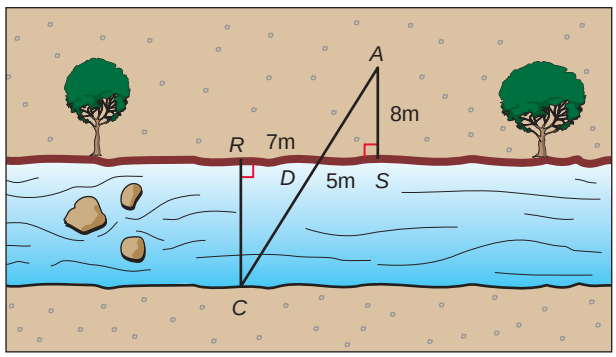


29. $(x, y) \rightarrow (^{-}3x, ^{-}3y)$



Reasoning and Problem Solving

30. A group of scouts formed the triangles with the measurements shown below, in hopes of finding the distance across the river. Will their method work, and if so, what is the width of the river?



31. During a football game, Beth and her friend were standing next to the goal posts and trying to estimate the height of the posts. Beth noticed that the length of the shadow of the posts was 10 yards, the length of the end zone, and her friend's shadow had a length of 6 feet. Knowing that her friend was 6 feet tall, she quickly computed the height of the goal posts.

a. What is the height of the posts in feet? Explain your reasoning.

b. In 1989, a change in football regulations allowed the height of the goal posts to be increased by 10 feet. If the posts Beth saw had been 10 feet higher, what would have been the length of their shadow in yards?

32. This photo shows a model of a 2005 VW Beetle and a Pilot pen.



a. What is the scale factor to the nearest tenth from the pen in the photo to the life-size version of the pen, if the pen's actual length is 5.25 inches?
b. Use the scale factor from part a to determine the length to the nearest tenth of an inch of the VW model.
c. Using your answer in part b and the fact that the scale factor from the model to the life-size VW car is 14, what is the length to the nearest tenth of a foot of the life-size car?
d. If the model car has a cargo volume of 17.0 cubic inches, what is the cargo volume to the nearest tenth of a cubic foot of the life-size car?

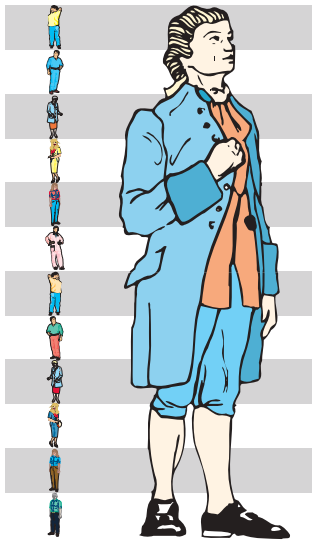
33. Engineers use models of planes to gain information about wing and fuselage (central body) designs. The scale factor from this model to the full-size plane is 15.



High-speed aircraft model in supersonic wind tunnel

a. The lift of an airplane depends on the surface area of its wings. The surface area of the wings of the full-size plane is how many times the surface area of the wings of its model?
b. The weight of a plane depends on its volume. The volume of the full-size plane is how many times the volume of its model?
c. If the tip of the wing of the model flaps a distance of 2 centimeters during a wind-tunnel test, what is the distance that the tip of the wing of the full-size plane flaps during flight?

34. In *Gulliver's Travels,* by Jonathan Swift, Gulliver went to the kingdom of Lilliput, where he found that he was 12 times the height of the average Lilliputian.



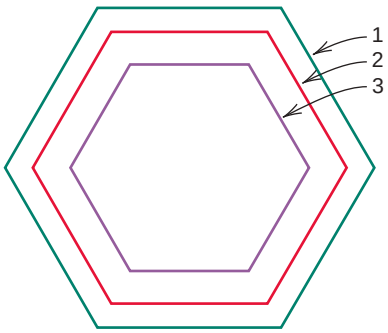
a. The Lilliputians computed Gulliver's surface area to make him a suit of clothes. The amount of material needed for his suit is about how many times the amount of material needed for one of theirs?
b. The Lilliputians computed Gulliver's volume to determine how much food he would need. The amount Gulliver required is about how many times the amount required by a Lilliputian?
c. Tiny people like the Lilliputians can exist only in fairy tales because in real life the ratio of their volume to their surface area would not allow them to maintain the proper body temperature. Would they be too warm or too cold?

35. Alisa is investigating projections by using different projection points, but keeping a constant scale factor of 2. She wonders if using a projection point inside a figure (point *M* in the figure on the next page) will produce the same-size image as using a projection point outside the figure (point *K* in the figure). She also wonders what will happen to the size of the image if the projection point is moved farther away from the figure. Sketch a few images of the pentagon for projections with a scale factor of 2, and form a conjecture regarding the location of the projection point.



36. Featured Strategy: Making an Organized List. The copier in Mr. Gary's print shop has five buttons for determining the size of a reproduction. If button 1 is used, the reproduction is congruent to the original. Buttons 2, 3, 4, and 5 reduce the original size by scale factors of .85, .76, .65, and .58, respectively. Using two buttons in sequence or just one button, what scale factors can Mr. Gary obtain on this copier?

a. Understanding the Problem. The hexagons below were reproduced on this copier. The length of the side of the red hexagon (2) is .85 times the length of the side of the green hexagon (1). The purple hexagon (3) was obtained by using button 2 on hexagon (1) and then using this reproduction and button 3. What is the scale factor from the green hexagon (1) to the purple hexagon (3)?

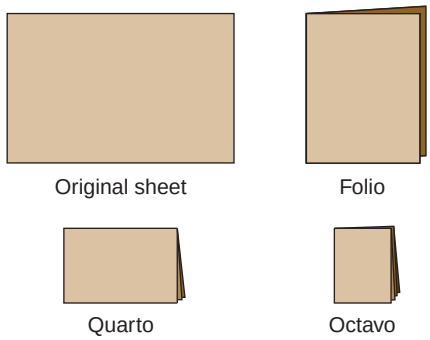


b. Devising a Plan. We could begin by *making an organized list* of the different ways that buttons 2, 3, 4, and 5 can be paired (a button can be paired with itself). What are they?

c. Carrying Out the Plan. List the different scale factors that can be obtained by using a single button or a combination of two buttons and rounding each scale factor to the nearest hundredth.

d. Looking Back. Mr. Gary also has a second copier that enlarges the size of a figure by a scale factor of 2. Using this machine once and a setting on the first copier once, what additional scale factors can Mr. Gary obtain?

37. The pages of a book come from the printing press as large, flat rectangular sheets of paper. Each sheet is fed through a series of rollers and folded in half several times. A sheet that has been folded once is called a *folio.* Half of a folio is a *quarto,* and half of a quarto is an *octavo.* Each fold is perpendicular to the previous fold.



a. Which pairs of these rectangles (original, folio, quarto, octavo) are similar? Explain why.

b. If this folding pattern is continued, which of the resulting rectangles will be similar?

38. The nickel in this photo shows that the other objects are miniature and not life-size.



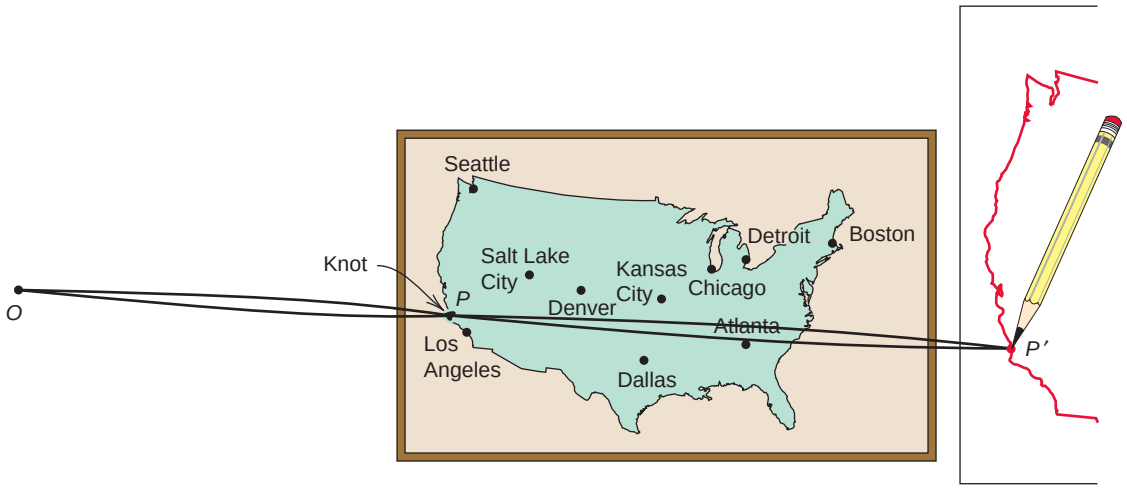
a. What is the scale factor to the nearest tenth from the objects in this photo to the miniature objects? (*Hint:* Measure the nickel on the table and use this measurement to find the scale factor for the photo.)

b. What is the length to the nearest centimeter of the front edge of the miniature table?

c. What is the diameter of the miniature plates to the nearest centimeter?

d. What is the height of the miniature cream pitcher to the nearest centimeter?

39. Loop or knot two elastic bands together, and hold one end fixed at point O. Stretch the bands so that as the knot at point P traces one figure, a pencil at point P' traces an enlargement. Use this method to enlarge the following map of the United States. Assume the distance from O to P' is 2.3 times the distance from O to P. How many times greater will the distance from



Denver to Kansas City be on the enlarged map than on the small map?

Teaching Questions

1. Addison says that all squares are similar and her partner agrees but says that all rectangles are similar to each other too. Is either student correct? Explain why or why not with examples.

2. One of your students wants to reduce a photo measuring 18.5 cm wide and 14.5 cm high to the maximum size that will fit into a space measuring 17 cm wide and 6.5 cm high. You know the copy machine enables you to enlarge or shrink an image by adjusting an enlarge-reduce setting in whole number percents between 0 percent and 200 percent. You know the 100 percent setting reproduces an image congruent to the original photo. How can you help this student determine what machine setting will be needed so the reduction can be done with one copy?

3. The relationship between the side lengths of a plane figure and its area is an important concept to develop. Design a sequence of pattern block activities, using no formulas, to help middle school students discover that figures enlarged by scale factors of 2, 3, or 4 have areas that increase by factors of 4, 9, and 16, respectively. Use diagrams to illustrate your activities.

4. Draw a small rectangle in the center of a sheet of grid paper. Place a point A anywhere inside the rectangle. With point A as the projection point, and using a scale factor of 3, draw the image of the rectangle. Put a point B anywhere on the perimeter of the original rectangle. Using B as the projection point and using a scale factor of 3, draw the image of the original rectangle. Put a projection point C anywhere outside the original rectangle and find a third image of the rectangle using projection point C and using a scale factor of 3. Compare the three images for the three projection points and form a conjecture regarding the placement of the projection points and the size of the resulting images.

Classroom Connections

1. In your library, find the article "On Being the Right Size," in *The World of Mathematics* by James R. Newman. This interesting essay tells you why animals are the size they are and why, in our world, a mouse the size of an elephant is impossible. Summarize the points in the article that you think would be of interest to elementary school students and would promote a discussion of the size and shape of living things.

2. The **Historical Highlight** in this section features the Greek mathematician Thales. Research Thales using the history links on the website or books in your library to learn more about this famous mathematician. Write a summary of historical information about Thales including a description of one or more of his accomplishments that would be appropriate for, and interesting to, middle school students.

3. Read the **Standards** statements on pages 789 and 791. What do these statements propose? Describe an elementary school activity to enable students to increase their knowledge of similarity, ratio, and proportion as suggested by these statements.

4. Based on your experience in doing the one-page **Math Activity** at the beginning of this section, design and write two pattern block activities that you could use in an elementary school classroom to help students construct similar figures. In your write-up, include diagrams and questions you can pose to promote student discussion about the characteristics of similar figures.

CHAPTER 11 REVIEW

1. **Mappings**
   a. A **mapping** is a function that assigns points to points such that to each point in one set there corresponds a unique point, called the **image,** in the second set.
   b. A mapping of $\triangle ABC$ to $\triangle RST$ such that R, S, and T are the images of A, B, and C, respectively, creates the following corresponding parts: **corresponding vertices,** $A \leftrightarrow R$, $B \leftrightarrow S$, $C \leftrightarrow T$; **corresponding sides,** $\overline{AB} \leftrightarrow \overline{RS}$, $\overline{BC} \leftrightarrow \overline{ST}$, $\overline{AC} \leftrightarrow \overline{RT}$; and **corresponding angles,** $\measuredangle A \leftrightarrow \measuredangle R$, $\measuredangle B \leftrightarrow \measuredangle S$, $\measuredangle C \leftrightarrow \measuredangle T$.
   c. A **translation** is a mapping that can be described as a sliding motion where each point is moved the same distance and in the same direction.
   d. A **reflection over a line** or **plane** is a mapping that can be described as folding in which each point on one side of the line or plane will coincide with its image on the other side of the line or plane (respectively).
   e. A **rotation** of a plane figure is a mapping that can be described as turning about a point, called the **center of rotation,** for a fixed angle.
   f. A translation followed by a reflection about a line that is parallel to the line of translation is called a **glide reflection.**
   g. Points that do not move for a mapping are called **fixed points.**
   h. If figure A is mapped to figure A' by one mapping and figure A' is mapped to figure A'' by a second mapping, the single mapping that maps A to A'' is called a **composition of mappings.**

2. **Congruence**
   a. Two **line segments** are **congruent** if they have the same length. Two **angles** are **congruent** if they have the same measure. The **included side** of two angles is the side that is common to the two angles.
   b. Two polygons are **congruent** if and only if there is a mapping from one to the other such that (1) corresponding sides are congruent and (2) corresponding angles are congruent.
   c. **Side-Side-Side Congruence Property (SSS)** If three sides of one triangle are congruent to three sides of another triangle, the two triangles are congruent.
   d. **Side-Angle-Side Congruence Property (SAS)** If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, the two triangles are congruent.
   e. **Angle-Side-Angle Congruence Property (ASA)** If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, the two triangles are congruent.
   f. For each of the following mappings, a figure and its image are congruent: **translation, reflection, rotation,** or **glide reflection.**

3. **Similarity**
   a. Two geometric figures are **similar** if and only if there exists a similarity mapping of one figure onto the other.
   b. Two polygons are **similar** if and only if there is a mapping from one to the other such that (1) corresponding angles are congruent and (2) lengths of corresponding sides have the same ratio.
   c. **Angle-Angle Similarity Property (AA)** If two angles of one triangle are congruent to two angles of another triangle, the two triangles are similar.
   d. **Side-Side-Side Similarity Property (SSS)** If the lengths of corresponding sides of two triangles are proportional, the two triangles are similar.
   e. If the **scale factor** for a similarity mapping is greater than 1, the image is an **enlargement.** If the scale factor is less than 1 and greater than zero, the image is a **reduction.**

4. **Geometric terms**
   a. A line that is perpendicular to a segment at its midpoint is called the **perpendicular bisector** of the segment.
   b. A circle that contains all the vertices of a polygon is called a **circumscribed circle.**
   c. A polygon is called an **inscribed polygon** if all its vertices are points of a circle.
   d. A **tangent** to a point P on a circle is a line through P that is perpendicular to the radius from the center of the circle to point P.
   e. A mapping for which the lengths of line segments is the same as their images is called a **distance-preserving mapping.**

5. **Geometric properties and theorems**
   a. **Triangle Inequality** The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

b. Perpendicular Bisector Theorem A point is on the perpendicular bisector of a line segment if and only if it is equidistant from the endpoints of the segment.

c. Two geometric figures are **congruent** if and only if there exists a translation, reflection, rotation, or glide reflection of one figure onto the other.

d. If a plane figure is similar to another figure by a scale factor of k, where k is any positive real number, then the second figure will have an area k^2 times the area of the first figure.

e. If a figure in space is similar to another figure by a scale factor of k, where k is any positive real number, then the second figure will have a volume k^3 times the volume of the first figure.

6. Constructions

a. A geometric figure produced with a straightedge and compass is called a **construction.**

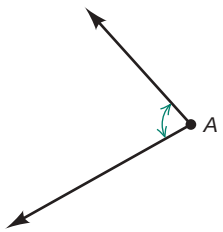
b. A few types of constructions:

Copying a line segment
Copying an angle
Bisecting a line segment
Bisecting an angle
Constructing a perpendicular to a line through a point not on the line
Constructing a line parallel to a given line through a point not on the line
Circumscribing a circle about a triangle

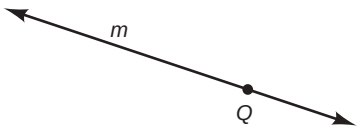
CHAPTER 11 TEST

1. Use a compass and straightedge to carry out each construction. Explain your steps.

a. The bisector of $\angle A$



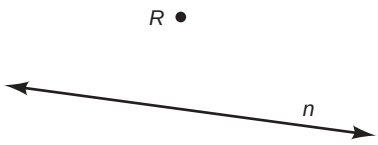
b. The perpendicular to line m through point Q



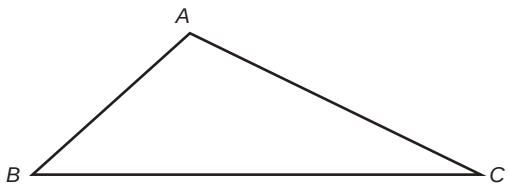
c. The line through point P that is parallel to line ℓ



d. The perpendicular to line n through point R



2. Construct the circumscribed circle about $\triangle ABC$.



3. Construct a triangle, if possible, whose sides are congruent to the given line segments. If it is not possible, explain why.

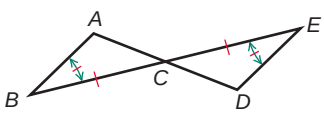
a.



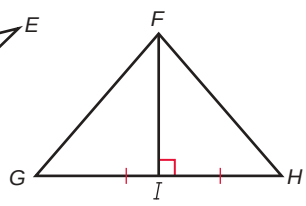
b.

4. Which of the following pairs of triangles are congruent? For each congruent pair, state the appropriate congruence property of triangles and write the congruence correspondence.

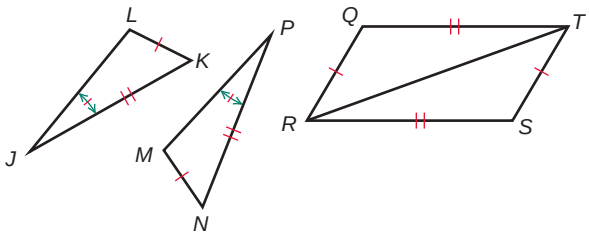
a.



b.

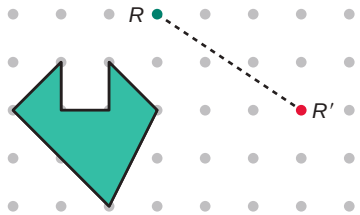


c. **d.**

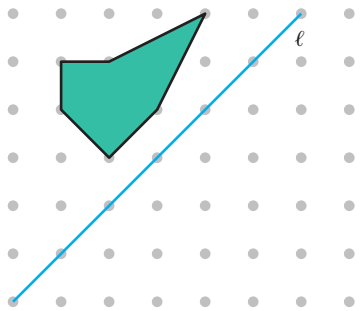


5. Sketch the image for each mapping.

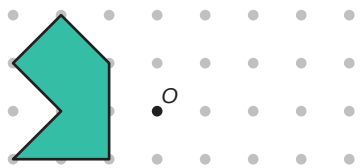
a. A translation that maps R to R'



b. A reflection about line ℓ



c. A 90° clockwise rotation about point O



6. Describe the single mapping that is equivalent to the composition of the two given mappings.

a. A clockwise rotation of 45° followed by a counterclockwise rotation of 70° about the same center of rotation

b. A reflection about line ℓ followed by a reflection about line m, when ℓ and m are parallel lines

c. A translation of P to P' that is to the right 12 units and up 8 units, followed by a translation of P' to P'' that is to the right 5 units and down 10 units

7. Determine the number of congruence mappings of each polygon onto itself and explain your reasoning.

a. Isosceles triangle

b. Regular hexagon

c. Rectangle

8. Show and explain how to alter the sides of the given polygon to create an Escher-type tessellation.

a. Equilateral triangle

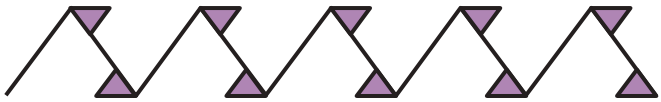
b. Parallelogram

9. Name all the transformations that will map each frieze pattern onto itself. (Assume that these patterns extend indefinitely to the right and left.)

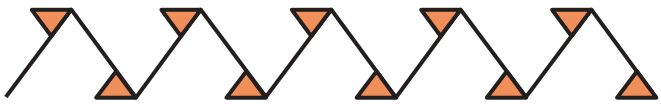
a.



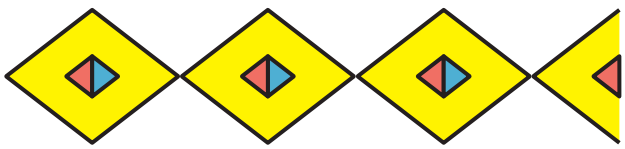
b.



c.

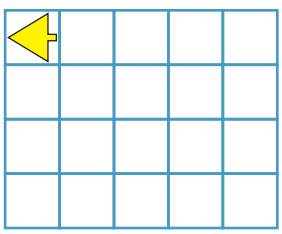


d.

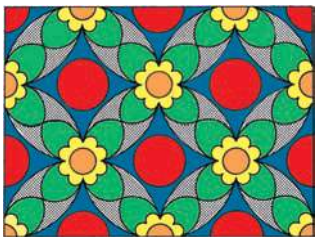


10. Complete the pattern in each grid square by carrying out the mappings on the basic figure in the small square in the upper left corner of the grid. (Copy rectangular grid paper from the website.)

Rows: Reflect about the right sides of the squares.
Columns: Reflect about the lower sides of the squares.

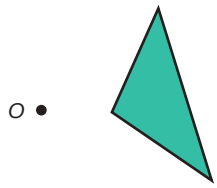


11. Mark off a grid and a square region that can be used as a basic figure to generate this wallpaper pattern. Describe the mappings for obtaining the rows and columns.



12. Using point O as the projection point, determine the image of each figure for the given scale factor.

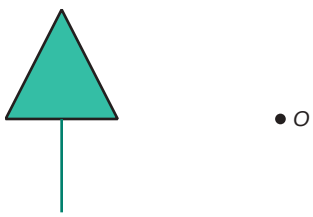
a. Scale factor of 2



b. Scale factor of $\frac{1}{3}$

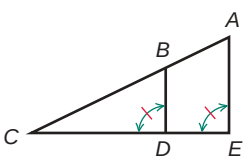


c. Scale factor of $\frac{-1}{2}$

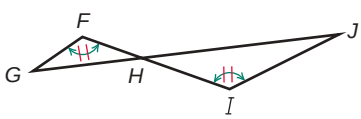


13. Which of the following pairs of triangles are similar? For each similar pair, state the appropriate similarity property of triangles and write the similarity correspondence.

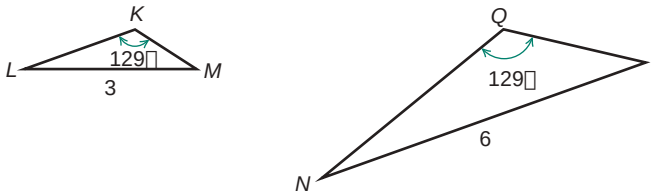
a.



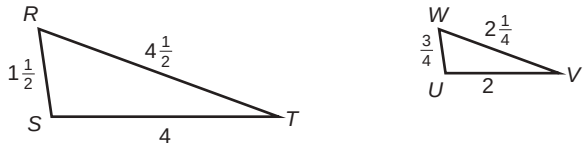
b.



c.



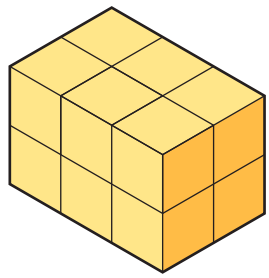
d.



14. If the figures listed here are similar, explain why. If not, sketch a counterexample with measurements.

a. Two rectangles
b. Two squares
c. Two right triangles
d. Two equilateral triangles
e. Two congruent quadrilaterals

15. The following figure has a volume of 12 cubic units and a surface area of 32 square units.



a. What is the volume of an enlargement of the preceding figure for a scale factor of 3?
b. What is the surface area of an enlargement with a scale factor of 3?
c. What is the volume of a reduction with a scale factor of $\frac{1}{2}$?
d. What is the surface area of a reduction with a scale factor of $\frac{1}{2}$?

16. The scale factor from a model to a life-size table is 4, and the model and table are both made of the same type of wood.

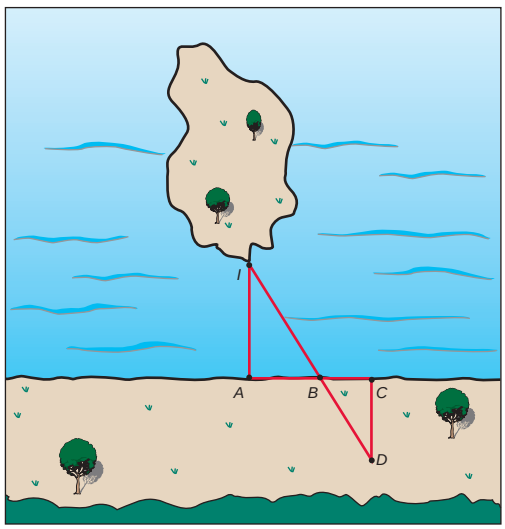
a. If the life-size table has a height of 28 inches, what is the height of the model?

b. If the model requires $\frac{1}{32}$ quart (1 fluid ounce) of stain, how many quarts of stain are needed for the life-size table?

c. If the model weighs $\frac{3}{4}$ pound, what is the weight of the life-size table?

17. A person 6 feet tall standing under a streetlight casts a 10-foot shadow on the ground. If the base of the streetlight is 45 feet from the tip of the shadow, what is the height of the streetlight?

18. Two campers on a beach by a lake want to determine the distance to a small island in the lake. They mark point A with a stake directly across from the island at point I, and walk perpendicular to $\overline{IA}$ to point B where they place another stake. Then they continue in a straight line to point C where they place another stake. Finally, they walk perpendicularly away from the lake to a point D such that points D, B, and I are on a line.



a. Are triangles AIB and CDB congruent? Are they similar? Explain your reasoning.

b. If $AB = 300$ feet, $BC = 24$ feet, and $CD = 40$ feet, what is the shortest distance from point A to the island?

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Answers to Selected Math Activities

Math Activity 1.1

3. For 3 tiles of each color, the minimum number of moves is 15 and the strategy is RGGRRRGGGRRRGGR.

5. For 5 pegs on each side of the board, the minimum number of moves is 35.

Math Activity 1.2

1. a. *Pattern:* Starting with a trapezoid, each successive figure is obtained by alternately adding a pair of parallelograms (one at each end) and a pair of trapezoids (one at each end).

b. 9 trapezoids and 10 parallelograms

2. b. The 20th figure would look like seven copies of the third figure, in a row, with the square removed from the right end.

3. a. After the first parallelogram you alternately attach triangles (two) and parallelograms (one). The 20th will look like five copies of the fourth figure and will contain 20 triangles and 10 parallelograms.

Math Activity 1.3

1. a.

Figure number	1	2	3	4	5
Green tiles	2	5	8	13	18
Yellow tiles	2	4	8	12	18

2. a.

Figure number	1	2	3	4	5	…	10	…	25
Red tiles	1	1	6	6	15	…	45	…	325
Blue tiles	0	3	3	10	10	…	55	…	300

Starting from the bottom right corner the reds form the number pattern $1 + 5 + 9 + 13 + \cdots$ and the blues form the pattern $3 + 7 + 11 + 15 + \cdots$

Math Activity 2.1

1. There are many ways to continue the sequence that has been started. Here are three ways:

(1) SYS, LYS, LYC, SYC, SRC
(2) SBH, SBT, LBT, LRT, LRH
(3) LYH, LYT, LBT, SBT, SBC

3. Here are the pieces in each region:

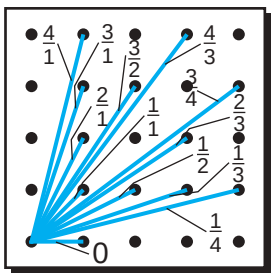
Region 1: LBS, SBS, LBC, SBC, LBT, SBT
Region 2: LBH, SBH
Region 3: LRH, SRH, LYH, SYH
Outside: LRS, SRS, LRT, SRT, LRC, SRC, LYS, SYS, LYT, SYT, LYC, SYC

Math Activity 2.2

1. a. $\frac{1}{4}$ **b.** $\frac{2}{4}$ **c.** $\frac{3}{4}$ **d.** $\frac{4}{4} = 1$

As the lines become steeper, the slopes increase.

4. The 12 line segments have the slopes of $\frac{4}{1}$, $\frac{3}{1}$, $\frac{2}{1}$, $\frac{3}{2}$, $\frac{4}{3}$, $\frac{1}{1}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and 0, beginning with the line segment on the left side of the board and moving clockwise to the right edge of the board. From least to greatest the slopes are $0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1, \frac{4}{3}, \frac{3}{2}, 2, 3, 4$.



Math Activity 2.3

1. Player A's number does not contain the digits 1, 2, or 3.

2. Player A's number contains two of the digits 4, 5, or 6, but no digit is in the correct place-value position.

3. Since two of the digits are correct but in the wrong position, switching positions will give us more information.

4. Now player B knows that one of the digits 6 or 4 is in the correct position, but cannot be sure which one. The digit 5 may still be the correct digit but in the incorrect position.

Math Activity 3.1

2.

	Long-Flats	Flats	Longs	Units
a.		3	1	4
b.	1	0	4	2
c.	2	0	3	2

4. The flat for the base-three piece is 3 by 3 and the fourth base-three piece contains 3 flats.
a. 121_{three} **b.** 68 units

Math Activity 3.2

1. 2 flats, 1 long, 1 unit
3. After 6 turns the player had 3 flats, 1 long, and 3 units. The player will need 1 flat, 3 longs, and 2 units to obtain 1 long-flat.

Math Activity 3.3

2. **a.** 1443_{five} **b.** 2101_{five} **c.** 3231_{five}
5. Base ten. $6_{ten} \times 373_{ten} = 2238_{ten}$

Math Activity 3.4

1. The minimal collection consists of 3 longs and 1 unit.
2. **a.** 14_{five}
3. **a.** 20_{five}

Math Activity 4.1

1. **a.** A three-digit base-five numeral tells how many flats, longs, and units are in a collection. When divided into groups of 4, there is 1 unit remaining for *each* flat and 1 for *each* long. So the sum of the digits in the numeral also tells us the total number of units remaining in the collection after flats and longs have been grouped by 4s.
b. When the units in a base-five long-flat are divided into groups of 4 there is 1 unit remaining. The number 1232_{five} is divisible by 4 because $1 + 2 + 3 + 2$ is divisible by 4. A base-five number is divisible by 4 if the sum of its digits is divisible by 4.

Math Activity 4.2

1. **a.** 20th is blue; 35th is yellow
b. Divide the figure number by 4. If there is no remainder, the color is blue. If the remainder is 1, 2, or 3, the color is red, green, or yellow, respectively.
c.

	Red	Green	Yellow	Blue
Figure 20	45	50	55	60
Figure 35	153	162	171	144

The following chart suggests one way to organize the data and look for patterns.

Figure	Red	Green	Yellow	Blue
1	1			
2	1	2		
3	1	2	3	
4	1	2	3	4
5	$1 + 5$	2	3	4
6	$1 + 5$	$2 + 6$	3	4
7	$1 + 5$	$2 + 6$	$3 + 7$	4
8	$1 + 5$	$2 + 6$	$3 + 7$	$4 + 8$
9	$1 + 5 + 9$	$2 + 6$	$3 + 7$	$4 + 8$

Math Activity 5.1

3. **a.** $^{-}7$ **b.** $^{-}1$ **c.** 3 **d.** $^{-}8$

Math Activity 5.2

3. **a.** $\frac{7}{12}$ **b.** $\frac{1}{4}$ or $\frac{3}{12}$ **c.** $\frac{5}{6}$ or $\frac{10}{12}$ **d.** $\frac{3}{4}$ or $\frac{9}{12}$

Math Activity 5.3

2. **a.** $\frac{5}{6} - \frac{1}{2} = \frac{2}{6}$ **b.** $\frac{7}{10} - \frac{2}{5} = \frac{3}{10}$ **c.** $\frac{5}{12} - \frac{1}{4} = \frac{2}{12}$
5. **a.** $\frac{8}{10} \div \frac{2}{5} = 2$ **b.** $\frac{5}{6} \div \frac{2}{12} = 5$ **c.** $\frac{5}{12} \div \frac{1}{4} = 1\frac{2}{3}$

Math Activity 6.1

1. **a.** Ten of any one of the pieces have the value of the next larger piece; the pattern of the shapes alternates between nonsquare rectangle and square; there are 100 one-hundredth squares in the unit square and 1000 one-thousandth regions in the unit square.
b. Here are a few of the many ways. The second row shows the minimum number of pieces.

Units	Tenths	Hundredths	Thousandths
1	4	12	
1	5	2	
0	15	2	
0	0	152	
0	0	0	1520
1	0	52	
1	1	42	
1	1	1	410

Math Activity 6.2

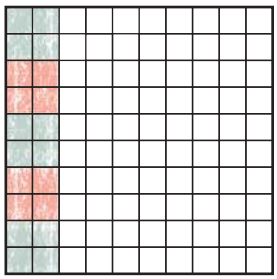
1. a. .1 + .9 = 1.0

Math Activity 6.3

2. 14 small squares (14 percent of 300) represents $14 \times 3 = 42$.
8 small squares (8 percent of 300) represents 24 years

3. One square represents 1.8 skateboards, so 45 percent (45 small squares) represents $45 \times 1.8 = 81$ skateboards. Or, 10 squares represent 18 skateboards, so 45 squares represent $4.5 \times 18 = 81$. Fifteen squares represent 27 skateboards.

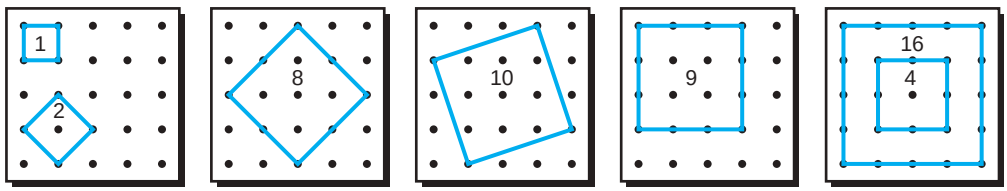
3. a.



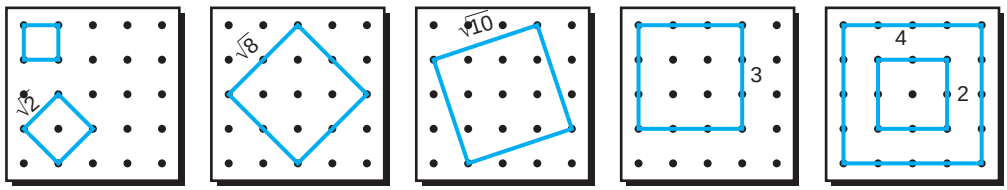
5 × .04 = .2

Math Activity 6.4

4.



5.



Math Activity 7.1

2. a. 1 path each to points *A* and *E*; 4 paths each to points *B* and *D*; 6 paths to point *C*, 16 paths all together.

Math Activity 7.2

1. First way: Take 6 tiles off the top of the column with 15 tiles so both columns now have the same height. Divide the difference by 2, and put these 3 on top of each column.
Second way: Count all the tiles and divide them equally into two columns.

a. First way: $15 - 9 = 6$ is the difference; $(15 - 9) \div 2 = 3$ is one-half the difference; and $9 + \frac{(15 - 9)}{2} = 12$ is the average.
Second way: $15 + 9 = 24$ is the total; $24 \div 2 = 12$ is the average.

b. Here are three ways: $\frac{y + (x - y)}{2}$; $\frac{x - (x - y)}{2}$; and $(x + y) \div 2$.

c. $\frac{y + (x - y)}{2} = \frac{2y}{2} + \frac{(x - y)}{2} = \frac{(x + y)}{2}$

Math Activity 7.3

1. e. Theoretically the average number of yogurts that must be purchased to obtain at least one of each type of cover is between 8 and 9.

Math Activity 8.1

1. b. The chances are slightly better than 60 percent that at least one sixth-grader will be chosen.

Math Activity 8.2

1. d. Because points are awarded for the total number of Xs, this is a fair game. The numbers 1, 2, 3, and 4 constitute 50 percent of the total outcomes, and the numbers 6, 8, 9, 12, and 16 constitute the other 50 percent.

Math Activity 9.1

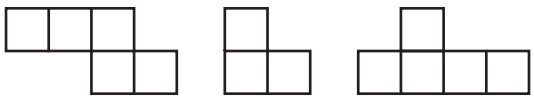
1. The triangle has three 60° interior angles. The square has four 90° interior angles. The hexagon has six 120° interior angles. The trapezoid has two 60° and two 120° interior angles. The blue parallelogram has two 60° and two 120° interior angles. The white parallelogram has two 30° and two 150° interior angles.

Math Activity 9.2

3. Different vertices have different numbers of shapes surrounding them. For example, one vertex has a code of 3, 3, 3, 3, 6, while another has code 3, 3, 6, 6.

Math Activity 9.3

1. a.



2. a.

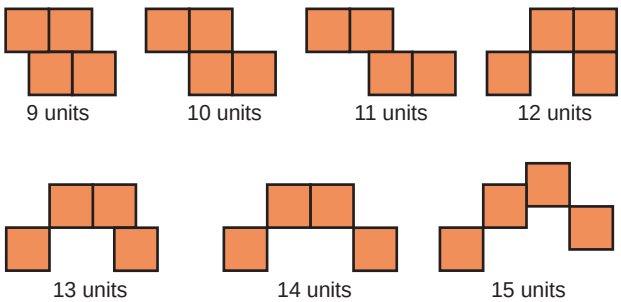
3	2	1
3	2	1
2	2	1
1	1	1

Math Activity 9.4

2. a. No rotation symmetries
b. Rotation symmetries of 60°, 120°, 180°, 240°, 300°, and 360°
c. Rotation symmetries of 90°, 180°, 270°, and 360°
d. Rotation symmetries of 180° and 360°

Math Activity 10.1

3.

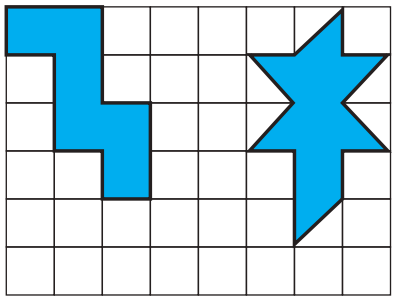


Math Activity 10.2

1. a. Trapezoid 3 units of area; hexagon 6 units of area
b. Square has slightly more than 2 units of area and white parallelogram slightly more than 1 unit of area

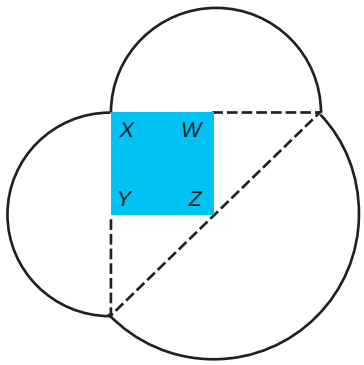
Math Activity 10.3

1. There are many patterns. Here are two:



Math Activity 11.1

1. c. Area of the blue tile is one-half the area of the large right triangle; the total area of the two small semicircles is π times the area of the blue tile; the total area of the two semicircles is equal to the area of the large semicircle.



Math Activity 11.2

1.

Triangle	1	2	3	4
Center of rotation	*A, G*	*A*	*A, F*	*F*

5	6	7	8	9	10	11
F, G	*G*	*E, M*		*B, S*	*G*	*G*

4. Any of the following: 2, 3, 4, 6, 7, 9, 10

Math Activity 11.3

2. The areas of these enlargements are 9 times the areas of the individual pattern blocks.

Answers to Odd-Numbered Exercises and Problems and Chapter Tests

Exercises and Problems 1.1

1. a. 4 feet; 2 feet
b. 6 feet
c. On day 9
d. If the snail climbs 4 feet during the day and slips back 3 feet at night, it will take 17 days for it to climb out of the well.

3. 52 and 78 feet

5. a. The old plan
b. 58¢
c. 15 checks
d. The difference decreases. 19 checks

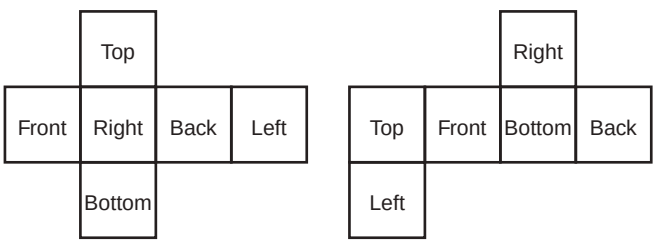
7. Nine letters

9. a. One possibility is 46 and 64.
b. For 64 and 46, the difference is 18.
c. 82 and 28
d. 93 and 39

11. a. Dumping 9 gallons into the 4-gallon container twice leaves 1 gallon in the 9-gallon container.
b. (1) Dump the 1 gallon from part a into the 4-gallon container.
(2) Fill the 9-gallon container and dump part of it into the 4-gallon container until the 4-gallon container is full.
(3) There will be 6 gallons left in the 9-gallon container.

13. a. The yellow tiles are touching at the corners.
b. Each of the other eight tiles touches the center tile at one or more points.
c. If only one new color is used for the remaining four tiles, then tiles of the same color will meet at their corners.
d. Yes, it can be done in two colors, with one color along the two diagonals.

15.



17. a. Girl *A* will have 20 chips, girl *B* will have 60 chips, and girl *C* will have 40 chips.
b, c.

	A	B	C
Beginning	65	35	20
End of first round	10	70	40
End of second round	20	20	80
End of third round	40	40	40

d. No, she would not have enough chips to double the other girls' supplies.

19. 35 tiles

21. The total of 7 squares has a value of 112, so each square has a value of 16. Thus, the numbers are 16, 32, and 64.

23. In the third row, a cup replaces the doughnut above it and the cost increases from \$4.10 to \$4.60. So a cup of coffee costs 50¢ more than a doughnut. Thus, replacing the cup in the second row by the doughnut in the first row decreases the cost by 50¢, so that 4 doughnuts cost \$3.60. Therefore, 1 doughnut costs 90¢, and a cup of coffee costs \$1.40.

25. 6 ships with 4 masts

27. 14 free movie DVDs.

29. The 90- and 110-pound people cross the river, and the 110-pound person returns. Then the 190-pound person crosses the river, and the 90-pound person returns. This requires 4 crossings of the boat. Similarly, another 4 crossings will get the 170-pound person across the river and leave the 110- and 90-pound people left to cross the river in the ninth crossing of the boat. A minimum of 9 crossings is required for the boat.

31. 2 is on back of disk 6; 5 is on back of disk 7; 9 is on back of disk 8.

33. One way: Start both hourglasses at the same time; put the vegetables in the water when the 7-minute hourglass finishes; turn the 11-minute hourglass over when it finishes; vegetables will be steamed in 15 minutes, when the 11-minute hourglass finishes.

35. a. 2 apples
b. 9 sheep
c. A quarter and a nickel
d. The cider costs \$2.73; the bottle costs 13 cents.
e. No dirt
f. 6 pounds
g. 10 children
h. Neither; the whites of the egg are white!

Exercises and Problems 1.2

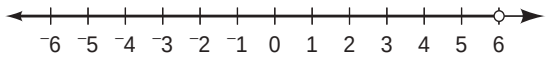
1. a. Repeats: ▲+○
b. Repeats: ✖✖+✖
c. Grows: ○✳○○○○○…
3. a. 26, 29, 32
b. 49, 56, 63
c. 29, 34, 32
5. a. 91 **b.** Yes
c. Ten layers of cannonballs: the base is 10×10; the next layers are 9×9, 8×8, etc., up to the top level, which has 1 cannonball. There are 385 cannonballs in the 10th pyramid.
7. a. Arithmetic
b. There are 58 cubes in the 20th figure, 39 placed in a row and 19 placed in a column on top of the row.
9. The sum divided by 3 is the middle number. The sum of 17, 18, and 19 is 54.
11. The sum divided by 9 is the center number of the array. The numbers 14, 15, 16, 21, 22, 23, 28, 29, and 30 form a 3×3 array whose sum is 198.
13. a. The missing sums equal 5×8, 8×13, and 13×21.
b. The sum equals the product of the last Fibonacci number times the next Fibonacci number.
15. a. 1, 5, 6, 11, 17, 28, 45, 73, 118, 191
b. 14, 6, 20, 26, 46, 72, 118, 190, 308, 498
c. Yes **d.** Yes
e. The sum of the first 10 numbers in a Fibonacci-type sequence equals 11 times the seventh number in the sequence.
17. a. $16 + 17 + 18 + 19 + 20 = 21 + 22 + 23 + 24$
b. $400 + 401 + 402 + \cdots + 420 = 421 + 422 + \cdots + 440$
19. a. 55 **b.** 220
21. 4096
23. a. Arithmetic; Add 5 to each number; 59
b. Geometric; Multiply each number by 2; 30,720
c. Arithmetic; Subtract 4 from each number; -20
d. Geometric; Multiply each number by 3; 708,588
25. a. Yes **b.** No
27. a. 187 **b.** 103
29. 25, 36, 49, and 10,000
31. a. 30 **b.** 420
33. a. Arithmetic **b.** Yes
35. Inductive reasoning
37. a. The ninth even number is 18 and can be illustrated by a 2×9 array.
b. 90
39. This procedure will produce a number that is divisible by 7 for all single-digit numbers but not for all multi-digit numbers. For example, beginning with 11, its double is 22, and 2211 is not divisible by 7. However, beginning with 14, its double is 28, and 2814 is divisible by 7.
41. a. The product of two odd whole numbers is not evenly divisible by 2.
b. 8 cannot be written as the sum of consecutive whole numbers.
43. a. $2 + 3 + 4 + 5$ is not divisible by 4. The sum of any four consecutive even numbers or four consecutive odd numbers is divisible by 4.
b. True
45. a. 729 **b.** 531,441
47. a. 78 **b.** 364
49. 344
51. 95

Exercises and Problems 1.3

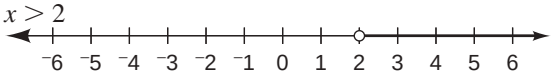
1. a. 19, 57.7, and 14.7 **b.** 45° and 65°
c. between 120 and 125
3. a. $28m$ **b.** $m + b$ **c.** $28m - 19b$
5. a. $6p = s$
b. One possible reason for writing $6s = p$ is that the statement says "6 times as many students."
7. a. 2 chips per box; $2x + 5 = 9$
b. 4 chips per box; $3x + 4 = x + 12$
9. a. Step 1: Simplification (distributive property)
Step 2: Addition property of equality
Step 3: Simplification
Step 4: Subtraction property of equality
Step 5: Simplification
Step 6: Division property of equality
Step 7: Simplification
b. Step 1: Simplification
Step 2: Subtraction property of equality
Step 3: Simplification
Step 4: Division property of equality
Step 5: Simplification
11. a. $x = 323$ **b.** $x = 11$ **c.** $x = 719$ **d.** $x = 19$
13. a. Replace each box by 0, 1, 2, or 3 chips; $2x + 5 < 12$.
b. Replace each box by 4 or more chips; $3x + 2 > 11$.

15. a. Step 1: Subtraction property of inequality
Step 2: Simplification
Step 3: Subtraction property of inequality
Step 4: Simplification
Step 5: Division property of inequality
Step 6: Simplification
b. Step 1: Subtraction property of inequality
Step 2: Simplification
Step 3: Multiplication property of inequality
Step 4: Simplification

17. a. $x < 6$



b. $x > 2$



19. a. $.28x$ **b.** $18 - x$ **c.** $.44(18 - x)$
d. $.28x + .44(18 - x) = 6.00$; $x = 12$
Marci mailed 12 postcards.

21. a. $10.5x$ **b.** $x + 3$ **c.** $8(x + 3)$
d. $10.5x + 8(x + 3) < 120$; $x < 5.189$
Merle bought either 1, 2, 3, 4, or 5 DVDs.

23. 60 feet. Let x equal the length of a side of the square.

$$4x + 110 = 350$$
$$x = 60$$

25. Let x equal the unknown number.

$$14 + x < 3x$$
$$7 < x$$

The statement is true for any number greater than 7.

27. a. $n + 1, n + 2, n + 3$
b. $n + (n + 1) + (n + 2) + (n + 3)$
c. $n = 86$ **d.** $350 = 68 + 69 + 70 + 71 + 72$, and 350 cannot be written as the sum of three consecutive whole numbers.

29. Let x equal an arbitrary number. Add 221:

$$x + 221$$

Multiply by 2652:

$$2652(x + 221) = 2652x + 586{,}092$$

Subtract 1326:

$$2652x + 586{,}092 - 1326 = 2652x + 584{,}766$$

Divide by 663:

$$\frac{2652x + 584{,}766}{663} = 4x + 882$$

Subtract 870:

$$4x + 882 - 870 = 4x + 12$$

Divide by 4:

$$\frac{4x + 12}{4} = x + 3$$

Subtract x:

$$(x + 3) - x = 3$$

Regardless of the original number, the result will always be 3.

31. 3 nails
33. 1200 miles
35. Yes, the 2744th figure has 8230 tiles.
37. a. Figure 1: 1 yellow, 8 red
Figure 2: 9 yellow, 16 red
Figure 3: 25 yellow, 24 red
Figure 4: 49 yellow, 32 red
Figure 5: 81 yellow, 40 red
b. 40,401 tiles: 39,601 yellow and 800 red
c. Number of yellow tiles: $(2n - 1)^2$
Number of red tiles: $8n$
Total number of tiles: $(2n + 1)^2$

Chapter 1 Test

1. Understanding the problem
Devising a plan
Carrying out the plan
Looking back

2. Making a drawing
Guessing and checking
Making a table
Using a model
Working backward
Finding a pattern
Solving a simpler problem
Using algebra

3. Sums: 3, 8, 21, 55. The sum of the first and third Fibonacci numbers is the fourth Fibonacci number; the sum of the first, third, and fifth Fibonacci numbers is the sixth Fibonacci number; and so forth.

4. 512, or 2^9

5. a. 243 **b.** 18
c. 30 **d.** 36
e. 53

6. a. Geometric **b.** Arithmetic
c. Arithmetic **d.** Neither
e. Neither

7. a. 91, 140, 204 **b.** 54, 77, 104

8. a. 15 **b.** 25 **c.** 35

9. $2 + 3 + 4 + 5 + 6 + 7 + 8 = 35$, which is not evenly divisible by 4.

10. a. 7 chips **b.** More than 2 chips

11. a. $x = 68$ **b.** $x = 12$

12. a. $x < 11$ **b.** $x > 12.5$

13. a. Step 1: Subtraction property of equality
Step 2: Simplification
Step 3: Addition property of equality
Step 4: Simplification
Step 5: Division property of equality
Step 6: Simplification

b. Step 1: Simplification (distributive property)
Step 2: Subtraction property of inequality
Step 3: Simplification
Step 4: Division property of inequality
Step 5: Simplification

14. 201 posts; making a drawing and/or solving a simpler problem

15. Working backward: $170 - 80 = 90$, $90 \times 2 = 180$, $180 - 50 = 130$, $130 \times 2 = 260$. She started with 260 chips.

16. 325 and $n(n + 1)/2$; solving a simpler problem and finding a pattern

17. a. 16 **b.** 200 **c.** $2(n - 1) + 2(n + 1)$ or $4n$; making a drawing, solving a simpler problem, and finding a pattern

18. 78 handshakes and n handshakes; making a drawing, solving a simpler problem, and finding a pattern

19. 9 crossings; making a drawing

20. a. 17 and 452 **b.** $3n + 2$

Exercises and Problems 2.1

1. Yes

3. a. Row 1

b. The numbers of marks in row 2 are prime numbers (see Chapter 4). Row 3 has numbers that are just before and just after 10 and 20.

5. *SY* and *SB*; *Y* and *L*

7. *Y* and *SB*; *SY* and *SB*; *SY* and *L*; *SB* and *L*

9. {*LYT*, *LYS*, *LYH*}

11. a. False **b.** True **c.** True

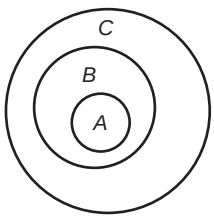
13. a. {*SYH*, *SBH*}

b. {*LBT, LBS, LBH, SBT, SBS, LYT, LYS, LYH, SYT, SYS*}

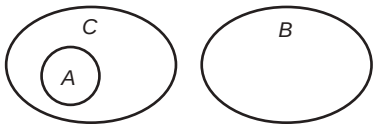
c. {*SYT, SYS, SYH, SBT, SBS, SBH, LYT, LYS, LYH*}

15. a. {4, 6} **b.** {0, 1, 2, 3, 5, 7, 8}

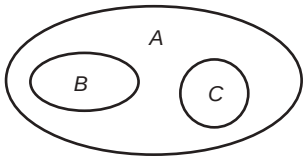
17. a.



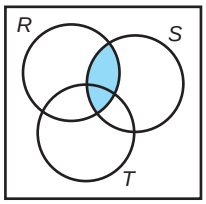
b.



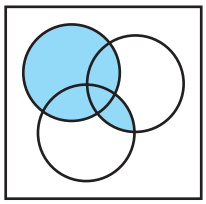
c.



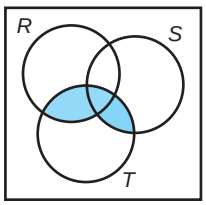
19. a.



b.

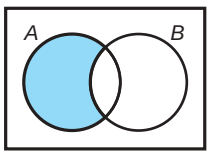


c.

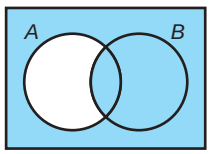


21. Maximum number for $A \cup B$ is 28.
Maximum number for $A \cap B$ is 13.

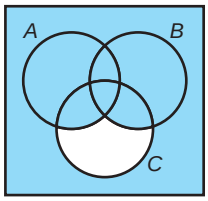
23. a.



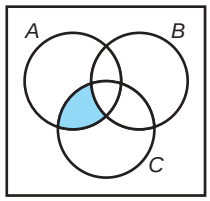
b.



c.



d.



25. a. $(A \cup B)'$ **b.** $(A \cap B)'$

27. a. $(A \cup B)'$ **b.** $C \cup (A \cap B)$

29. a. *d*, *c* **b.** *j*, *i*

31. 2

33. 400

35. 39

37. 43

39. 767

41. a. 8 **b.** 37

Exercises and Problems 2.2

1. As the level of difficulty increases, the level of motivation decreases.

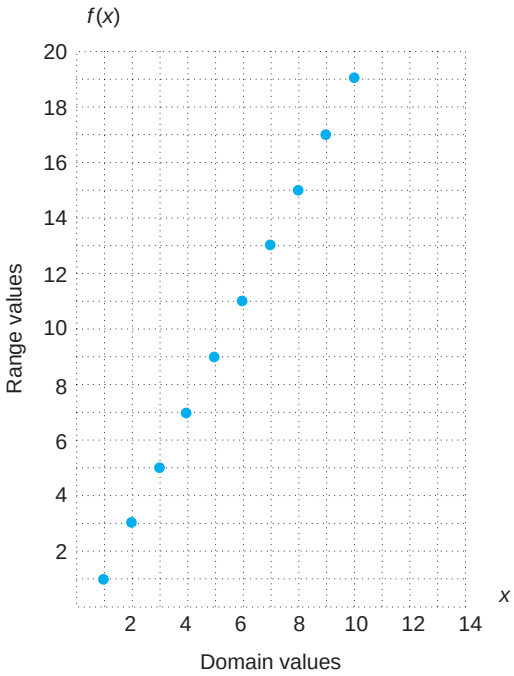
3. Each element of the domain gets assigned to an element of the range that is 1 less than twice its value.

a. $f(x) = 2x - 1$

b.

x	1	2	3	4	5	6	7	8	9	10
$f(x)$	1	3	5	7	9	11	13	15	17	19

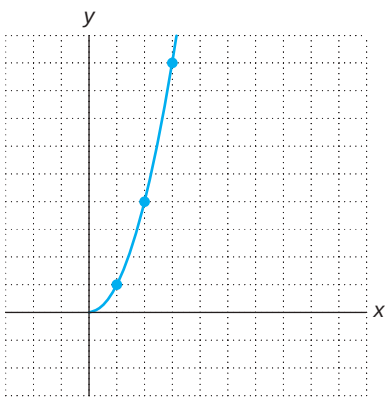
c.



5. a. Function
b. Function
c. Not a function, as a person may have more than one telephone number.

7. a. $f(x) = x + 17$, where x is a whole number
b. $f(x) = 3x - 2$, where x is a whole number

9. a. The range values are 1, 4, 9, 16, and 25, respectively.
b.

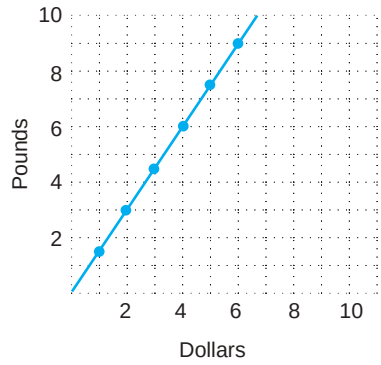


c. $f(x) = x^2$
d. Nonlinear

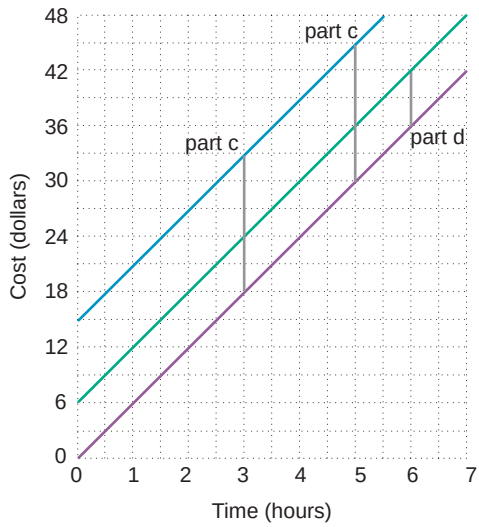
11. a. Slope, $^-2$; equation, $y = {}^-2x + 2$
b. Slope, $\frac{1}{4}$; equation, $y = \frac{x}{4} + 2$

13. a. Line i, 10; line ii, 1
b. Yes. One possibility: $y = 20x$
c. No. The slope m in $y = mx + b$ can be arbitrarily large.
d. $y = x + 5$

15. a. \$7.50
b. 17 pounds
c. $c(x) = 1.5x$
d.

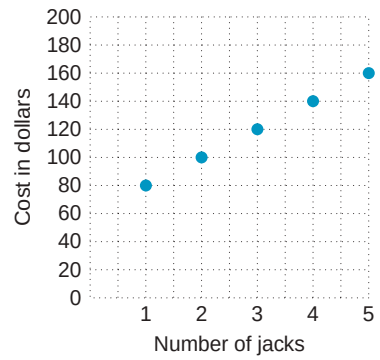


17. a. Middle
b. \$15 more for both 3 hours and 5 hours
c.



d. \$6

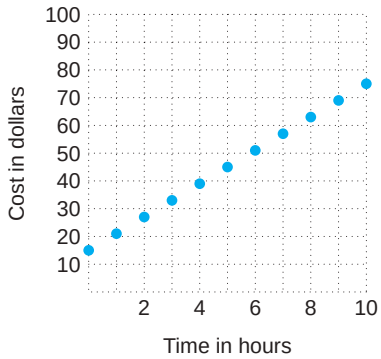
19. a. $c(x) = 20x + 60$
b. \$160
c.



21. a. $g(x) = 6x + 15$

b. $99

c.



23. a. 12 miles per hour

b. At the 10- and 20-minute points

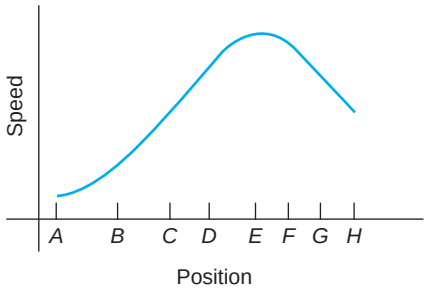
c. 0 to 3 minutes, 6 to 7 minutes, 10 to 13 minutes, and 15 to 16 minutes

25. a. I, Bob; II, Joan; III, Mary; IV, Joel

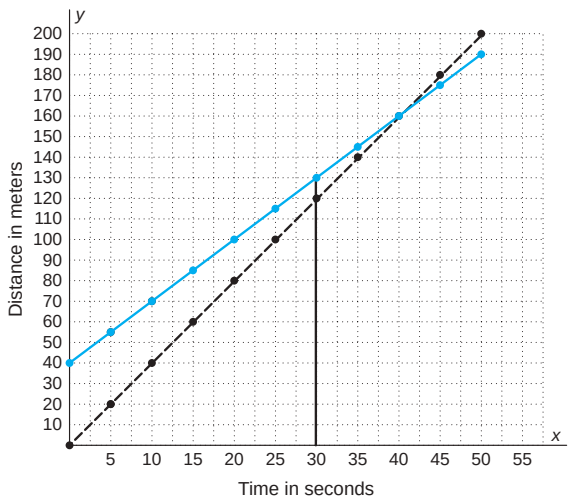
b. Joel **c.** Mary **d.** Bob **e.** Joel

27. a. II **b.** III **c.** I **d.** I

29.



31. a.



b. Pat's distance is 120 meters, and Hal's distance is 130 meters.

c. 40 seconds **d.** Pat **e.** 10 meters

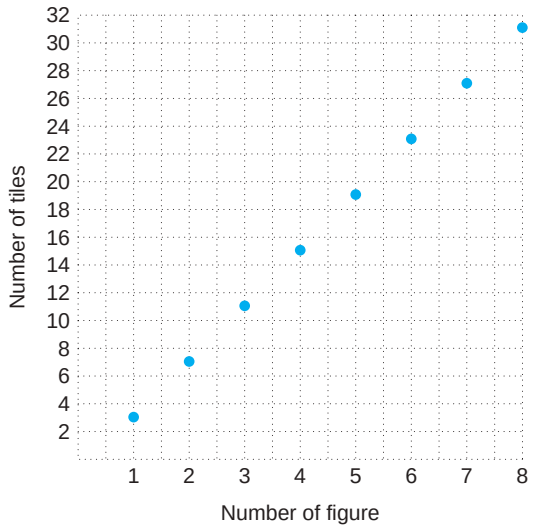
33. a. 4.60 seconds **b.** .12 second

c. Approximately .68 second. This is a pulse rate of approximately 88 beats per minute.

35. a.

Number of figure	1	2	3	4	5	6	7	8
Number of tiles	3	7	11	15	19	23	27	31

b.



Patterns: The points lie on a line. For each horizontal increase of 1 unit, there is a vertical increase of 4 units. With the scales of the axes as shown above, moving over 2 spaces and up 2 spaces from a given point on the graph locates another point on the graph.

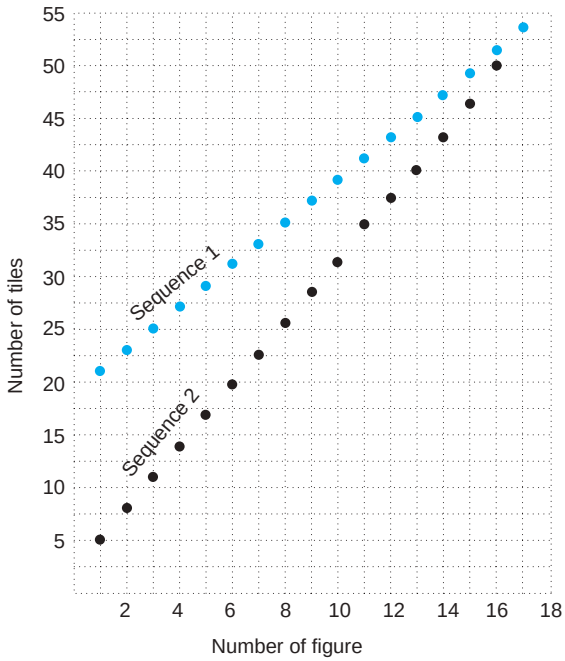
c. $f(20) = 79$

d. $f(n) = 4n - 1$ and $f(350) = 1399$

37. a. Sequence 1: The number of tiles in the nth figure is $2n + 19$.

Sequence 2: The number of tiles in the nth figure is $3n + 2$.

b.



c. $n = 17$

d. The 17th figures in both sequences have the same number of tiles. For $n < 17$, the nth figure in sequence 1 has more tiles than the nth figure in sequence 2. For $n > 17$, the nth figure in sequence 2 has more tiles than the nth figure in sequence 1.

Exercises and Problems 2.3

1. Yes

3. a. If a person takes a hard line with a bill collector, then it may lead to a lawsuit.

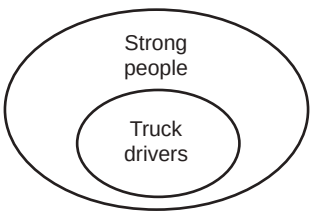
b. If a person is an employee of the Tripak Company, then he or she must retire by age 65.

c. If the class is to meet only twice a week, then there must be 2-hour class sessions.

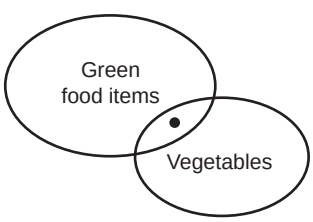
d. If a person is a pilot, then she or he must have a physical examination every 6 months.

5.

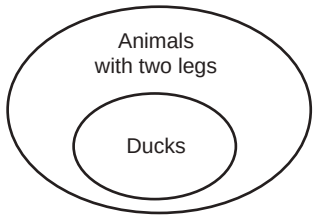
a.



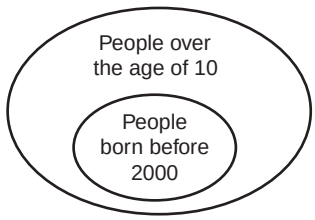
b.



c.



d.



7. **Converse:** If you itemize your deductions, then you take a deduction for your home office. **Inverse:** If you do not take a deduction for your home office, then you do not itemize your deductions. **Contrapositive:** If you do not itemize your deductions, then you do not take a deduction for your home office.

9. **Converse:** If the camera focus is on manual, switch B was pressed. **Inverse:** If switch B is not pressed, the camera focus is not on manual. **Contrapositive:** If the camera focus is not on manual, then switch B was not pressed.

11. **c.**

13. a. If the computer does not reject your income tax return, then you did not subtract \$750 for each dependent.

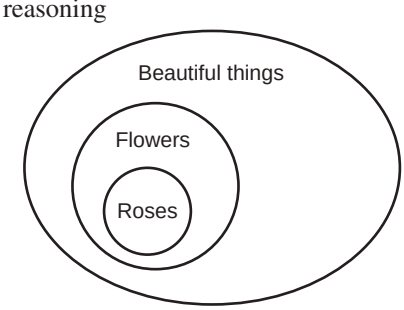
b. If the cards are not dealt again, then there was an opening bid.

c. If the books are not returned at the end of the week's free sing-along, then you were delighted with them.

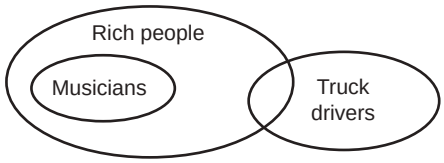
15. You pay the Durham poll tax if and only if you are age 18 or older.

17. If Robinson is hired, then she meets the conditions set by the board. If Robinson meets the conditions set by the board, then she will be hired.

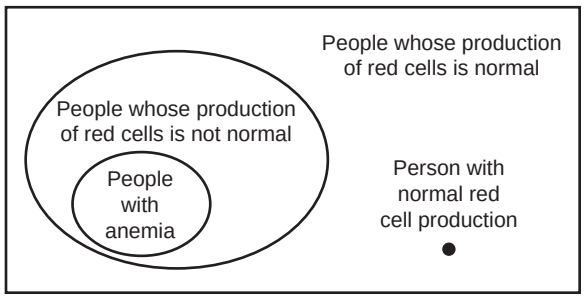
19. Valid reasoning



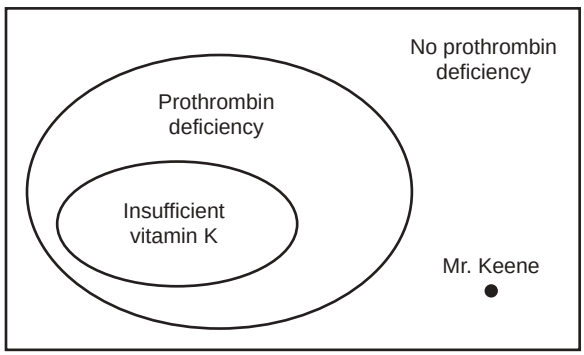
21. Invalid reasoning. The following Venn diagram satisfies the given conditions, but there is not necessarily an intersection between the set of truck drivers and the set of musicians.



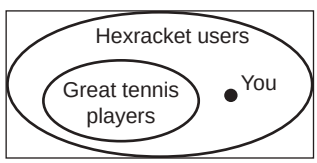
23. This patient does not have anemia.



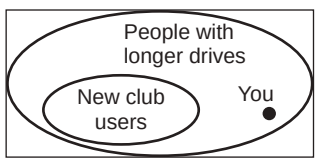
25. Mr. Keene has sufficient vitamin K in his body.



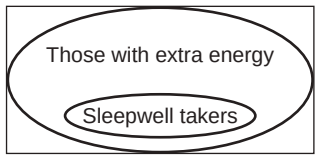
27. Invalid



29. Invalid



31. Valid



33. Dow is a female cook; Eliot is a male singer; Finley is a male appraiser; Grant is a male broker; Hanley is a female painter.

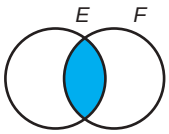
35. No, the proof is arranged in logical order.

Chapter 2 Test

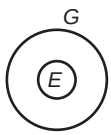
1. a. $\{SYH\}$ **b.** $\{SBT, SBR, SBH, SYT\}$ **c.** $\{SBT, SBH\}$

2. a. $A \cap B = \{2, 4\}$ **b.** $A \cup B = \{1, 2, 3, 4, 6\}$
c. $A' \cap B = \{1, 3\}$ **d.** $A \cup B' = \{0, 2, 4, 5, 6\}$

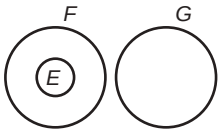
3. a.



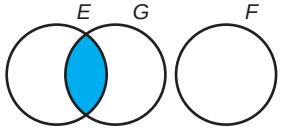
b.



c.



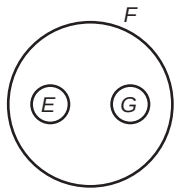
d.



e.

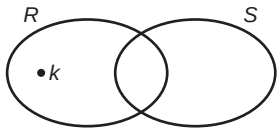


f.

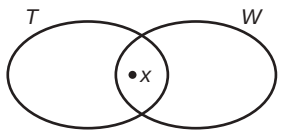


4. a. $A \cap B'$ or $(A' \cup B)'$ **b.** $A' \cup B'$ or $(A \cap B)'$

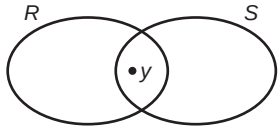
5. a. Not necessarily



b. Yes

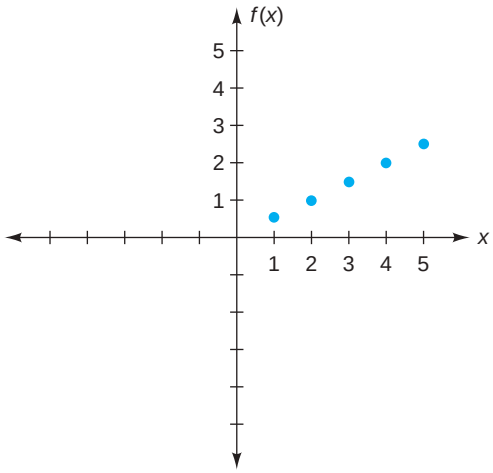


c. No



6. a. For the domain values 1, 2, 3, 4, and 5, the corresponding range values are .5, 1, 1.5, 2, and 2.5, respectively.

b.



c. $y = \frac{1}{2}x$ **d.** Linear

7. a. $-\frac{1}{2}$ **b.** $\frac{1}{2}$ **c.** $y = -\frac{1}{2}x + 4$

8. $y = 3x + 4$

9. a. $y = 55x + 120$ **b.** \$670 **c.** 13 days

10. a. $95x$ **b.** $60 - x$
c. $75(60 - x)$ **d.** 18 hours

11. Monday, jogged; Tuesday, walked; Wednesday, biked; and Thursday, skateboarded.

12. 5 cars

13. 20 men

14. a. If you are denied credit, then you have the right to protest to the credit bureau.
b. If a child was absent yesterday, then the child was absent today.
c. If you were at the party, then you received a gift.

15. a. **Converse:** If her husband goes with her, then Mary goes fishing. **Inverse:** If Mary does not go fishing, then her husband does not go with her. **Contrapositive:** If her husband does not go with her, then Mary does not go fishing.
b. **Converse:** If you receive five free books, then you will join the book club. **Inverse:** If you do not join the book club, then you will not receive five free books. **Contrapositive:** If you do not receive five free books, then you have not joined the book club.

16. Statement 3

17. There will be peace talks if and only if the prisoners are set free.

18. a. Invalid **b.** Valid **c.** Invalid

19. a. The people in ward *B* are not healthy.
b. The game pieces should be set up as they were before the illegal move was made.

20. a. Invalid **b.** Valid

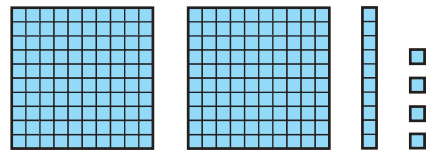
Exercises and Problems 3.1

1. 1241

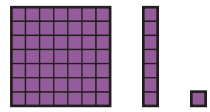
3. Boolla Boolla Neecha (5)
Boolla Boolla Boolla (6)

5. a. 4 hands and 2
b. Hand of hands, 2 hands, and 2

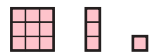
7.



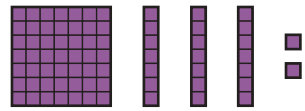
9. a.



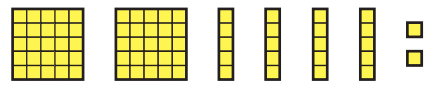
b.



11. a. 132_{seven}



b. 242_{five}



13. a. 98 units **b.** 389 units

15. In each numeration system, the symbol for 1 is repeated to create the symbols for 2, 3, and 4. In the Babylonian and Egyptian systems, the symbol for 1 is repeated in the symbols for 2 through 9. Grouping by 5s occurs in the Roman and Mayan systems. In these systems, a symbol for 5 is used with the symbols for 1, 2, 3, and 4 to form the symbols for 6, 7, 8, and 9; the symbol for 10 can be formed by combining two symbols for the number 5.

17. a.



b. MDCCLXXVI

c.



d.



19. Base ten

Hindu-Arabic	1	4	8	16	26
Attic-Greek	Ⅰ	ⅠⅠⅠⅠ	ΓⅠⅠⅠ	ΔΓⅠ	ΔΔΓⅠ

Hindu-Arabic	32	52	57	206	511
Attic-Greek	ΔΔΔⅡ	ΓⅡ	ΓΓⅡ	HHΓⅠ	ΠΔⅠ

21. a. $7(10^6) + 0(10^5) + 8(10^4) + 2(10^3) + 5(10^2) + 5(10^1) + 5(1)$ or
$7 \times 1{,}000{,}000 + 0 \times 100{,}000 + 8 \times 10{,}000 + 2 \times 1000 + 5 \times 100 + 5 \times 10 + 5 \times 1$

b. $1(5^4) + 4(5^3) + 3(5^2) + 2(5^1) + 1(1)$

23. a. The value is 75; the place value is 5^2 or 25.

b. The value is 0; the place value is thousands.

c. The value is 2,000,000; the place value is millions.

25. a. Four thousand forty

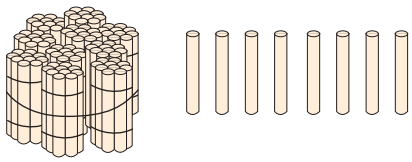
b. Seven hundred ninety-three million, four hundred twenty-eight thousand, five hundred eleven

c. Thirty million, one hundred ninety-seven thousand, seven hundred thirty-three

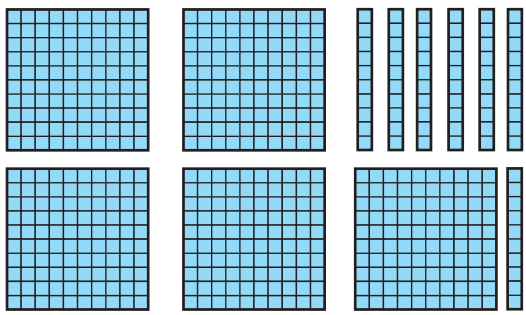
d. Five billion, two hundred ten million, nine hundred ninety-nine thousand, six hundred seventeen

27. a. 43,700,000 **b.** 43,670,000
c. 43,669,000 **d.** 43,668,900

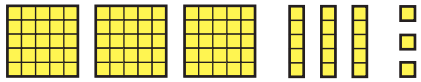
29. a. 108



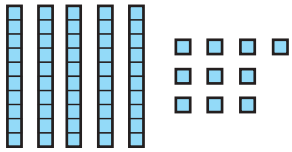
b.



c.



d.



31. a. + 6,000,000

b. + 200,000 + 300

c. + 3000 − 50,000 + 300,000 − 7,000,000

33. a. 123,456,789

b. 111,111,111

35. a. 707,007

b. 12,832

37. Yes, this is true.

39. 345

41. 160

Exercises and Problems 3.2

1. a. Units wheel and hundreds wheel

b. Yes

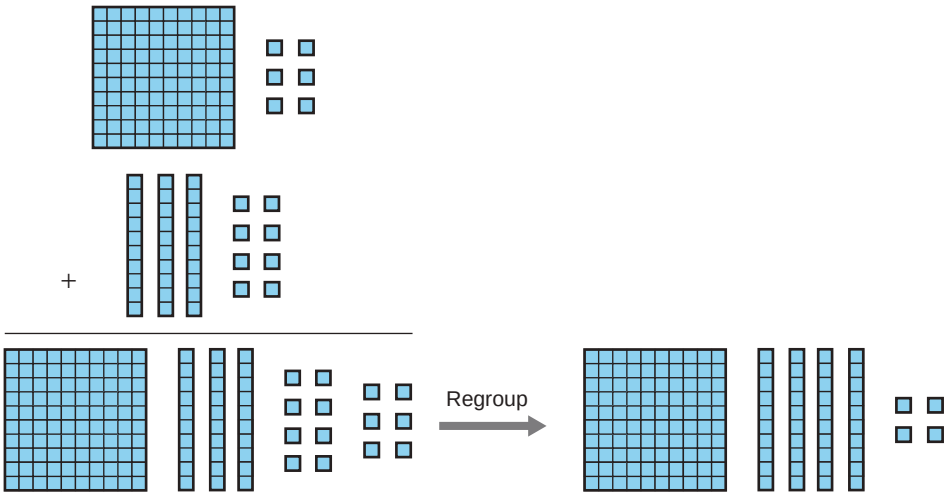
3. a. $232_{five} + 123_{five} = 410_{five}$

b. $852_{twelve} + 295_{twelve} = E27_{twelve}$, where E represents eleven

5. a. 2 flats, 3 longs, 6 units; $523_{eight} - 265_{eight} = 236_{eight}$

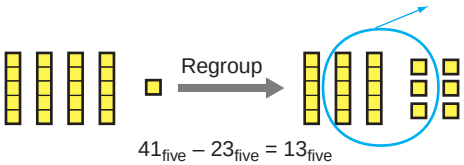
b. 2 flats, 0 longs, 3 units; $342_{five} - 134_{five} = 203_{five}$

7. a. $106 + 38 = 144$



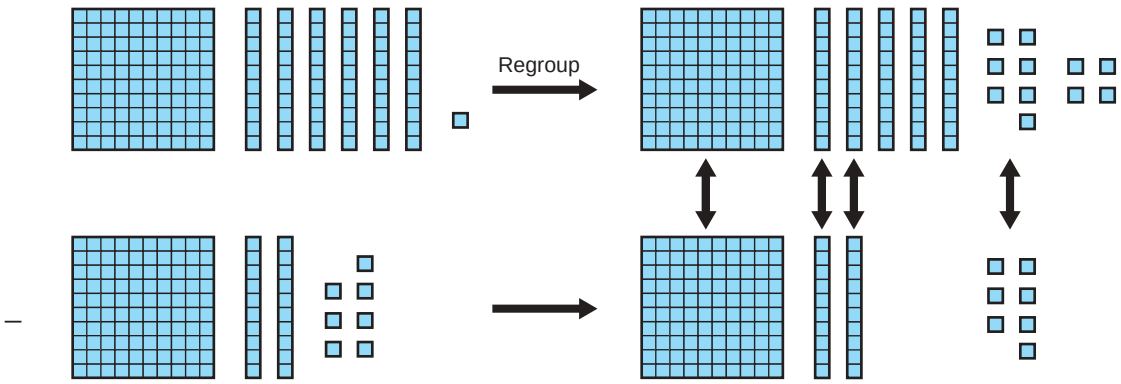
$106 + 38 = 144$

b. $41_{five} - 23_{five} = 13_{five}$



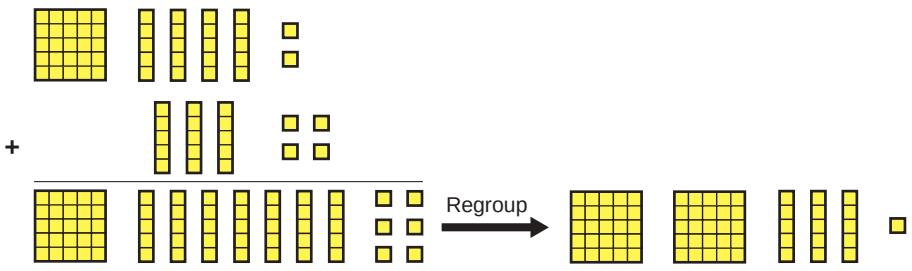
$41_{five} - 23_{five} = 13_{five}$

c. $161 - 127 = 34$



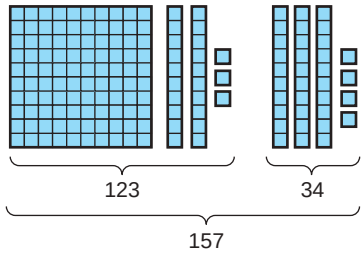
$161 - 127 = 34$

d. $142_{five} + 34_{five} = 231_{five}$

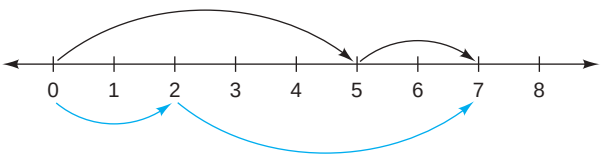


$142_{five} + 34_{five} = 231_{five}$

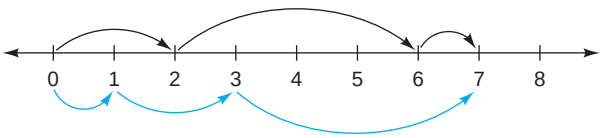
e. $157 - 123 = 34$



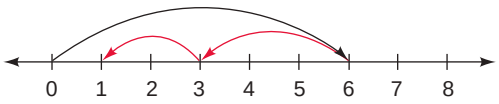
9. a.



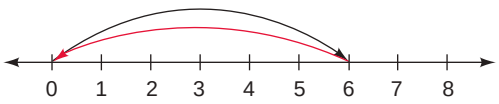
b.



11. a.



b.



13. a. An advantage of this method is that when the digits of highest place value are added first, a subsequent error will affect only the digits of lower place value.

$$\begin{array}{r} 726 \\ +\ 508 \\ \hline 1224 \\ {\scriptstyle 3} \end{array}$$

b. An advantage of this method is that all digits of column sums are recorded before regrouping. This eliminates the need to add and regroup in the same step.

$$\begin{array}{r} 974 \\ +\ 382 \\ \hline 6 \\ 15 \\ 12 \\ \hline 1356 \end{array}$$

15. a. Associative property for addition
b. Commutative property for addition

17. a. No. For example, $3 - 5 \neq 5 - 3$.
b. Yes; the sum of two even numbers is another even number.

19. a. Ten was not regrouped (1 was not carried) to the tens column.
b. The sum of the units digits is 14. Instead of recording a 4 in the units column and carrying the 1, a 1 was recorded and 4 was carried to the tens column.

21. a. The student computed $6 - 4$ (that is, subtracted the smaller number from the larger).
b. After a 10 in the tens place was regrouped to units, the 5 was not reduced to 4.

23. Other compatible numbers or substitutions are possible.

a. $\begin{aligned} 23 + 25 + 28 &= 25 + 23 + (2 + 26) \\ &= 25 + 25 + 26 \\ &= 50 + 26 \\ &= 76 \end{aligned}$

b. $\begin{aligned} 128 - 15 + 27 - 50 &= 128 + 12 - 50 \\ &= 140 - 50 \\ &= 90 \end{aligned}$

c. $\begin{aligned} 83 + 50 - 13 + 24 &= (83 - 13) + 50 + 24 \\ &= 70 + 74 \\ &= 144 \end{aligned}$

25. Other combinations are possible.
a. $6502 - 152 = 6500 - 150 = 6350$
b. $894 - 199 = 895 - 200 = 695$
c. $14{,}200 - 2700 = 14{,}000 - 2500 = 11{,}500$

27. Other combinations are possible.

a. $185 + 15 = 200$
$200 + 200 = 400$
So $185 + 215 = 400$

b. $250 + 250 = 500$
$500 + 35 = 535$
So $250 + 285 = 535$

c. $47 + 53 = 100$
$100 + 35 = 135$
So $47 + 88 = 135$

29. a. $100 + 40 + 20 = 160$
b. $30 + 40 + 60 = 130$

31. Other compatible numbers are possible.

a. $\begin{aligned} 359 - 192 &\approx 360 - 200 \\ &= 160 \end{aligned}$

b. $\begin{aligned} 712 + 293 &\approx 700 + 300 \\ &= 1000 \end{aligned}$

c. $\begin{aligned} 882 + 245 &\approx 900 + 245 \\ &= 1145 \end{aligned}$

d. $\begin{aligned} 1522 - 486 &\approx 1500 - 500 \\ &= 1000 \end{aligned}$

33. a. 1600, since $3 + 4 + 9 = 16$
b. 160, since $1 + 4 + 8 + 3 = 16$
c. 20,000, since $7 + 5 + 8 = 20$

35. a. $2800 **b.** No

37. a. 8723, 8823, 8923, 9023, 9123, 9223, 9323, 9423

b. 906, 896, 886, 876, 866, 856, 846

39. a. 2859, 3004, 3149, 3294, 3439, 3584

b. 4164

c. 2569, 2424, 2279, 2134, 1989, 1844

41. a. 2 ⊞ 2 ⊟ ⊟ ⊟ ⊟ ⊟

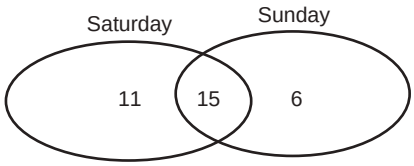
b. 30 ⊟ 3 ⊟ ⊟ ⊟ ⊟ ⊟

c. 20 ⊞ 5 ⊟ ⊟ ⊟ ⊟ ⊟

43. a. 30 cars **b.** 52 cars

c. 35 cars **d.** Case b

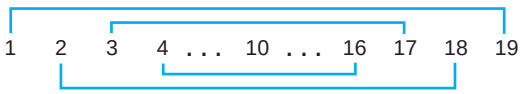
45. There must be 15 students who watched the Olympics on both Saturday and Sunday to satisfy the given conditions. Therefore, there are 32 students in the class.



47. a. 665 **b.** 724 **c.** 1143

d. 831 **e.** 1289 **f.** 572

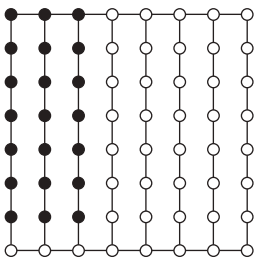
49. Matching pairs of numbers as indicated produces nine pairs, each with a sum of 20. One of each pair can be placed opposite the other on opposite sides of the "circle," and the remaining number 10 can be placed in the center circle to produce sums of 30.



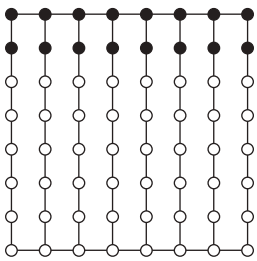
51.
$$\begin{array}{r} 999 \\ +\ \ 2 \\ \hline 1001 \end{array}$$

Exercises and Problems 3.3

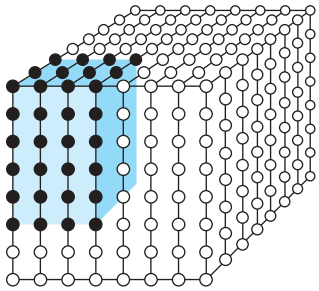
1. a.



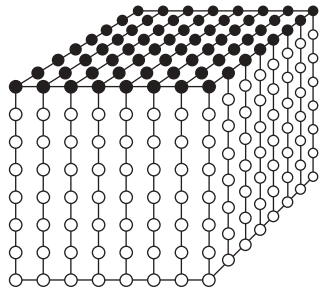
b.



3. a.



b.



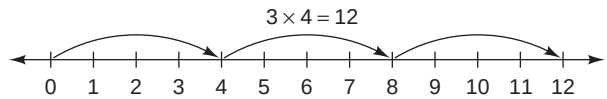
5. a. Three copies of the base-ten pieces that represent 168 have a total of 3 flats, 18 longs, and 24 units. The 24 units regroup to 2 longs and 4 units; and the (18 + 2) longs regroup to 2 flats. The final minimal set contains 5 flats, 0 longs, and 4 units.

b. Four copies of the base-ten pieces that represent 209 have a total of 8 flats and 36 units. The 36 units regroup to 3 longs and 6 units. The final minimal set contains 8 flats, 3 longs, and 6 units.

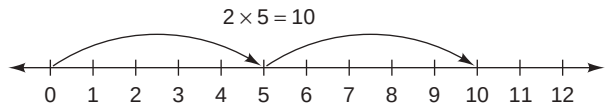
c. Three copies of the base-five pieces that represent 423_{five} have a total of 12 flats, 6 longs, and 9 units. The 9 units regroup to 1 long and 4 units; the (6 + 1) longs regroup to 1 flat and 2 longs. The (12 + 1) flats regroup to 2 long-flats and 3 flats. The final minimal set contains 2 long-flats, 3 flats, 2 longs, and 4 units.

d. Five copies of the base-eight pieces that represent 47_{eight} have a total of 20 longs and 35 units. The 35 units regroup to 4 longs and 3 units; and the (20 + 4) longs regroup to 3 flats. The final minimal set contains 3 flats, 0 longs, and 3 units.

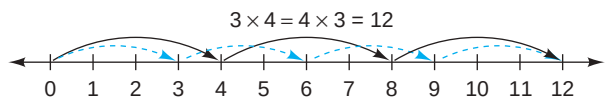
7. a.



b.



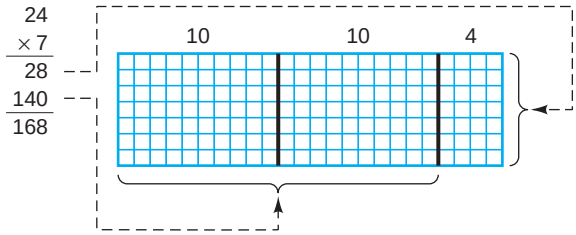
c.



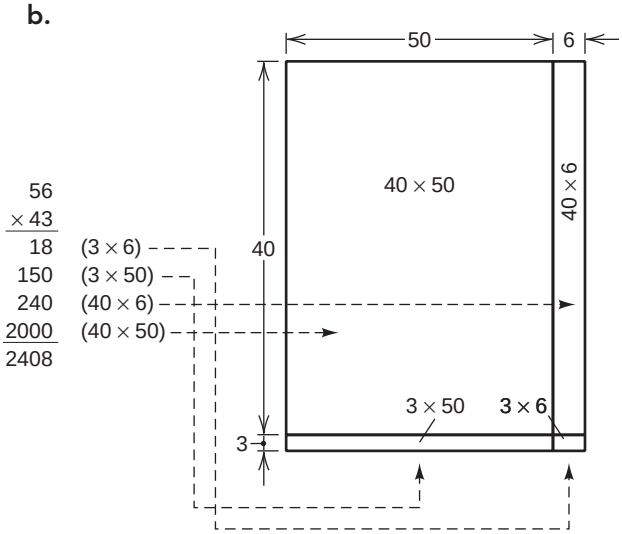
9. a. The 2 that was carried was either multiplied by or added to the 2 in the tens column.

b. The 2 and 1 in the tens column were added; or the 2 that was carried was ignored.

11. a.



b.



13. a. Commutative property for multiplication
b. Associative property for multiplication
c. Distributive property for multiplication

15. a. Closed; the product of two even numbers is another even number.
b. Not closed. For example, $2 \times 60 > 100$.
c. Closed; the product of two whole numbers whose unit digit is 6 is another whole number whose unit digit is 6 since $6 \times 6 = 36$.

17. Other combinations and compatible numbers are possible.
a. 8300 (multiply 83 by 100)
b. 210 (multiply 21 by 10)

19. a. $25 \times 12 = 25 \times (10 + 2) = 250 + 50 = 300$
b. $15 \times 106 = 15 \times (100 + 6) = 1500 + 90 = 1590$

21. a. $35 \times 19 = 35(20 - 1) = 700 - 35 = 665$
b. $30 \times 99 = 30(100 - 1) = 3000 - 30 = 2970$

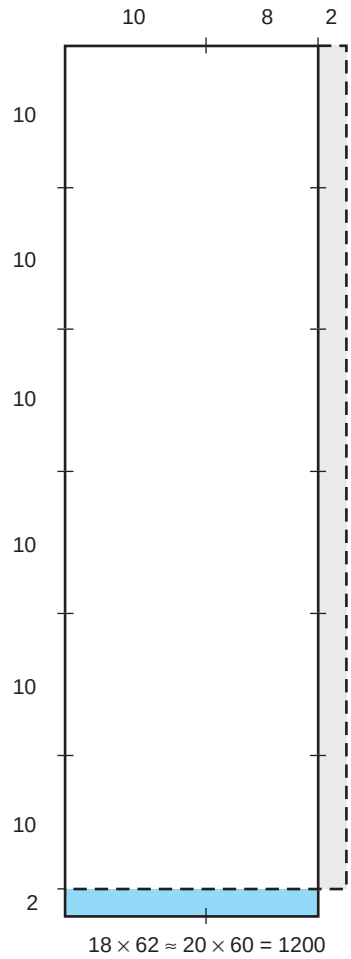
23. Other products are possible.
a. Divide 24 by 4 and multiply 25 by 4.
$24 \times 25 = 6 \times 100 = 600$
b. Divide 35 by 5 and multiply 60 by 5.
$35 \times 60 = 7 \times 300 = 2100$

25. Other rounded-number replacements are possible.
a. $22 \times 17 \approx 20 \times 20 = 400$ (Too big; estimate could be improved by subtracting 20.)
b. $83 \times 31 \approx 80 \times 30 = 2400$ (Too small; estimate could be improved by adding $3 \times 30 = 90$, or $80 \times 1 = 80$, or adding both.)

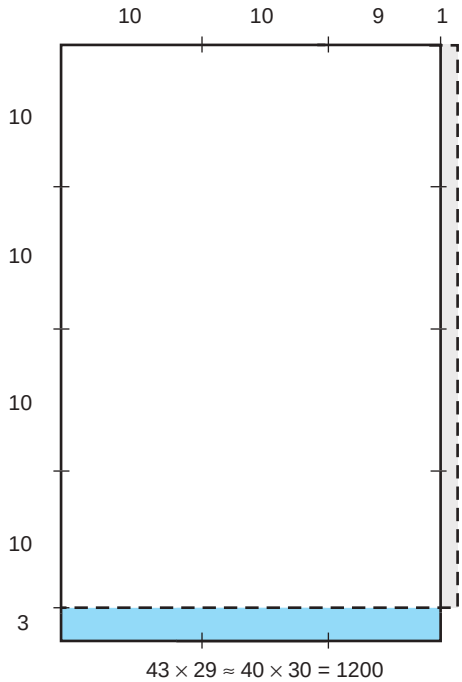
27. Other compatible numbers and combinations are possible.
a. $4 \times 76 \times 24 \approx 4 \times 25 \times 76$
$= 100 \times 76$
$= 7600$
This product is greater than the actual product.
b. $3 \times 34 \times 162 \approx 100 \times 162 = 16{,}200$
Since $100 < 3 \times 34$, the estimated product of 16,200 is less than the actual product.

29. a. Front-end estimation:
$3 \times 5 = 15$, so $36 \times 58 \approx 1500$
Combinations of tens and units digits:
$36 \times 58 \approx 30 \times 50 + (6 \times 50) + (8 \times 30) = 2040$
b. Front-end estimation:
$4 \times 2 = 8$, so $42 \times 27 \approx 800$
Combinations of tens and units digits:
$42 \times 27 \approx 40 \times 20 + (2 \times 20) + (7 \times 40) = 1120$

31. a. $18 \times 62 \approx 20 \times 60 = 1200$
The gray region shows the increase due to rounding 18 to 20, and the blue region shows the decrease due to rounding 62 to 60. Since the increase is greater than the decrease, the estimated product of 1200 is greater than the actual product.



b. $43 \times 29 \approx 40 \times 30 = 1200$
The gray region shows the increase due to rounding 29 to 30, and the blue region shows the decrease due to rounding 43 to 40. Since the increase is less than the decrease, the estimated product of 1200 is less than the actual product.



33. a. $62 \otimes 45 + 14 \otimes 29$
$\approx 60 \times 50 + 10 \times 30$
$= 3000 + 300$
$= 3300$ Exact answer: 3196
b. $36 \times 18 \otimes 40 + 15$
$\approx 40 + 20 \times 40 + 15$
$= 40 + 800 + 15$
$= 855$ Exact answer: 771

35. a. 5, 15, 45, 135, 405, 1215, 3645, 10,935, 32,805
b. 20, 25, 30, 35, 40, 45, 50, 55, 60, 65

37. a. 405, 2025, 10,125, 50,625
b. 476, 1904, 7616, 30,464

39. a. 25 or 26 **b.** 18

41. a. Each row increases by a constant amount (each row is an arithmetic sequence). Each column increases by a constant amount. The table is symmetric about the diagonal from upper left to lower right.
b. The sum of the digits in each product is 9. The tens digits in the products (18, 27, 36, . . . , 81) increased from 1 to 8 while the units digits decreased from 8 to 1.

43. $1570

45. 16

47. This pattern holds for the first nine equations. It does not hold for the 10th equation.
$12{,}345{,}678{,}910 \times 9 + 10 = 111{,}111{,}110{,}200$

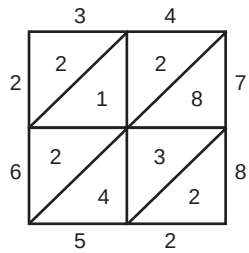
49. a. One pattern: The product of 99 and a two-digit number greater than 10 is a four-digit number *abcd* such that *ab* is 1 less than the two-digit number and $ab + cd = 99$. Similarly, the product of 999 and a two-digit number greater than 10 is a five-digit number *abcde* such that *ab* is 1 less than the two-digit number and $ab + cde = 999$.
b, c. Conjectures will vary.

51. 50

53. 8

55. The raised fingers represent 5 tens, or 50, and the product of the numbers of closed fingers is $2 \times 3 = 6$.

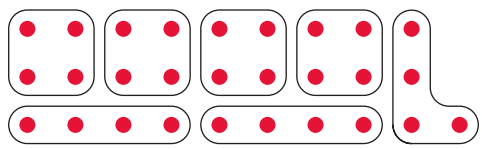
57.



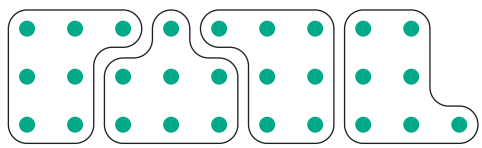
$34 \times 78 = 2652$

Exercises and Problems 3.4

1. a. Partitive (sharing) concept



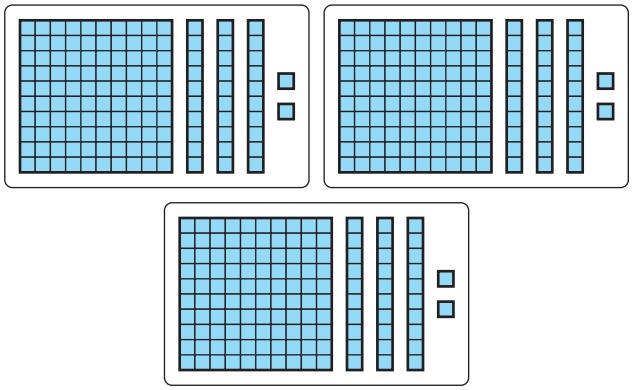
b. Measurement (subtractive) concept



3. a. $68 = 17 \times 4$
b. $414 = 23 \times 18$

5. a. $336 \div 14 = 24$
b. $72 \div 8 = 9$

7. a. Measure off 1 flat, 3 longs, and 2 units into each group. There are three groups: $396 \div 132 = 3$.



b. Regroup the 3 flats into 15 longs, and regroup 3 of the longs to units so that there are 12 longs and 16 units. Then form 4 groups of 3 longs and 4 units; $301_{\text{five}} \div 4_{\text{five}} = 34_{\text{five}}$.

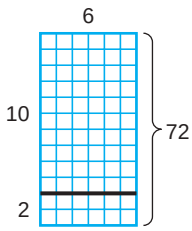
9. a. 392 is represented by 3 flats, 9 longs, and 2 units. Regroup the 3 flats to 30 longs so that there is a total of 39 longs. The 39 longs are divided into 7 groups of 5 longs with 4 longs remaining. This 5 is recorded in the tens place of the quotient. Then the 4 longs are regrouped to 40 units, and the total of 42 units is divided into 7 groups of 6 units each. This 6 is recorded in the units place of the quotient.

$$\begin{array}{r} 56 \\ 7\overline{)392} \end{array}$$

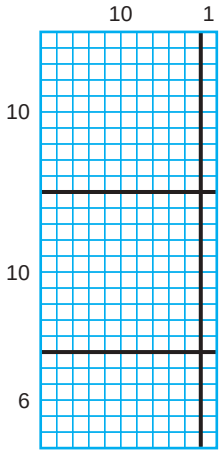
b. 320 is represented by 3 flats, 2 longs, and 0 units. Regroup the 3 flats to 30 longs so that there is a total of 32 longs. The 32 longs are divided into 5 groups of 6 longs with 2 longs remaining. This 6 is recorded in the tens place of the quotient. Then the 2 longs are regrouped to 20 units that are divided into 5 groups of 4 units each. This 4 is recorded in the units place of the quotient.

$$\begin{array}{r} 64 \\ 5\overline{)320} \end{array}$$

11. a. $72 \div 12 = 6$



b. $286 \div 26 = 11$



13. a. Regroup 3 flats into 30 longs to obtain 3 flats, 30 longs, and 8 units. Then regroup 1 long into 10 units to obtain 3 flats, 29 longs, and 18 units. These pieces will form a 32×19 rectangle.
Multiplication fact: $32 \times 19 = 608$
Division fact: $608 \div 32 = 19$

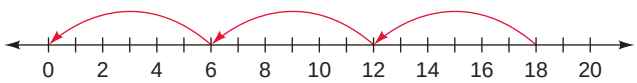
b. Regroup 1 flat into 10 longs to obtain 1 flat, 12 longs, and 1 unit. Then regroup 2 longs into 20 units to obtain 1 flat, 10 longs, and 21 units. These pieces will form a 13×17 rectangle.
Multiplication fact: $13 \times 17 = 221$
Division fact: $221 \div 13 = 17$

c. These pieces will form a 21×14 rectangle.
Multiplication fact: $21 \times 14 = 294$
Division fact: $294 \div 21 = 14$

15. a. $0 \div 4 = 0$ **b.** Undefined **c.** Undefined

17. a. $15 \div 5 = 3$ using the measurement concept, or $15 \div 3 = 5$ using the sharing concept.

b.



19. a. In the second step of the division algorithm, $5 \div 8$ is 0 with a remainder of 5. The 0 should have been placed in the quotient.

b. The 6 and 8 in the quotient were placed in the wrong columns.

21. a. The two sides of the equation are equal. Division is distributive over addition.

b. Division is not commutative; $8 \div 4 \neq 4 \div 8$

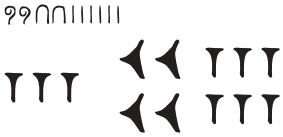
23. a. Not closed; for example, $\frac{3}{11}$ is not an odd whole number.

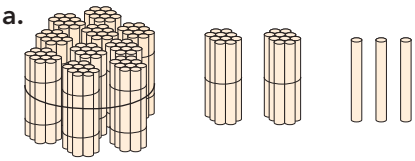
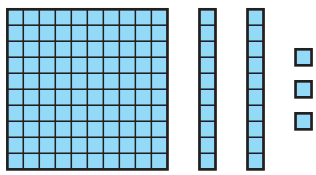
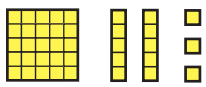
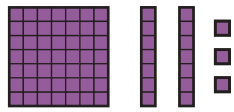
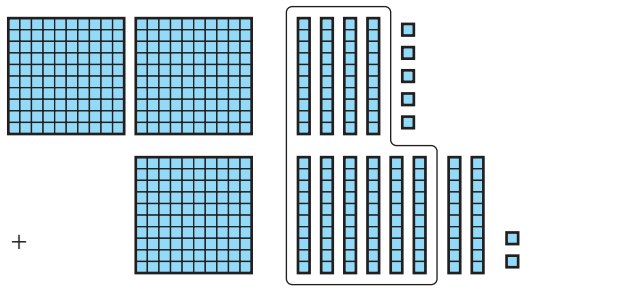
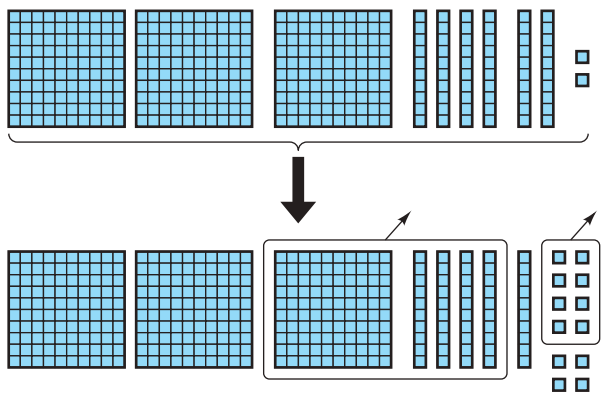
b. Not closed; $\frac{0}{1} = 0$, $\frac{1}{1} = 1$, but $\frac{0}{0}$ and $\frac{1}{0}$ are undefined.

c. Not closed; $\frac{20}{10} = 2$

25. a. 70 remainder 28 **b.** 118 remainder 12
c. 2411 remainder 381
27. Other quotients are possible.
a. Divide both numbers by 9; $90 \div 18 = 10 \div 2 = 5$
b. Divide both numbers by 2; $84 \div 14 = 42 \div 7 = 6$
29. Other number replacements are possible.
a. $250 \div 46 \approx 250 \div 50 = 5$ (less than the exact quotient)
b. $82 \div 19 \approx 80 \div 20 = 4$ (less than the exact quotient)
c. $486 \div 53 \approx 500 \div 50 = 10$ (greater than the exact quotient)
31. a. $623 \div 209 \approx 6 \div 2 = 3$
b. $7218 \div 1035 \approx 7 \div 1 = 7$
33. a. 5^{34} **b.** 10^2
35. a. 35 **b.** 38 **c.** 240 **d.** 73
37. a. 10^{12} **b.** 10^{15}
39. Yes, the correct answer is obtained.
41. This sequence produces the correct answer if the calculator follows the rules for the order of operations.
43. a. 1. Q, 4 and R, 4
2. Q, 3 and R, 5
3. Q, 4 and R, 6
4. Q, 4 and R, 2
5. Q, 3 and R, 6
6. Q, 3 and R, 3
b. A total of 27 vans
45. a. 1. 1647086 **2.** 235298
3. 33614 **4.** 4802
5. 686 **6.** 98
b. 8, the ninth number is less than 1.
47. a. Q, 510 and R, 13 **b.** Q, 12 and R, 406
49. a. 500,014 **b.** 6812
51. 4th row $4^2 + 5^2 + 20^2 = 21^2$
12th row $12^2 + 13^2 + 156^2 = 157^2$
53. $133 + 135 + 137 + 139 + 141 + 143 + 145 + 147 + 149 + 151 + 153 + 155 = 1728$
55. a. 8 **b.** 6
57. a. First method. Receiving \$1 + \$2 + \$4, etc., for 22 weeks equals \$4,194,303.
b. \$2,194,303
59. 3 quarts

Chapter 3 Test

1. a. 𓎆𓎆∩|||||||
b. CCXXVI
c.

d. $\frac{\div}{-}$

2. a. The value is 4 million; the place value is millions.
b. The value is 0; the place value is ten thousands.
3. a. 6,300,000 **b.** 6,281,500
c. 6,281,000
4. a.

b.

c.

d.

5. a. $245 + 182 = 427$

b. $362 - 148 = 214$


6. a.
$$\begin{array}{r} 483 \\ +\ 274 \\ \hline 657 \\ \hline \end{array}$$
7

b.
$$\begin{array}{r} 864 \\ +\ 759 \\ \hline 13 \\ 11 \\ 15 \\ \hline 1623 \end{array}$$

7. a. $65 - 19 = 66 - 20 = 46$

b. $843 - 97 = 846 - 100 = 746$

8. a. $321 + 435 + 106 \approx 300 + 400 + 100 = 800$

b. $7410 - 2563 + 4602 \approx 7000 - 2000 + 4000 = 9000$

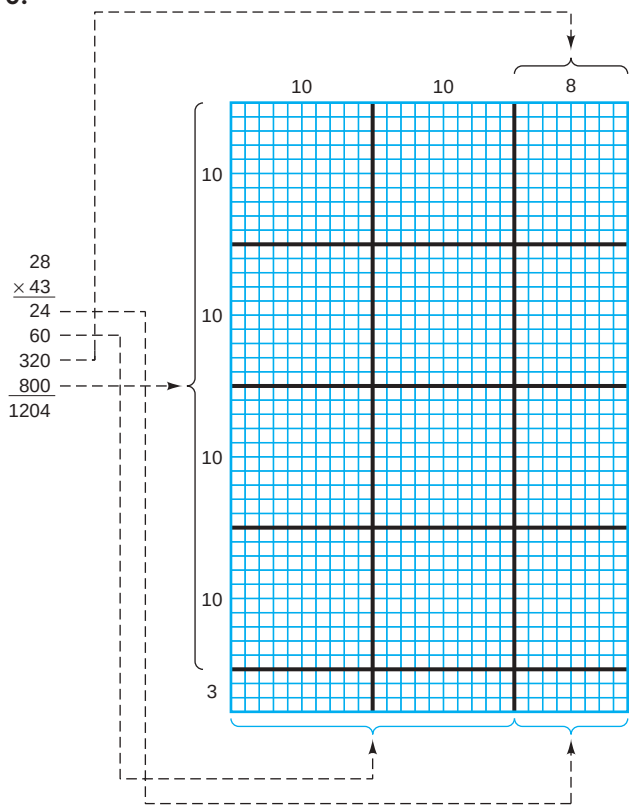
c. $32 \times 56 \approx 30 \times 50 = 1500$

d. $3528 \div 713 \approx 3000 \div 700 \approx 4$

9. a. $18 \times 5 = 3 \times 30 = 90$

b. $25 \times 28 = 100 \times 7 = 700$

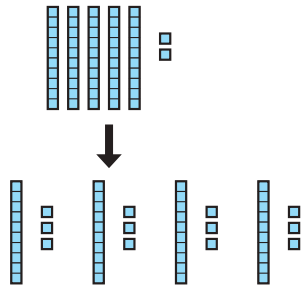
10.



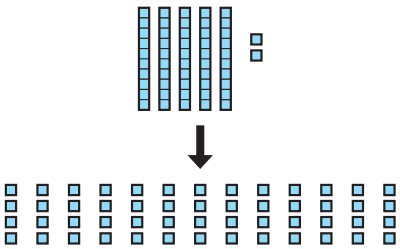
11. a. 117 **b.** 32 **c.** 53

12. a. 3^8 **b.** 7^{10}

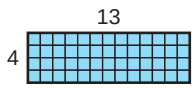
13. a. Share 52 in 4 groups of 13 each.



b. Measure off 4 units at a time to make 13 groups.



c. Use 5 longs and 2 units to form a rectangular array with one dimension of 4. The quotient is 13.



14. Other answers are possible.

a. $473 + 192 \approx 500 + 200 = 700$

b. $534 - 203 \approx 500 - 200 = 300$

c. $993 \times 42 \approx 1000 \times 40 = 40{,}000$

d. $350 \div 49 \approx 350 \div 50 = 7$

15. a. True **b.** True **c.** False

d. False **e.** False

16. $21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$

$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$

$7230 = 7230$

The pattern holds for the fourth equation.

17. 6 players **18.** 24 types of pizza

Exercises and Problems 4.1

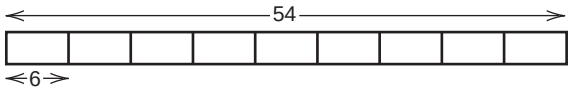
1. a. Take the escalator to the elevator that serves the even-numbered floors, and deliver to the 26th and 48th floors. Then walk down to the 47th floor, and use the elevator that serves the odd-numbered floors to deliver to the 35th and 11th floors. Return to the street level on the elevator that serves the odd-numbered floors.

b. Use the top deck elevator to deliver to floors 20, 22, 24, 26, 28, and 30. Then walk down one flight, deliver to floor 29, and use the bottom-deck elevator to deliver to floors 27, 25, 23, and 21.

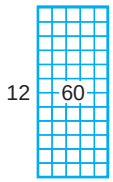
3. a. True **b.** False **c.** False

5. a. 7|63
b. 8|40
c. 13|39
d. 12|36

7. a.



b.



9. a. White, green, purple, dark green. These rods show that 1, 3, 4, and 6 are factors of 12.
b. White, green, yellow. These rods show that 1, 3, and 5 are factors of 15.
c. One train. The number 1 is the only factor of a prime which is less than the prime.

11. a. Prime numbers
b. 15, 30, 17
c. Each array will be a rectangle whose sides have two different lengths.
d. 4. The smallest whole number with eight factors is 24.

13. Both numbers in parts a and b are divisible by 3. When a number is divided by 3, the remainder is equal to the remainder when the sum of the digits is divided by 3.

15. a. No. Remainder is 3.
b. No. Remainder is 2.

17. a. Not necessarily, 3 divides 6 but 9 does not divide 6.
b. Yes, because 3×3 is a factor, so 3 is a factor.

19. a. Not divisible by 4. Remainder is 2.
b. Divisible by 4

21. The base-ten pieces show that each long-flat and each flat can always be divided into four equal parts. So, to determine whether the entire collection of base-ten pieces can be divided into four equal parts, we only need to look at the longs and units.

23. a. If a divides b and a does not divide c, then a does not divide the difference $(b - c)$. True.
b. If a does not divide b and a does not divide c, then a does not divide the sum $(b + c)$. False: $2 \nmid 5$ and $2 \nmid 7$ but $2|(5 + 7)$.
c. If a divides b and b divides c, then a divides c. True.

25. No number less than 13 divides 173. But if a number n greater than 13 divided 173, there would have to be another number m less than 13 that divided 173. Why?

27. 277 and 683

29. Carry out the process of circling and crossing out multiples until the prime number 17 has been circled. Since every composite number less than 300 has at least one prime factor less than or equal to 17, the process ends when 17 is circled.

31. a. Divisible by 11 **b.** Not divisible by 11
c. Divisible by 11 **d.** Yes

33. 47, 59, 73, 89, 107, 127, 149, 173, 199, 227, 257, 289

35. 41, 43, 47, 53, 61, 71, 83, 97

37. True

39. 235 (Every even number between 31 and 501 can be paired with an odd number that is 1 greater. There are 235 odd numbers between 32 and 502).

41. $3 \times 5 \times 7 \times 11 \times 13 = 15{,}015$

43. Using the fact that if $a|b$ and $a|c$, then $a|(b + c)$, we see that 2 is a factor of $2 \times 3 \times 4 \times 5 \times 6 + 2$ because it is a factor of both $2 \times 3 \times 4 \times 5 \times 6$ and 2. Similarly, 3 is a factor of the next number; 4 is a factor of the next number; etc.
a. The following 10 numbers have factors of 2, 3, 4, . . . , 11, respectively. Other sequences are possible.

$$2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 + 2$$
$$2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 + 3$$
$$2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 + 4$$
$$\vdots$$
$$2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 + 11$$

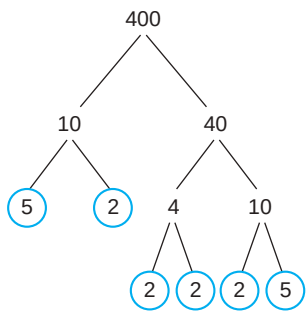
b. Let n be the product of the whole numbers from 2 through 101. The numbers $n + 2$, $n + 3$, $n + 4$, . . . , $n + 101$ form a sequence of 100 consecutive composite numbers: $n + 2$ is divisible by 2; $n + 3$ is divisible by 3; $n + 4$ is divisible by 4; etc.

45. 67,713

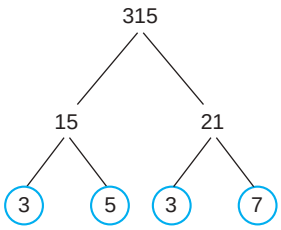
Exercises and Problems 4.2

1. a. $126 = 2 \times 3 \times 3 \times 7 = 2 \times 3^2 \times 7$
b. $308 = 2 \times 2 \times 7 \times 11 = 2^2 \times 7 \times 11$
c. $245 = 5 \times 7 \times 7 = 5 \times 7^2$

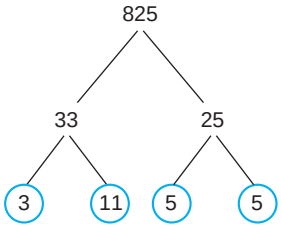
3. a.



b.



c.



5. 1,000,000,000 has a unique factorization containing only 2s and 5s. Since there is no other factorization, 7 is not a factor.

7. a. 1, 2, 4, 5, 10, 20, 25, 50, 100, 125, 250, 500
b. 1, 3, 7, 11, 21, 33, 77, 231
c. 1, 5, 7, 35, 49, 245

9. a. 1, 2, 5, 10
b. 1
c. 1, 2, 7, 14

11. a. 56 **b.** 1 **c.** 33

13. a. 28, 56, 84, 112, 140
b. 24, 48, 72, 96, 120
c. 204, 408, 612, 816, 1020

15. a. 616 **b.** 912 **c.** 210

17. a. 6 and 4 have common multiple 12 and the division lines also show that 12 is their least common multiple.
b. 4 is a common factor of 12 and 24.

19. a. 5 brown and 4 orange rods. This shows LCM(8, 10) = 40.
b. 7 is a common factor of 35 and 63.

21. a. 6480 seconds, which is greater than 1 hour
b. If you started counting after the lights flashed together, there would be 648 births, 405 deaths, and 80 immigrants. This would be a gain in population of 323 people.
c. 32,400 seconds, or 540 minutes, or 9 hours

23. 18 teams and 28 students on a team

25. a. 12 cookies
b. 25 piles
c. 22 piles

27. 150 minutes later, or at 7:30

29. If Cindy and Nicole go together on the first day, then there will be 59 days out of 180 on which neither uses the club.

31. 24 (5 occurs as a factor 24 times and 2 occurs as a factor at least 24 times)

33. a. Yes, 6 is a perfect number.
b. More deficient numbers

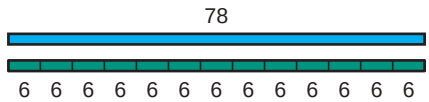
35. a. $\frac{99}{105}$ will be replaced by $\frac{33}{35}$ because their GCF is 3.
b. $\frac{102}{275}$ will be replaced by $\frac{102}{275}$, since the numerator and denominator are relatively prime.

37. The prime numbers less than the square root of 211 are 2, 3, 5, 7, 11, 13. The product of these numbers is 30,030, and the second view screen shows that 211 and 30,030 have no common factors other than 1.

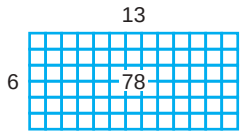
Chapter 4 Test

1. a. False **b.** True **c.** True **d.** False
2. a. $3|45$ **b.** $12|60$ **c.** $20|140$ **d.** $17|102$

3. a.



b.



4. a. Exactly one array
b. Two or more arrays
c. One or more arrays, one of which is a square

5. a. False **b.** True **c.** False **d.** False

6. a. Prime **b.** Composite **c.** Composite

7. $1836 = 2^2 \times 3^3 \times 17$

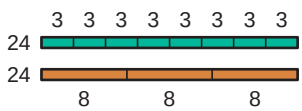
8. 1, 3, 7, 13, 21, 39, 91, 273

9. a. True **b.** False: $2|(5 + 7)$ but $2 \nmid 5$ and $2 \nmid 7$
c. True **d.** False: $2|(3 \times 6)$ but $2 \nmid 3$

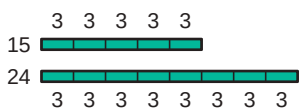
10. a. 1, 2, 5, 10
b. The four smallest common multiples are 60, 120, 180, and 240.
c. 1, 3, 5, 15
d. The five smallest common multiples are 260, 520, 780, 1040, and 1300.

11. a. 1 **b.** 154 **c.** 420
d. 5 **e.** 2 **f.** 390

12. a.



b.



13. 60 seconds

14. 428

15. 14 inches

Exercises and Problems 5.1

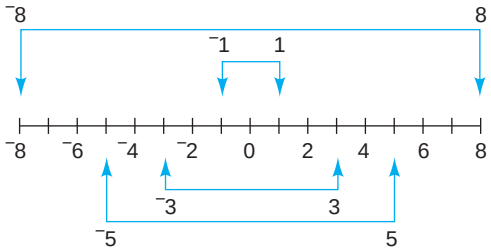
1. $^{-}17°$

3. a. $^{-}3 + 1 = {}^{-}2$ $\quad {}^{-}3 < {}^{-}2$

b. $^{-}14 + 17 = 3$ $\quad 3 > {}^{-}14$

c. $^{-}7 + 8 = 1$ $\quad {}^{-}7 < 1$

5.

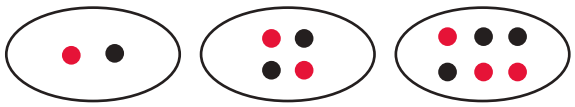


7. a. $^{-}7°$ **b.** $^{-}32$ billion dollars

9. a.



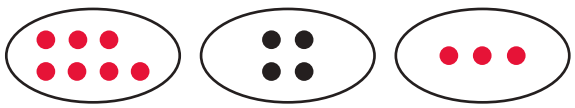
b.



c.



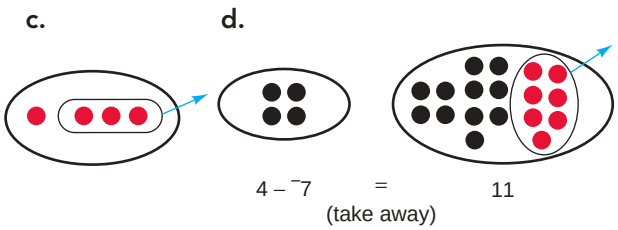
11. a.



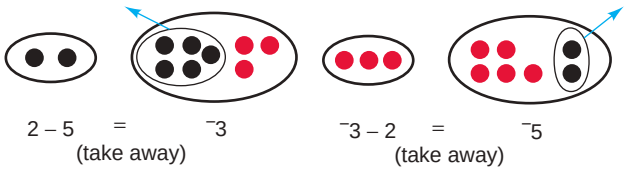
b.



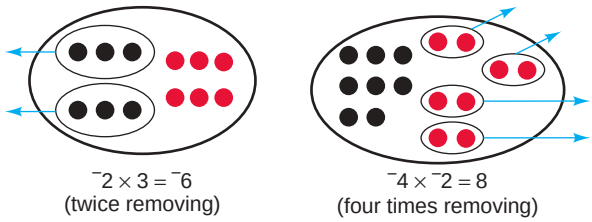
c. **d.**



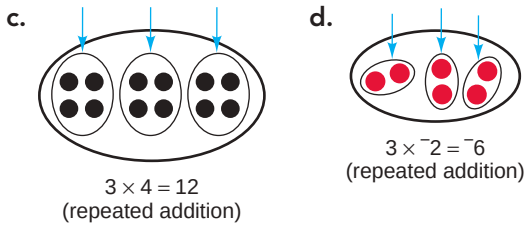
e. **f.**



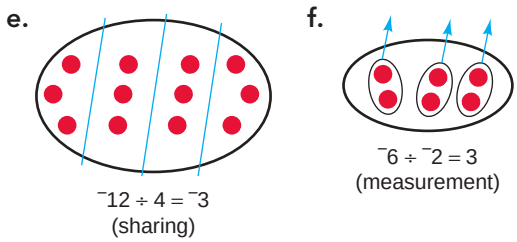
13. a. **b.**



c. **d.**



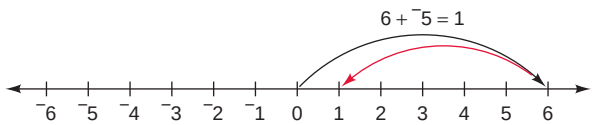
e. **f.**



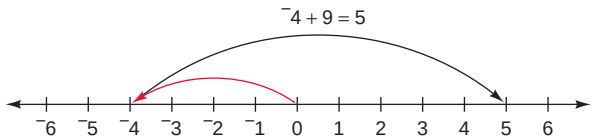
15. a. $42°$; $^{-}2 \times {}^{-}6 = 12$

b. $^{-}12°$; $^{-}6 \times 4 = {}^{-}24$

17. a.



b.



19. a. $^{-}322$ **b.** $^{-}12$

c. 12,788 **d.** 27

21. a. $^{-}14$ **b.** $^{-}4$

c. 2 **d.** 3

23. a. Associative property for multiplication

b. Commutative property for multiplication

25. a. Closed; the difference of two integers is another integer.

b. Not closed; example $\frac{-2}{5}$ is not an integer.

27. Other compatible numbers and substitutions are possible.

a. Compatible numbers:

$$\begin{aligned} -125 + 17 + -25 + 13 &= -125 + -25 + 17 + 13 \\ &= -150 + 30 \\ &= -120 \end{aligned}$$

b. Substitution:

$$\begin{aligned} 700 + -298 + 135 &= 700 + -300 + 2 + 135 \\ &= 400 + 137 \\ &= 537 \end{aligned}$$

29. Other equal products or equal quotients are possible.

a. Divide 24 by 4 and multiply -25 by 4:
$24 \times -25 = 6 \times -100 = -600$

b. Divide both numbers by 9:
$-90 \div 18 = -10 \div 2 = -5$

c. Multiply 5 by 2 and divide -28 by 2:
$-28 \times 5 = -14 \times 10 = -140$

d. Divide both numbers by 4:
$400 \div -16 = 100 \div -4 = -25$

31. a. 20 **b.** -100

33. Other compatible number replacements are possible.

a. $-241 \div 60 \approx -240 \div 60 = -4$

b. $64 \times -11 \approx 64 \times -10 = -640$

35. a. Positive. There are an even number of negative numbers in the product.

b. Negative. The product of the first three numbers is negative, and this negative number is less than -50, so its sum with 50 is a negative number.

37. a.
$5 \times -1 = -5$
$5 \times -2 = -10$
$5 \times -3 = -15$
A positive number times a negative number equals a negative number.

b.
$-1 \times 6 = -6$
$-2 \times 6 = -12$
$-3 \times 6 = -18$
A negative number times a positive number equals a negative number.

39. a. $-217 - 366 = -583$

b. $-483 + 225 = -258$

c. $2257 \div -37 = -61$

d. $-1974 \div 42 = -47$

41. a. Compute $487 + 653$ and negate the answer:
$-487 + -653 = -(487 + 653) = -1140$

b. Compute $360 + 241$:
$360 - -241 = 360 + 241 = 601$

c. Compute 32×14 and negate the answer:
$32 \times -14 = -(32 \times 14) = -448$

d. Compute $336 \div 16$ and negate the answer:
$336 \div -16 = -(336 \div 16) = -21$

43. a. $-17, -34, -51, -68, -85, -102$

b. -52, 1352, $-35{,}152$, 913,952, $-23{,}762{,}752$

45. Part b produces the correct answer, -9, and part c produces the correct answer on some calculators.

47. a. 51 days **b.** 17 days 17 hours **c.** 29 hours

49. $-8°F$

51. $12 - 34 + 5 + 67 = 50$

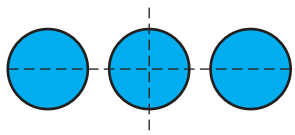
Exercises and Problems 5.2

1. a. Some possibilities: $\frac{1}{2}, \frac{1}{3}, \frac{3}{4}, \frac{1}{8}$

b. Answers vary.

3. a. $\frac{4}{9}$ shaded and $\frac{5}{9}$ unshaded.

b. $3 \div 4 = \frac{3}{4}$. Each of the four regions is $\frac{3}{4}$ of a circle.



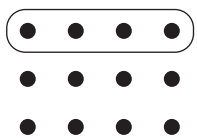
5. a. $\frac{2}{3}$, ratio concept

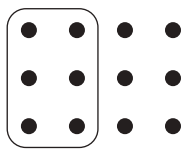
b. $\frac{6}{7}$ foot, fraction-quotient concept

c. $\frac{4}{5}$, part-to-whole concept

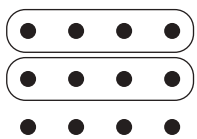
7. a. The brown rod **b.** $\frac{5}{7}$

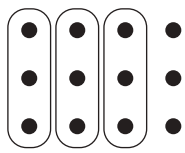
9. a. $\frac{1}{3} = \frac{4}{12}$ **b.** $\frac{6}{12} = \frac{1}{2}$





c. $\frac{2}{3} = \frac{8}{12}$ **d.** $\frac{9}{12} = \frac{3}{4}$

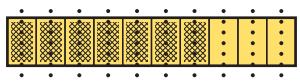




11. a. 28 **b.** 5 **c.** 25 **d.** 4

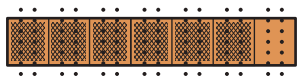
13. a. $\frac{2}{9}$ **b.** $\frac{4}{9}$ **c.** $\frac{1}{3}$ **d.** $\frac{-2}{3}$

15. a.



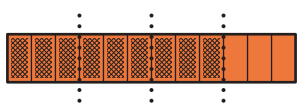
$\frac{7}{10} = \frac{14}{20}$

b.

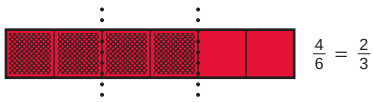


$\frac{6}{7} = \frac{18}{21}$

17. a.



$\frac{9}{12} = \frac{3}{4}$

b.  $\frac{4}{6} = \frac{2}{3}$

19. a. $\frac{2}{3} = \frac{10}{15}$ $\quad \frac{4}{5} = \frac{12}{15}$ **b.** $\frac{1}{6} = \frac{2}{12}$ $\quad \frac{-7}{12} = \frac{-7}{12}$

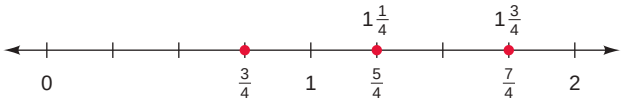
21. a. $\frac{3}{7} < \frac{5}{9}$ **b.** $\frac{1}{4} > \frac{1}{6}$ **c.** $\frac{-5}{6} > \frac{-7}{8}$

23. a. $\frac{3}{40}$ **b.** $\frac{9}{16}$

25. a. $1\frac{2}{3}$ **b.** 1 **c.** $4\frac{1}{6}$

27. a. $\frac{7}{4}$ **b.** $\frac{-11}{5}$ **c.** $\frac{14}{3}$

29.



31. a. A bar with 12 equal parts has larger parts than a bar with 20 equal parts. So $\frac{1}{12} > \frac{1}{20}$.

b. A bar with 10 equal parts has smaller parts than a bar with 8 equal parts. Thus, since $\frac{1}{10} < \frac{1}{8}$, $\frac{9}{10}$ is closer to 1 than $\frac{7}{8}$. So $\frac{7}{8} < \frac{9}{10}$.

c. The bar for $\frac{5}{12}$ is less than half shaded, and the bar for $\frac{6}{11}$ is more than half shaded. So $\frac{5}{12} < \frac{6}{11}$.

33. a. 0 **b.** 1 **c.** 4

35. Fractions for view screens b, c, and d: $\frac{30}{315}$, $\frac{10}{105}$, $\frac{2}{21}$

37. a. $2\frac{3}{4}$ **b.** $2\frac{1}{5}$

39. 67.5 seconds or 1 minute 7.5 seconds

41. $\frac{6}{16}$

43. More iron

45. $\frac{40}{1040} = \frac{1}{26}$

47. 125, since $\frac{1}{125} > \frac{1}{250}$

49. a. Yes **b.** No

51. a. No **b.** No

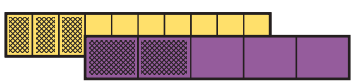
53. $\frac{5}{13}, \frac{1}{2}, \frac{3}{8}$

Exercises and Problems 5.3

1. a. 36 inches **b.** 1 foot

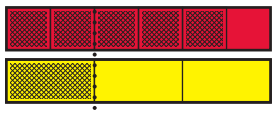
3. a. 15 inches **b.** 12 inches

5. a.



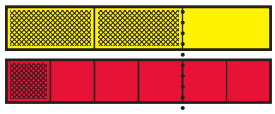
$\frac{3}{10} + \frac{2}{5} = \frac{7}{10}$

b.



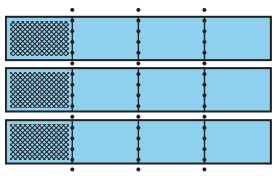
$\frac{5}{6} - \frac{1}{3} = \frac{3}{6}$

c.



$\frac{2}{3} \div \frac{1}{6} = 4$

d.



$\frac{1}{4} \times 3 = \frac{3}{4}$

e.



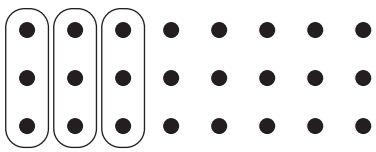
$\frac{1}{12}$ $\quad \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$

f.

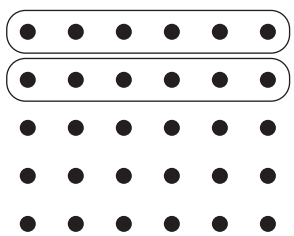


$\frac{1}{18}$ $\quad \frac{1}{3} \times \frac{1}{6} = \frac{1}{18}$

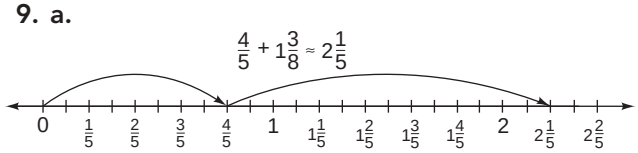
7. a. $\frac{3}{8} \times 24 = 9$



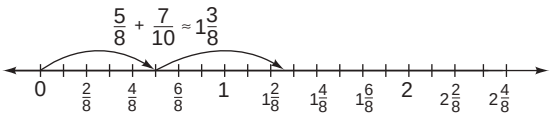
b. $\frac{2}{5} \times 30 = 12$



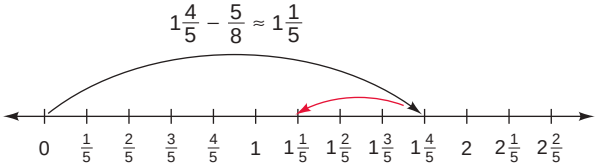
9. a.



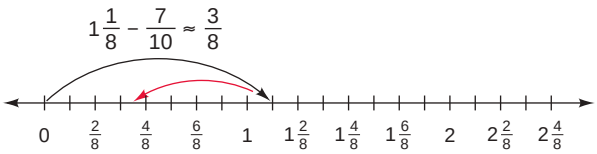
b.



11. a.



b.



13. a. $1\frac{5}{12}$ **b.** $\frac{13}{24}$ **c.** 4
d. $\frac{-3}{10}$ **e.** $3\frac{7}{12}$ **f.** $1\frac{2}{3}$
g. $7\frac{7}{8}$ **h.** $6\frac{4}{9}$ **i.** $\frac{13}{24}$

15.

Number	$\frac{7}{8}$	$^{-}4$	$\frac{-1}{2}$	10
Opposite	$\frac{-7}{8}$	4	$\frac{1}{2}$	$^{-}10$
Reciprocal	$\frac{8}{7}$	$\frac{-1}{4}$	$^{-}2$	$\frac{1}{10}$

17. a. Obtained common denominators by adding 5 to the numerator and denominator of $\frac{1}{3}$, and adding 4 to the numerator and denominator of $\frac{1}{4}$
b. Subtracted numerators and subtracted denominators
c. Computed 2×4 for the numerator and 1×11 for the denominator
d. Divided 11 by 3 and multiplied by 1

19. a. Commutative property for addition
b. Inverse for multiplication
c. Associative property for addition

21. a. Closed; the sum of two positive fractions is another positive fraction.
b. Not closed; example $\frac{-1}{2} \times \frac{-3}{2} > 0$

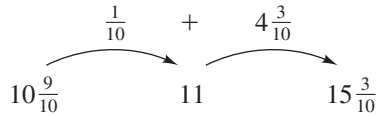
23. a. $10\frac{2}{3}$ (Combine mixed numbers with equal denominators.)
b. 7 (First find the difference between the mixed numbers with a denominator of 8.)
c. $^{-}32$ (First compute $\frac{-1}{3} \times 12$.)
d. 50 (Follow the order of operations by first computing $\frac{7}{4} \times 24$.)

25. Other substitutions are possible.
a. $215\frac{1}{5} - 11 + \frac{1}{5} = 215\frac{2}{5} - 11 = 204\frac{2}{5}$
b. $4 \times (10 - \frac{1}{7}) = 40 - \frac{4}{7} = 39\frac{3}{7}$
c. $86 + \frac{6}{12} + \frac{5}{12} + 10\frac{1}{2} = 97\frac{5}{12}$

27. Other methods are possible.
a. Equal differences (add $\frac{1}{7}$ to both numbers):

$$8 - 3\tfrac{6}{7} = 8\tfrac{1}{7} - 4 = 4\tfrac{1}{7}$$

b. Adding up:



so $15\frac{3}{10} - 10\frac{9}{10} = 4\frac{4}{10}$

c. Equal quotients (multiply both numbers by 3):

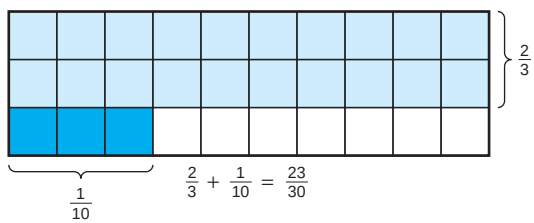
$$\tfrac{5}{8} \div \tfrac{1}{3} = 3\left(\tfrac{5}{8}\right) \div 3\left(\tfrac{1}{3}\right) = \tfrac{15}{8} \div 1 = 1\tfrac{7}{8}$$

29. a. 9 **b.** 14

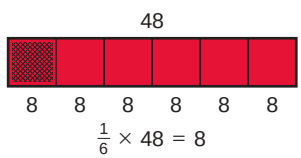
31. a. $4\frac{1}{3} \times 6\frac{1}{2} \approx 4 \times 6 + \frac{1}{3} \times 6 + \frac{1}{2} \times 4$
$= 24 + 2 + 2$
$= 28$
b. $5\frac{1}{4} \times 8\frac{2}{5} \approx 5 \times 8 + \frac{1}{4} \times 8 + \frac{2}{5} \times 5$
$= 40 + 2 + 2$
$= 44$

33. a. $\frac{6}{7} \times 34 \approx \frac{6}{7} \times 35$
$= \frac{6}{\cancel{7}_{1}} \times \cancel{35}^{5}$
$= 30$
b. $9\frac{4}{5} + 5\frac{1}{6} \approx 9\frac{4}{5} + 5\frac{1}{5}$
$= 15$

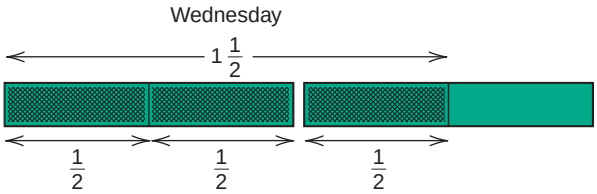
35. $\frac{23}{30}$



37. 8 million pounds



39. $\frac{1}{2}$ inch. Divide $1\frac{1}{2}$ inches into 3 equal parts:



41. **a.** $1\frac{1}{28}$ **b.** $\frac{5}{24}$ **c.** $\frac{2}{7}$ **d.** $\frac{20}{21}$

43. Because of order of operations, the calculator steps in Jan's calculation will produce the correct answer. First, 3 is divided by 5 to produce .6, and then 80 is divided by .6 to give 133 or 133.33. Carl's sequence of steps will not produce the correct answer because $80 \div \frac{3}{5} \neq (80 \div 3) \div 5$. Carl's calculator follows the rules for order of operations, first dividing 80 by 3 and then dividing the result by 5.

45. **a.** \$7333.33
b. \$2400.00
c. \$9733.33
d. \$688.00
e. \$2752.00
f. \$5070.00

47. **a.** 2 cups nonfat cottage cheese; 2 cups nonfat plain yogurt; $1\frac{1}{3}$ cups low-fat buttermilk; 1 cup Roquefort cheese; and 4 teaspoons white pepper.
b. $\frac{5}{12}$ cup

49. **a.** D, E, A, B
b. The pairs of notes that are separated by black keys correspond to the pairs of strings in part a in which one string is $\frac{8}{9}$ times the length of the other.

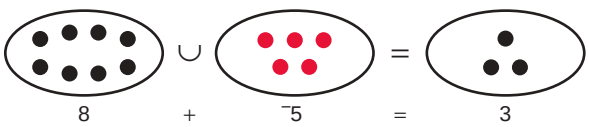
51. 24 students

53. 160 DVDs

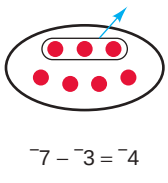
55. **Conjecture:** This holds for all pairs of fractions except those that are equal.

Chapter 5 Test

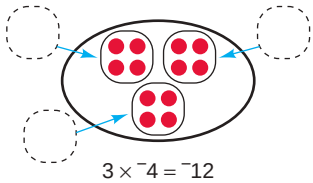
1. **a.**



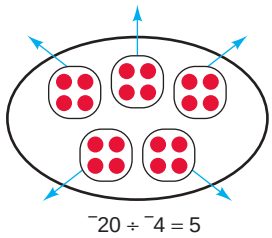
b.



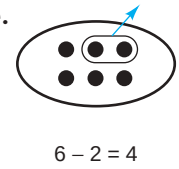
c.



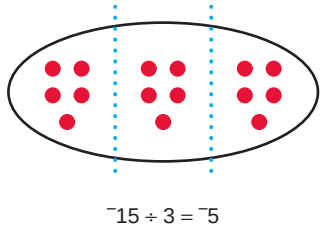
d.



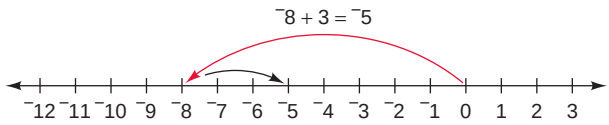
e.



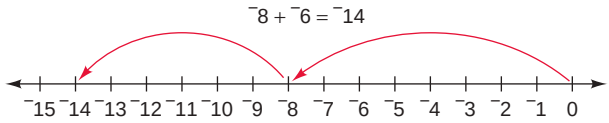
f.



2. **a.**



b.

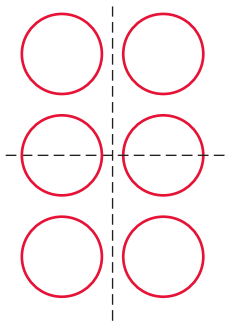


3. **a.** 42 **b.** ⁻6 **c.** ⁻80 **d.** 5

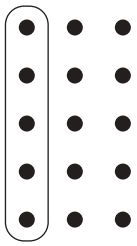
4. **a.** $^{-}16 \times 25 = {}^{-}16 \times \frac{1}{4} \times 4 \times 25 = {}^{-}4 \times 100 = {}^{-}400$
b. $800 \div {}^{-}16 = 200 \div {}^{-}4 = 100 \div {}^{-}2 = {}^{-}50$

5. **a.** $^{-}271 \div 30 \approx {}^{-}270 \div 30 = {}^{-}9$
b. $\frac{1}{8} \times 55 \approx \frac{1}{8} \times 56 = 7$
c. $4 \times 6\frac{1}{5} \approx 4 \times (6 + \frac{1}{4}) = 24 + 1 = 25$
d. $11 \times {}^{-}34 \approx 10 \times {}^{-}35 = {}^{-}350$

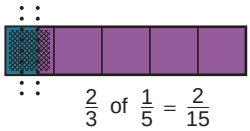
6. **a.** $6 \div 4 = 1\frac{1}{2}$



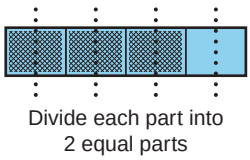
b. $\frac{1}{3} \times 15 = 5$



c. $\frac{2}{3} \times \frac{1}{5} = \frac{2}{15}$



d. $\frac{3}{4} = \frac{6}{8}$



7. a. $\frac{3}{14} = \frac{24}{112}$ $\frac{5}{16} = \frac{35}{112}$ **b.** $\frac{1}{24} = \frac{1}{24}$ $\frac{-7}{8} = \frac{-21}{24}$

8. a. $\frac{6}{11} < \frac{5}{9}$ **b.** $\frac{3}{5} > \frac{6}{11}$ **c.** $\frac{-4}{9} < \frac{-3}{7}$

9. a. $\frac{1}{8} > \frac{1}{10}$. For two figures of the same size, 1 out of 10 equal parts is less than 1 out of 8 equal parts.

b. $\frac{4}{7} > \frac{5}{12}$, $\frac{4}{7}$ is greater than $\frac{1}{2}$, $\frac{5}{12}$ is less than $\frac{1}{2}$.

c. $\frac{1}{2} < \frac{7}{12}$, $\frac{7}{12}$ is greater than $\frac{6}{12} = \frac{1}{2}$.

d. $\frac{5}{6} < \frac{7}{8}$. A whole with $\frac{1}{8}$ missing is greater than a whole with $\frac{1}{6}$ missing.

10. a. $6\frac{1}{2}$ **b.** $4\frac{7}{15}$ **c.** $8\frac{7}{24}$ **d.** $5\frac{11}{30}$

11. a. $14\frac{1}{4}$ **b.** $14\frac{12}{85}$ **c.** $^{-}9\frac{1}{3}$ **d.** 4

12. a. Inverse for multiplication
b. Distributive property
c. Commutative property for multiplication
d. Inverse for addition

13. a. False **b.** True **c.** True
d. False **e.** False

14. a. 126 inches ($10\frac{1}{2}$ feet)
b. 27 inches (2 feet, 3 inches)

15. $3\frac{1}{3}$ hours

Exercises and Problems 6.1

1. a. .001 **b.** .000001

3. a. 7 ⓪ 4 ① 6 ②

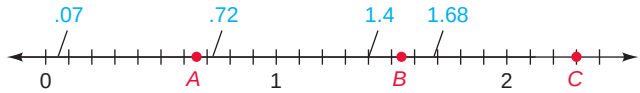
b. The 0 was placed between the digits where we now place the decimal point. A small zero could have suggested a point.

5. a. $\frac{7}{100}$ **b.** $\frac{6}{10{,}000}$ **c.** $\frac{3}{1000}$ **d.** $\frac{9}{10}$

7. a. .33; Thirty-three hundredths
b. .0392; Three hundred ninety-two ten-thousandths
c. .054; Fifty-four thousandths
d. 748.1; Seven hundred forty-eight and one-tenth

9. a. Three hundred forty-seven and ninety-six hundredths dollars
b. Twenty-three and fifty hundredths dollars
c. One thousand one hundred forty-four and three hundredths dollars

11. *A*, .65 *B*, 1.55 *C*, 2.3

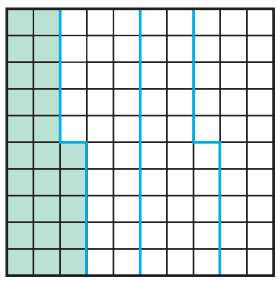


13. a. .40; a square with 100 equal parts, 40 of which are shaded
b. .470; a square with 1000 equal parts, 470 of which are shaded

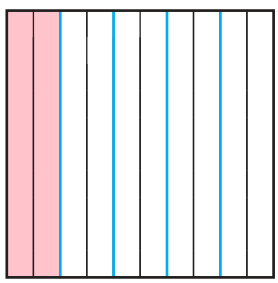
15. a. 7 parts out of 10 is equal to 70 parts out of 100. Both decimals are represented by seven shaded columns.
b. 43 parts out of 100 is equal to 430 parts out of 1000.
c. 45 parts out of 100 is less than five full columns and 6 parts out of 10 is six full columns.

17. a. 247 **b.** 2.47

19. a. Divide a Decimal Square for hundredths into four equal parts, and shade one of these parts.



b. Divide a Decimal Square for tenths into five equal parts, and shade one of these parts.



21. **a.** .375 **b.** $.\overline{18}$
c. $.3\overline{48}$ **d.** 3.75

23. $\frac{1}{3} = .\overline{3}$ $.25 = \frac{1}{4}$

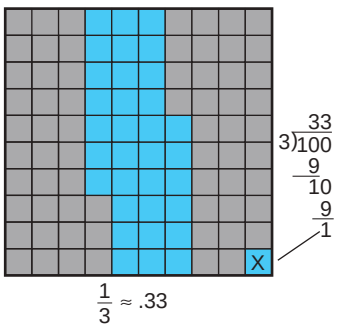
$\frac{5}{6} = .8\overline{3}$ $.41\overline{6} = \frac{5}{12}$

$.875 = \frac{7}{8}$ $.0\overline{6} = \frac{1}{15}$

$\frac{3}{10} = .3$ $\frac{34}{99} = .\overline{34}$

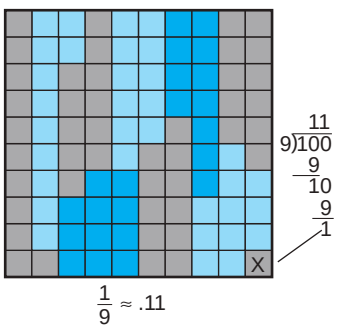
25. The fractions in a and b have repeating decimals. The fraction in c has a terminating decimal.

27. **a.** Dividing 100 small squares by 3 produces 3 equal groups of 33 small squares each, with 1 small square remaining. Thus, the decimal for $\frac{1}{3}$ begins with .33; and since 1 (the number of remaining squares) is less than half of the divisor 3, the decimal for $\frac{1}{3}$ to two decimal places is .33.



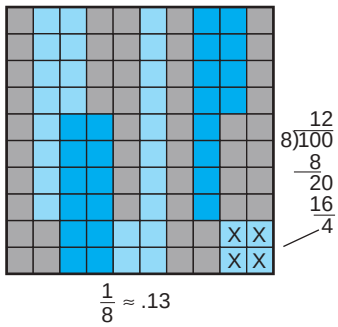
$\frac{1}{3} \approx .33$

b. Dividing 100 small squares by 9 produces 9 equal groups of 11 small squares each, with 1 small square remaining. Thus, the decimal for $\frac{1}{9}$ begins with .11; and since 1 (the number of remaining squares) is less than half of the divisor 9, the decimal for $\frac{1}{9}$ to two decimal places is .11.



$\frac{1}{9} \approx .11$

c. Dividing 100 small squares by 8 produces 8 equal groups of 12 small squares each, with 4 small squares remaining. Thus, the decimal for $\frac{1}{8}$ begins with .12, and since 4 (the number of remaining squares) is equal to one-half of the divisor 8, the decimal for $\frac{1}{8}$ to three decimal places is .125, or rounded to two decimal places is .13.



$\frac{1}{8} \approx .13$

29. **a.** .004, or .005 if rounded to the leading nonzero digit
b. .4, which gives the same result if we round to the leading nonzero digit
c. .07, which gives the same result if we round to the leading nonzero digit
d. .002, or .003 if rounded to the leading nonzero digit

31. $\frac{19}{34}, \frac{11}{17}, \frac{38}{52}, \frac{16}{20}, \frac{21}{25}$

33. **a.** .0625 **b.** .0938
c. .1094 **d.** .5469

35. **a.** .37 **b.** .1
c. 14.372 **d.** .3

37. **a.** .07 is the smallest. Its Decimal Square has 7 parts shaded out of 100. The Decimal Square for .075 has 75 parts shaded out of 1000 or $7\frac{1}{2}$ parts shaded out of 100; the Decimal Square for .08 has 8 parts shaded out of 100; and the Decimal Square for .3 has 3 parts shaded out of 10, or 30 parts shaded out of 100. 1.003 is represented by 1 whole shaded square and 3 parts shaded out of 1000.
b. Students may have believed that the more decimal places there are, the smaller the decimal.

39. $.05 = \frac{1}{20}$ $\frac{2}{3} = .\overline{6}$

$.125 = \frac{1}{8}$ $.4 = \frac{2}{5}$

$.58\overline{3} = \frac{7}{12}$ $\frac{7}{18} = .3\overline{8}$

$.\overline{532} = \frac{532}{999}$ $.2\overline{6} = \frac{4}{15}$

41. **a.** 579 **b.** 207

43. **a.** 17.3, 16.3, 15.9, 28.1, 22.6
b. Knees **c.** .85

45. **a.** $\frac{1}{8}$ **b.** No. Her largest drill size is .25.
c. $\frac{1}{16}$

47. a. 6.92 feet **b.** 6.79 feet

49. The pattern continues to hold for $\frac{4}{9}$, $\frac{5}{9}$, $\frac{6}{9}$, $\frac{7}{9}$, $\frac{8}{9}$, and $\frac{9}{9}$. Note that $\frac{9}{9} = .9999\ldots$, which also equals 1. If we try to continue the pattern, it requires that

$$\frac{10}{9} = .101010\ldots$$

which is not true. However, there is a pattern that continues:

$$\frac{10}{9} = 1.1111\ldots$$
$$\frac{11}{9} = 1.2222\ldots$$
$$\vdots$$

Exercises and Problems 6.2

1. a. $^{-}11.9°$ C **b.** 4.4° C ($^{-}11.9 - {}^{-}16.3$)

3. a. 3 parts out of 10 has the same amount of shading as 30 parts out of 100. So 3 parts out of 10 plus 45 parts out of 100 equals 75 parts out of 100.

b. 2 parts out of 10 has the same amount of shading as 200 parts out of 1000. So 350 parts out of 1000 minus 200 parts out of 1000 equals 150 parts out of 1000, or 15 parts out of 100.

c. Use a Decimal Square for .4 and divide the shaded part into 10 equal parts. Three of these parts is .12 of a whole square.

d. The Decimal Square for .37 has 37 shaded parts. Ten of these squares have 370 shaded parts, which is 3 whole squares and 70 parts out of 100 and equals 3 whole squares and 7 parts out of 10.

e. The Decimal Square for .45 has 45 parts and the Decimal Square for .15 has 15 parts. Since all these parts have the same size, mark off 3 groups of 15 shaded parts each.

f. The Decimal Square for .30 has 30 parts. Dividing these parts into 10 equal groups results in 3 parts per group, which is .03 of a whole square.

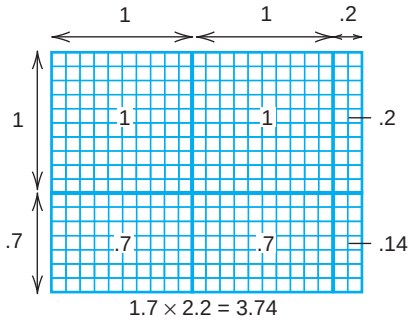
5. a.

$$\begin{array}{r} {\scriptstyle 1} \\ 4.821 \\ +\ 61.73 \\ \hline 66.551 \end{array} \qquad \frac{8}{10} + \frac{7}{10} = \frac{15}{10} = \frac{10}{10} + \frac{5}{10} = 1 + \frac{5}{10}$$

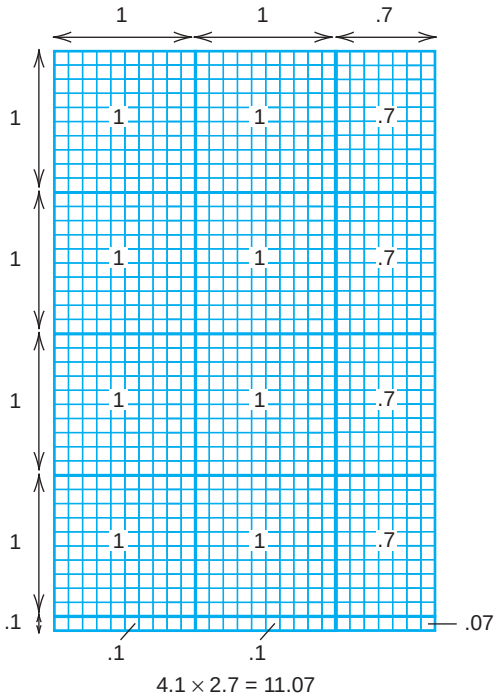
b A 1 from the hundredths column is used to obtain $\frac{10}{1000}$ to increase the 6 in the thousandths column to 16 thousandths.

$$\begin{array}{r} {\scriptstyle 3} \\ .0\not{4}6 \\ -\ .018 \\ \hline .028 \end{array} \qquad \frac{4}{100} = \frac{3}{100} + \frac{1}{100} = \frac{3}{100} + \frac{10}{1000}$$

7. a.



b.



9. a. 24.96; multiply 32 times 78 and count off two decimal places.

b. 6.43; divide 141.46 by 22.

11. a. .076; move the decimal point two places to the left

b. .034; move the decimal point three places to the left

c. .0003; move the decimal point two places to the left

d. .004; move the decimal point one place to the left

13. a. $\frac{5}{9}$ **b.** $\frac{14}{99}$ **c.** $\frac{14}{90}$

15. Other answers are possible.

a. .65 **b.** .0055

17. a. Add 6 thousandths to both numbers to obtain .343 − .300, and then subtract .300 to obtain .043.

b. Compute 4.2 + .8 = 5.0 and then add .1 to obtain 5.1.

c. Compute $2.6 \times 100 = 260$ and add 2.6 to obtain 262.6.

d. Divide both numbers by 3 to obtain $4.3 \div 100$, which equals .043.

19. a. $.25 \times 48 = \frac{1}{4} \times 48 = 12$

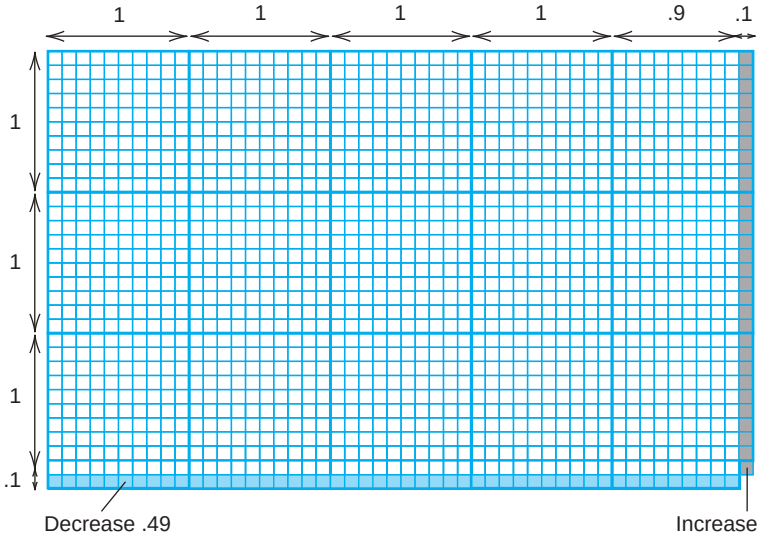
b. $.5 \times 40.8 = \frac{1}{2} \times 40.8 = 20.4$

c. $5.5 \times .2 = 5.5 \times \frac{1}{5} = 1.1$

21. a. Front-end estimation: \$90; rounding to leading digit: \$110

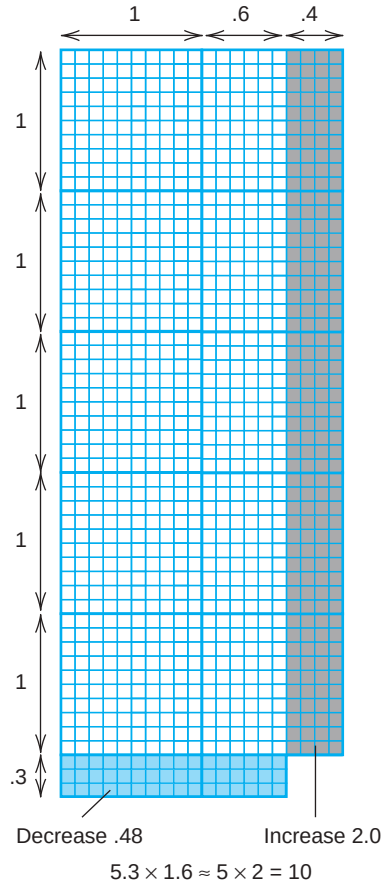
b. Front-end estimation: \$300; rounding to leading digit: \$400

23. a. $3.1 \times 4.9 \approx 3 \times 5 = 15$



This estimation is less than the actual product because the decrease due to rounding 3.1 to 3 is greater than the increase due to rounding 4.9 to 5.

b. $5.3 \times 1.6 \approx 5 \times 2 = 10$



This estimation is greater than the actual product since the increase due to rounding 1.6 to 2 is greater than the decrease due to rounding 5.3 to 5.

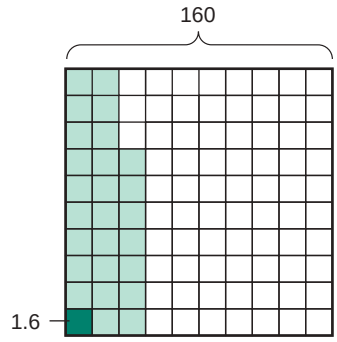
25. a. $8 \div .5 = 16$ **b.** $11.60 + .4 = 12$

c. $\frac{1}{3} \times 120 = 40$ **d.** $\frac{1}{4} \times 80 = 20$

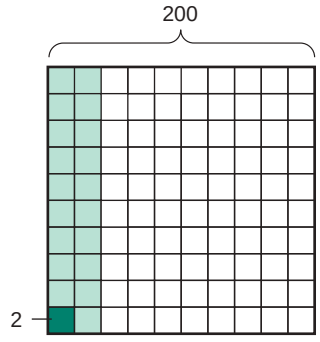
27. a. 78 percent
b. By ignoring the decimal points and estimating $304 \times 5 \approx 1600$.
29. a. The 2 should have been written in the tenths column and the 1 in the ones column.
b. The 6 in the hundredths column was not subtracted (possibly 0 was subtracted from 6). This error caused a subsequent error in the tenths column.
c. The decimal point in the product was placed under the decimal point in the two given numbers. Perhaps one decimal place was counted off rather than two.
d. The remainder of 2 was recorded in the quotient.
31. a. Add .0007.
b. Subtract 70.0006.
c. Subtract .503.
33. a. .41667 **b.** 2.08333 **c.** 217.39130
35. a. View screen 4; 35.9936; view screen 5; $^{-}57.58976$
b. View screen 4; 440.2; view screen 5; 423.9
37. a. Geometric with common ratio 1.8; 44.08992
b. Arithmetic with common difference .7; 13.5
39. $4677.98
41. a. $12.66 **b.** $3.76
43. a. 2004 **b.** 2005–2011, 15.0%
c. 2011, coal and petroleum increased
45. a. $39.88 **b.** $1.58 **c.** $117.79
47. a. 6.04 seconds **b.** 3.33 seconds
49. a. .5 cent **b.** $325
c. $3250 **d.** $2925

Exercises and Problems 6.3

1. a. $\frac{4}{5}$ **b.** 2,250,000
3. a. 111 **b.** 150
5. a. $17.36 **b.** $6.65 **c.** $1.40
7. a. .375; a Decimal Square having $37\frac{1}{2}$ shaded parts out of 100.
b. .065; a Decimal Square having $6\frac{1}{2}$ shaded parts out of 100
c. .283; a Decimal Square having $28\frac{1}{3}$ shaded parts out of 100
9. a. 60 percent (60 parts shaded out of 100)
b. 6 percent (6 parts shaded out of 100)
c. 25.6 percent (25 and 6 tenths parts shaded out of 100, or 256 parts shaded out of 1000)
11. a. 80 percent **b.** 83.3 percent **c.** 175 percent
13. a. Since one small square represents 1.6, 27 small squares represent $27 \times 1.6 = 43.2$. So 27 percent of 160 is 43.2.



b. Since each small square represents 2, it will require 20 small squares to represent 40. So 40 is 20 percent of 200.



c. 40 **d.** 91 **e.** 150 percent
15. a. To determine 10 percent of $128.50, move the decimal point one place to the left to obtain $12.85.
b. 75 percent of 32 is $\frac{3}{4} \times 32 = 24$.
c. 10 percent of $60 is $6, so 90 percent of $60 is $54.
d. 10 percent of 80 is 8, so 110 percent of 80 is 88.
17. a. 9 percent of $30.75 $\approx$ 10 percent of $30.75 = $3.075 $\approx$ $3.08
b. 19 percent of 60 $\approx \frac{1}{5} \times 60 = 12$
c. 4.9 percent of 128 $\approx \frac{1}{2} \times$ 10 percent of $128 = \frac{1}{2}$ of $12.8 = 6.4$
d. 15 percent of 241 $\approx$ 10 percent of 240 + 5 percent of $240 = 24 + 12 = 36$
19. a. $\frac{2}{19} \approx \frac{2}{20} = 10$ percent
b. $\frac{408}{1210} \approx \frac{400}{1200} = 33\frac{1}{3}$ percent
c. $\frac{100}{982} \approx \frac{100}{1000} = 10$ percent

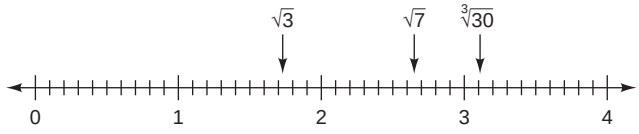
21. **a.** 4.6×10^{9} **b.** 2.7×10^{-8}
23. **a.** .0000000012 **b.** 31,556,900
25. **a.** 1.58436×10^{18} miles **b.** 1.49×10^{3} seconds
27. **a.** 51 **b.** 71
29. **a.** Maine, 11.5 to 1; Missouri, 13.9 to 1; Oregon, 20.6 to 1; Wyoming, 13.3 to 1
b. Maine
c. Oregon
31. **a.** \$59.97 **b.** 20 percent
c. \$55,900 **d.** 78.6 percent
33. **a.** 8.34 cents per ounce
b. The larger package
c. 4 batches
35. The identity property for multiplication in the first equation and the distributive property for multiplication over addition in the second equation
a. \$178.08
b. \$123.16
c. \$61.92
37. **a.** \$21.80 **b.** \$118.08
c. 8 years **d.** \$32,992.31
39. **a.** \$13.70
b. \$28.77
c. 1276.28 pounds
41.

Mercury	36,002,000	.4 unit
Venus	6.7273×10^{7}	.7 unit
Earth	93,003,000	1.0 unit
Mars	1.41709×10^{8}	1.5 units
Jupiter	483,881,000	5.2 units
Saturn	8.87151×10^{8}	9.5 units
Uranus	1,784,838,000	19.2 units
Neptune	2.796693×10^{9}	30.1 units

43. 0, 3, 6, 12, 24, 48, 96
.4, .7, 1.0, 1.6, 2.8, 5.2, 10.0
a. 2.8 astronomical units
b. 19.6 astronomical units

Exercises and Problems 6.4

1. The numbers in b and d are irrational.
3. **a.** $\sqrt{33} \approx 5.7$ **b.** $\sqrt{113} \approx 10.6$
5. **a.** $\frac{1}{4}$ **b.** 4 **c.** 3.1 **d.** $^{-}5$
7. **a.** Irrational, 4.2
b. Rational, 6
c. Rational, $\frac{1}{3}$
9. **a.** $\sqrt{7} \approx 2.6$
b. $\sqrt[3]{30} \approx 3.1$
c. $\sqrt{3} \approx 1.7$



11. **a.** $a = 4, b = 3, c = 5$
b. $a = 12, b = 5, c = 13$
c. $a = 60, b = 11, c = 61$
13.

	Whole Numbers	Integers	Rational Numbers	Real Numbers
$^{-}3$		✓	✓	✓
$\frac{1}{8}$			✓	✓
$\sqrt{3}$				✓
π				✓
14	✓	✓	✓	✓
$\frac{1.6}{4}$			✓	✓
$.\overline{82}$			✓	✓

15. **a.** Not closed, $2 - 5$ is not a whole number
b. Closed
17. **a.** Commutative property for multiplication
b. Distributive property
c. Commutative property for addition
19. **a.** Irrational **b.** Irrational **c.** Rational, 12
21. **a.** $3\sqrt{5}$ **b.** $4\sqrt{3}$ **c.** $2\sqrt{15}$
23. **a.** True
b. False
$\sqrt{9} - \sqrt{4} = 3 - 2 = 1 \neq \sqrt{9 - 4} = \sqrt{5} \approx 2.2$
25. **a.** $\frac{4\sqrt{7}}{7}$ **b.** $\frac{\sqrt{6}}{4}$
27. If the calculator has eight places for digits, 1.0905077 is displayed in step 4. Eventually the number in the view screen will be 1.
29. 1
31. **a.** $\sqrt[4]{81} = 3$ **b.** $\sqrt{2.25} = 1.5$
c. $\sqrt{3.0625} = 1.75$ **d.** $\sqrt[3]{^{-}2.197} = {}^{-}1.3$
33. 217 steps
35. 13 feet
37. 17.0×17.0 inches
39. **a.** 1.5 hours
b. In this position the satellite takes approximately 24 hours to make 1 orbit of Earth. Therefore, the satellite stays in the same position relative to Earth.

41. a. Here are the first nine ratios for consecutive pairs of Fibonacci numbers: 1, 2, 1.5, $1.\overline{6}$, 1.6, 1.625, 1.6153846, 1.6190476, 1.6176471
b. $233 \div 144 \approx 1.6180556$

Chapter 6 Test

1. a. The Decimal Square for .4 has four full columns shaded, but the Decimal Square for .27 has less than three full columns shaded.
b. The Decimal Square with 7 parts shaded out of 10 has the same amount of shading as the Decimal Square with 70 parts shaded out of 100.
c. The Decimal Square for .225 has less than three full columns shaded, but the Decimal Square for .35 has more than three full columns shaded.
d. The Decimal Square for .09 has less than one full column shaded, but that for .1 has one full columm shaded.

2. a. .75 **b.** .07 **c.** $.\overline{6}$
d. .875 **e.** $.\overline{4}$ **f.** .24

3. a. $\frac{278}{1000}$ **b.** $\frac{35}{99}$ **c.** $\frac{3}{100}$ **d.** $\frac{7319}{9990}$

4. a. .88 **b.** .4 **c.** .510 **d.** .6667

5. a. 1.3: The total amount of shading in 2 squares, one with 6 columns shaded out of 10 and one with 7 columns shaded out of 10, is 13 shaded columns, or 1 completely shaded square and 3 shaded columns.
b. 1.2: The total amount of shading in 3 squares, each with 4 columns shaded out of 10, is 12 shaded columns, or 1 completely shaded square and 2 shaded columns.
c. .14: If a square has 62 parts shaded out of 100 and 48 of the shaded parts are taken away, then 14 shaded parts out of 100 remain.
d. 16: If a square has 80 parts shaded out of 100 and 5 of the 80 shaded parts are removed at a time, the process can be done 16 times, or the 80 shaded parts can be divided into 16 groups, each containing 5 shaded parts.

6. a. .186 **b.** .0496 **c.** .703 **d.** 319

7. a. Move the decimal point in .073 a total of two places to the right to obtain 7.3.
b. Compute $7 \times 6 = 42$ and count off one decimal place to obtain 4.2.
c. Move the decimal point in 4.9 a total of three places to the left to obtain .0049.
d. Move the decimal point in 372 a total of two places to the left to obtain 3.72.
e. 10 percent of 260 is 26, and one-half of this is 13. So 15 percent of 260 is $26 + 13 = 39$.
f. 25 percent of 36 is $\frac{1}{4} \times 36 = 9$.

8. a. $\frac{1}{2} \times 310 = 155$ **b.** $\frac{1}{4} \times 416 = 104$
c. $\frac{1}{3} \times 60 = 20$ **d.** $\frac{3}{4} \times 40 = 30$

9. a. 16.6 **b.** 37.5 percent **c.** 62.5
d. 147.5 **e.** 140 percent

10. a. 4.378×10^2 **b.** 1.06×10^{-4}

11. a. Irrational **b.** Rational **c.** Irrational
d. Irrational **e.** Rational **f.** Irrational

12. a. 5.8 **b.** 2.6

13. a. Closed by the closure property for rational numbers
b. Not closed ($\sqrt{2} \times \sqrt{8} = 4$)
c. Not closed ($^{-}\sqrt{3} + \sqrt{3} = 0$)

14. a. $9\sqrt{5}$ **b.** $2\sqrt{6}$

15. a. $2\sqrt{13}$ **b.** $\sqrt{319}$

16. $220

17. 16-ounce glass

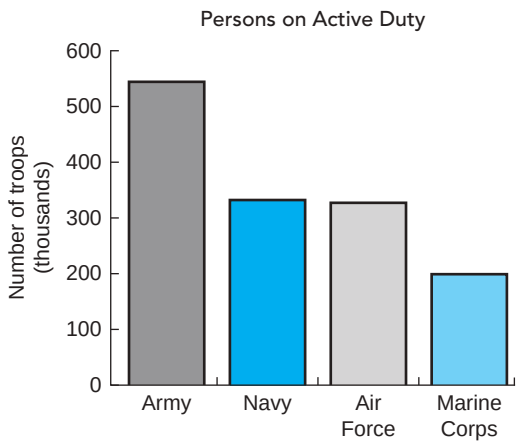
18. 67.1 feet

19. $7.20

20. 15,664 students

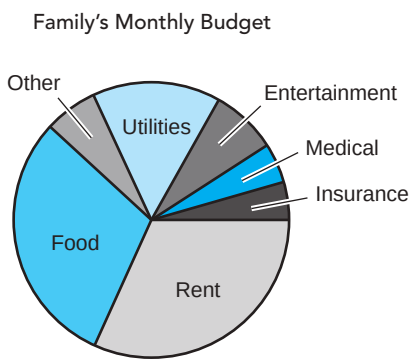
Exercises and Problems 7.1

1. a. 2.7 **b.** 133,000



3. a. 2000, 9 percent
b. The prime rate increased by 2% in 2005 and 2006.

5. a.



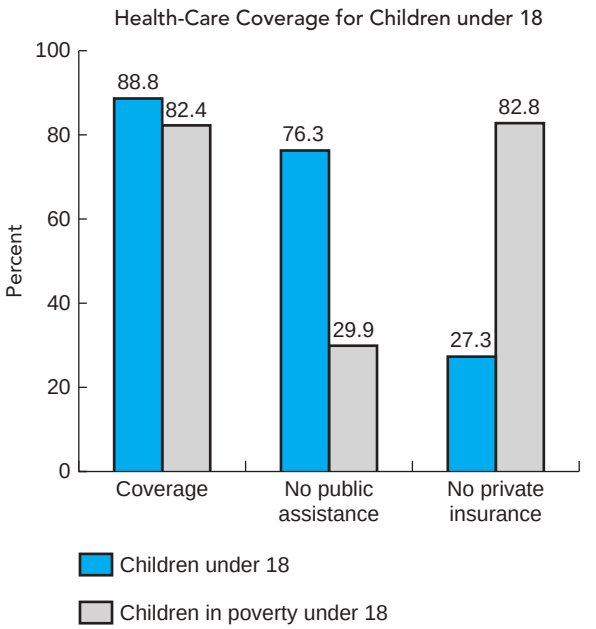
b. Rent, 115°; food, 108°; utilities, 54°; insurance, 14°; medical, 18°; entertainment, 29°; other, 22°

7. a. Kansas City
b. Kansas City, 36 inches; Portland, 39 inches

9. a. Bicycle riding, camping, fishing (fresh water), running/jogging and weightlifting
b. Fishing (fresh water)
c. Aerobic exercising

11. a. Hispanic
b. Asian; about 39 per thousand females
c. Indian

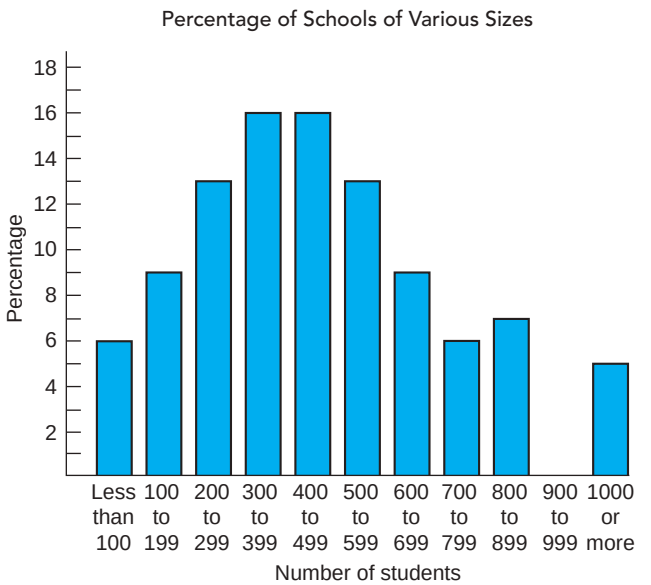
13. a.



b. 46.4 percent

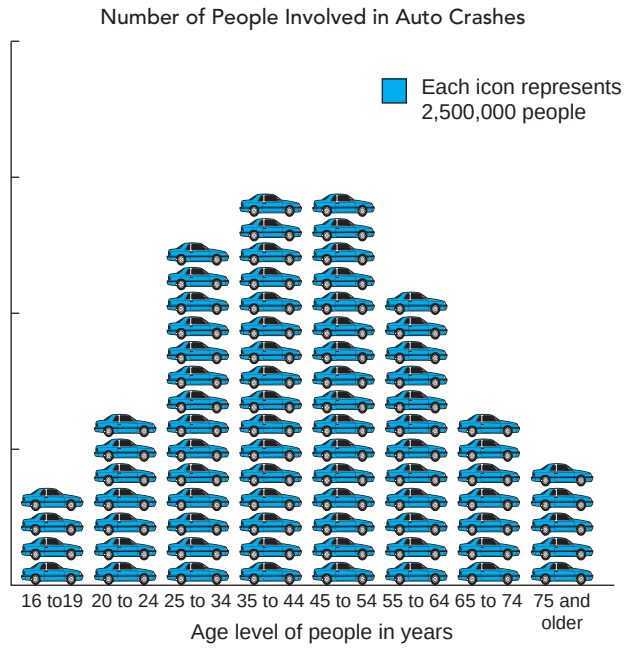
15. a. 14.7 percent **b.** 14.4

17. a.



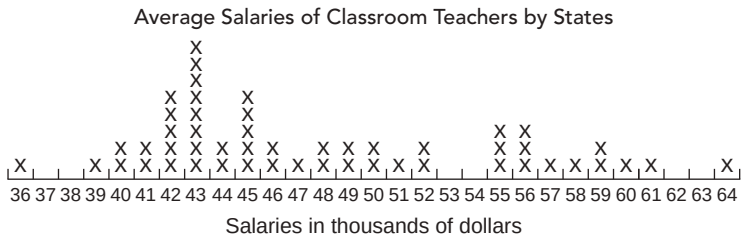
b. 45 percent

19. a.



b. Fewer people had crashes in these two age groups combined than in the third age group.

21. a.



b. $43,000–$43,999
c. 54.9 percent

23. a.

Stem	Leaf
4	6, 9
5	3, 6, 7, 7, 8
6	0, 0, 3, 3, 4, 4, 5, 6, 7, 7, 7, 8
7	0, 1, 1, 1, 2, 3, 4, 7, 8, 8, 9
8	0, 1, 3, 5, 8
9	0, 0, 3

b. 21.1 percent **c.** 18.4 percent

25. a.

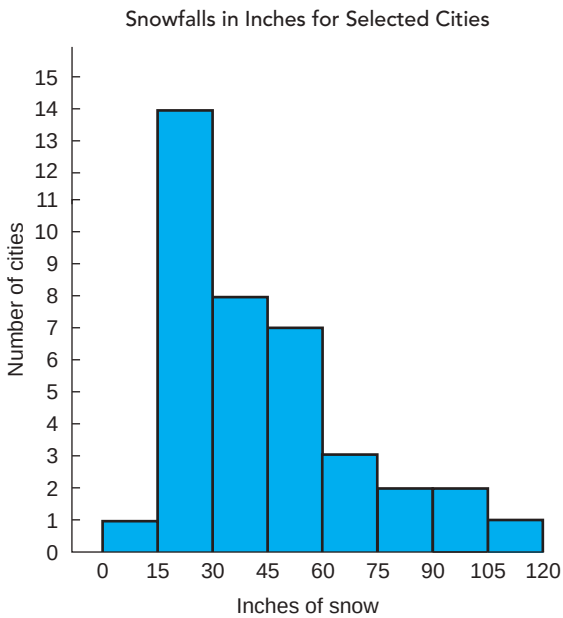
Stem	Leaf
27	7
26	
25	1, 3, 0
24	6, 6, 6, 2
23	8, 4, 1, 2, 0, 0
22	3, 5, 7, 0, 1, 0
21	8, 6, 5, 6, 0, 4, 2, 2
20	2, 5, 3, 1, 6, 4, 0, 8, 1, 6
19	3, 7, 0, 9, 7, 8, 1
18	2, 2, 1, 6
17	0, 2, 3
16	6

b. 20 **c.** 27.7 and 16.6 kilograms

27. a.

Interval	Frequency
0–15	1
16–30	14
31–45	8
46–60	7
61–75	3
76–90	2
91–105	2
106–120	1

b.

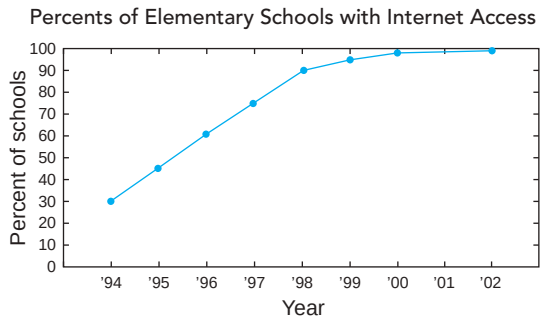


c. 16–30 **d.** 8

29. a. Decreasing
b. 2 percent
c. 1992
d. 2000 to 2003; 2 percent

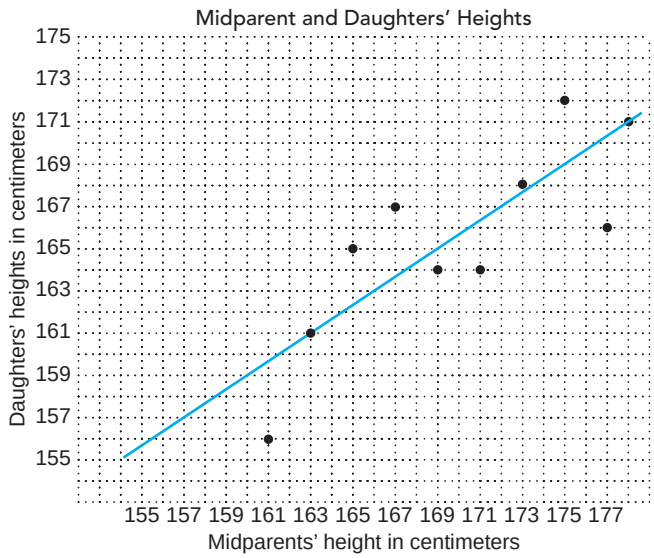
31. a. 2003, 2004, and 2005
b. Between 6 and 7 percent
c. Between 7 and 8 percent
d. Approximately 2 percent in 1998, 2000, 2006 and 2007

33. a.



b. 1994 to 1996

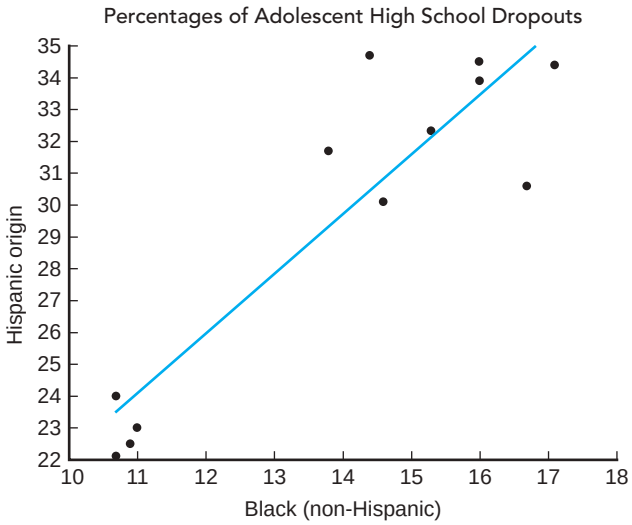
35. a.



b. Between 158 and 160 centimeters; between 167 and 169 centimeters
c. Between 165 and 167 centimeters; between 175 and 177 centimeters

37. a. 3.7 percent

b. 3 times



c. Positive

d. Between 13 and 14 percent

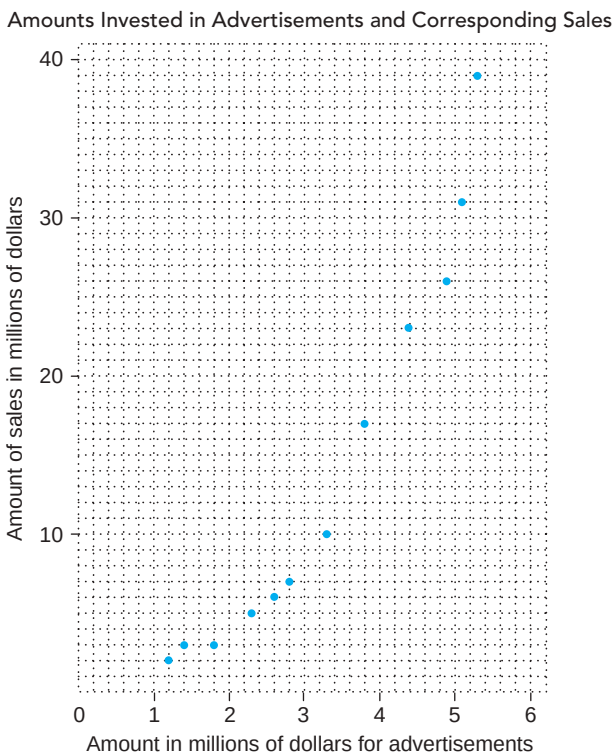
39. a. Greatest diameter is 8 inches and oldest age is 42 years

b. Positive

c. Approximately 5 to 6 inches and 6 to 7 inches

d. Approximately 44 to 50 years old

41. a.



b. Exponential curve

c. Between \$12 and \$14 million

d. Between \$4.0 and \$4.5 million

Exercises and Problems 7.2

1. a. Mean, $4.\overline{6}$; median, 4.5; mode, 4

b. Mean, 1.5; median, 1; mode, 0

3. a. Mode **b.** Mode **c.** Median

5. a. 8,587 megawatts

b. 20, but this average is inflated by the large number of reactors in Russia, France, Japan, and the United States.

7. a. Mean, 49.7; median, 48; mode, 47

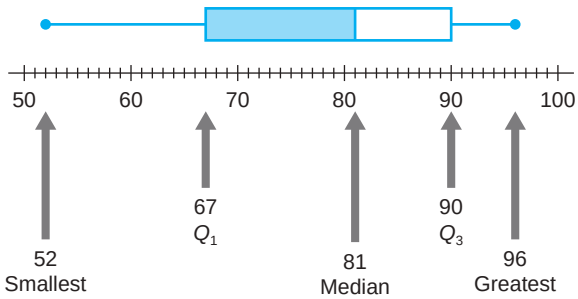
b. Mean, 48.2; median, 48; mode, 43, 47, 48, 52, 54. and 56

c. In 8 out of the 20 seasons

d. The home run leaders for the two leagues have about the same records. The means differ by 1.5 and the medians are equal. The American League's home run leaders hit more home runs in 8 years, and the National League's home run leaders hit more home runs in 10 years. The home run leaders were tied for the remaining 2 out of 20 seasons. However, in the past 12 years from 1998 to 2009, the National League home run leaders have hit more home runs in 9 years.

9. a. 65 **b.** 10 **c.** 30 **d.** 16

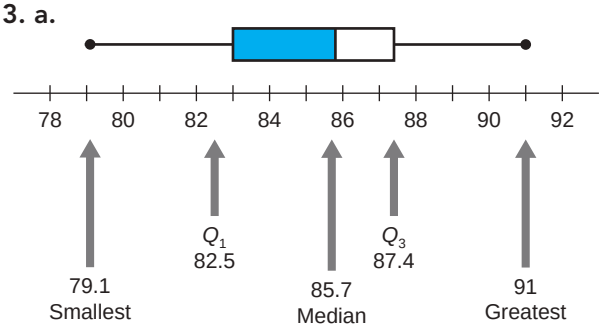
11.



a. 44

b. Approximately 25 percent of the data are greater than or equal to 90, and the middle 50 percent are approximately between 67 and 90.

13. a.



b. The median of 85.7 indicates that approximately half of these states have less than 85.7 percent of their students completing high school.

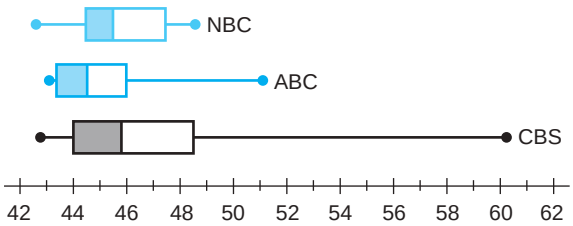
c. The upper quartile of 87.4 indicates that approximately 25 percent of these states have more than 87.4 percent of their students completing high school.

d. The lower quartile of 82.5 indicates that approximately 25 percent of these states have less than 82.5 percent of their students completing high school.

15. a. Tracer **b.** Civic and Golf **c.** Justy

d. Golf. It has the greatest lower quartile and the greatest median, and its ratings have less variability than those of the Civic, which has the second-best set of ratings.

17. a. CBS has the best rating. It has the greatest upper quartile, and most of the upper 25 percent of its ratings are above the highest rating received by NBC. ABC has the poorest ratings. It has the smallest lower and upper quartiles.



b. The interquartile range for CBS is 4.5, and since 60.2 is greater than $48.5 + 1.5(4.5) = 55.25$, the maximum data value for CBS is an outlier, and indicates an exceptional rating. The maximum data value for ABC is barely an outlier.

19. a. The data for set *B* are less spread out and should have the smaller standard deviation.

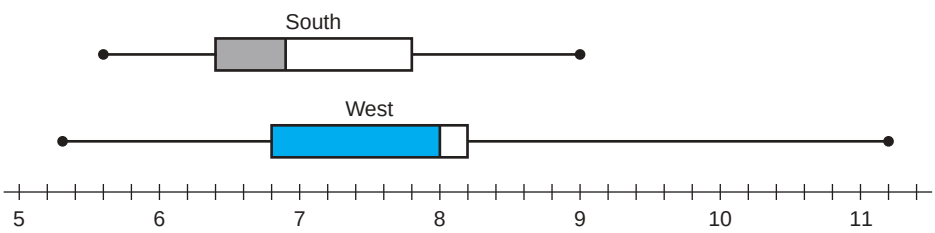
b. Standard deviation for set *A* is 4, and standard deviation for set *B* is 2.

c. Yes, the standard deviation for set *A* is twice the standard deviation for set *B*.

21. a. 3.6 percent **b.** 5.5 percent

23. a. South: lower quartile, 6.4; median, 6.9; upper quartile, 7.8. West: lower quartile, 6.8; median, 8.0; upper quartile, 8.2.

b.



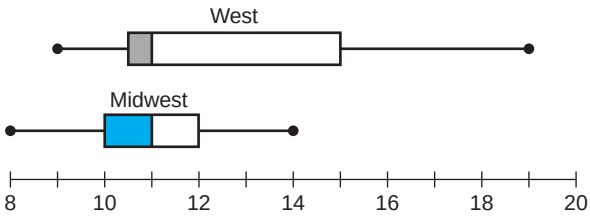
The quartiles and median for the south are all lower than their corresponding measures for the west. The data for the west are more spread out than for the south, especially in the top 25 percent.

c. The median for the west is greater than the upper quartile for the south. So 75 percent of the states in the south spent less per student than the median amount spent per student by the states in the west.

d. The interquartile range for the west is equal to that for the south, so both regions have the same amount of variability in money spent per student for their midrange states.

25. a. West: lower quartile, 10.5; median, 11; upper quartile, 15. Midwest: lower quartile, 10; median, 11; upper quartile, 12.

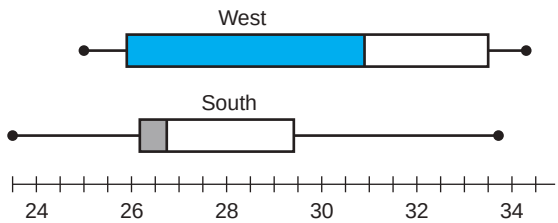
b.



c. The medians for both regions are equal, but the west has higher quartiles. The upper quartile for the west is greater than the top value for the midwest, which means that for 25 percent of the states in the west, the percent of students not finishing high school is above the percentages for all of the states in the midwest.

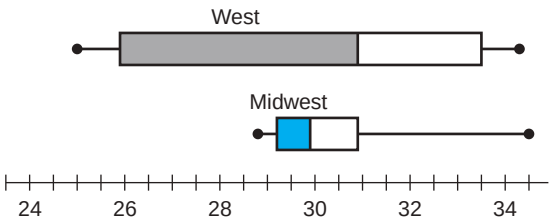
d. There are no outliers.

27. a.



b. West: 7.6; south: 3.3. The smaller interquartile range for the south shows that incomes for the states in the middle 50 percent of the south are less variable.
c. The upper 50 percent of the average incomes for the west are above the upper quartile for the average incomes of the south.
d. The median income for the south.
e. The west has more variability in the middle 50 percent and the south has more variability at both extremes. In at least half of the states in the west the average income is higher than in $\frac{3}{4}$ of the states in the south.

29. a.



b. About 4.5 times.
c. The lowest 25 percent of average incomes for the west are well below all the average incomes for the midwest.
d. Average incomes in the midwest are less variable, especially in the lower three quarters. Incomes for all states in the midwest are above the incomes for the lowest quarter of states in the west. The range of incomes for the top half of the states is almost the same in each region.

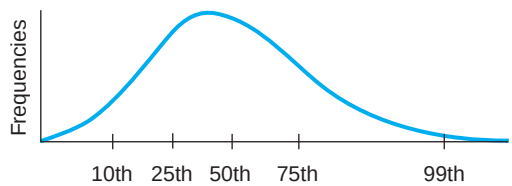
31. 13 grams
33. 69
35. a. 5.8 grams **b.** .46 gram **c.** 60 percent
37. At least 75 percent of Peter's arrow strikes are within 26 inches of the center of the target, and at least 75 percent of Sally's arrow strikes are within 24 inches of the center of the target.

Exercises and Problems 7.3

1. a. 8908 and 9042 **b.** 8841 and 9109
3. a. Number the names from 1 to 9. Randomly select a number from the table, and continue in the table until you obtain two different numbers less than or equal to 9.
b. Number the questions from 1 to 60. Randomly select a number from the table, and continue in the table until you obtain 10 different numbers less than or equal to 60.
5. Grade K, 10; grade 1, 16; grade 2, 18; grade 3, 16; grade 4, 20
7. a. The majority of scores will be low, and the distribution will be skewed to the right.
b. The majority of scores will be high, and the distribution will be skewed to the left.
9. a. Skewed left; most but not all amounts will be high with a few low values on the left.
b. Symmetric due to general normal distribution in population.
c. Skewed left; most but not all will be a large size.
11. a. 68 percent
b. 16 percent
c. 2.5 percent
13. a. 1
b. 7
c. 7
15. a. A normal distribution
b.

Diameter (inches)	No. of Trees
7	2
8	5
9	8
10	10
11	13
12	26
13	12
14	9
15	8
16	4
17	3

c. 70 percent
17.



a. 30 percent **b.** 10 percent
19. a. 54 percent **b.** 68 percent **c.** 96 percent
d. The score for mathematics comprehension
e. A lower local percentile means that at the local level fewer students scored below the given student on a given subtest than at the national level.
21. a. 86 percent **b.** 78 percent
c. 90 percent **d.** 80 percent

23. a. 5

b. 69th percentile is stanine 6; 62nd percentile is stanine 6; 83rd percentile is stanine 7; 45th percentile is stanine 5.

25. Test A, $z \approx -.14$ and test B, $z \approx -.53$. So, A is the better test.

27. Each number from 1 to 6 represents that outcome on the roll of a die. The numbers 0, 7, 8, and 9 are disregarded. The first few random numbers from the table are listed here, and the circled numbers represent the outcomes for 10 rolls of a die.

⑥ ① ④ ④ ③ ④ 0 ③ 0 9 0 ⑤ ⑥ ④

29. a. The z scores to the nearest .01 are ACT, 1.74 and SAT, 1.06.

b. The student performed better on the American College Test.

31. 5 percent

33. 16 percent

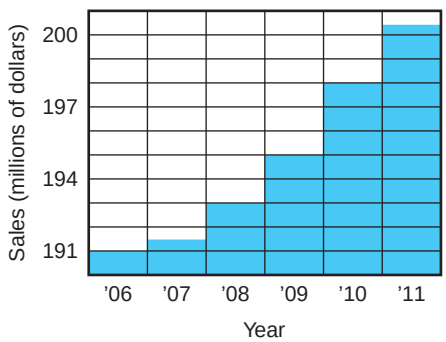
35. Since .3 is more than 2 standard deviations below the mean, we may be suspicious of the company's claim.

37. Average number of boxes will vary. One method: label each of five slips of paper with the name of a different color. Place the slips in a container, and randomly select one at a time (with replacement). Compute the average number of selections needed to obtain all five colors.

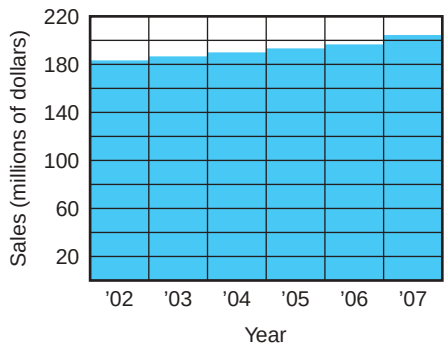
39. The average number of children a couple must have to be sure of having a child of each sex is 3. Label one slip of paper *boy* and one slip of paper *girl*. Place them in a container, and randomly select one at a time (with replacement). Compute the average number of selections required to select each slip of paper once.

41. Graph (i) gives the impression of substantial increases in sales from 2006 to 2011, whereas graph (ii) gives the impression of slight increases. The top graph is misleading; it suggests that sales doubled from 2007 to 2011 and more than doubled from 2008 to 2010. The second graph below better illustrates the true sales increases.

i.



ii.



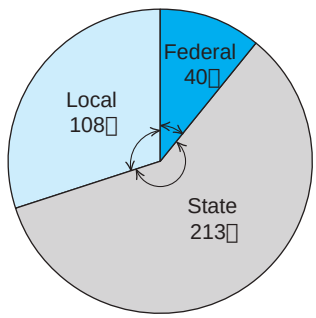
43. "It is remarkable that a science which began with the consideration of games of chance should be elevated to the rank of the most important subjects of human knowledge."

Chapter 7 Test

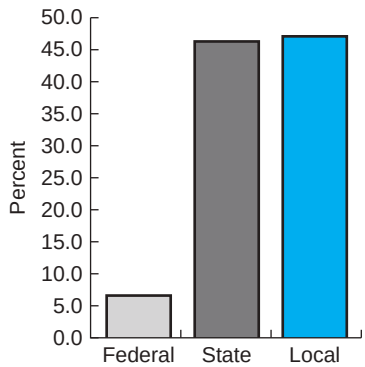
1. a. Hawaii **b.** Illinois

c. 62.3 percent **d.** 47.35 percent

2. a. Sources of revenue for California public schools



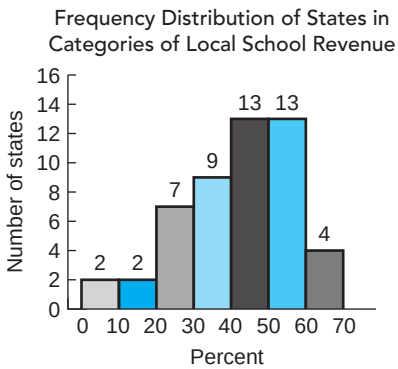
b. Sources of revenue for Iowa public schools



3.

Stem	Leaf
3	2 3
5	4 7
6	4 4 5 6 8 9
7	3 3 4 5 5 5 7 7
8	0 0 1 1 4 5 6 6 9 9
9	3 4 5 9
10	0 1 4 4
11	0 1 2 2
12	3 4 6 6
13	0 9
14	2
15	3 5
17	3

4.

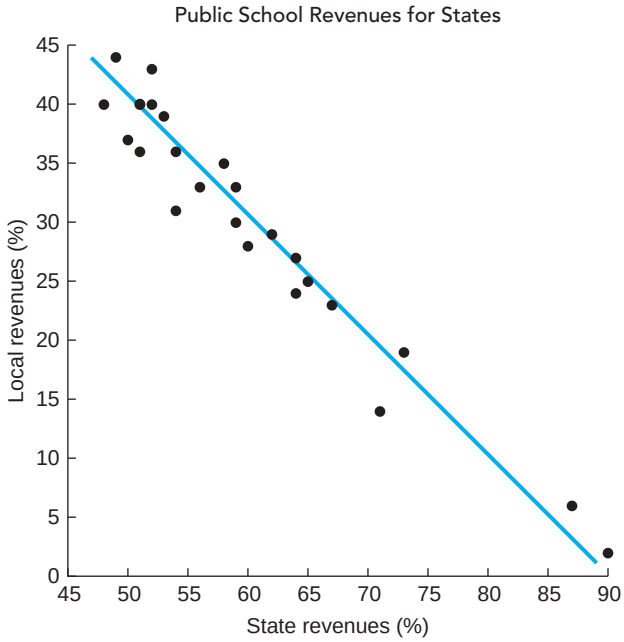


a. 7 **b.** 13

c. The intervals from 40 to 49.95 and 50 to 50.95 both contain 13 values.

5. a. 60.6 percent **b.** 12.7 percent

6. Using the following best-fit line, if 35 percent of the revenue for schools is from local sources, then approximately 54 to 55 percent is from state sources and if 75 percent of the revenue is from state sources, then between 15 and 16 percent is from local sources. Yes, there appears to be a strong negative correlation between the two types of revenue.



7. a. Set *B* **b.** Set *A* **c.** Set *A*

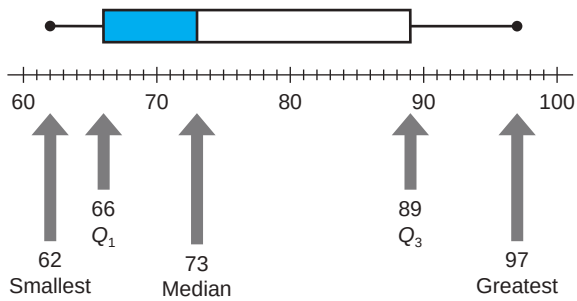
8. Because 4 of the 6 rides have a length of less than 10 minutes.

9. Graph A suggests the company is extremely profitable. Ask students to compare the two different vertical profit scales and explain how the Graph A scale gives a misleading graph.

10. The scale on the vertical axis is not marked off from 0 to 15. So, for example, if 16 students chose biology as their favorite class, it appears then about 3 times that number (or 48) chose math. In reality, the difference is $35 - 16 = 19$ (not 32).

11. $1.5(Q_3 - Q_1) + Q_3 = 1.5(6 - 1) + 6 = 13.5$. The number of gold medals for China and the United States are both outliers; in fact, the number of gold medals for Australia, Germany, and Great Britain are also outliers.

12. The interquartile range of 23 is greater than half of the range of the data, which is 35.



13. a. .8 **b.** 1.3 **c.** On the PSAT
14. a. Skewed to the right **b.** Normal
15. a. 82.8 percent **b.** 93 percent
c. On the Total Math subtest, the scores of 60 percent of local students were lower than this student's score.
d. This student is performing at grade levels much higher than the fourth grade.
16. a. 68 percent **b.** 95 percent
17. 70.3
18. 90

Exercises and Problems 8.1

1.

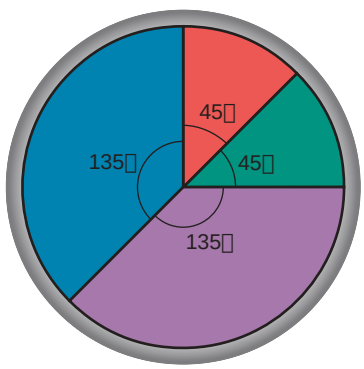
Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

a. $\frac{5}{12}$ **b.** $\frac{5}{12}$
3. a. $\frac{2}{3}$ **b.** 7
5. a. $\frac{4}{7}$ **b.** $\frac{4}{7}$ **c.** $\frac{4}{7}$
7. The sample space is the set whose elements are the 18 chips.
a. $\frac{1}{9}$ **b.** $\frac{7}{18}$ **c.** $\frac{8}{9}$

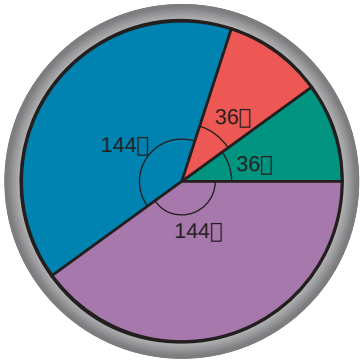
9.

	Tetra-hedron	Cube	Octa-hedron	Dodeca-hedron	Icosa-hedron
a.	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{20}$
b.	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{17}{20}$

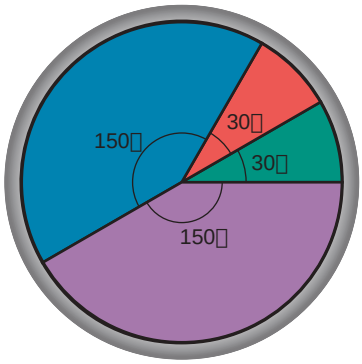
11. a. $\frac{5}{12}$ **b.** $\frac{3}{4}$
c. $\frac{1}{4}$ **d.** $\frac{1}{2}$
13. a. A, B A, C A, D B, C B, D C, D
b. $\frac{1}{2}$ **c.** $\frac{1}{6}$
15. a. BB, BR, BY, BG, BW, BP, RB, RR, RY, RG, RW, RP, YB, YR, YY, YG, YW, YP, GB, GR, GY, GG, GW, GP, WB, WR, WY, WG, WW, WP, PB, PR, PY, PG, PW, PP
b. $\frac{1}{36}$ **c.** $\frac{11}{36}$ **d.** $\frac{25}{36}$
17. a. $\frac{2}{3}$ and $\frac{5}{6}$
b. $\frac{5}{6}$ and 1
c. None. The two sets must be disjoint.
19. a. $\frac{1}{13}$ **b.** $\frac{1}{4}$ **c.** $\frac{1}{2}$ **d.** $\frac{11}{26}$ **e.** $\frac{3}{52}$
21. a. $\frac{12}{13}$ **b.** $\frac{10}{13}$ **c.** $\frac{3}{4}$ **d.** $\frac{23}{26}$
23. a. $\frac{1}{4}$; 1 to 3 **b.** $\frac{2}{13}$; 2 to 11
c. $\frac{1}{2}$; 1 to 1 **d.** $\frac{1}{2}$; 1 to 1
25. a. 7 to 3 **b.** 3 to 2 **c.** 1 to 499 **d.** 4 to 1
27. a. $\frac{10}{13}$ **b.** $\frac{1}{5}$
29. a. .21 **b.** .97
31. a. $\frac{6{,}800{,}531}{7{,}698{,}698} \approx .88$
b. $\frac{9{,}781{,}958}{10{,}000{,}000} \approx .98$
c. $\frac{5{,}592{,}012}{10{,}000{,}000} \approx .56$
d. The probability to five decimal places of a 28-year-old person's not reaching age 29 is .00203 $\left(\frac{19{,}324}{9{,}519{,}442}\right)$. Since .00203 × 7000 = 14.21, the insurance company should be prepared to pay approximately 14 death claims.
33. Let even digits represent heads and odd digits represent tails. Arbitrarily select a sequence of 10 digits from the table, and count the number of heads. Carry out this experiment repeatedly, recording the number of heads in each sequence of 10 digits. The probability is approximately .62.
35. Label 3 slips of paper *B* and 2 slips of paper *G*, and put them in a hat. Randomly select these slips one at a time without replacing them, and record the sequence of *B*s and *G*s. Carry out this experiment many times, and divide the number of times that 3 *B*s occur in succession by the total number of experiments. The theoretical probability of having 3 boys in succession is .3.
37. Label 3 slips of paper *H* (for hit) and 7 slips of paper *O* (for out), and put them in a sack. Randomly select 5 slips without replacement and record the number of hits. Carry out this experiment many times. Divide the total number of experiments in which 3, 4, or 5 of the slips of paper represented a hit by the total number of experiments. The theoretical probability is approximately .16.
39. a. Use a table of random digits, letting even digits represent heads and odd digits represent tails. Check sequences of 6 digits, and compute the experimental probability of having exactly 3 heads. The theoretical probability is .3125.
b. Less than .5. The theoretical probability of obtaining exactly 10 heads in 20 tosses is approximately .18.
41. a.



b.



c.



43. .20

45. $\frac{1}{3}$

47. .85

49. **a.** .49 **b.** .23 **c.** .64

51. **a.** .12 **b.** .17 **c.** .37

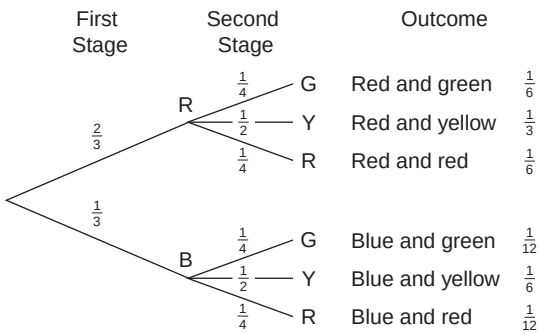
Exercises and Problems 8.2

1. **a.** $\frac{1}{4}$ **b.** $\frac{1}{32}$ **c.** $\frac{1}{1024}$

3. **a.** $\frac{1}{6}$ **b.** $\frac{1}{4}$ **c.** $\frac{1}{6}$

5. **a.** $\frac{1}{6}$ **b.** $\frac{3}{4}$ **c.** $\frac{1}{2}$

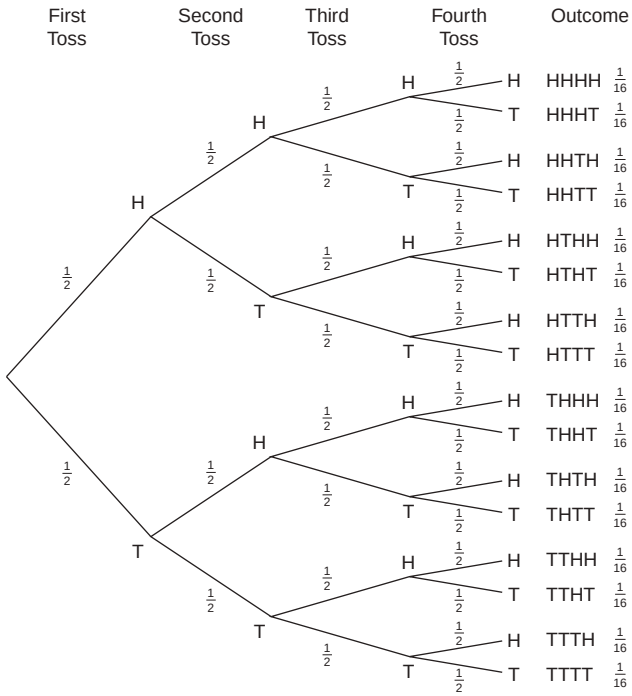
d.



7. **a.** $\frac{1}{8}$ **b.** $\frac{7}{8}$ **c.** $\frac{1}{2}$

9. **a.** $\frac{1}{4}$ **b.** $\frac{11}{16}$

c.



11. **a.** $\frac{1}{25}$ **b.** $\frac{7}{75}$ **c.** $\frac{1}{15}$

13. **a.** $\frac{1}{35}$ **b.** $\frac{1}{10}$ **c.** $\frac{1}{14}$

15. **a.** $\frac{1}{2}$ **b.** $\frac{1}{6}$

17. **a.** Independent, $\frac{1}{36}$ **b.** Dependent, $\frac{5}{14}$

19. **a.** $\frac{11}{12}$ **b.** $\frac{5}{6}$ **c.** $\frac{121}{144} \approx .84$

21. **a.** Approximately .53
b. Approximately .80

23. **a.** $\frac{1}{20} \times \frac{3}{20} \times \frac{1}{20} = .000375$

b. $\frac{5}{20} \times \frac{1}{20} \times \frac{1}{20} = .000625$

25. $\frac{7}{18}$

27. If G_1, G_2, G_3, and G_4 are the four good flashbulbs and B is the bad bulb, the sample space is

G_1G_2 G_1G_3 G_1G_4 G_1B G_2G_3 G_2G_4
G_2B G_3G_4 G_3B G_4B

a. $\frac{3}{5}$ **b.** $\frac{2}{5}$

29. **a.** $\frac{1}{365}$ **b.** $\left(\frac{1}{365}\right)^4 \approx 5.6 \times 10^{-11}$

31. Approximately .59; that is $1 - \left(\frac{4}{5}\right)^4$. A simulated probability can be obtained by using a random device in which a given number or object has a $\frac{1}{5}$ probability of occurring. Many experiments in which this device is used 4 times will produce a simulated probability.

33. **a.** $\left(\frac{1}{6}\right)(\$1 + \$2 + \$3 + \$4 + \$5 + \$6) = \$3.50$

b. \$3.50

35. a. $\frac{1}{1000}(\$100) + \frac{1}{500}(\$50) + \frac{1}{200}(\$20) + \frac{1}{5}(\$1) + \frac{1}{100}(\$10) = \$.60$

b. No

37. a. $\frac{37}{38}$ **b.** The expected values are equal.

39. a. The probability of winning is $\frac{1}{10}$.

b. .0975 (There is a $\frac{19}{20}$ chance of not winning on the first draw and a $\frac{19}{20}$ chance of not winning on the second draw. Thus, the chance of not winning is $\frac{19}{20} \times \frac{19}{20} = \frac{361}{400} = .9025$, and the chance of winning is .0975.) The chances of winning are better in part a.

41. No

43. .996

45. a. .224 **b.** .216 **c.** .856

47. $\frac{1}{20{,}736} \approx .00005$

49. a. 210 **b.** 126

51. a. 495 **b.** 336

53. 11,880

55. 66

57. a. ${}_{52}C_5 = 2{,}598{,}960$ **b.** ${}_{12}C_5 = 792$

c. ${}_{12}C_5 \div {}_{52}C_5 = 792 \div 2{,}598{,}960 = .0003$

d. ${}_{40}C_5 \div {}_{52}C_5 = 658{,}008 \div 2{,}598{,}960 = .25318$

59. a. ${}_{10}P_4 = 5040$ **b.** ${}_{9}P_3 = 504$

c. $504 \div 5040 = .1$

Chapter 8 Test

1. a. AB, AC, AD, AE, AF, BC, BD, BE, BF, CD, CE, CF, DE, DF, EF

b. $\frac{1}{15}$ **c.** $\frac{1}{3}$

2. a. $\frac{1}{3}$ **b.** $\frac{3}{4}$ **c.** $\frac{2}{3}$

3. a. G_1G_2, G_1G_3, G_1O_1, G_1O_2, G_2G_3, G_2O_1, G_2O_2, G_3O_1, G_3O_2, O_1O_2

b. $\frac{3}{10}$ **c.** $\frac{1}{10}$ **d.** $\frac{3}{5}$

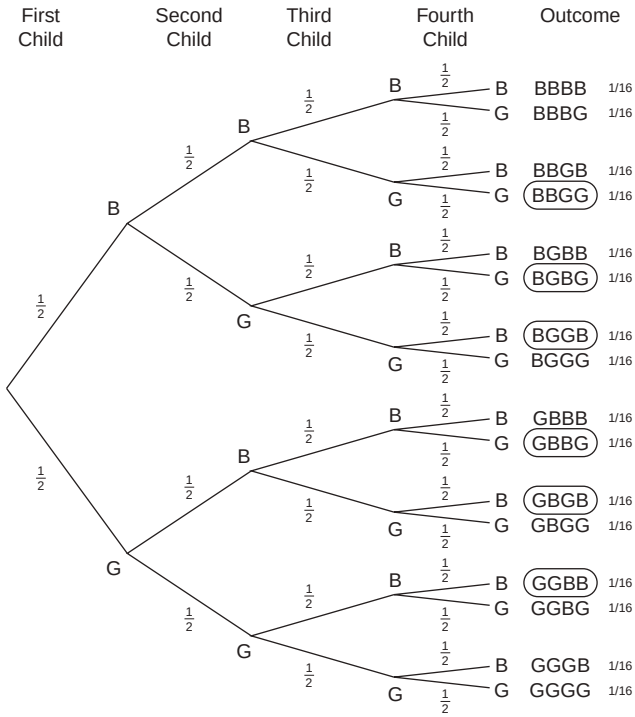
4. a. 5 to 7 **b.** $\frac{7}{12}$

5. a. 1 **b.** $\frac{2}{5}$ **c.** $\frac{4}{5}$ **d.** $\frac{1}{5}$

6. a. $\frac{4}{9}$ **b.** $\frac{2}{9}$ **c.** $\frac{5}{9}$

7. a. $\frac{2}{5}$ **b.** $\frac{4}{15}$ **c.** $\frac{3}{5}$

8. a.



b. $\frac{3}{8}$ **c.** $\frac{11}{16}$

9. $\frac{11}{21}$

10. .488

11. a. $\$2\left(\frac{1}{2}\right) + \$1\left(\frac{1}{6}\right) + \$3\left(\frac{1}{6}\right) + \$5\left(\frac{1}{6}\right) = \$2.50$

b. $2.50

12. .039, or approximately 4 percent $[1 - (.99)^4]$

13. a. .72 **b.** .648 **c.** .998

14. Yes

15. a. .63 **b.** .69

16. a. 5! **b.** 4!

c. $\frac{4!}{5!} = .2$ **d.** $1 - .2 = .8$

17. a. ${}_{45}C_5 = 1{,}221{,}759$ **b.** $1 \times {}_{44}C_4 = 135{,}751$

c. $135{,}751 \div 1{,}221{,}759 = .\overline{1}$

d. $1 - .\overline{1} = .\overline{8}$

Exercises and Problems 9.1

1. Answers will vary.

a. Two in the triangle to the right of center

b. Figure at right edge near center

c. Angle whose vertex is peak of roof (in foreground) on building at left

d. Front face (in foreground) of building on left

3. *Undefined terms* are words that are undefined, but information can be obtained about these terms from the axioms. They are used in definitions to define other words. *Definitions* use undefined words and other defined words

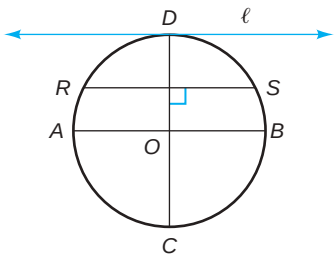
to define new words. *Axioms* are statements that are assumed to be true. *Theorems* are statements that are proved true by using deductive reasoning.

5. a. Edge of a ruler or stretch a piece of string
b. Angle supports in buildings and bridges or the rack for racking pool balls
c. Top of a table or wall of a room

7. a. None
b. Angles A, C, and D
c. Angles B and E

9. a. $\measuredangle HOK$ and $\measuredangle KOJ$; $\measuredangle KOJ$ and $\measuredangle JOI$; $\measuredangle JOI$ and $\measuredangle IOH$
b. $\measuredangle HOI$ and $\measuredangle KOJ$; $\measuredangle HOK$ and $\measuredangle IOJ$
c. $\measuredangle HOK$ and $\measuredangle IOJ$; $\measuredangle HOI$ and $\measuredangle KOJ$

11. a. Diameter $\overline{CD} \perp$ chord $\overline{RS}$
b. Line ℓ is tangent to radius $\overline{OD}$.
c. Chords $\overline{AB}$ and $\overline{CD}$ bisect each other.



13. a. 145.5° **b.** 34.5°
c. 145.5° **d.** 34.5°

15. a. 120° **b.** 60° **c.** 7 minutes

17. a. Simple
b. None of these
c. Closed

19. a. Concave **b.** Convex
c. Concave

21. a. 90° **b.** 60°

23. 0, 1, 4, 5, 6, 7, 8, 9, 10

25. a. True
b. False; the resulting figure will be a parallelogram if the rectangle is not a square.
c. True

27. 13

29. 4851

31. a. Yes **b.** No

33. a. No **b.** No **c.** No **d.** Yes

Exercises and Problems 9.2

1. a. Hexagons; no, they are not regular.
b. Hexagons and heptagons and they alternate.

3. a. 1440° **b.** 2340°

5. a. Not all angles are congruent.
b. Not all angles are congruent, and not all sides are congruent.
c. Not all angles are congruent.

7.

No. of sides	3	4	5	6	7	8	9	10	20	100
Central angle	120°	90°	72°	60°	51.4°	45°	40°	36°	18°	3.6°

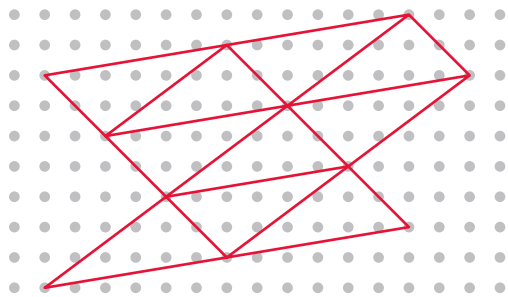
9. a. Regular hexagon **b.** Equilateral triangle

11. a. 15 **b.** 18 **c.** 5

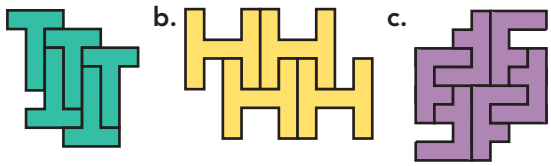
13. a. True
b. True
c. False; beginning with an isosceles triangle whose sides are 20 inches, 20 inches, and 1 inch, connecting the midpoints of the sides will produce a triangle with one side that is shorter than the other two sides.

15. a. The regular hexagon tessellates since the measure of the vertex angles is a divisor of 360°.
b. The regular heptagon does not tessellate.
c. The regular octagon does not tessellate.

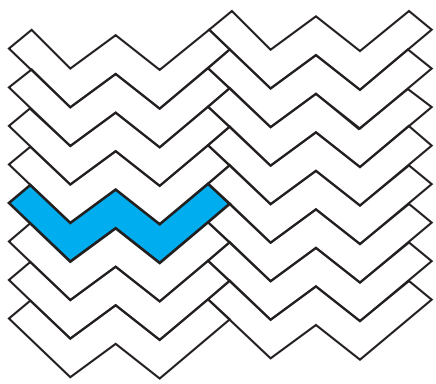
17. Every triangle will tessellate because the sum of the measures of the three angles of a triangle is 180° and each angle is used twice at each vertex point of the tessellation.



19. a. **b.** **c.**



21. One possibility



23. a. Not semiregular; some vertices are surrounded by two squares and three triangles (4, 4, 3, 3, 3), and other vertices are surrounded by two squares, one triangle, and one hexagon (4, 4, 3, 6).
b. Semiregular; each vertex has the same arrangement of polygons (4, 6, 12).

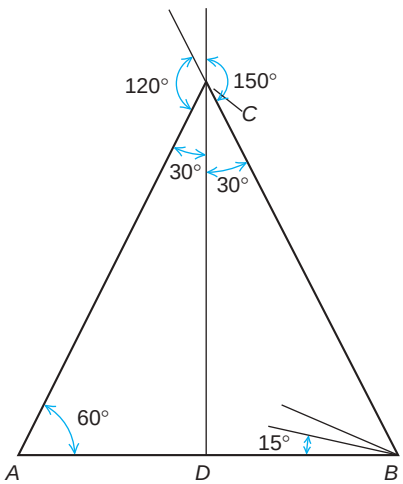
25. Here are three methods: (1) Draw a line segment one inch long and use a protractor to draw a 150° angle at one endpoint. Then continue to draw sides for the dodecagon and angles of 150°. (2) Construct a circle with a compass, and draw the central angle for the dodecagon of 30°. Then "pace off" 12 chords of equal length around the circle with a compass. (3) Draw a circle and an inscribed regular hexagon. Then draw the perpendicular bisectors of the sides of the hexagon, and connect the 12 points on the circle.

27. There are six semiregular tessellations that each use two regular polygons. The arrangements of the polygons are octagon, octagon, square; dodecagon, dodecagon, equilateral triangle; square, square, equilateral triangle, equilateral triangle, equilateral triangle; square, equilateral triangle, square, equilateral triangle, equilateral triangle; hexagon and four equilateral triangles; and hexagon, equilateral triangle, hexagon, equilateral triangle. In addition to the one in Figure 9.39a, there is another semiregular tessellation that uses three regular polygons. It is shown in part b of exercise 23.

29. 121

31. The word LOVE

33. Fold paper to obtain $\overline{CD}$ and then place B to coincide with C to obtain an equilateral triangle: $AB = AC = BC$. The desired angles are marked in the following figure. To obtain a 15° angle, bisect twice at $\measuredangle DBC$.



Exercises and Problems 9.3

1. a. Hexagon **b.** Hexagonal prisms

3. Part a is the only polyhedron.

5. a. Concave **b.** Convex
c. Convex

7. a. Square, triangle, square, triangle
b. Decagon, square, hexagon

9. a. Square pyramid
b. Cylinder (or right cylinder)
c. Triangular prism

11. a. Right cone
b. Oblique cylinder
c. Right pentagonal pyramid

13. a. *ABCDEF* **b.** *GLFA*
c. 90° **d.** 120°

15. a. Plane
b. Cylindrical
c. Conic

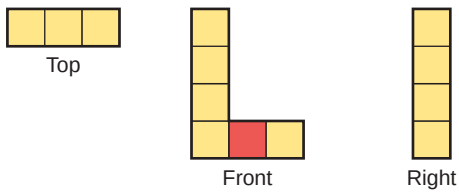
17. a. Hands of the clock
b. Windows
c. Window on the right side of the front of the building
d. Windows below the clock
e. Faces of the roof of the tower

19. a. (20°S, 60°E)
b. (30°N, 100°W); the United States

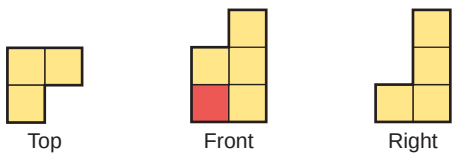
21. a. Pentagon **b.** Rectangle

23. *CFGH*, *AEFH*, *ABCF*, and *ACDH*

25. a.



b.



27. a. 1 through 7 **b.** 26

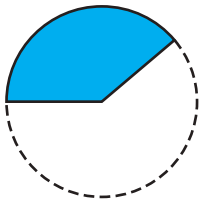
29. a. 12 edges; 6 **b.** 10 vertices; 12 **c.** 5 faces; 4

31. September 15 (32.5°N, 48°W); September 23 (31°N, 64.5°W); September 30 (35°N, 76°W)

33. Two possibilities: (1) A cylinder that just fits inside a square prism. These objects share a vertical cross section. (2) A sphere that just fits inside a cylinder. These objects share a horizontal cross section.

35. a. Roll up a rectangular sheet of paper and tape the opposite edges.

b. Cut out and roll up a sector of a disk and tape the radii.

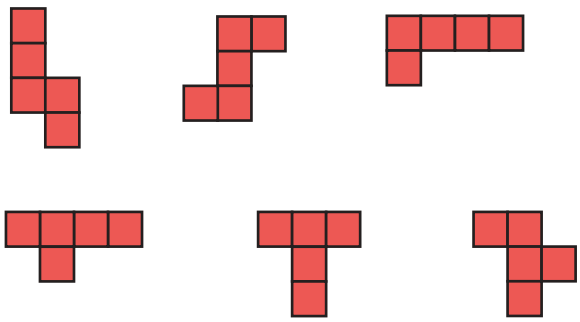


c. Hold the right cylinder from part a at an angle, dip the ends at the same angle into a liquid, and cut off the moistened part. Cutting along the taped edges produces the following pattern.



37. a. The first two pentominoes will fold into an open-top box.

b. Here are the remaining six pentominoes that will fold into an open-top box.



39. a. The number of faces in a cube equals the number of vertices in an octahedron, and the number of faces in an octahedron equals the number of vertices in a cube.

b. The dodecahedron and icosahedron are duals.

c. A tetrahedron

41. a. No.

b. Angle 1 is largest. Angles 2 and 3 are right angles.

Exercises and Problems 9.4

1. a. In general, there are many objects to the left of the center of the photograph that have a corresponding object to the right.

b. The structure at the top left of the fortress and the rectangular window on the right side of the fortress do not have images. There are other windows and openings on the fortress that do not have images for the vertical plane of symmetry.

c. The rectangular windows on the fortress and the surface of the pool have horizontal lines of symmetry.

3. a. Two lines of reflection and two rotation symmetries

b. Five lines of reflection and five rotation symmetries

5. a. Four lines of symmetry and four rotation symmetries

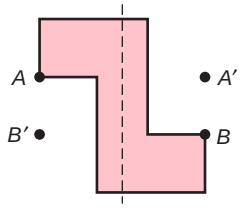
b. No lines of symmetry and two rotation symmetries

c. Five lines of symmetry and five rotation symmetries

d. One line of symmetry

Polygon c has the highest rating and polygons b and d have the lowest rating.

7. The image of A is not on the figure.



9. Figures a and c do not have lines of symmetry.

11. a. H, X, O, and I have two lines of symmetry.

b. N, S, and Z have two rotation symmetries but no lines of symmetry.

13. Figure d has no line of symmetry

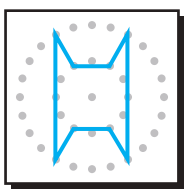
a. Two lines of symmetry and two rotation symmetries, 180° and 360°

b. One line of symmetry

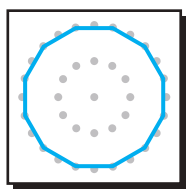
c. Three lines of symmetry and three rotation symmetries, 120°, 240°, and 360°

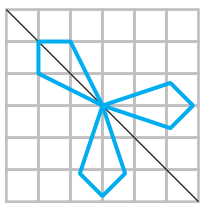
d. Two rotation symmetries, 180° and 360°

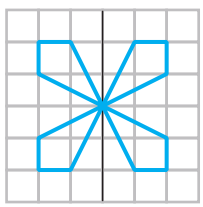
15.

a. 

b. 

c. 

17. a. 

b. 

19. a. 16 rotation symmetries and 16 planes of symmetry
b. (i) 2 rotation symmetries and 2 planes of symmetry
(ii) 2 rotation symmetries and 2 planes of symmetry
21. a. Right cylinder, equilateral prism, sphere, and cube
b. Right cylinder, 2; equilateral prism, 2; sphere, infinite number; cube, 4
c. Rectangular pyramid, 2; equilateral prism, 3; cube, 4
23. Five planes of symmetry and five rotation symmetries
25. a. Four rotation symmetries and four lines of symmetry
b. Eight rotation symmetries and no lines of symmetry
27. a. 6 **b.** 5
29. a. Four rotation symmetries (JPMorganChase Bank)
b. One line of symmetry or, if the letter "T" is disregarded, there are five lines of symmetry and five rotation symmetries (Texaco)
c. Two rotation symmetries (Two-way traffic)
d. Three rotation symmetries and three lines of symmetry (Biohazard sign)
e. Two rotation symmetries (Roadside junction)
31. a. One line of symmetry
b. Three lines of symmetry and three rotation symmetries
c. Four lines of symmetry and four rotation symmetries
d. One line of symmetry
e. No lines of symmetry and six rotation symmetries
33. 42 feet
35. 48 points. This can be accomplished by beanbags in cups 3, 4, 6, 7, and 8.
37. a. Three possibilities:



b. Three possibilities:



c. Three possibilities:



Chapter 9 Test

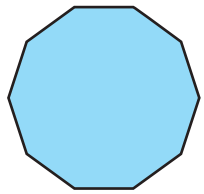
1. a. (iv) **b.** (iii) **c.** (ii)
d. (i) **e.** (vi) **f.** (i) and (v)
2. a.



b.



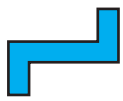
c.



d.

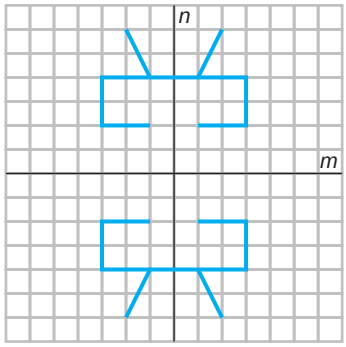


3. a. *C* and *E* **b.** *D*
c. *B* **d.** *A*
4. a. True, the pairs of opposite sides are parallel and of equal length and all are right angles.
b. True, scalene refers to side length, not the angles.
c. False, many do not have all right angles.
d. True, the pairs of opposite sides are parallel and of equal length.
e. False, 60°–60°–60° triangles cannot have 90° angles.
5. a. 45° **b.** 120° **c.** 72°
6. a. Not all angles are congruent.
b. Not all sides are congruent.
c. Not all sides are congruent.
d. Not all angles are congruent.
7. a. No **b.** Yes **c.** Yes
d. Yes **e.** No
8. No. The measure of a vertex angle of a regular octagon is 135°, and 135 cannot be combined with multiples of 60 and 90 (the degrees in the angles of the equilateral triangle and the square) to equal 360°.
9. a. Right pentagonal pyramid
b. Right rectangular prism
c. Right hexagonal prism
d. Oblique cylinder
e. Right cone
f. Oblique triangular pyramid
10. a. Nonpolyhedron **b.** Polyhedron
c. Polyhedron **d.** Nonpolyhedron
e. Polyhedron **f.** Polyhedron
11. a. 12 **b.** 24
12. a. An equilateral triangle
b.



c. A regular pentagon
13. a. 9 **b.** An infinite number **c.** 5

14.



15. a. Two lines of symmetry and two rotation symmetries
b. Seven lines of symmetry and seven rotation symmetries
c. Three lines of symmetry and three rotation symmetries
d. Two rotation symmetries

16. 21

17. 171°

18. a. 11 **b.** 56

19. a. Three lines of symmetry and three rotation symmetries
b. Four lines of symmetry and four rotation symmetries
c. Two lines of symmetry and two rotation symmetries
d. Four lines of symmetry and four rotation symmetries

Exercises and Problems 10.1

1. a. 92 kilometers per hour **b.** 40 kilometers

3. a. Approximately $2\frac{1}{2}$ to $2\frac{2}{3}$ paper clips
b. Approximately 5 eraser heads

5. a. 202,500 grams
b. 202,500,000 milligrams
c. 446 pounds

7. a. $\frac{1}{32}$ inch **b.** $\frac{1}{4}$ inch
c. $\frac{11}{16}$ inch **d.** $6\frac{3}{16}$ inches
e. $8\frac{1}{2}$ inches

9. a. 8 pints **b.** 16 cups **c.** 1 cup

11. a. 4.2 yards **b.** 3.5 pounds **c.** 2.5 gallons

13. a. 200 centimeters **b.** 75 kilograms **c.** 48 liters

15. a. Most cubits are less than 52.5 centimeters
b. Length, 158 meters; breadth, 26 meters; height, 16 meters. Or length, 517 feet; breadth, 86 feet; height, 52 feet

17.

1 1	2 6	5		3 4		
	3	4 3	1	5	0	
5 1	2	0		0		6 4
				0		3
			7 1		8 9	2
		9 7	9	2	0	
10 5	5	5	0		0	

19. a. Approximately 450 grams
b. Approximately 60 to 90 feet for a step of 2 to 3 feet
c. Approximately 8 feet

21. a. 17 millimeters
b. $\frac{17}{200}$ millimeter = .085 millimeter
c. 85 micrometers **d.** 85

23. a. 111.5 to 112.5 kilograms
b. 38.15 to 38.25° C
c. 48.25 to 48.35 centimeters
d. 3.455 to 3.465 kilograms

25. 11.6 kilograms

27. a. Rounding to the nearest multiple of 10 gives $40 + 30 + 50 + 40 + 30 = 190$, and $\frac{8}{10}$ of $\$190 \approx 8 \times \$20 \approx \$160$
b. \$160.38

29. 320 days; 5 cents per day

31. a. 7.2 cubic centimeters **b.** 32 injections

33. a. 299,792,458 meters
b. 299,792.458 kilometers per second
c. Yes

35. 13 centimeters

Exercises and Problems 10.2

1. a. The width of these rectangles is greater than the width of the outstretched arms of the average adult, and the height of these rectangles is greater than the height of an average room.
b. 106 meters
c. 40 meters
d. 4240 square meters
e. 165,000 square feet, which to the nearest whole acre is 4 acres

3. a. 8 **b.** 6

5. Approximately 4 plastic fasteners

7. a. 27,878,400 square feet
b. 640 acres
c. 671,360 acres

9. a. 10 units **b.** 8 units

11. a. 10,000 ares **b.** 100 hectares

13. a. Area 1375 square millimeters = 13.75 square centimeters; perimeter 160 millimeters
b. Area 1800 square millimeters = 18 square centimeters; perimeter 183 millimeters

15. a. Area 2350 square millimeters = 23.50 square centimeters; circumference 172 millimeters
b. Area 1480 square millimeters = 14.80 square centimeters; perimeter 157 millimeters

17. a. Area 1536 square millimeters = 15.36 square centimeters; perimeter 215 millimeters
b. Area 1062 square millimeters = 10.62 square centimeters; perimeter 135 millimeters

19. 15.5 square centimeters
21. a. 140 square centimeters
b. 198 square centimeters
23. The circumferences are $2\pi \approx 6.3$; $4\pi \approx 12.6$; $8\pi \approx 25.1$; $16\pi \approx 50.3$. When the radius is doubled, the circumference is doubled. When the radius is tripled, the circumference is tripled.
25. Between 45 and 55 square centimeters
27. a. Square, 120 millimeters; circle, 120 millimeters
b. 246 square millimeters
29. a. 6.8×10^8 **b.** 10 directories
31. Type *A*
33. $8.28
35. Approximately 1428.6 square meters
37. a. 90,675 square centimeters **b.** Two rolls
39. a. $66,791.30 **b.** $4875.76
41. Approximately 31.8 centimeters $\times$ 31.8 centimeters
43. a. 982 square meters **b.** 125 meters
c. 111 meters **d.** 14 meters
e. 3080 square meters
45. 4 revolutions

Exercises and Problems 10.3

1. a. 166,762 cubic centimeters **b.** 1167 kilograms
3. a. Unit (i): 54 square units, 27 cubic units
Unit (ii): $13\frac{1}{2}$ square units, $3\frac{3}{8}$ cubic units
b. Unit (i): 68 square units, 30 cubic units
Unit (ii): 17 square units, $3\frac{3}{4}$ cubic units
5. a. 46,656 cubic inches
b. 1,000,000,000 cubic millimeters
7. a. Volume, 48 cubic centimeters; surface area, 96 square centimeters
b. Volume, 60 cubic centimeters; surface area, 124 square centimeters
9. a. Volume, 113 cubic centimeters; surface area, 113 square centimeters
b. Volume, 196 cubic centimeters; surface area, 196 square centimeters
11. a. Volume, 601 cubic centimeters; surface area, 478 square centimeters
b. Volume, 145 cubic centimeters; surface area, 219 square centimeters
13. a. 9.4 cubic centimeters
b. 12 cubic centimeters
15. a. 37.5 liters **b.** 1250 cubic centimeters
c. 12 **d.** 5750 square centimeters
17. 21,000 Btu
19. Type *B*
21. 10 gallons
23. a. 25 **b.** 5
25. a. 509 cubic meters **b.** 25.5 hours
27. a. 7.85 square meters **b.** 722.2 kilograms
29. 512 of the small cubes will be unpainted.

$10 \times 10 \times 10$ cube

No. of painted faces	3	2	1	0
No. of cubes	8	12×8	6×8^2	8^3

Here are the results for an n by n by n cube with $n \geq 2$.

No. of painted faces	3	2	1	0
No. of cubes	8	$12(n-2)$	$6(n-2)^2$	$(n-2)^3$

31. a. 2 **b.** 4
c. 2^{20} (more than 1 million)
33. a. 679 cubic inches **b.** 8.8 inches ($\sqrt[3]{679} \approx 8.8$)
35. a. 88 feet **b.** 2462 square feet
c. 88 square feet **d.** $\frac{1}{4}$ inch

Chapter 10 Test

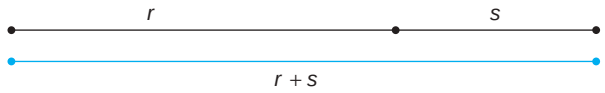
1. a. Gram **b.** Milliliter **c.** Meter
d. Kilogram **e.** Square meter **f.** Cubic meter
2. a. 42 **b.** 9 **c.** 13.6
d. 80 **e.** 67.5 **f.** 44
3. a. 1600 **b.** 470 **c.** 5200
d. 2.5 **e.** 160 **f.** 1,000,000
4. a. 1600 **b.** 0° **c.** 55
d. 2 **e.** $\frac{3}{5}$ **f.** 2.2
5. a. Minimum, 5.25 kilograms; maximum, 5.35 kilograms
b. Minimum, 84.5 grams; maximum, 85.5 grams
c. Minimum, 4.115 ounces; maximum, 4.125 ounces
6. Area using unit (i): 26 square units; area using unit (ii): 6.5 square units
7. a. 21 square centimeters; 22 centimeters
b. 168 square centimeters; 64 centimeters
c. 18 square centimeters; 20 centimeters
d. 12.5664 square centimeters; 12.5664 centimeters
8. 6.4 centimeters
9. a. 54 square centimeters
b. 16 square centimeters
10. Unit (i): 18 cubic units; 48 square units
Unit (ii): 2.25 cubic units; 12 square units
11. a. 27 cubic centimeters; 66 square centimeters
b. 54 cubic centimeters; 114 square centimeters
c. 27 cubic centimeters; 66 square centimeters
12. a. 1568 cubic centimeters
b. 4834.9 cubic centimeters
c. 1231.5 cubic centimeters
d. 3990 cubic centimeters
13. a. 1522 square centimeters
b. 1583.4 square centimeters
c. 2463.0 square centimeters
d. 896 square centimeters

14. 3.6 cubic yards
15. Type *A*
16. $670

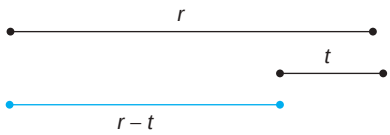
Exercises and Problems 11.1

1. a. $\angle T$ **b.** $\angle D$ **c.** $\angle R$
d. $\overline{DT}$ **e.** $\overline{TR}$ **f.** $\overline{DR}$
g. $\triangle TDR$

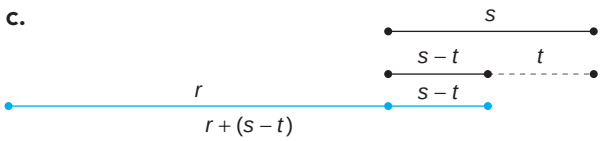
3. a.



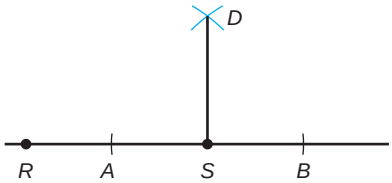
b.



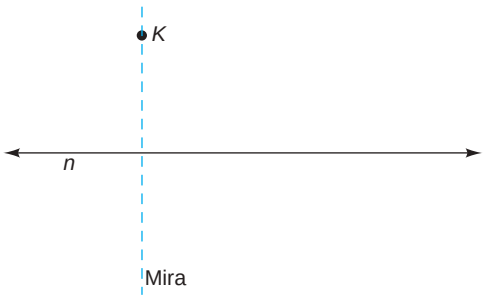
c.



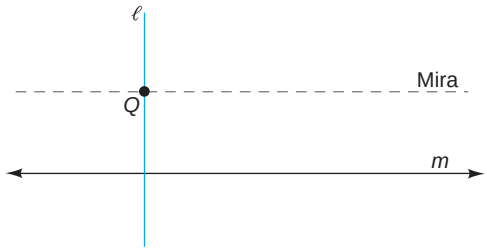
5. See the text of this section for the steps to construct an angle that is congruent to a given angle.
7. See steps for bisecting a line segment, pages 742 to 743.
9. See steps for constructing a perpendicular to a line through a given point, pages 745 to 746.
11. Extend $\overline{RS}$ to point *B*; use a compass and locate point *A* so that $AS = SB$; use a compass to draw arcs intersecting at point *D* so that *D* is equidistant from *A* and *B*; $\overline{DS} \perp \overline{RS}$.



13. Place the edge of the Mira on point *K* and across line *n* so that the reflection from one side of line *n* coincides with the other side of line *n*.

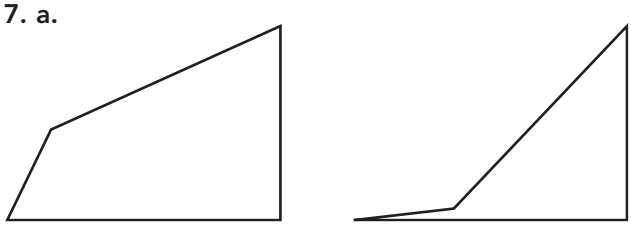


Use the Mira to draw a perpendicular line ℓ through point *Q* to line *m*. Then place the edge of the Mira on point *Q* so that it is parallel to line *m*. The edge of the Mira and line *m* will be parallel when the reflection of one side of line ℓ coincides with the other side of line ℓ.



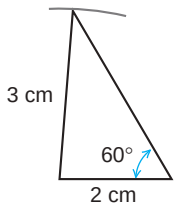
15. For both polygons, construct the perpendicular bisectors of any two sides of the polygon. Their intersection is the center of the circumscribed circle whose radius is the distance from the center to any vertex of the polygon.

17. a.

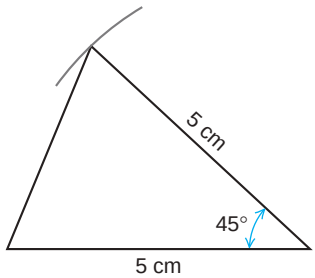


b. An infinite number of quadrilaterals can be constructed.

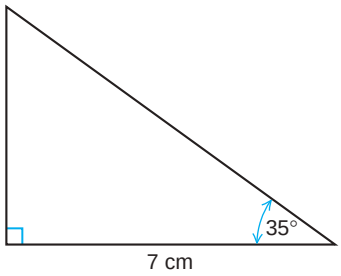
19. a.

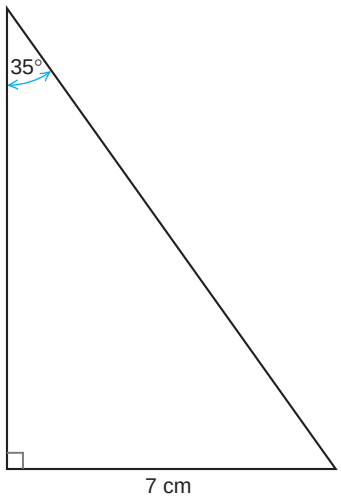


b.

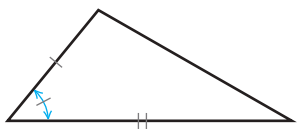


c.

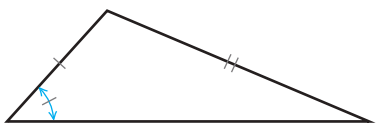




21. a.



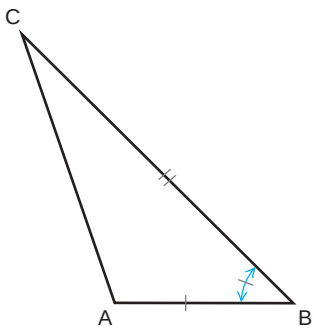
b. This is one of two possibilities:

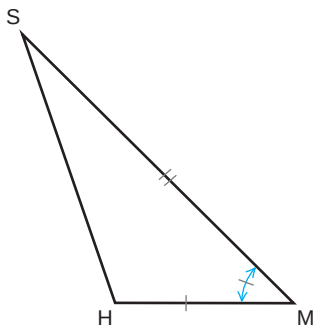


c. No, the triangles are not congruent.

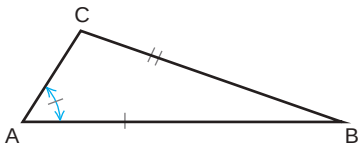
d. Two sides and one angle of a triangle may be congruent to two sides and one angle of a second triangle, and yet the triangles are not necessarily congruent.

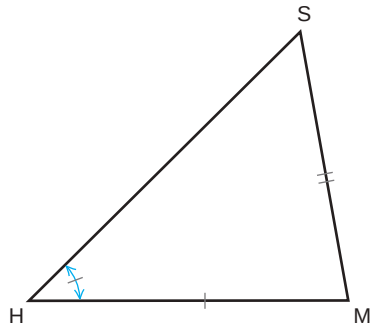
23. a. $\triangle ABC \cong \triangle HMS$ by the SAS congruence property.



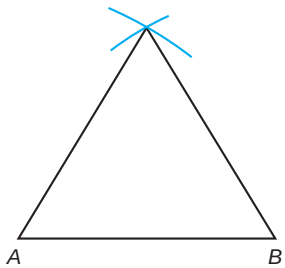


b. One possibility to show $\triangle ABC$ is not necessarily congruent to $\triangle HMS$.

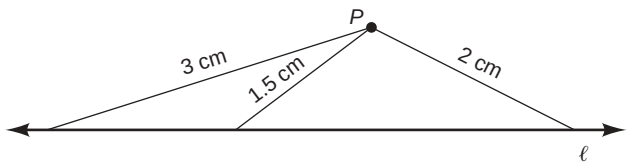




25. Construct $\overline{AB}$ and then with a compass open to span $\overline{AB}$, swing arcs from both A and B. The arcs intersect in the third vertex of the triangle.



27. a. One of two possible points on line ℓ, for each distance.



b. 1.1 centimeters

c. The shortest distance from a point to a line is the length of the perpendicular line segment from the line to the point.

29. a. The triangles are congruent by SAS.

b. The triangles are congruent by SSS.

31. a. The triangles are congruent by the SAS congruence property.

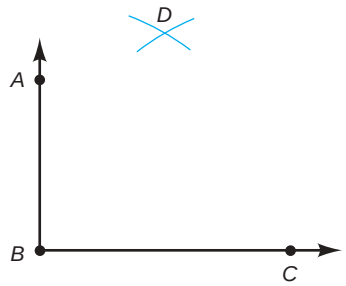
b. The triangles are not necessarily congruent because the congruent angles are not the included angles for the pairs of congruent sides.

33. a. In the answer to exercise 23b above, it was possible to form two noncongruent triangles with the given angle and two given sides in the same relative positions of the two triangles. However, in exercise 33a, it is not possible to form two noncongruent triangles with the given angle and the two given sides in the same relative positions. So, these triangles are congruent.

b. The triangles are not necessarily congruent.

35. Since $\overline{RT} \cong \overline{ST}$, $\overline{RK} \cong \overline{SK}$, and $\overline{KT}$ is common to both triangles, $\triangle RKT \cong \triangle SKT$ by the SSS congruence property. Since $\triangle RKT \cong \triangle SKT$, the triangles are congruent, and so $\measuredangle R \cong \measuredangle S$ by corresponding parts.

37. To trisect right $\measuredangle ABC$, open a compass to span $\overline{BC}$ and draw an arc with B as the center. With the same compass opening, draw an arc with C as the center and that intersects the first arc at point D. Then $\triangle BDC$ is an equilateral triangle, and $\measuredangle DBC$ has a measure of 60°. Bisecting this angle will provide a trisection of $\measuredangle ABC$.



39. Since $\overline{AB} \cong \overline{CB}$, $\measuredangle RAB$ and $\measuredangle DCB$ are right angles, and $\measuredangle RBA \cong \measuredangle DBC$ (because they are vertical angles), the triangles are congruent by the ASA congruence property. Thus, $\overline{CD}$ can be measured, and by corresponding parts, $\overline{CD} \cong \overline{AR}$.

41. 144 feet

Exercises and Problems 11.2

1. a. Answers will vary: Seven large metal rods coming down from the baskets to the wheels; one of the sets of seven small metal rods on the sides of the carts; many small rods forming the sides of the baskets.

b. Seven congruent right angles near the wheels. Seven obtuse angles just above the wheels. Angles in the triangular metal frames on the wheels.

c. Translation mapping

3. a., b. They are equal.

5. a. 90°

b. The distances are equal.

c. All points on ℓ

7. a. 90°

b., c. They are equal.

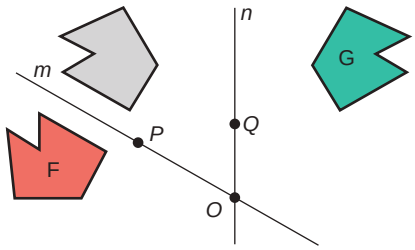
9. a. Right 6 and up 1

b. Quadrilateral $ABCD$

11. a. A translation 8 units horizontally to the left

b. The distance between any point and its image is 8 units, twice the distance between lines m and n.

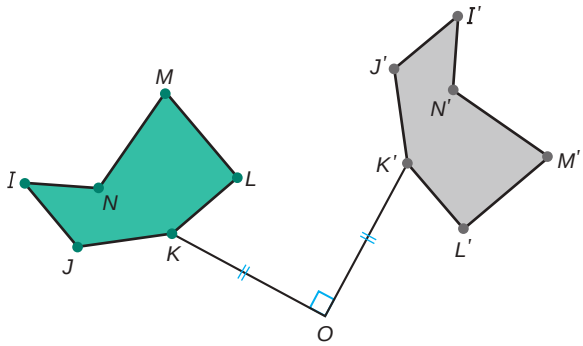
13.



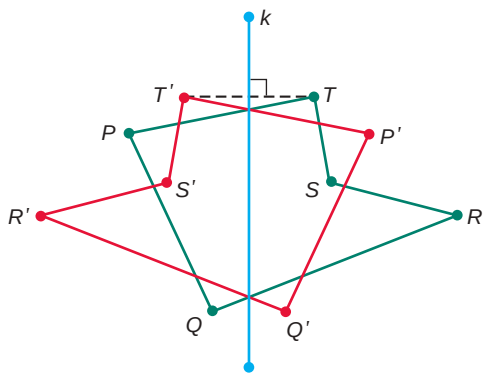
a. Hexagon G

b. The number of degrees in the angle of rotation is twice the number of degrees in $\measuredangle POQ$.

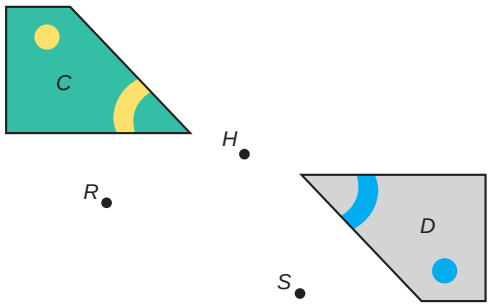
15. a. Draw $\overline{KO}$ and then construct a perpendicular to $\overline{KO}$ through point O. Use a compass to locate K' on the perpendicular so that $KO = OK'$. Then K' is the image of K. Obtain the images of the remaining five vertex points in a similar manner.



b. Construct the perpendicular through point T to line k. Use a compass to locate T' on the perpendicular so that the distance from T' to line k equals the distance from T to line k. Then T' is the image of T, and images of the remaining four points can be obtained in a similar manner as shown in the following diagram.

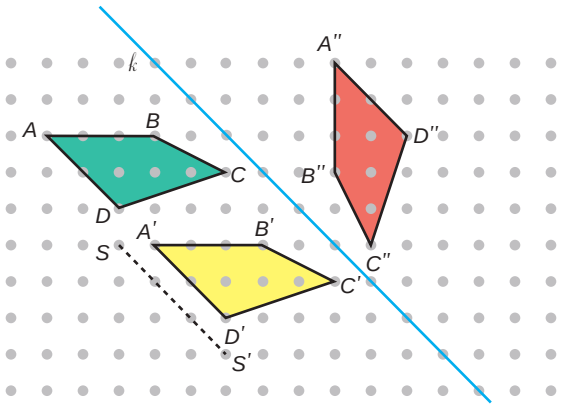


17. a.

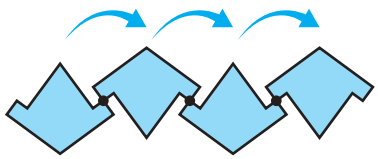


b. Figure C can be mapped to figure D by a 180° rotation about point H.

19. $A''B''C''D''E''$ is the final image. The order of composition does not matter.

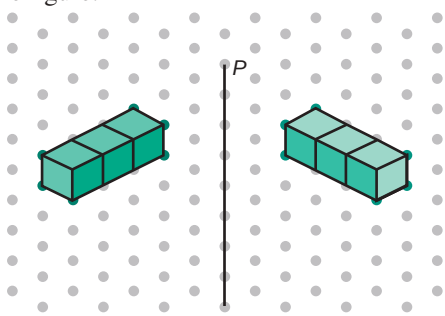


21. a. The centers of rotation are shown in the following figure.



b. Reflections, or 180° rotations about the lower point of the figure.

23.

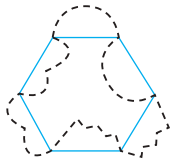


25. a. Four, because a rhombus has rotation symmetries of 180° and 360° and two reflection symmetries.

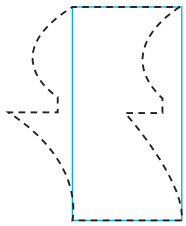
b. Six; three rotation symmetries and three reflection symmetries

c. Sixteen; eight rotation symmetries and eight reflection symmetries

27. Regular hexagon

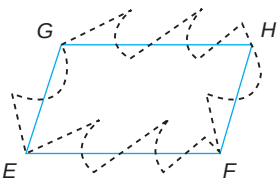


29. Rectangle

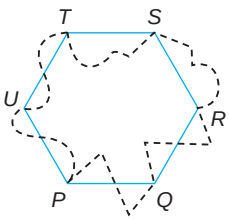


31. See reflection tessellations, pages 773–774.

33. Create a curve from E to F and translate it to side $\overline{GH}$. Then create a curve from F to H and translate it to side $\overline{EG}$. This figure can be used to produce a translation tessellation.



35. Create a curve from P to Q and rotate it about Q so that P maps to R. Similarly, create a curve from R to S and rotate it about S, and create a curve from T to U and rotate it about U. This figure can be used to produce a rotation tessellation.



37. a. Rotation about point O, which is the intersection of the perpendicular bisectors of two line segments whose endpoints are pairs of points on the figure and their images.

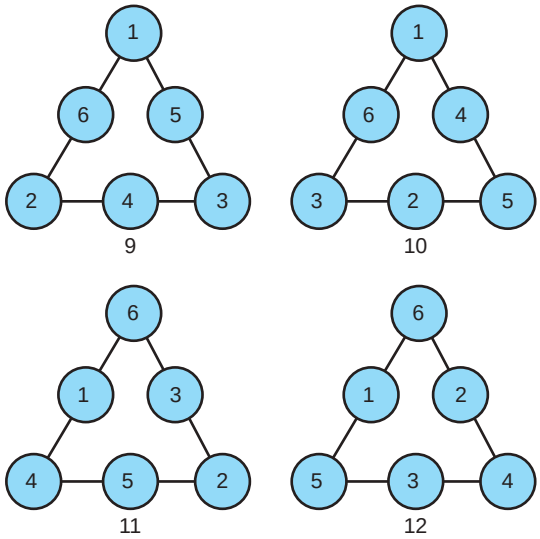
b. Rotation about point Q, which is the intersection of the perpendicular bisectors of $\overline{AA'}$ and $\overline{BB'}$.

39. a. D' $(3, -5)$, E' $(1, -2)$, F' $(3, -3)$, G' $(5, -1)$

b. D'' $(-3, -5)$, E'' $(-1, -2)$, F'' $(-3, -3)$, G'' $(-5, -1)$

c. A 180° rotation about the origin

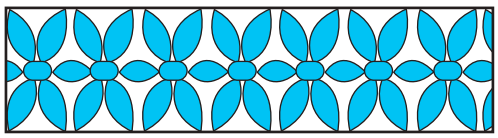
41. There are four solutions that cannot be obtained from each other by rotations or reflections. The sums along their sides are 9, 10, 11, and 12.



43. a. A translation

b. A translation, horizontal reflection, or glide reflection

45.



47. Eight, counting the given magic square: four from rotations of 90°, 180°, 270°, and 360°; one from a reflection about a vertical line; one from a reflection about a horizontal line; and two by reflections about the diagonals.

49. a. 3;



b.



c.

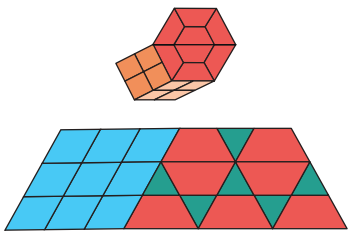


Exercises and Problems 11.3

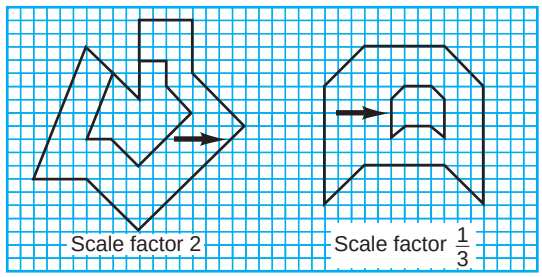
1. a. 2 **b.** $\frac{-1}{2}$

c.

d.



3.



5. For scale factors of 2, 3, and $\frac{1}{2}$, the point (2, 6) is mapped to (4, 12), (6, 18), and (1, 3), respectively. In general, for a scale factor of k, point (a, b) will be mapped to (ka, kb).

7. a. $RV = 6$, $US = 7.5$, $XW = 3$

b. $AD = 4.6$, $NP = 3$, $NO = 6.6$

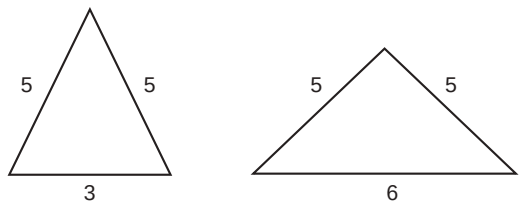
9. a. Similar by the AA similarity property: $\triangle ABC \sim \triangle EFD$

b. Since vertical angles are congruent, the two triangles are similar by the AA similarity property: $\triangle HIG \sim \triangle KIJ$.

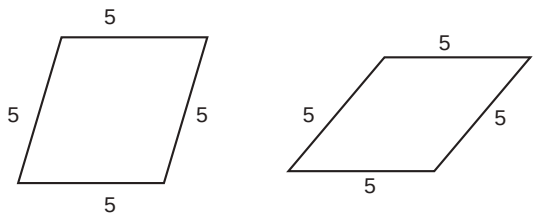
c. Not necessarily similar

11. a. Similar (corresponding sides proportional and all angles 90°)

b. Not necessarily similar. For example:

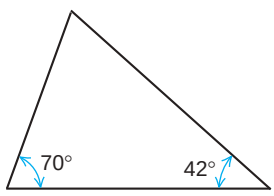


c. Not necessarily similar. For example:

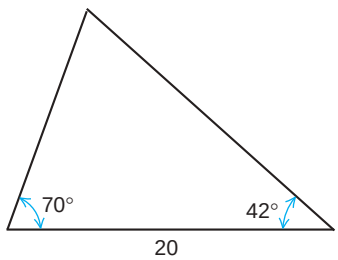


d. Similar (corresponding sides proportional and all angles 135°)

13. a.



b.



c. Yes

d. If two angles of one triangle are congruent to two angles of another triangle, the two triangles are similar.

15. $\triangle ABC$ is similar to $\triangle AEF$. Therefore, $\frac{AB}{AE} = \frac{BC}{EF} = \frac{AC}{AF}$. Also, $\triangle ACD$ is similar to $\triangle AFG$, so $\frac{AD}{AG} = \frac{DC}{GF} = \frac{AC}{AF}$. All angles in both rectangles equal 90°. Therefore, the corresponding sides of *ABCD* and *AEFG* are proportional, and their corresponding angles are equal.

17. a. Since there are 180° in every triangle, the third angles of these triangles are also congruent. Thus, the triangles are similar by the AA similarity property.

b. 14 meters

19. a. 9 **b.** $\frac{1}{4}$

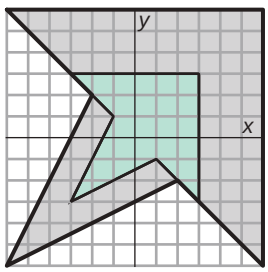
21. a. 4 square units **b.** 144 square units

23.

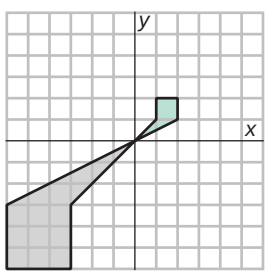
Scale Factor	Surface Area (square units)	Volume (cubic units)
1	26	7
2	104	56
3	234	189
4	416	448
5	650	875

25. 3

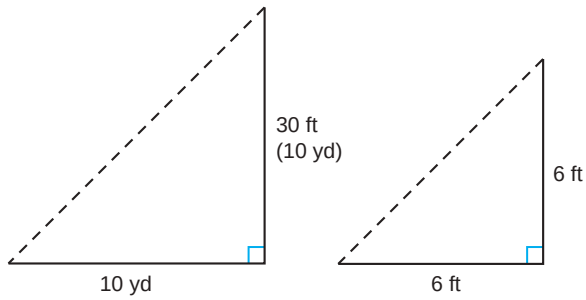
27. Scale factor = 2 (Image includes total figure.)



29. Scale factor = $^{-}3$



31. a. Since the ratio of the height of her friend to the length of her friend's shadow is $\frac{6}{6} = 1$, the ratio of the height of the goal post to its shadow is also 1. Thus, the goal posts are 10 yards or 30 feet in height.



b. $13\frac{1}{3}$ yards

33. a. 225

b. 3375

c. 30 centimeters

35. If the scale factor is 2, all images of the pentagon will be congruent pentagons, regardless of where the projection point is located.

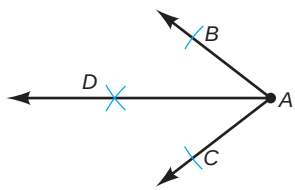
37. a. The original sheet is similar to the quarto with a scale factor of $\frac{1}{2}$, because the two adjacent edges of the original sheet were both folded in half to obtain the quarto; and the folio is similar to the octavo with a scale factor of $\frac{1}{2}$ because two adjacent edges of the folio were both folded in half.

b. Every other rectangle will be similar to the original sheet; and beginning with the folio, every other rectangle will be similar to the folio.

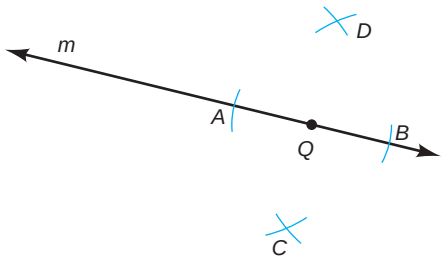
39. 2.3 times greater

Chapter 11 Test

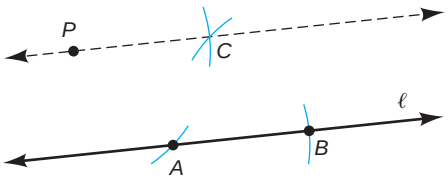
1. a. Using any compass opening and *A* as the center, draw arcs intersecting the sides of $\measuredangle A$ in points *B* and *C*. With the same compass opening and points *B* and *C* as centers, draw intersecting arcs at point *D*. Line segment $\overline{DA}$ is the bisector.



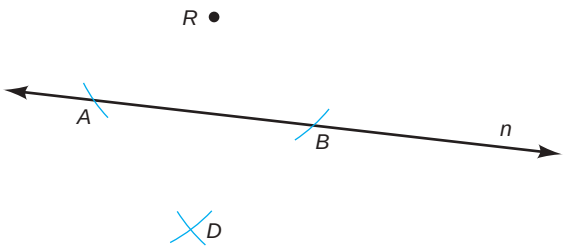
b. Using the same compass opening and point Q as center, locate points A and B on line m so that $AQ = BQ$. With the same compass opening and A and B as centers, draw arcs intersecting at point D in one half-plane and point C in the other half-plane. Then $\overline{DC} \perp m$.



c. Using the same compass opening throughout, draw arcs so that $PA = AB = BC = PC$, as shown in the following figure. $\overleftrightarrow{PC} \parallel \ell$.

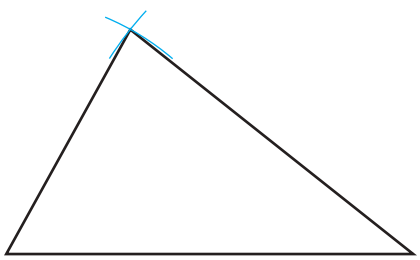


d. With R as center, draw arcs intersecting line n so that $RA = RB$. With A and B as centers, draw arcs intersecting at D so that $AD = BD$. Line segment $\overline{RD} \perp n$.



2. Construct the perpendicular bisectors of $\overline{BA}$ and $\overline{AC}$. The intersection of these bisectors is the center of the circle whose radius is the distance from the center to any vertex of the triangle.

3. a.



b. A triangle cannot be constructed with the given segments because the sum of the lengths of the two shorter segments is less than the length of the third segment.

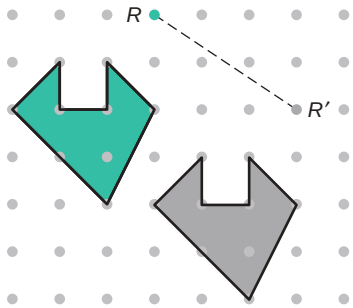
4. a. $\triangle ABC \cong \triangle DEC$ by the ASA congruence property.

b. $\triangle FGI \cong \triangle FHI$ by the SAS congruence property.

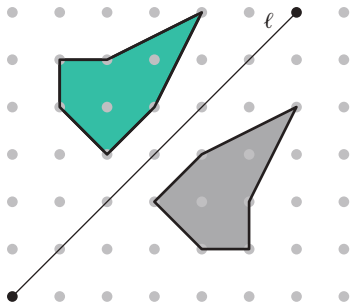
c. Since the two congruent angles are not included between the pairs of congruent sides, the triangles are not necessarily congruent.

d. $\triangle RQT \cong \triangle TSR$ by the SSS congruence property.

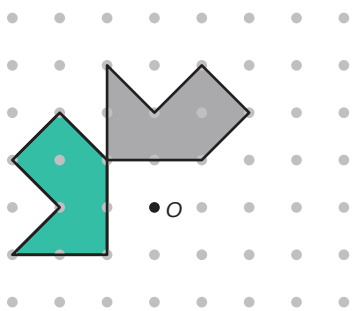
5. a.



b.



c.



6. a. A counterclockwise rotation of 25°

b. A translation twice the distance between lines ℓ and m

c. A translation of P to the right 17 units and down 2 units

7. The number of congruence mappings of each figure onto itself is its total number of symmetries.

a. 2; An isosceles triangle has one line of symmetry and one rotation symmetry.

b. 12; A regular hexagon has 6 lines of symmetry and 6 rotation symmetries.

c. 4; A rectangle has 2 lines of symmetry and 2 rotation symmetries.

8. a. Alter one side of the triangle by drawing a curve from one vertex to another. Then rotate this curve about either vertex to the adjacent side. Mark to midpoint on the third side, and alter this side between the midpoint and vertex by drawing a new curve. Then rotate this curve about the midpoint to the other half of this side.

b. Alter one side of the parallelogram by drawing a curve and then translate this curve to the opposite side. Then alter one of the remaining sides of the parallelogram by a curve and translate this curve to the opposite side.

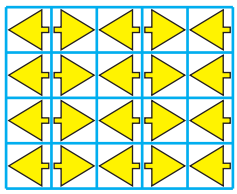
9. a. Translation or reflection about a vertical line

b. Translation or a 180° rotation

c. Translation or a glide reflection

d. Translation, 180° rotation, reflections about horizontal or vertical lines, or glide reflection

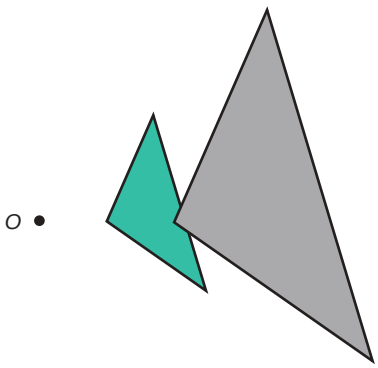
10.



11. One possibility for the grid shown below:
Rows: Reflect about the right edge of the small square.
Columns: Reflect about the lower edge of the small square.



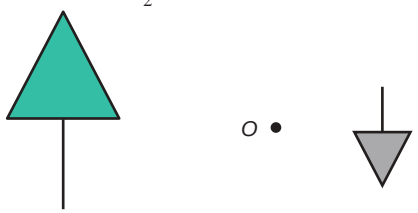
12. a. Scale factor of 2



b. Scale factor of $\frac{1}{3}$



c. Scale factor of $\frac{-1}{2}$



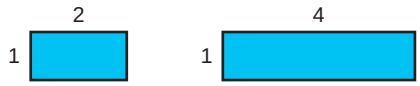
13. a. $\triangle BDC \sim \triangle AEC$ by the AA similarity property.

b. $\triangle FGH \sim \triangle IJH$ by the AA similarity property.

c. These triangles are not necessarily similar.

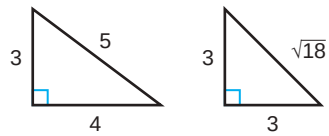
d. $\triangle RST \sim \triangle WUV$ by the SSS similarity property.

14. a. Not necessarily similar because sides need not be proportional.



b. Similar because corresponding sides are proportional and corresponding angles are congruent.

c. Not necessarily similar because sides need not be proportional, and not all pairs of corresponding angles need be congruent.



d. Similar because corresponding sides are proportional and corresponding angles are congruent.

e. Similar because corresponding sides have a proportion of 1 and corresponding angles are congruent

15. a. 324 cubic units **b.** 288 square units
c. 1.5 cubic units **d.** 8 square units

16. a. 7 inches **b.** $\frac{1}{2}$ quart **c.** 48 pounds

17. 27 feet

18. a. The triangles are not necessarily congruent and the information in part b shows they are not congruent. The triangles are similar.

b. 500 feet

Credits

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National Council of Teachers of Mathematics
Table of Standards and Expectations

PROCESS STANDARDS

Problem Solving

Instructional programs from prekindergarten through grade 12 should enable all students to—

- Build new mathematical knowledge through problem solving
- Solve problems that arise in mathematics and in other contexts
- Apply and adapt a variety of appropriate strategies to solve problems
- Monitor and reflect on the process of mathematical problem solving

Communication

Instructional programs from prekindergarten through grade 12 should enable all students to—

- Organize and consolidate their mathematical thinking through communication
- Communicate their mathematical thinking coherently and clearly to peers, teachers, and others
- Analyze and evaluate the mathematical thinking and strategies of others
- Use the language of mathematics to express mathematical ideas precisely

Representation

Instructional programs from prekindergarten through grade 12 should enable all students to—

- Create and use representations to organize, record, and communicate mathematical ideas
- Select, apply, and translate among mathematical representations to solve problems
- Use representations to model and interpret physical, social, and mathematical phenomena

Reasoning and Proof

Instructional programs from prekindergarten through grade 12 should enable all students to—

- Recognize reasoning and proof as fundamental aspects of mathematics
- Make and investigate mathematical conjectures
- Develop and evaluate mathematical arguments and proofs
- Select and use various types of reasoning and methods of proof

Connections

Instructional programs from prekindergarten through grade 12 should enable all students to—

- Recognize and use connections among mathematical ideas
- Understand how mathematical ideas interconnect and build on one another to produce a coherent whole
- Recognize and apply mathematics in contexts outside of mathematics

CONTENT STANDARDS for Grades Pre-K–2

Data Analysis and Probability

In prekindergarten through grade 2 all students should—

Formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them and should—

- Pose questions and gather data about themselves and their surroundings;
- Sort and classify objects according to their attributes and organize data about their objects;
- Represent data using concrete objects, pictures, and graphs

Select and use appropriate statistical methods to analyze data and should—

- Describe parts of the data and the set of data as a whole to determine what the data show

Develop and evaluate inferences and predictions that are based on data and should—

- Discuss events related to student's experiences as likely or unlikely

Number and Operations

In prekindergarten through grade 2 all students should—

Understand numbers, ways of representing numbers, relationships among numbers, and number systems and should—

- Count with understanding and recognize "how many" in sets of objects;
- Use multiple models to develop initial understandings of place value and the base-ten number system;
- Develop understanding of the relative position and magnitude of whole numbers and of ordinal and cardinal numbers and their connections;
- Develop a sense of whole numbers and represent and use them in flexible ways, including relating, composing, and decomposing numbers;
- Connect number words and numerals to the quantities they represent, using various physical models and representations;
- Understand and represent commonly used fractions, such as ¼, ⅓, and ½

Understand meanings of operations and how they relate to one another and should—

- Understand various meanings of addition and subtraction of whole numbers and the relationship between the two operations;
- Understand the effects of adding and subtracting whole numbers;
- Understand situations that entail multiplication and division, such as equal groupings of objects and sharing equally

Compute fluently and make reasonable estimates and should—

- Develop and use strategies for whole-number computations, with a focus on addition and subtraction;
- Develop fluency with basic number combinations for addition and subtraction;
- Use a variety of methods and tools to compute, including objects, mental computation, estimation, paper and pencil, and calculators

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Algebra

In prekindergarten through grade 2 all students should—

Understand patterns, relations, and functions and should—

- Sort, classify, and order objects by size, number, and other properties;
- Recognize, describe, and extend patterns such as sequences of sounds and shapes or simple numeric patterns and translate from one representation to another;
- Analyze how both repeating and growing patterns are generated

Represent and analyze mathematical situations and structures using algebraic symbols and should—

- Illustrate general principles and properties of operations, such as commutativity, using specific numbers;
- Use concrete, pictorial, and verbal representations to develop an understanding of invented and conventional symbolic notations

Use mathematical models to represent and understand quantitative relationships and should—

- Model situations that involve the addition and subtraction of whole numbers, using objects, pictures, and symbols

Analyze change in various contexts and should—

- Describe qualitative change, such as a student's growing taller;
- Describe quantitative change, such as a student's growing two inches in one year

Geometry

In prekindergarten through grade 2 all students should—

Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships and should—

- Recognize, name, build, draw, compare, and sort two- and three-dimensional shapes;
- Describe attributes and parts of two- and three-dimensional shapes;
- Investigate and predict the results of putting together and taking apart two- and three-dimensional shapes

Specify locations and describe spatial relationships using coordinate geometry and other representational systems and should—

- Describe, name, and interpret relative positions in space and apply ideas about relative position;
- Describe, name, and interpret direction and distance in navigating space and apply ideas about direction and distance;
- Find and name locations with simple relationships such as "near to" and in coordinate systems such as maps

Apply transformations and use symmetry to analyze mathematical situations and should—

- Recognize and apply slides, flips, and turns;
- Recognize and create shapes that have symmetry

Use visualization, spatial reasoning, and geometric modeling to solve problems and should—

- Create mental images of geometric shapes using spatial memory and spatial visualization;
- Recognize and represent shapes from different perspectives;
- Relate ideas in geometry to ideas in number and measurement;
- Recognize geometric shapes and structures in the environment and specify their location

Measurement

In prekindergarten through grade 2 all students should—

Understand measurable attributes of objects and the units, systems, and processes of measurement and should—

- Recognize the attributes of length, volume, weight, area, and time;
- Compare and order objects according to these attributes;
- Understand how to measure using the nonstandard and standard units;
- Select an appropriate unit and tool for the attribute being measured

Apply appropriate techniques, tools, and formulas to determine measurements and should—

- Measure with multiple copies of units of the same size, such as paper clips laid end to end;
- Use repetition of a single unit to measure something larger than the unit, for instance, measuring the length of a room with a single meterstick;
- Use tools to measure;
- Develop common referents for measures to make comparisons and estimates

CONTENT STANDARDS for Grades 3–5

Number and Operations

In grades 3–5 all students should—

Understand numbers, ways of representing numbers, relationships among numbers, and number systems and should—

- Understand the place-value structure of the base-ten number system and be able to represent and compare whole numbers and decimals;
- Recognize equivalent representations for the same number and generate them by decomposing and composing numbers;
- Develop understanding of fractions as parts of unit wholes, as parts of a collection, as locations on number lines, and as divisions of whole numbers;
- Use models, benchmarks, and equivalent forms to judge the size of fractions;
- Recognize and generate equivalent forms of commonly used fractions, decimals, and percents;
- Explore numbers less than 0 by extending the number line and through familiar applications;
- Describe classes of numbers according to characteristics such as the nature of their factors

Understand meanings of operations and how they relate to one another and should—

- Understand various meanings of multiplication and division;
- Understand the effects of multiplying and dividing whole numbers;
- Identify and use relationships between operations, such as division and the inverse of multiplication, to solve problems;
- Understand and use properties of operations, such as the distributivity of multiplication over addition

Compute fluently and make reasonable estimates and should—

- Develop fluency with basic number combinations for multiplication and division and use these combinations to mentally compute related problems, such as 30×50;
- Develop fluency in adding, subtracting, multiplying, and dividing whole numbers;
- Develop and use strategies to estimate the results of whole-number computations and to judge the reasonableness of such results
- Develop and use strategies to estimate computations involving fractions and decimals in situations relevant to students;
- Use visual models, benchmarks, and equivalent forms to add and subtract commonly used fractions and decimals;
- Select appropriate methods and tools for computing with whole numbers from among mental computation, estimation, calculators, and paper and pencil according to the context and nature of the computation and use the selected method or tool

Algebra

In grades 3–5 all students should—

Understand patterns, relations, and functions and should—

- Describe, extend, and make generalizations about geometric and numeric patterns;
- Represent and analyze patterns and functions, using words, tables, and graphs

Represent and analyze mathematical situations and structures using algebraic symbols and should—

- Identify such properties as commutativity, associativity, and distributivity and use them to compute with whole numbers;
- Represent the idea of a variable as an unknown quantity using a letter or a symbol;
- Express mathematical relationships using equations

Use mathematical models to represent and understand quantitative relationships and should—

- Model problem situations with objects and use representations such as graphs, tables, and equations to draw conclusions

Analyze change in various contexts and should—

- Investigate how a change in one variable relates to a change in a second variable;
- Identify and describe situations with constant or varying rates of change and compare them

Geometry

In grades 3–5 all students should—

Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships and should—

- Identify, compare, and analyze attributes of two- and three-dimensional shapes and develop vocabulary to describe the attributes;
- Classify two- and three-dimensional shapes according to their properties and develop definitions of classes of shapes such as triangles and pyramids;
- Investigate, describe, and reason about the results of subdividing, combining, and transforming shapes;
- Explore congruence and similarity;
- Make and test conjectures about geometric properties and relationships and develop logical arguments to justify conclusions

Specify locations and describe spatial relationships using coordinate geometry and other representational systems and should—

- Describe location and movement using common language and geometric vocabulary;
- Make and use coordinate systems to specify locations and to describe paths;
- Find the distance between points along horizontal and vertical lines of a coordinate system

Apply transformations and use symmetry to analyze mathematical situations and should—

- Predict and describe the results of sliding, flipping, and turning two-dimensional shapes;
- Describe a motion or a series of motions that will show that two shapes are congruent;
- Identify and describe line and rotational symmetry in two- and three-dimensional shapes and designs

Use visualization, spatial reasoning, and geometric modeling to solve problems and should—

- Build and draw geometric objects;
- Create and describe mental images of objects, patterns, and paths;
- Identify and build a three-dimensional object from a two-dimensional representation of that object;
- Identify and build a two-dimensional representation of a three-dimensional object;
- Use geometric models to solve problems in other areas of mathematics, such as number and measurement;
- Recognize geometric ideas and relationships and apply them to other disciplines and to problems that arise in the classroom or in everyday life

Measurement

In grades 3–5 all students should—

Understand measurable attributes of objects and the units, systems, and processes of measurement and should—

- Understand such attributes as length, area, weight, volume, and size of angle and select the appropriate type of unit for measuring each attribute;
- Understand the need for measuring with standard units and become familiar with standard units in the customary and metric systems;
- Carry out simple unit conversions, such as from centimeters to meters, within a system of measurement;
- Understand that measurements are approximations and understand how differences in units affect precision;
- Explore what happens to measurements of a two-dimensional shape such as its perimeter and area when the shape is changed in some way

Apply appropriate techniques, tools, and formulas to determine measurements and should—

- Develop strategies for estimating the perimeters, areas, and volumes of irregular shapes;
- Select and apply appropriate standard units and tools to measure length, area, volume, weight, time, temperature, and the size of angles;
- Select and use benchmarks to estimate measurements;
- Develop, understand, and use formulas to find the area of rectangles and related triangles and parallelograms;
- Develop strategies to determine the surface areas and volumes of rectangular solids

Data Analysis and Probability

In grades 3–5 all students should—

Formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them and should—

- Design investigations to address a question and consider how data-collection methods affect the nature of the data set;
- Collect data using observations, surveys, and experiments;
- Represent data using tables and graphs such as line plots, bar graphs, and line graphs;
- Recognize the difference in representing categorical and numerical data

Select and use appropriate statistical methods to analyze data and should—

- Describe the shape and important features of a set of data and compare related data sets, with an emphasis on how the data are distributed;
- Use measures of center, focusing on the median, and understand what each does and does not indicate about the data set;
- Compare different representations of the same data and evaluate how well each representation shows important aspects of the data

Develop and evaluate inferences and predictions that are based on data and should—

- Propose and justify conclusions and predictions that are based on data and design studies to further investigate the conclusions or predictions

Understand and apply basic concepts of probability and should—

- Describe events as likely or unlikely and discuss the degree of likelihood using such words as certain, equally likely, and impossible;
- Predict the probability of outcomes of simple experiments and test the predictions;
- Understand that the measure of the likelihood of an event can be represented by a number from 0 to 1